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Research article

Global exponential stability of pseudo almost automorphic solutions of octonion-valued stochastic high-order Hopfield neural networks with delays

Yuwei Cao and Yongkun Li*

Department of Mathematics, Yunnan University, Kunming, Yunnan 650091, China

* **Correspondence:** Email: yklie@ynu.edu.cn.

Abstract: Octonion-valued neural networks (OVNNs) provide a powerful framework for modeling and processing high-dimensional data due to their eight-dimensional normed division algebraic structure. However, their inherent non-commutativity and non-associativity, coupled with ubiquitous time delays and stochastic disturbances in real-world systems, make the analysis of their dynamical behavior a significant challenge. This paper focuses on the complex oscillatory dynamics, specifically pseudo almost automorphy, within a class of stochastic higher-order Hopfield neural networks (NNs) based on octonions. To adequately characterize the stochastic processes involved, a novel concept of pseudo almost automorphic stochastic processes in finite-dimensional distributions is first proposed. Subsequently, by fixed point theorems and employing inequality techniques, sufficient criteria are established for the existence and global exponential stability of pseudo almost automorphic solutions in finite-dimensional distributions for the considered octonion-valued stochastic higher-order Hopfield NNs with time-varying delays. The obtained results are not only new for the octonionic system, but also remain novel even when the system degenerates to its real-valued counterpart. Furthermore, the analytical framework developed herein offers a general methodology applicable to studying pseudo almost automorphic dynamics in other types of complex-valued NNs.

Keywords: octonion-valued neural networks; stochastic neural networks; pseudo almost automorphic solutions in finite-dimensional distributions; global exponential stability

1. Introduction

As an eight-dimensional normed division algebra, octonions possess unique algebraic properties of non-commutativity and non-associativity. Their extended dimensionality and algebraic structure provide new mathematical tools for high-dimensional data processing and complex system modeling. Introducing octonions into the field of NNs gives rise to octonion-valued NNs (OVNNs) [1], which not only break through the dimensional limitations of traditional real-valued, complex-valued, and quaternion-valued NNs, but also offer a new paradigm for solving high-dimensional complex tasks. An in-depth exploration of their dynamical behavior is a core prerequisite for ensuring stable network operation and expanding application boundaries.

Traditional real-valued, complex-valued, and quaternion-valued NNs have inherent limitations when handling ultra-high-dimensional data. Complex-valued and quaternion-valued NNs struggle to handle eight-dimensional and higher-dimensional data such as hyperspectral images and seven-degree-of-freedom physical systems [2]. In contrast, octonions, as an eight-dimensional number system, can directly model high-dimensional signals through their rich algebraic structure while capturing complex correlations among components [2]. For example, in signal processing tasks such as hyperspectral fluorescence data fusion and salient object detection, octonions can integrate multi-dimensional features, enhancing the completeness and efficiency of data processing [2, 3]. Moreover, compared to lower-dimensional NNs, OVNNs have greater storage capacity and information-carrying capability [3], giving them a natural advantage in high-dimensional feature recognition and representation tasks. They can effectively reduce information loss and computational redundancy caused by decomposing high-dimensional data [3].

The high-dimensional modeling capability of OVNNs demonstrates unique value in multiple engineering fields. In robot control, octonion-valued feedforward NNs can directly generate control inputs for robotic manipulators, enabling trajectory tracking of the end-effector in three-dimensional space. Their learning performance is comparable to that of quaternion-valued NNs, showing potential for practical engineering applications [4]. In secure communication, chaos masking techniques based on the fixed-time synchronization properties of OVNNs can encrypt and decrypt information signals, enhancing communication security [3].

Recently, various dynamical behaviors of OVNNs have been continuously revealed. For example, reference [5] studied the Hopf bifurcation problem of fractional-order OVNNs, reference [3, 6] investigated the finite-time synchronization of OVNNs, reference [7] examined the existence and stability of almost automorphic solutions for stochastic fuzzy OVNNs with delays, and reference [8] studied Weyl almost periodic synchronization of hypercomplex-valued NNs that include OVNNs as a special case. Reference [2] investigated the existence of (μ_1, μ_2)-pseudo almost periodic solutions and μ-stability for a class of OVNNs with mixed delays. However, there are currently no results on the existence of pseudo almost automorphic solutions for stochastic OVNNs. As we know, the dynamical behavior of OVNNs is crucial for realizing their engineering applications. The coupling of non-commutative, non-associative algebraic properties with practical factors such as delays and impulses makes dynamical analysis a key challenge and focus in OVNN research.

On the one hand, almost automorphy is a generalization of almost periodicity, describing oscillation phenomena more complex than almost periodic oscillations. Pseudo-almost automorphy further generalizes almost automorphy. There has been extensive research on pseudo almost automorphic dynamics in deterministic systems [9–15]. Meanwhile, real-world systems are inevitably subject to numerous random disturbances, which are important causes of oscillation and instability [16–18]. In recent years, the study of dynamical behavior in stochastic NNs has become a hotspot in the field of NN research [19–23]. As noted in [24], for stochastic differential equations, it is more reasonable to study almost automorphic solutions in the distribution sense. However, results on pseudo almost automorphic dynamics in the distribution sense for stochastic are still very scarce. It is well known that the one-dimensional distribution of a stochastic process can only very limitedly characterize the process, whereas the finite-dimensional distributions of a stochastic process can completely determine it. Therefore, studying pseudo almost automorphic oscillations in the finite-dimensional distributions of stochastic NNs is of great significance.

On the other hand, higher-order Hopfield NNs achieve a qualitative leap by introducing higher-order interactions between neurons into the energy function compared to classical models that only support pairwise connections. Their core advantages include an exponential increase in storage capacity, more accurate modeling of complex constraints, significant reduction of spurious attractors, and stronger representation capabilities for complex patterns. The main application scenarios include associative memory systems requiring high-density storage and precise retrieval (such as image or semantic pattern recovery) and solving combinatorial optimization problems with higher-order dependencies among variables. Moreover, their theoretical framework provides an important perspective for understanding the memory dynamics of modern attention mechanisms and Transformer models.

Inspired by the above observations and considering the fact that time delays are unavoidable in real-world situations, the main purpose of this paper is to propose a concept of pseudo almost automorphic stochastic processes in the finite-dimensional sense. Taking a class of octonion-valued stochastic higher-order Hopfield neural networks as an example, we establish the existence and stability of their pseudo almost automorphic solutions in finite-dimensional distributions.

The main contributions of this paper can be summarized as follows:

(1) A definition of pseudo almost automorphic stochastic processes in finite-dimensional distributions is proposed.

(2) For octonion-valued stochastic high-order Hopfield NNs with time-varying delays, the existence and stability of pseudo-almost automorphic solutions in finite-dimensional distributions are established. This shows that even under simultaneous random disturbances and time-delay effects, the solutions can retain regular restorative oscillations and dynamic stability. Mathematically, this confirms the model's robustness, reliability, and behavioral predictability. Together, these properties provide a critical foundation for advancing the theoretical model toward practical engineering applications.

(3) The results of this paper are novel even when the considered system degenerates to a real-valued system.

(4) The methods presented in this paper can be used to study the pseudo almost automorphic dynamics of other types of NNs.

The remainder of this paper is structured as follows: Section 2 introduces some definitions, lemmas, and provides the model description. Section 3 studies the existence and the global exponential stability of pseudo almost automorphic solutions in finite-dimensional distributions for octonion-valued stochastic higher-order Hopfield NNs with time-varying delays. Section 4 provides an example to illustrate the validity of the theoretical results. Section 5 gives a brief conclusion.

2. Model description and preliminaries

The algebra of octonions is defined as

$$\mathbb{O} = \left\{ x = \sum_{l=0}^{7} x^l e_l \middle| x^l \in \mathbb{R}, l = 0, 1, 2, \ldots, 7 \right\},$$

where e_l represent the octonion units, $0 \le l \le 7$ and satisfy the following multiplication table:

Table 1. Multiplication table of octonions.

$\times$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

From Table 1, it is easy to see that the multiplication of $\mathbb{O}$ satisfies neither the commutative law nor the associative law.

The conjugate of an octonion x is defined as $\bar{x} = x^0 e_0 - \sum\limits_{l=1}^{7} x^l e_l$ and its norm is defined by $\|x\|_{\mathbb{O}} = \sqrt{x\bar{x}} = \sqrt{\bar{x}x} = \sqrt{\sum\limits_{l=0}^{7}(x^l)^2}$. Thus, $(\mathbb{O}, \|\cdot\|_{\mathbb{O}})$ forms a Banach space.

The addition of octonions is expressed as $x + y = \sum\limits_{l=0}^{7}(x^l + y^l)e_l$, where $x = \sum\limits_{l=0}^{7} x^l e_l$ and $y = \sum\limits_{l=0}^{7} y^l e_l$. Scalar multiplication is defined by $ax = \sum\limits_{l=0}^{7}(ax^l)e_l$. An octonion-valued function $z(t) \in \mathbb{O}$ can be written as $z(t) = \sum\limits_{l=0}^{7} z^l(t)e_l$, where each $z^l(t)$ is a real-valued function. The derivative of $z(t)$ is given by $\dot{z}(t) = \sum\limits_{l=0}^{7} \dot{z}^l(t)e_l$. For $x = \sum\limits_{l=0}^{7} x^l e_l$, we denote $x^c = \sum\limits_{l \neq 0}^{7} x^l e_l$ and set $x^\phi = x - x^c$. For a vector $x \in \mathbb{O}^n$, its norm is defined as $\|x\|_0 = \max_{1 \le i \le n}\{\|x_i\|_{\mathbb{O}}\}$. Then $(\mathbb{O}^n, \|\cdot\|_0)$ is also a Banach space.

Readers interested in octonion algebra may refer to [25].

The model considered in this paper is the following octonion-valued stochastic high-order Hopfield NN with time-varying delays:

$$\begin{aligned}
\mathrm{d}x_i(t) =& \Big[- a_i(t)x_i(t) + \sum_{j=1}^{n} b_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{n} c_{ij}(t)g_j(x_j(t - \eta_{ij}(t))) \\
&+ \sum_{j=1}^{n}\sum_{l=1}^{n} Q_{ijl}(t)r_j(x_j(t - \theta_{ijl}(t)))r_l(x_l(t - \xi_{ijl}(t))) + I_i(t)\Big]\mathrm{d}t \\
&+ \sum_{j=1}^{n} h_{ij}(t)\delta_{ij}(x_j(t - v_{ij}(t)))\mathrm{d}w_j(t), i = 1, 2, \cdots, n, \qquad (2.1)
\end{aligned}$$

where $x_i = x_i^0 e_0 + x_i^1 e_1 + x_i^2 e_2 + x_i^3 e_3 + x_i^4 e_4 + x_i^5 e_5 + x_i^6 e_6 + x_i^7 e_7 \in \mathbb{O}$ denotes the state of the ith unit at time t; $\mathbb{O}$ is an octonion-algebra; $a_i(t) \in \mathbb{O}$ represents the self-feedback coefficient at time t; $b_{ij}(t)$,

$c_{ij}(t)$, $Q_{ijl}(t)$, and $h_{ij}(t) \in \mathbb{O}$ are the connection weights; f_j, g_j, r_j, $r_l : \mathbb{O} \to \mathbb{O}$ stand for the activation functions; $I_i(t) \in \mathbb{O}$ signifies the external input to the ith unit; η_{ij}, θ_{ijl}, ξ_{ijl}, $v_{ij} \in \mathbb{R}^+$ are the transmission delays; $w(t) = (w_1(t), w_2(t), \cdots, w_n(t))$ denotes a two sided n-dimensional Brownian motion specified later; and $\delta_{ij} : \mathbb{O} \to \mathbb{O}$ is a Borel measurable function.

Remark 2.1. *In system* (2.1)*, take* $e_0 = 1$*. If* $x^l = 0$ *for all* $l = 1, 2, \ldots, 7$*, then system* (2.1) *degenerates into a real-valued system; if* $x^l = 0$ *for all* $l = 2, 3, \ldots, 7$*, then system* (2.1) *degenerates into a complex-valued system; and, if* $x^l = 0$ *for all* $l = 3, 4, \ldots, 7$*, then system* (2.1) *degenerates into a quaternion-valued system. In a word, system* (2.1) *contains real-valued, complex-valued, and quatenion-valued systems as its special cases.*

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Let $\{B^+(t)\}_{t\geq 0}$ and $\{B^-(t)\}_{t\geq 0}$ be two independent n-dimensional standard Brownian motions defined on this space, each satisfying $B^+(0) = B^-(0) = 0$. The two-sided n-dimensional Brownian motion $\{\omega(t)\}_{t\in\mathbb{R}}$ is defined as follows:

$$\omega(t) := \begin{cases} B^+(t), & t \geq 0, \\ B^-(-t), & t < 0. \end{cases}$$

We write $\omega(t) = (\omega_1(t), \omega_2(t), \ldots, \omega_n(t))$, where the components are mutually independent. Define the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ by

$$\mathcal{F}_t := \bigcap_{u>t} \sigma(\omega(s) : s \leq u) \vee \mathcal{N},$$

where $\sigma(\omega(s) : s \leq u)$ denotes the σ-algebra generated by all random vectors $\{\omega(s) : s \leq u\}$, $\mathcal{N} := \{N \subset \Omega : \exists A \in \mathcal{F} \text{ with } N \subset A \text{ and } P(A) = 0\}$ is the collection of P-null sets, $\mathcal{G}_1 \vee \mathcal{G}_2$ denotes the smallest σ-algebra containing both $\mathcal{G}_1$ and $\mathcal{G}_2$, and $\bigcap_{u>t}$ ensures right-continuity of the filtration.

Then, $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ is a complete and right-continuous filtration (i.e., it satisfies the *usual conditions*), and $\{\omega(t), \mathcal{F}_t\}_{t\in\mathbb{R}}$ is a two-sided n-dimensional Brownian motion with the following properties:

(a) $\omega(0) = 0$ almost surely;
(b) For any $s < t$, the increment $\omega(t) - \omega(s)$ is independent of $\mathcal{F}_s$ and follows an n-dimensional normal distribution $N(0, |t-s| I_n)$, where I_n denotes the $n \times n$ identity matrix;
(c) The sample paths are almost surely continuous.

For any $a, b, c \in \mathbb{O}$, it is stipulated throughout this paper that $abc = (ab)c$. For convenience, the following notations are introduced.

$$\underline{a}_i^{\phi} = \inf_{t\in\mathbb{R}} a_i^{\phi}(t), \quad \bar{a}_i^{c} = \sup_{t\in\mathbb{R}} \|a_i^c(t)\|_{\mathbb{O}}, \quad \check{b}_{ij} = \sup_{t\in\mathbb{R}} \|b_{ij}(t)\|_{\mathbb{O}}, \quad \check{c}_{ij} = \sup_{t\in\mathbb{R}} \|c_{ij}(t)\|_{\mathbb{O}},$$

$$\check{Q}_{ijl} = \sup_{t\in\mathbb{R}} \|Q_{ijl}(t)\|_{\mathbb{O}}, \quad \check{h}_{ij} = \sup_{t\in\mathbb{R}} \|h_{ij}(t)\|_{\mathbb{O}}, \quad \eta_{ij}^+ = \sup_{t\in\mathbb{R}} \eta_{ij}(t), \quad \theta_{ijl}^+ = \sup_{t\in\mathbb{R}} \theta_{ijl}(t),$$

$$\xi_{ijl}^+ = \sup_{t\in\mathbb{R}} \xi_{ijl}(t), \quad v_{ij}^+ = \sup_{t\in\mathbb{R}} v_{ij}(t), \quad I_i^+ = \sup_{t\in\mathbb{R}} \|I_i(t)\|_{\mathbb{O}}.$$

We denote by $BC_{\mathcal{F}0}([-\rho, 0], \mathbb{O}^n)$ the set of all bounded, $\mathcal{F}_0$-measurable random variables taking values in $C([-\rho, 0], \mathbb{O}^n)$, where $\rho = \max\limits_{1\leq i,j,l\leq n} \{\eta_{ij}^+, \theta_{ijl}^+, \xi_{ijl}^+, v_{ij}^+\}$.

System (2.1) is supplemented with initial values given by

$$x_i(s) = \psi_i(s) \in \mathbb{O}, \ \ s \in [-\rho, 0], \tag{2.2}$$

where $\psi_i \in BC_{\mathcal{F}_0}([-\rho, 0], \mathbb{O})$.

Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space. We denote by $BC(\mathbb{R}, \mathbb{X})$ the collection of all bounded and continuous functions from $\mathbb{R} \to \mathbb{X}$. For a random variable $X : (\Omega, \mathcal{F}, P) \to \mathbb{X}$, we signify by $\mu(X) := P \circ X^{-1}$ its law and by EX its expectation. Let $\mathcal{L}^2(\Omega, \mathbb{X})$ be the space of all $\mathbb{X}$-valued random variables such that $E\|X\|^2 = \int_\Omega \|X\|^2 \mathrm{d}P < \infty$.

Definition 2.1. *[26] A function $g \in BC(\mathbb{R}, \mathbb{O}^n)$ is termed almost automorphic if, for every sequence of real numbers $(s'_k)_{k\in\mathbb{N}}$, one can find a subsequence $(s_k)_{k\in\mathbb{N}}$ such that*

$$\tilde{g}(t) = \lim_{k\to\infty} g(t + s_k)$$

for each $t \in \mathbb{R}$, and

$$\lim_{k\to\infty} \tilde{g}(t - s_k) = g(t)$$

for every $t \in \mathbb{R}$. The set of all such functions is denoted by $AA(\mathbb{R}, \mathbb{O}^n)$.

Let $PAA_0(\mathbb{R}, \mathbb{O}^n) = \Big\{g \in BC(\mathbb{R}, \mathbb{O}^n) | \lim\limits_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \|g(t)\|_0 \mathrm{d}t = 0\Big\}$.

Definition 2.2. *[27] A function $g \in BC(\mathbb{R}, \mathbb{O}^n)$ is said to be pseudo almost automorphic if it can be expressed as $g = g^1 + g^0$, where $g^1 \in AA(\mathbb{R}, \mathbb{O}^n)$ and $g^0 \in PAA_0(\mathbb{R}, \mathbb{O}^n)$. The collection of all such functions will be denoted by $PAA(\mathbb{R}, \mathbb{O}^n)$.*

Remark 2.2. *[26] Let $g \in BC(\mathbb{R}, \mathbb{O}^n)$ be almost automorphic. Then, $\tilde{g}$ is bounded and* $\sup_{t\in\mathbb{R}} \|\tilde{g}\|_{\mathbb{O}^n} = \sup_{t\in\mathbb{R}} \|g\|_{\mathbb{O}^n}$*, where $\tilde{g}$ is mentioned in Definition 2.1.*

Definition 2.3. *[28] A stochastic process $X : \mathbb{R} \to \mathcal{L}^2(\Omega, \mathbb{O}^n)$ is said to be $\mathcal{L}^2$-continuous if for any $t_0 \in \mathbb{R}$,*

$$\lim_{t\to t_0} E\|X(t) - X(t_0)\|_0^2 = 0.$$

It is said to be $\mathcal{L}^2$-bounded if $\sup_{t\in\mathbb{R}} E\|X(t)\|_0^2 < \infty$.

Definition 2.4. *[29] A stochastic process $X : \mathbb{R} \to \mathcal{L}^2(\Omega, \mathbb{O}^n)$ is said to be square-mean almost automorphic if for every sequence of real numbers $(s'_k)_{k\in\mathbb{N}}$, there exists a subsequence $(s_k)_{k\in\mathbb{N}}$ such that*

$$\lim_{k\to\infty} E\|X(t + s_k) - \tilde{X}(t)\|_0^2 = 0$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{k\to\infty} E\|\tilde{X}(t - s_k) - X(t)\|_0^2 = 0$$

for each $t \in \mathbb{R}$. The family of all such functions is denoted by $SAA(\mathbb{R}, \mathcal{L}^2(\Omega, \mathbb{O}^n)$.

Set $SPAA_0(\mathbb{R}, \mathcal{L}^2(\Omega, \mathbb{O}^n)) = \Big\{X \in BC(\mathbb{R}, \mathcal{L}^2(\Omega, \mathbb{O}^n)) | \lim\limits_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} E\|X(t)\|_0^2 \mathrm{d}t = 0\Big\}$.

Definition 2.5. *[29] A stochastic process $X : \mathbb{R} \to \mathcal{L}^2(\Omega, \mathbb{O}^n)$ is said to be square-mean pseudo almost automorphic if it can be decomposed as $X = X^1 + X^0$, where $X^1 \in SAA(\mathbb{R}, \mathcal{L}^2(\Omega, \mathbb{O}^n))$ and $X^0 \in SPAA_0(\mathbb{R}, \mathcal{L}^2(\Omega, \mathbb{O}^n))$. Denote by $SPAA(\mathbb{R}, \mathcal{L}^2(\Omega, \mathbb{O}^n)$ the set of all such functions.*

Let $\mathbb{E}$ be a separable Banach space. For any fixed positive integer number m, let $\mathcal{B}(\mathbb{E}^m)$ be the σ-algebra of Borel sets of $\mathbb{E}^m$ and $\mathcal{P}(\mathbb{E}^m)$ the set of all probability measures on $\mathcal{B}(\mathbb{E}^m)$. We denote by $C_b(\mathbb{E}^m, \mathbb{R})$ the family of all bounded and continuous functions $g : \mathbb{E}^m \to \mathbb{R}$ with $\|g\|_\infty := \sup_{x\in\mathbb{E}^m}\{|g(x)|\} < \infty$.

For any fixed positive integer number m, let $\mathcal{B}(\mathbb{O}^m)$ be the σ-algebra of Borel sets of $\mathbb{O}^m$ and $\mathcal{P}(\mathbb{O}^m)$ the set of all probability measures on $\mathcal{B}(\mathbb{O}^m)$. We denote by $C_b(\mathbb{O}^m, \mathbb{R})$ the family of all bounded and continuous functions $g : \mathbb{O}^m \to \mathbb{R}$ with $\|g\|_\infty := \sup_{x\in\mathbb{O}^m}\{|g(x)|\} < \infty$.

For $g \in C_b(\mathbb{E}^m, \mathbb{R})$ and $\alpha, \beta \in \mathcal{P}(\mathbb{E}^m)$, we define

$$\|g\|_L = \sup_{u\neq v} \frac{|g(u) - g(v)|}{d(u, v)}, \ \|g\|_{BL} = \max\{\|g\|_\infty, \|g\|_L\}, \ d_{BL}(\alpha,\beta) = \sup_{\|g\|_{BL}\leq 1} \left| \int_{\mathbb{E}^m} gd(\alpha - \beta)\right|.$$

Then, $(\mathbb{E}^m, d_{BL})$ is a Polish space [30].

Definition 2.6. *[31] A stochastic process $X : \mathbb{R} \to \mathcal{L}^2(\Omega, \mathbb{O}^n)$ is said to be almost automorphic in finite-dimensional distributions if for all finite points $t_1, t_2, \cdots, t_m \in \mathbb{R}$ and every sequence of real numbers $(s'_k)_{k\in\mathbb{N}}$, there exists a subsequence $(s_k)_{k\in\mathbb{N}}$ such that*

$$\lim_{k\to\infty} d_{BL}(M_X(t + s_k), M_X(t)) = 0$$

for each $t \in \mathbb{R}$ and

$$\lim_{k\to\infty} d_{BL}(\tilde{M}_X(t - s_k), M_X(t)) = 0$$

for every $t \in \mathbb{R}$, where M_X and $\tilde{M}_X : \mathbb{R} \to (\mathcal{P}((\mathbb{O}^n)^m), d_{BL}(\cdot, \cdot))$ are defined by $M_X(t) = \mu(X(t + t_1), X(t + t_2), \cdots, X(t + t_m))$ and $\tilde{M}_X = \mu(\tilde{X}(t + t_1), \tilde{X}(t + t_2), \cdots, \tilde{X}(t + t_m))$, respectively. The collection of all such functions will be denoted by $DAA(\mathbb{R}, \mathcal{L}^2(\Omega, \mathbb{O}^n))$.

Lemma 2.1. *[32] Assume that $X : \mathbb{R} \to \mathcal{L}^2(\Omega, \mathbb{O}^n)$ is an $\mathcal{L}^2$-bounded stochastic process. Then, for all finite numbers $t_1, t_2, \ldots, t_m \in \mathbb{R}$ and each $\tau \in \mathbb{R}$, it holds that*

$$d^p_{BL}(M_X(t + \tau), M_X(t)) \leq m \sum_{i=1}^{m} E\|X(t_i + t + \tau) - X(t_i + t)\|^2_{\mathbb{X}},$$

in which $M_X(t)$ is mentioned in Definition 2.6.

An $\mathcal{L}^2$-bounded stochastic process belongs to $DPAA_0(\mathbb{R}, \mathcal{L}^2(\Omega, \mathbb{O}^n))$ if and only if

$$\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} d_{BL}(\mu(X(t)), \delta_0)dt = 0,$$

where δ_0 is the distribution of random variable $\mathbf{0}$.

Lemma 2.2. *[33] If $X \in SPAA_0(\mathbb{R}, \mathcal{L}^2(\Omega, \mathbb{O}^n))$, then $X \in DPAA_0(\mathbb{R}, \mathcal{L}^2(\Omega, \mathbb{O}^n))$.*

The definition of pseudo almost automorphic stochastic processes in finite-dimensional distributions introduced in this paper is as follows.

Definition 2.7. *A stochastic process $X : \mathbb{R} \to \mathcal{L}^2(\Omega, \mathbb{O}^n)$ is said to be pseudo almost automorphic in finite-dimensional distributions if it can be expressed as*

$$X = X^1 + X^0,$$

where $X^1 \in DAA(\mathbb{R}, \mathcal{L}^2(\Omega, \mathbb{O}^n))$ and $X^0 \in DPAA_0(\mathbb{R}, \mathcal{L}^2(\Omega, \mathbb{O}^n))$. The collection of all such functions will be denoted by $DPAA(\mathbb{R}, \mathcal{L}^2(\Omega, \mathbb{O}^n)$.

Lemma 2.3. *[31] Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying, for all $t \in \mathbb{R}$,*

$$0 \le \varphi(t) \le a(t) + b \int_{-\infty}^{t} e^{-c(t-s)} \varphi(s) \mathrm{d}s,$$

where $a : \mathbb{R} \to \mathbb{R}$ is locally integrable, b and c are positive constants with $c > b$, and the integral on the right-hand side is assumed to be convergent. Then, for any $d \in (0, c - b]$ such that $\int_{-\infty}^{0} e^{ds} a(s) \mathrm{d}s < \infty$, it holds that

$$\varphi(t) \le a(t) + b \int_{-\infty}^{t} e^{-d(t-s)} a(s) \mathrm{d}s. \tag{2.3}$$

The assumptions used in this paper are as follows:

(A_1) For $i, j, l = 1, 2, \cdots, n$, $a_i^{\phi} \in AA(\mathbb{R}, \mathbb{R}^+)$ with $\underline{a}_i^{\phi} > 0$, $a_i^c \in AA(\mathbb{R}, \mathbb{O}), b_{ij}, c_{ij}, Q_{ijl}, h_{ij}, I_i \in PAA(\mathbb{R}, \mathbb{O})$, i.e., $b_{ij} = b_{ij}^1 + b_{ij}^0$, $c_{ij} = c_{ij}^1 + c_{ij}^0$, $Q_{ijl} = Q_{ijl}^1 + Q_{ijl}^0$, $h_{ij} = h_{ij}^1 + h_{ij}^0$, where b_{ij}^1, c_{ij}^1, Q_{ijl}^1, $h_{ij}^1 \in AA(\mathbb{R}, \mathbb{O})$, b_{ij}^0, c_{ij}^0, Q_{ijl}^0, $h_{ij}^0 \in PAA_0(\mathbb{R}, \mathbb{O})$; η_{ij}, θ_{ijl}, ξ_{ijl}, $v_{ij} \in AA(\mathbb{R}, \mathbb{R}^+) \cap C^1(\mathbb{R}, \mathbb{R})$, there exist constants $\dot{\eta}_{ij}^+$, $\dot{\theta}_{ijl}^+$, $\dot{\xi}_{ijl}^+$, $\dot{v}_{ij}^+$ such that $\dot{\eta}_{ij} \le \dot{\eta}_{ij}^+ < 1$, $\dot{\theta}_{ijl} \le \dot{\theta}_{ijl}^+ < 1$, $\dot{\xi}_{ijl} \le \dot{\xi}_{ijl}^+ < 1$, $\dot{v}_{ij} \le \dot{v}_{ij}^+ < 1$.

(A_2) For $j = 1, 2, \cdots, n$, f_j, g_j, r_j, $\delta_{ij} \in C(\mathbb{O}, \mathbb{O})$, and there exist positive constants $L_j^f, L_j^g, L_j^r, L_{ij}^{\delta}$, and M_j^r such that for all $x, y \in \mathbb{O}$,

$$\|f_j(x) - f_j(y)\|_{\mathbb{O}} \le L_j^f \|x - y\|_{\mathbb{O}}, \quad \|g_j(x) - g_j(y)\|_{\mathbb{O}} \le L_j^g \|x - y\|_{\mathbb{O}},$$

$$\|r_j(x) - r_j(y)\|_{\mathbb{O}} \le L_j^r \|x - y\|_{\mathbb{O}}, \quad \|\delta_{ij}(x) - \delta_{ij}(y)\|_{\mathbb{O}} \le L_{ij}^{\delta} \|x - y\|_{\mathbb{O}}, \quad \|r_j(x)\|_{\mathbb{O}} \le M_j^r;$$

moreover, $f_j(0) = g_j(0) = r_j(0) = \delta_{ij}(0) = 0$.

(A_3) The positive constant

$$\begin{aligned} R :=& 5 \max_{1 \le i \le n} \Bigg\{ \frac{1}{(\underline{a}_i^{\phi})^2} \Bigg[(\bar{a}_i^c)^2 + \sum_{j=1}^{n} (\check{b}_{ij})^2 \sum_{j=1}^{n} (L_j^f)^2 + \sum_{j=1}^{n} (\check{c}_{ij})^2 \sum_{j=1}^{n} (L_j^g)^2 \\ &+ \sum_{j=1}^{n} \Bigg(\sum_{l=1}^{n} (\check{Q}_{ijl})^2 \sum_{l=1}^{n} (M_j^r)^2 \Bigg) \sum_{j=1}^{n} (L_j^r)^2 + \frac{\underline{a}_i^{\phi}}{2} \sum_{j=1}^{n} (\check{h}_{ij})^2 \sum_{j=1}^{n} (L_{ij}^{\delta})^2 \Bigg] \Bigg\} < \frac{1}{4}. \end{aligned}$$

(A_4) The positive constant

$$\begin{aligned} R' :=& 5 \max_{1 \le i \le n} \Bigg\{ \frac{1}{(\underline{a}_i^{\phi})^2} \Bigg[(\bar{a}_i^c)^2 + \sum_{j=1}^{n} (\check{b}_{ij}^1)^2 \sum_{j=1}^{n} (L_j^f)^2 + \sum_{j=1}^{n} (\check{c}_{ij}^1)^2 \sum_{j=1}^{n} (L_j^g)^2 \\ &+ \sum_{j=1}^{n} \Bigg(\sum_{l=1}^{n} (\check{Q}_{ijl}^1)^2 \sum_{l=1}^{n} (M_j^r)^2 \Bigg) \sum_{j=1}^{n} (L_j^r)^2 + \frac{\underline{a}_i^{\phi}}{2} \sum_{j=1}^{n} (\check{h}_{ij}^1)^2 \sum_{j=1}^{n} (L_{ij}^{\delta})^2 \Bigg] \Bigg\} < \frac{1}{4}, \end{aligned}$$

where $b^1_{ij}, c^1_{ij}, Q^1_{ijl}, h^1_{ij}$ are defined in (H_1), $\check{b}^1_{ij} = \sup_{t\in\mathbb{R}} \|b^1_{ij}(t)\|_{\mathbb{O}}$, $\check{c}^1_{ij} = \sup_{t\in\mathbb{R}} \|c^1_{ij}(t)\|_{\mathbb{O}}$, $\check{Q}^1_{ijl} = \sup_{t\in\mathbb{R}} \|Q^1_{ijl}(t)\|_{\mathbb{O}}$, $\check{h}^1_{ij} = \sup_{t\in\mathbb{R}} \|h^1_{ij}(t)\|_{\mathbb{O}}$.

(A_5) The positive constant

$$\begin{aligned}\Upsilon := \max_{1\le i\le n}\Big\{\frac{17}{\underline{a}^\phi_i}\Big[(\bar{a}^c_i)^2 + \sum_{j=1}^n (\check{b}^1_{ij})^2 \sum_{j=1}^n (L^f_j)^2 + \sum_{j=1}^n (\check{c}^1_{ij})^2 \sum_{j=1}^n (L^g_j)^2 \frac{e^{\underline{a}^\phi_i \eta^+_{ij}}}{1-\dot{\eta}^+_{ij}} \\ + 4n \sum_{j=1}^n \Big[\sum_{l=1}^n (\check{Q}^1_{ijl})^2 \sum_{l=1}^n (M^r_j L^r_l)^2\Big] \frac{e^{\underline{a}^\phi_i \xi^+_{ijl}}}{1-\dot{\xi}^+_{ijl}} + 4n \sum_{j=1}^n \Big[\sum_{l=1}^n (\check{Q}^1_{ijl})^2 \sum_{l=1}^n (M^r_l L^r_j)^2\Big] \\ \times \frac{e^{\underline{a}^\phi_i \theta^+_{ijl}}}{1-\dot{\theta}^+_{ijl}} + 2\underline{a}^\phi_i \sum_{j=1}^n (\check{h}^1_{ij})^2 \sum_{j=1}^n (L^\delta_{ij})^2 \frac{e^{2\underline{a}^\phi_i v^+_{ij}}}{1-\dot{v}^+_{ij}}\Big]\Big\} < \min_{1\le i\le n}\{\underline{a}^\phi_i\},\end{aligned}$$

where $\check{b}^1_{ij}, \check{c}^1_{ij}, \check{Q}^1_{ijl}, \check{h}^1_{ij}$ are the same as those defined in (A_4).

3. Pseudo almost automorphic solutions in finite-dimensional distributions

Let $\mathbb{H}$ denote the set of $\mathcal{L}^2$-bounded and $\mathcal{L}^2$-uniformly continuous functions from $\mathbb{R}$ to $\mathcal{L}^2(\Omega, \mathbb{O}^n)$. For $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_n) \in \mathbb{H}$, define the norm $\|\varphi\|_{\mathbb{H}} = \big(\sup_{t\in\mathbb{R}}(E\|\varphi(t)\|_0^2)\big)^{\frac{1}{2}}$. Then, $(\mathbb{H}, \|\cdot\|_{\mathbb{H}})$ is a Banach space.

Let

$$\begin{aligned}\varphi_0 =& \Big(\int_{-\infty}^t e^{-\int_s^t a^\phi_i(u)\mathrm{d}u} I_1(s)\mathrm{d}s, \int_{-\infty}^t e^{-\int_s^t a^\phi_i(u)\mathrm{d}u} I_2(s)\mathrm{d}s, \cdots, \int_{-\infty}^t e^{-\int_s^t a^\phi_i(u)\mathrm{d}u} I_n(s)\mathrm{d}s\Big), \\ \varphi^1_0 =& \Big(\int_{-\infty}^t e^{-\int_s^t a^\phi_i(u)\mathrm{d}u} I^1_1(s)\mathrm{d}s, \int_{-\infty}^t e^{-\int_s^t a^\phi_i(u)\mathrm{d}u} I^1_2(s)\mathrm{d}s, \cdots, \int_{-\infty}^t e^{-\int_s^t a^\phi_i(u)\mathrm{d}u} I^1_n(s)\mathrm{d}s\Big)\end{aligned}$$

and take a constant $K \ge \max\{\|\varphi_0\|_{\mathbb{H}}, \|\varphi^1_0\|_{\mathbb{H}}\}$.

Definition 3.1. *An $\mathcal{F}_t$-progressively measurable stochastic process is called a mild solution of system (2.1), if $x(t)$ satisfies the following stochastic integral equation:*

$$\begin{aligned}x_i(t) =& x_i(t_0)e^{-\int_{t_0}^t a^\phi_i(u)\mathrm{d}u} + \int_{t_0}^t e^{-\int_s^t a^\phi_i(u)\mathrm{d}u}\Big[-a^c_i(s)x_i(s) + \sum_{j=1}^n b_{ij}(s)f_j(x_j(s)) \\ &+ \sum_{j=1}^n c_{ij}(s)g_j(x_j(s-\eta_{ij}(s))) + \sum_{j=1}^n\sum_{l=1}^n Q_{ijl}(s)r_j(x_j(s-\theta_{ijl}(s)))r_l(x_l(s-\xi_{ijl}(s))) \\ &+ I_i(s)\Big]\mathrm{d}s + \int_{t_0}^t e^{-\int_s^t a^\phi_i(u)\mathrm{d}u}\sum_{j=1}^n h_{ij}(s)\delta_{ij}(x_j(s-v_{ij}(s)))\mathrm{d}w_j(s). \qquad (3.1)\end{aligned}$$

Theorem 3.1. *If conditions (A_1)–(A_5) are satisfied, then system (2.1) has a unique pseudo almost automorphic solution in finite-dimensional distributions. More precisely, this solution belongs to the closed ball $\mathbb{H}_0 = \{\varphi \in \mathbb{H} \mid \|\varphi - \varphi_0\|_{\mathbb{H}} \le K\}$.*

Proof. According to Definition 3.1, taking the limit as $t_0 \to -\infty$, we obtain

$$\begin{aligned} x_i(t) = &\int_{-\infty}^{t} e^{-\int_s^t a_i^{\phi}(u)\mathrm{d}u}\Big[-a_i^c(s)x_i(s) + \sum_{j=1}^{n} b_{ij}(s)f_j(x_j(s)) + \sum_{j=1}^{n} c_{ij}(s)g_j(x_j(s-\eta_{ij}(s))) \\ &+ \sum_{j=1}^{n}\sum_{l=1}^{n} Q_{ijl}(s)r_j(x_j(s-\theta_{ijl}(s)))r_l(x_l(s-\xi_{ijl}(s))) + I_i(s)\Big]\mathrm{d}s \\ &+ \int_{-\infty}^{t} e^{-\int_s^t a_i^{\phi}(u)\mathrm{d}u} \sum_{j=1}^{n} h_{ij}(s)\delta_{ij}(x_j(s-v_{ij}(s)))\mathrm{d}w_j(s), \end{aligned} \tag{3.2}$$

which is also a mild solution of system (2.1).

Define an operator $\Gamma : \mathbb{H}_0 \to \mathbb{H}_0$ by $\Gamma\varphi = (\Gamma_1\varphi, \cdots, \Gamma_n\varphi)$, where for $t \in \mathbb{R}$, $i = 1, \cdots, n$,

$$\begin{aligned} (\Gamma_i\varphi)(t) = &\int_{-\infty}^{t} e^{-\int_s^t a_i^{\phi}(u)\mathrm{d}u}\Big[-a_i^c(s)\varphi_i(s) + \sum_{j=1}^{n} b_{ij}(s)f_j(\varphi_j(s)) + \sum_{j=1}^{n} c_{ij}(s)g_j(\varphi_j(s-\eta_{ij}(s))) \\ &+ \sum_{j=1}^{n}\sum_{l=1}^{n} Q_{ijl}(s)r_j(\varphi_j(s-\theta_{ijl}(s)))r_l(\varphi_l(s-\xi_{ijl}(s))) + I_i(s)\Big]\mathrm{d}s \\ &+ \int_{-\infty}^{t} e^{-\int_s^t a_i^{\phi}(u)\mathrm{d}u} \sum_{j=1}^{n} h_{ij}(s)\delta_{ij}(\varphi_j(s-v_{ij}(s)))\mathrm{d}w_j(s). \end{aligned}$$

We will divide the rest of the proof into three steps.

Step 1. We will prove that system (2.1) has a unique solution in $\mathbb{H}_0$. To begin with, we will prove that the mapping Γ is a self-mapping from $\mathbb{H}_0$ to $\mathbb{H}_0$. In fact, for each $\varphi \in \mathbb{H}_0$, we have

$$\|\varphi\|_{\mathbb{H}} \le \|\varphi - \varphi_0\|_{\mathbb{H}} + \|\varphi_0\|_{\mathbb{H}} \le 2K \tag{3.3}$$

and

$$\begin{aligned} \|\Gamma\varphi - \varphi_0\|_{\mathbb{H}}^2 \le &5 \sup_{t\in\mathbb{R}} \max_{1\le i\le n} \Big\{ E\Big\| \int_{-\infty}^{t} e^{-\int_s^t a_i^{\phi}(u)\mathrm{d}u} a_i^c(s)\varphi_i(s)\mathrm{d}s \Big\|_{\mathbb{O}}^2 \Big\} \\ &+ 5 \sup_{t\in\mathbb{R}} \max_{1\le i\le n} \Big\{ E\Big\| \int_{-\infty}^{t} e^{-\int_s^t a_i^{\phi}(u)\mathrm{d}u} \sum_{j=1}^{n} b_{ij}(s)f_j(\varphi_j(s))\mathrm{d}s \Big\|_{\mathbb{O}}^2 \Big\} \\ &+ 5 \sup_{t\in\mathbb{R}} \max_{1\le i\le n} \Big\{ E\Big\| \int_{-\infty}^{t} e^{-\int_s^t a_i^{\phi}(u)\mathrm{d}u} \sum_{j=1}^{n} c_{ij}(s)g_j(\varphi_j(s-\eta_{ij}(s)))\mathrm{d}s \Big\|_{\mathbb{O}}^2 \Big\} \\ &+ 5 \sup_{t\in\mathbb{R}} \max_{1\le i\le n} \Big\{ E\Big\| \int_{-\infty}^{t} e^{-\int_s^t a_i^{\phi}(u)\mathrm{d}u} \sum_{j=1}^{n}\sum_{l=1}^{n} Q_{ijl}(s)r_j(\varphi_j(s-\theta_{ijl}(s)))r_l(\varphi_l(s-\xi_{ijl}(s)))\mathrm{d}s \Big\|_{\mathbb{O}}^2 \Big\} \\ &+ 5 \sup_{t\in\mathbb{R}} \max_{1\le i\le n} \Big\{ E\Big\| \int_{-\infty}^{t} e^{-\int_s^t a_i^{\phi}(u)\mathrm{d}u} \sum_{j=1}^{n} h_{ij}(s)\delta_{ij}(\varphi_j(s-v_{ij}(s)))\mathrm{d}w_j(s) \Big\|_{\mathbb{O}}^2 \Big\} \\ :=&J_1 + J_2 + J_3 + J_4 + J_5. \end{aligned} \tag{3.4}$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} J_1 \leq& \sup_{t\in\mathbb{R}} \max_{1\leq i\leq n} \left\{ \left[\int_{-\infty}^{t} e^{-\int_s^t a_i^{\phi}(u)\mathrm{d}u} \mathrm{d}s \right] \left[\int_{-\infty}^{t} e^{-\int_s^t a_i^{\phi}(u)\mathrm{d}u} E\|a_i^c(s)\varphi_i(s)\|_{\mathbb{O}}^2 \mathrm{d}s \right] \right\} \\ \leq& 5 \max_{1\leq i\leq n} \left\{ \frac{(\bar{a}_i^c)^2}{(\underline{a}_i^{\phi})^2} \right\} \|\varphi\|_{\mathbb{H}}^2. \end{aligned} \tag{3.5}$$

Similarly, we can get

$$J_2 \leq 5 \max_{1\leq i\leq n} \left\{ \frac{1}{(\underline{a}_i^{\phi})^2} \sum_{j=1}^{n} (\check{b}_{ij})^2 \sum_{j=1}^{n} (L_j^f)^2 \right\} \|\varphi\|_{\mathbb{H}}^2, \tag{3.6}$$

$$J_3 \leq 5 \max_{1\leq i\leq n} \left\{ \frac{1}{(\underline{a}_i^{\phi})^2} \sum_{j=1}^{n} (\check{c}_{ij})^2 \sum_{j=1}^{n} (L_j^g)^2 \right\} \|\varphi\|_{\mathbb{H}}^2, \tag{3.7}$$

$$J_4 \leq 5 \max_{1\leq i\leq n} \left\{ \frac{1}{(\underline{a}_i^{\phi})^2} \sum_{j=1}^{n} \left(\sum_{l=1}^{n} (\check{Q}_{ijl})^2 \sum_{l=1}^{n} (M_l^r)^2 \right) \sum_{j=1}^{n} (L_j^r)^2 \right\} \|\varphi\|_{\mathbb{H}}^2. \tag{3.8}$$

By the Itô isometry, we can obtain

$$J_5 \leq 5 \max_{1\leq i\leq n} \left\{ \frac{1}{2\underline{a}_i^{\phi}} \sum_{j=1}^{n} (\check{h}_{ij})^2 \sum_{j=1}^{n} (L_{ij}^{\delta})^2 \right\} \|\varphi\|_{\mathbb{H}}^2. \tag{3.9}$$

Substituting (3.5)–(3.9) into (3.4), by (3.3) and (A_3), we obtain that

$$\begin{aligned} \|\Gamma\varphi - \varphi_0\|_{\mathbb{H}}^2 \leq& 5 \max_{1\leq i\leq n} \left\{ \frac{1}{(\underline{a}_i^{\phi})^2} \left[(\bar{a}_i^c)^2 + \sum_{j=1}^{n} (\check{b}_{ij})^2 \sum_{j=1}^{n} (L_j^f)^2 + \sum_{j=1}^{n} (\check{c}_{ij})^2 \sum_{j=1}^{n} (L_j^g)^2 \right. \right. \\ & \left. \left. + \sum_{j=1}^{n} \left(\sum_{l=1}^{n} (\check{Q}_{ijl})^2 \sum_{l=1}^{n} (M_l^r)^2 \right) \sum_{j=1}^{n} (L_j^r)^2 + \frac{\underline{a}_i^{\phi}}{2} \sum_{j=1}^{n} (\check{h}_{ij})^2 \sum_{j=1}^{n} (L_{ij}^{\delta})^2 \right] \right\} \|\varphi\|_{\mathbb{H}}^2 \\ \leq& \frac{1}{4} (2K)^2 = K^2. \end{aligned}$$

Therefore, we arrive at $\|\Gamma\varphi - \varphi_0\|_{\mathbb{H}} \leq K$ for every $\varphi \in \mathbb{H}_0$.

To prove that $\Gamma\varphi$ is uniformly $\mathcal{L}^2$-continuous, for every $\varphi \in \mathbb{H}_0$, $t_1, t_2 \in \mathbb{R}$ with $t_1 > t_2$, we infer that

$$\begin{aligned} & E\|(\Gamma\varphi)(t_1) - (\Gamma\varphi)(t_2)\|_0^2 \\ =& \max_{1\leq i\leq n} \left\{ E \left\| \int_{-\infty}^{t_2} \left[e^{-\int_s^{t_1} a_i^{\phi}(u)\mathrm{d}u} - e^{-\int_s^{t_2} a_i^{\phi}(u)\mathrm{d}u} \right] \left[-a_i^c(s)\varphi_i(s) + \sum_{j=1}^{n} b_{ij}(s) f_j(\varphi_j(s)) \right. \right. \right. \\ & + \sum_{j=1}^{n} c_{ij}(s) g_j(\varphi_j(s - \eta_{ij}(s))) + \sum_{j=1}^{n} \sum_{l=1}^{n} Q_{ijl}(s) r_j(\varphi_j(s - \theta_{ijl}(s))) r_l(\varphi_l(s - \xi_{ijl}(s))) \\ & \left. + I_i(s) \right] \mathrm{d}s + \int_{t_2}^{t_1} e^{-\int_s^{t_1} a_i^{\phi}(u)\mathrm{d}u} \left[-a_i^c(s)\varphi_i(s) + \sum_{j=1}^{n} b_{ij}(s) f_j(\varphi_j(s)) \right. \end{aligned}$$

$$\begin{aligned}
&+\sum_{j=1}^{n} c_{ij}(s)g_j(\varphi_j(s-\eta_{ij}(s)))+\sum_{j=1}^{n}\sum_{l=1}^{n} Q_{ijl}(s)r_j(\varphi_j(s-\theta_{ijl}(s)))r_l(\varphi_l(s-\xi_{ijl}(s)))\\
&+I_i(s)\bigg]\mathrm{d}s+\int_{-\infty}^{t_2}\Big[e^{-\int_s^{t_1}a_i^{\phi}(u)\mathrm{d}u}-e^{-\int_s^{t_2}a_i^{\phi}(u)\mathrm{d}u}\Big]\sum_{j=1}^{n} h_{ij}(s)\delta_{ij}(\varphi_j(s-v_{ij}(s)))\mathrm{d}w_j(s)\\
&+\int_{t_2}^{t_1} e^{-\int_s^{t_1}a_i^{\phi}(u)\mathrm{d}u}\sum_{j=1}^{n} h_{ij}(s)\delta_{ij}(\varphi_j(s-v_{ij}(s)))\mathrm{d}w_j(s)\bigg\|_{\mathbb{O}}^2\bigg\}\\
\le&12\max_{1\le i\le n}\bigg\{\bigg[\int_{-\infty}^{t_2}\Big|e^{-\int_s^{t_1}a_i^{\phi}(u)\mathrm{d}u}-e^{-\int_s^{t_2}a_i^{\phi}(u)\mathrm{d}u}\Big|\mathrm{d}s\bigg]\bigg[\int_{-\infty}^{t_2}\Big|e^{-\int_s^{t_1}a_i^{\phi}(u)\mathrm{d}u}-e^{-\int_s^{t_2}a_i^{\phi}(u)\mathrm{d}u}\Big|\\
&\times\Big(E\|a_i^c(s)\varphi_i(s)\|_{\mathbb{O}}^2+E\Big\|\sum_{j=1}^{n} b_{ij}(s)f_j(\varphi_j(s))\Big\|_{\mathbb{O}}^2+E\Big\|\sum_{j=1}^{n} c_{ij}(s)g_j(\varphi_j(s-\eta_{ij}(s)))\Big\|_{\mathbb{O}}^2\\
&+E\Big\|\sum_{j=1}^{n}\sum_{l=1}^{n} Q_{ijl}(s)r_j(\varphi_j(s-\theta_{ijl}(s)))r_l(\varphi_l(s-\xi_{ijl}(s)))\Big\|_{\mathbb{O}}^2+E\|I_i(s)\|_{\mathbb{O}}^2\Big)\mathrm{d}s\bigg]\bigg\}\\
&+12\max_{1\le i\le n}\bigg\{\bigg[\int_{t_2}^{t_1} e^{-\int_s^{t_1}a_i^{\phi}(u)\mathrm{d}u}\mathrm{d}s\bigg]\bigg[\int_{t_2}^{t_1} e^{-\int_s^{t_1}a_i^{\phi}(u)\mathrm{d}u}\Big(E\|a_i^c(s)\varphi_i(s)\|_{\mathbb{O}}^2\\
&+E\Big\|\sum_{j=1}^{n} b_{ij}(s)f_j(\varphi_j(s))\Big\|_{\mathbb{O}}^2+E\Big\|\sum_{j=1}^{n} c_{ij}(s)g_j(\varphi_j(s-\eta_{ij}(s)))\Big\|_{\mathbb{O}}^2\\
&+E\Big\|\sum_{j=1}^{n}\sum_{l=1}^{n} Q_{ijl}(s)r_j(\varphi_j(s-\theta_{ijl}(s)))r_l(\varphi_l(s-\xi_{ijl}(s)))\Big\|_{\mathbb{O}}^2+E\|I_i(s)\|_{\mathbb{O}}^2\Big)\mathrm{d}s\bigg]\bigg\}\\
&+12\max_{1\le i\le n}\bigg\{\int_{-\infty}^{t_2}\Big[e^{-\int_s^{t_1}a_i^{\phi}(u)\mathrm{d}u}-e^{-\int_s^{t_2}a_i^{\phi}(u)\mathrm{d}u}\Big]^2E\Big\|\sum_{j=1}^{n} h_{ij}(s)\delta_{ij}(\varphi_j(s-v_{ij}(s)))\Big\|_{\mathbb{O}}^2\mathrm{d}s\bigg\}\\
&+12\max_{1\le i\le n}\bigg\{\int_{t_2}^{t_1} e^{-2\int_s^{t_1}a_i^{\phi}(u)\mathrm{d}u}E\Big\|\sum_{j=1}^{n} h_{ij}(s)\delta_{ij}(\varphi_j(s-v_{ij}(s)))\Big\|_{\mathbb{O}}^2\mathrm{d}s\bigg\}\\
\le&\max_{1\le i\le n}\bigg\{12\bigg[(\bar{a}_i^c2K)^2+\sum_{j=1}^{n}(\check{b}_{ij}L_j^f2K)^2+\sum_{j=1}^{n}(\check{c}_{ij}L_j^g2K)^2+\sum_{j=1}^{n}\sum_{l=1}^{n}(\check{Q}_{ijl}L_j^r2K)^2+(I_i^+)^2\bigg]\\
&\times\bigg[\Big(\int_{-\infty}^{t_2} e^{-\underline{a}_i^{\phi}(t_2-s)}\Big|\int_s^{t_2}a_i^{\phi}(u)\mathrm{d}u-\int_s^{t_1}a_i^{\phi}(u)\mathrm{d}u\Big|\mathrm{d}s\Big)^2+\Big(\int_{t_2}^{t_1} e^{-\int_s^{t_1}a_i^{\phi}(u)\mathrm{d}u}\mathrm{d}s\Big)^2\bigg]\bigg\}\\
&+\max_{1\le i\le n}\bigg\{12\sum_{j=1}^{n}(\check{h}_{ij}L_{ij}^{\delta}2K)^2\bigg[\int_{-\infty}^{t_2} e^{-2\underline{a}_i^{\phi}(t_2-s)}\Big|\int_s^{t_2}a_i^{\phi}(u)\mathrm{d}u-\int_s^{t_1}a_i^{\phi}(u)\mathrm{d}u\Big|^2\mathrm{d}s+\int_{t_2}^{t_1} e^{-2\int_s^{t_1}a_i^{\phi}(u)\mathrm{d}u}\mathrm{d}s\bigg]\bigg\}\\
\le&\max_{1\le i\le n}\bigg\{12\bigg[(\bar{a}_i^c2K)^2+\sum_{j=1}^{n}(\check{b}_{ij}L_j^f2K)^2+\sum_{j=1}^{n}(\check{c}_{ij}L_j^g2K)^2+\sum_{j=1}^{n}\sum_{l=1}^{n}(\check{Q}_{ijl}L_j^r2K)^2+(I_i^+)^2\bigg]\\
&\times\bigg[\Big(\frac{\bar{a}_i^{\phi}}{\underline{a}_i^{\phi}}\Big)^2|t_1-t_2|^2+|t_1-t_2|^2\bigg]\bigg\}+\max_{1\le i\le n}\bigg\{12\sum_{j=1}^{n}(\check{h}_{ij}L_{ij}^{\delta}2K)^2\bigg[\frac{(\bar{a}_i^{\phi})^2}{2\underline{a}_i^{\phi}}|t_1-t_2|^2+|t_1-t_2|\bigg]\bigg\}\\
\le&\max_{1\le i\le n}\bigg\{12\bigg[(\bar{a}_i^c2K)^2+\sum_{j=1}^{n}(\check{b}_{ij}L_j^f2K)^2+\sum_{j=1}^{n}(\check{c}_{ij}L_j^g2K)^2+\sum_{j=1}^{n}\sum_{l=1}^{n}(\check{Q}_{ijl}L_j^r2K)^2+(I_i^+)^2\bigg]\bigg[\Big(\frac{\bar{a}_i^c}{\underline{a}_i^{\phi}}\Big)^2+1\bigg]
\end{aligned}$$

$$+\frac{6(\bar{a}_i^c)^2}{\underline{a}_i^\phi}\sum_{j=1}^{n}(\check{h}_{ij}L_{ij}^{\delta}2K)^2\Big\}|t_1-t_2|^2+\max_{1\le i\le n}\Big\{12\sum_{j=1}^{n}(\check{h}_{ij}L_{ij}^{\delta}2K)^2\Big\}|t_1-t_2|.$$

Therefore, we find that $E\|\Gamma\varphi(t_1)-\Gamma\varphi(t_2)\|_0^2\to 0$ as $t_1\to t_2$, which implies that $T\varphi$ is uniformly $\mathcal{L}^2$-continuous. As a result, we reach the conclusion that $\Gamma(\mathbb{H}_0)\subset\mathbb{H}_0$.

Now, we prove that Γ is a contracting mapping. Indeed, for any $\varphi,\psi\in X_0$, we can deduce that

$$\begin{aligned}
&\big\|\Gamma\varphi-\Gamma\psi\big\|_{\mathbb{H}}^2\\
\le&5\sup_{t\in\mathbb{R}}\max_{1\le i\le n}\Big\{E\Big\|\int_{-\infty}^{t}e^{-\int_s^t a_i^\phi(u)\mathrm{d}u}a_i^c(s)[\varphi_i(s)-\psi_i(s)]\mathrm{d}s\Big\|_{\mathbb{O}}^2\Big\}\\
&+5\sup_{t\in\mathbb{R}}\max_{1\le i\le n}\Big\{E\Big\|\int_{-\infty}^{t}e^{-\int_s^t a_i^\phi(u)\mathrm{d}u}\sum_{j=1}^{n}b_{ij}(s)[f_j(\varphi_j(s))-f_j(\psi_j(s))]\mathrm{d}s\Big\|_{\mathbb{O}}^2\Big\}\\
&+5\sup_{t\in\mathbb{R}}\max_{1\le i\le n}\Big\{E\Big\|\int_{-\infty}^{t}e^{-\int_s^t a_i^\phi(u)du}\sum_{j=1}^{n}c_{ij}(s)[g_j(\varphi_j(s-\eta_{ij}(s)))-g_j(\psi_j(s-\eta_{ij}(s)))]\mathrm{d}s\Big\|_{\mathbb{O}}^2\Big\}\\
&+5\sup_{t\in\mathbb{R}}\max_{1\le i\le n}\Big\{E\Big\|\int_{-\infty}^{t}e^{-\int_s^t a_i^\phi(u)\mathrm{d}u}\sum_{j=1}^{n}\sum_{l=1}^{n}Q_{ijl}(s)r_j(\varphi_j(s-\theta_{ijl}(s)))r_l(\varphi_l(s-\xi_{ijl}(s)))\mathrm{d}s\Big\|_{\mathbb{O}}^2\Big\}\\
&+5\sup_{t\in\mathbb{R}}\max_{1\le i\le n}\Big\{E\Big\|\int_{-\infty}^{t}e^{-\int_s^t a_i^\phi(u)du}\sum_{j=1}^{n}h_{ij}(s)[\delta_{ij}(\varphi_j(s-v_{ij}(s)))-\delta_{ij}(\psi_j(s-v_{ij}(s)))]\mathrm{d}s\Big\|_{\mathbb{O}}^2\Big\}\\
\le&5\max_{1\le i\le n}\Big\{\frac{1}{(\underline{a}_i^\phi)^2}\Big[(\bar{a}_i^c)^2+\sum_{j=1}^{n}(\check{b}_{ij})^2\sum_{j=1}^{n}(L_j^f)^2+\sum_{j=1}^{n}(\check{c}_{ij})^2\sum_{j=1}^{n}(L_j^g)^2\\
&+\sum_{j=1}^{n}\Big(\sum_{l=1}^{n}(\check{Q}_{ijl})^2\sum_{l=1}^{n}(M_l^r)^2\Big)\sum_{j=1}^{n}(L_j^r)^2+\frac{\underline{a}_i^\phi}{2}\sum_{j=1}^{n}(\check{h}_{ij})^2\sum_{j=1}^{n}(L_{ij}^{\delta})^2\Big]\Big\}\|\varphi-\psi\|_{\mathbb{H}}^2,
\end{aligned}$$

which combined with (A_3) yields that

$$\big\|\Gamma\varphi-\Gamma\psi\big\|_{\mathbb{H}}\le\sqrt{R}\big\|\varphi-\psi\big\|_{\mathbb{H}}<\frac{1}{2}\big\|\varphi-\psi\big\|_{\mathbb{H}},$$

i.e., Γ is a contraction mapping. Hence, Γ has a unique fixed point x in $\mathbb{H}_0$. Therefore, system (2.1) possesses a unique solution $x\in\mathbb{H}_0$.

Step 2. We will show that the system

$$\begin{aligned}
y_i(t)=&\int_{-\infty}^{t}e^{-\int_s^t a_i^\phi(u)\mathrm{d}u}\Big[-a_i^c(s)y_i(s)+\sum_{j=1}^{n}b_{ij}^1(s)f_j(y_j(s))+\sum_{j=1}^{n}c_{ij}^1(s)g_j(y_j(s-\eta_{ij}(s)))\\
&+\sum_{j=1}^{n}\sum_{l=1}^{n}Q_{ijl}^1(s)r_j(y_j(s-\theta_{ijl}(s)))r_l(y_l(s-\xi_{ijl}(s)))+I_i^1(s)\Big]\mathrm{d}s\\
&+\int_{-\infty}^{t}e^{-\int_s^t a_i^\phi(u)\mathrm{d}u}\sum_{j=1}^{n}h_{ij}^1(s)\delta_{ij}(y_j(s-v_{ij}(s)))\mathrm{d}w_j(s),
\end{aligned}\tag{3.10}$$

possesses a unique solution y that is almost automorphic in finite-dimensional distributions.

In fact, similar to Step 1, by using conditions (A_1), (A_2), and (A_4), one can show that system (3.10) does have a unique solution $y \in \mathbb{H}_0$, which satisfies (3.10). We will prove that y is also almost automorphic in finite-dimensional distributions.

By (3.10), we can get

$$\begin{aligned} y_i(t+s_k) = {} & \int_{-\infty}^{t} e^{-\int_s^t a_i^{\phi}(u+s_k)\mathrm{d}u}\Big[-a_i^c(s+s_k)y_i(s+s_k) + \sum_{j=1}^{n} b_{ij}^1(s+s_k)f_j(y_j(s+s_k)) \\ & + \sum_{j=1}^{n} c_{ij}^1(s+s_k)g_j(y_j(s+s_k-\eta_{ij}(s+s_k))) + \sum_{j=1}^{n}\sum_{l=1}^{n} Q_{ijl}^1(s+s_k) \\ & \times r_j(y_j(s+s_k-\theta_{ijl}(s+s_k)))r_l(y_l(s+s_k-\xi_{ijl}(s+s_k))) + I_i^1(s+s_k)\Big]\mathrm{d}s \\ & + \int_{-\infty}^{t} e^{-\int_s^t a_i^{\phi}(u+s_k)\mathrm{d}u} \sum_{j=1}^{n} h_{ij}^1(s+s_k)\delta_{ij}(y_j(s+s_k-v_{ij}(s+s_k)))\mathrm{d}[w_j(s+s_k)-w_j(s_k)], \end{aligned}$$

where $i = 1,\dots,n$, $w_j(s+s_k) - w_j(s_k)$ is a Brownian motion with the same distribution as $w_j(s)$. Let us consider the processes

$$\begin{aligned} y_i(t+s_k) = {} & \int_{-\infty}^{t} e^{-\int_s^t a_i^{\phi}(u+s_k)\mathrm{d}u}\Big[-a_i^c(s+s_k)y_i(s+s_k) + \sum_{j=1}^{n} b_{ij}^1(s+s_k)f_j(y_j(s+s_k)) \\ & + \sum_{j=1}^{n} c_{ij}^1(s+s_k)g_j(y_j(s+s_k-\eta_{ij}(s+s_k))) + \sum_{j=1}^{n}\sum_{l=1}^{n} Q_{ijl}^1(s+s_k) \\ & \times r_j(y_j(s+s_k-\theta_{ijl}(s+s_k)))r_l(y_l(s+s_k-\xi_{ijl}(s+s_k))) + I_i^1(s+s_k)\Big]\mathrm{d}s \\ & + \int_{-\infty}^{t} e^{-\int_s^t a_i^{\phi}(u+s_k)\mathrm{d}u} \sum_{j=1}^{n} h_{ij}^1(s+s_k)\delta_{ij}(y_j(s+s_k-v_{ij}(s+s_k)))\mathrm{d}w_j(s) \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \tilde{y}_i(t) = {} & \int_{-\infty}^{t} e^{-\int_s^t \tilde{a}_i^{\phi}(u)\mathrm{d}u}\Big[-\tilde{a}_i^c(s)\tilde{y}_i(s) + \sum_{j=1}^{n} \tilde{b}_{ij}^1(s)f_j(\tilde{y}_j(s)) + \sum_{j=1}^{n} \tilde{c}_{ij}^1(s)g_j(\tilde{y}_j(s-\tilde{\eta}_{ij}(s))) \\ & + \sum_{j=1}^{n}\sum_{l=1}^{n} \tilde{Q}_{ijl}^1(s)r_j(\tilde{y}_j(s-\tilde{\theta}_{ijl}(s)))r_l(\tilde{y}_l(s-\tilde{\xi}_{ijl}(s))) + \tilde{I}_i^1(s)\Big]\mathrm{d}s \\ & + \int_{-\infty}^{t} e^{-\int_s^t \tilde{a}_i^{\phi}(u)du} \sum_{j=1}^{n} \tilde{h}_{ij}^1(s)\delta_{ij}(\tilde{y}_j(s-\tilde{v}_{ij}(s)))\mathrm{d}w_j(s). \end{aligned} \tag{3.12}$$

Under conditions (A_1), (A_2), and (A_4), according to Remark 2.2, in the same way as in Step 1, one can prove that (3.12) possesses a unique solution $\tilde{y} \in \mathbb{H}_0$.

In view of (A_1), for every sequence of real numbers $(s'_k)_{k\in\mathbb{N}}$, there exists a subsequence $(s_k)_{k\in\mathbb{N}}$ such that for each $t \in \mathbb{R}$, there hold

$$\lim_{k\to\infty} |a_i^{\phi}(t+s_k) - \tilde{a}_i^{\phi}(t)| = 0, \quad \lim_{k\to\infty} |\tilde{a}_i^{\phi}(t-s_k) - a_i^{\phi}(t)| = 0,$$

$$\begin{aligned}
&\lim_{k\to\infty}\|a_i^c(t+s_k)-\tilde{a}_i^c(t)\|_{\mathbb{O}}^2=0, && \lim_{k\to\infty}\|\tilde{a}_i^c(t-s_k)-a_i^c(t)\|_{\mathbb{O}}^2=0,\\
&\lim_{k\to\infty}\|b_{ij}^1(t+s_k)-\tilde{b}_{ij}^1(t)\|_{\mathbb{O}}^2=0, && \lim_{k\to\infty}\|\tilde{b}_{ij}^1(t-s_k)-b_{ij}^1(t)\|_{\mathbb{O}}^2=0,\\
&\lim_{k\to\infty}\|c_{ij}^1(t+s_k)-\tilde{c}_{ij}^1(t)\|_{\mathbb{O}}^2=0, && \lim_{k\to\infty}\|\tilde{c}_{ij}^1(t-s_k)-c_{ij}^1(t)\|_{\mathbb{O}}^2=0,\\
&\lim_{k\to\infty}\|Q_{ijl}^1(t+s_k)-\tilde{Q}_{ijl}^1(t)\|_{\mathbb{O}}^2=0, && \lim_{k\to\infty}\|\tilde{Q}_{ijl}^1(t-s_k)-Q_{ijl}^1(t)\|_{\mathbb{O}}^2=0,\\
&\lim_{k\to\infty}\|I_i^1(t+s_k)-\tilde{I}_i^1(t)\|_{\mathbb{O}}^2=0, && \lim_{k\to\infty}\|\tilde{I}_i^1(t-s_k)-I_i^1(t)\|_{\mathbb{O}}^2=0,\\
&\lim_{k\to\infty}\|h_{ij}^1(t+s_k)-\tilde{h}_{ij}^1(t)\|_{\mathbb{O}}^2=0, && \lim_{k\to\infty}\|\tilde{h}_{ij}^1(t-s_k)-h_{ij}^1(t)\|_{\mathbb{O}}^2=0,\\
&\lim_{k\to\infty}|\eta_{ij}(t+s_k)-\tilde{\eta}_{ij}(t)|^2=0, && \lim_{k\to\infty}|\tilde{\eta}_{ij}(t-s_k)-\eta_{ij}(t)|^2=0,\\
&\lim_{k\to\infty}|\theta_{ijl}(t+s_k)-\tilde{\theta}_{ijl}(t)|^2=0, && \lim_{k\to\infty}|\tilde{\theta}_{ijl}(t-s_k)-\theta_{ijl}(t)|^2=0,\\
&\lim_{k\to\infty}|\xi_{ijl}(t+s_k)-\tilde{\xi}_{ijl}(t)|^2=0, && \lim_{k\to\infty}|\tilde{\xi}_{ijl}(t-s_k)-\xi_{ijl}(t)|^2=0,\\
&\lim_{k\to\infty}|v_{ij}(t+s_k)-\tilde{v}_{ij}(t)|^2=0, && \lim_{k\to\infty}|\tilde{v}_{ij}(t-s_k)-v_{ij}(t)|^2=0.
\end{aligned}$$

Since $\tilde{y}\in UCB(\mathbb{R},\mathcal{L}^2(\Omega,\mathbb{O}^n))$, for any $\{t_k\}$, $\{t'_k\}\subset\mathbb{R}$, we have that if $|t_k-t'_k|\to 0$ as $k\to\infty$, then $E\|\tilde{y}(t_k)-\tilde{y}(t'_k)\|_0^2\to 0$ as $k\to\infty$. Thus, for every sequence of real numbers of $(s'_k)_{k\in\mathbb{N}}$, there exists a subsequence $(s_k)_{k\in\mathbb{N}}$ of $(s'_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty}E\|\tilde{y}(s-\eta_{ij}(s+s_k))-\tilde{y}(s-\tilde{\eta}_{ij}(s))\|_0^2=0, \tag{3.13}$$

$$\lim_{k\to\infty}E\|\tilde{y}(s-\theta_{ijl}(s+s_k))-\tilde{y}(s-\tilde{\theta}_{ijl}(s))\|_0^2=0, \tag{3.14}$$

$$\lim_{k\to\infty}E\|\tilde{y}(s-\xi_{ijl}(s+s_k))-\tilde{y}(s-\tilde{\xi}_{ijl}(s))\|_0^2=0, \tag{3.15}$$

$$\lim_{k\to\infty}E\|\tilde{y}(s-v_{ij}(s+s_k))-\tilde{y}(s-\tilde{v}_{ij}(s))\|_0^2=0. \tag{3.16}$$

For every $t_q\in\mathbb{R}$, by (3.11) and (3.12), we get

$$\begin{aligned}
&E\|y(t_q+t+s_k)-\tilde{y}(t_q+t)\|_0^2\\
\le& 17\max_{1\le i\le n}E\left\|\int_{-\infty}^{t_q+t}e^{-\int_s^{t_q+t}a_i^{\phi}(u+s_k)\mathrm{d}u}a_i^c(s+s_k)[y_i(s+s_k)-\tilde{y}_i(s)]\mathrm{d}s\right\|_{\mathbb{O}}^2\\
&+17\max_{1\le i\le n}E\left\|\int_{-\infty}^{t_q+t}e^{-\int_s^{t_q+t}a_i^{\phi}(u+s_k)\mathrm{d}u}[a_i^c(s+s_k)-\tilde{a}_i^c(s)]\tilde{y}_i(s)\mathrm{d}s\right\|_{\mathbb{O}}^2\\
&+17\max_{1\le i\le n}E\left\|\int_{-\infty}^{t_q+t}\left(e^{-\int_s^{t_q+t}a_i^{\phi}(u+s_k)du}-e^{-\int_s^{t_q+t}\tilde{a}_i^{\phi}(u)\mathrm{d}u}\right)\tilde{a}_i^c(s)\tilde{y}_i(s)\mathrm{d}s\right\|_{\mathbb{O}}^2\\
&+17\max_{1\le i\le n}E\left\|\int_{-\infty}^{t_q+t}e^{-\int_s^{t_q+t}a_i^{\phi}(u+s_k)\mathrm{d}u}\sum_{j=1}^{n}b_{ij}^1(s+s_k)[f_j(y_j(s+s_k))-f_j(\tilde{y}_j(s))]\mathrm{d}s\right\|_{\mathbb{O}}^2
\end{aligned}$$

$$
\begin{aligned}
&+17\max_{1\le i\le n}E\Big\|\int_{-\infty}^{t_q+t}e^{-\int_s^{t_q+t}a_i^{\phi}(u+s_k)\mathrm{d}u}\sum_{j=1}^{n}(b_{ij}^1(s+s_k)-\tilde{b}_{ij}^1(s))f_j(\tilde{y}_j(s))\mathrm{d}s\Big\|_{\mathbb{O}}^2\\
&+17\max_{1\le i\le n}E\Big\|\int_{-\infty}^{t_q+t}\Big(e^{-\int_s^{t_q+t}a_i^{\phi}(u+s_k)\mathrm{d}u}-e^{-\int_s^{t_q+t}\tilde{a}_i^{\phi}(u)\mathrm{d}u}\Big)\sum_{j=1}^{n}\tilde{b}_{ij}^1(s)f_j(\tilde{y}_j(s))\mathrm{d}s\Big\|_{\mathbb{O}}^2\\
&+17\max_{1\le i\le n}E\Big\|\int_{-\infty}^{t_q+t}e^{-\int_s^{t_q+t}a_i^{\phi}(u+s_k)\mathrm{d}u}\sum_{j=1}^{n}c_{ij}^1(s+s_k)[g_j(y_j(s+s_k-\eta_{ij}(s+s_k)))-g_j(\tilde{y}_j(s-\tilde{\eta}_{ij}(s)))]\mathrm{d}s\Big\|_{\mathbb{O}}^2\\
&+17\max_{1\le i\le n}E\Big\|\int_{-\infty}^{t_q+t}e^{-\int_s^{t_q+t}a_i^{\phi}(u+s_k)\mathrm{d}u}\sum_{j=1}^{n}(c_{ij}^1(s+s_k)-\tilde{c}_{ij}^1(s))g_j(\tilde{y}_j(s-\tilde{\eta}_{ij}(s)))\mathrm{d}s\Big\|_{\mathbb{O}}^2\\
&+17\max_{1\le i\le n}E\Big\|\int_{-\infty}^{t_q+t}\Big(e^{-\int_s^{t_q+t}a_i^{\phi}(u+s_k)\mathrm{d}u}-e^{-\int_s^{t_q+t}\tilde{a}_i^{\phi}(u)\mathrm{d}u}\Big)\sum_{j=1}^{n}\tilde{c}_{ij}^1(s)g_j(\tilde{y}_j(s-\tilde{\eta}_{ij}(s)))\mathrm{d}s\Big\|_{\mathbb{O}}^2\\
&+17\max_{1\le i\le n}E\Big\|\int_{-\infty}^{t_q+t}e^{-\int_s^{t_q+t}a_i^{\phi}(u+s_k)\mathrm{d}u}\sum_{j=1}^{n}\sum_{l=1}^{n}Q_{ijl}^1(s+s_k)\Big[r_j(y_j(s+s_k-\theta_{ijl}(s+s_k)))\\
&\times r_l(y_l(s+s_k-\xi_{ijl}(s+s_k)))-r_j(\tilde{y}_j(s-\tilde{\theta}_{ijl}(s)))r_l(\tilde{y}_l(s-\tilde{\xi}_{ijl}(s)))\Big]\mathrm{d}s\Big\|_{\mathbb{O}}^2\\
&+17\max_{1\le i\le n}E\Big\|\int_{-\infty}^{t_q+t}e^{-\int_s^{t_q+t}a_i^{\phi}(u+s_k)\mathrm{d}u}\sum_{j=1}^{n}\sum_{l=1}^{n}(Q_{ijl}^1(s+s_k)-\tilde{Q}_{ijl}^1(s))r_j(\tilde{y}_j(s-\tilde{\theta}_{ijl}(s)))\\
&\times r_l(\tilde{y}_j(s-\tilde{\xi}_{ijl}(s)))\mathrm{d}s\Big\|_{\mathbb{O}}^2\\
&+17\max_{1\le i\le n}E\Big\|\int_{-\infty}^{t_q+t}\Big(e^{-\int_s^{t_q+t}a_i^{\phi}(u+s_k)\mathrm{d}u}-e^{-\int_s^{t_q+t}\tilde{a}_i^{\phi}(u)\mathrm{d}u}\Big)\sum_{j=1}^{n}\sum_{l=1}^{n}\tilde{Q}_{ijl}^1(s)r_j(\tilde{y}_j(s-\tilde{\theta}_{ijl}(s)))\\
&\times r_l(\tilde{y}_j(s-\tilde{\xi}_{ijl}(s)))\mathrm{d}s\Big\|_{\mathbb{O}}^2\\
&+17\max_{1\le i\le n}E\Big\|\int_{-\infty}^{t_q+t}e^{-\int_s^{t_q+t}a_i^{\phi}(u+s_k)\mathrm{d}u}[I_i^1(s+s_k)-\tilde{I}_i^1(s)]\mathrm{d}s\Big\|_{\mathbb{O}}^2\\
&+17\max_{1\le i\le n}E\Big\|\int_{-\infty}^{t_q+t}\Big(e^{-\int_s^{t_q+t}a_i^{\phi}(u+s_k)\mathrm{d}u}-e^{-\int_s^{t_q+t}\tilde{a}_i^{\phi}(u)\mathrm{d}u}\Big)\tilde{I}_i^1(s)\mathrm{d}s\Big\|_{\mathbb{O}}^2\\
&+17\max_{1\le i\le n}E\Big\|\int_{-\infty}^{t_q+t}e^{-\int_s^{t_q+t}a_i^{\phi}(u+s_k)\mathrm{d}u}\sum_{j=1}^{n}h_{ij}^1(s+s_k)[\delta_{ij}(y_j(s+s_k-\nu_{ij}(s+s_k)))\\
&-\delta_{ij}(\tilde{y}_j(s-\tilde{\nu}_{ij}(s)))]\mathrm{d}w_j(s)\Big\|_{\mathbb{O}}^2\\
&+17\max_{1\le i\le n}E\Big\|\int_{-\infty}^{t_q+t}e^{-\int_s^{t_q+t}a_i^{\phi}(u+s_k)\mathrm{d}u}\sum_{j=1}^{n}(h_{ij}^1(s+s_k)-\tilde{h}_{ij}^1(s))\delta_{ij}(\tilde{y}_j(s-\tilde{\nu}_{ij}(s)))\mathrm{d}w_j(s)\Big\|_{\mathbb{O}}^2\\
&+17\max_{1\le i\le n}E\Big\|\int_{-\infty}^{t_q+t}\Big(e^{-\int_s^{t_q+t}a_i^{\phi}(u+s_k)\mathrm{d}u}-e^{-\int_s^{t_q+t}\tilde{a}_i^{\phi}(u)\mathrm{d}u}\Big)\sum_{j=1}^{n}\tilde{h}_{ij}^1(s)\delta_{ij}(\tilde{y}_j(s-\tilde{\nu}_{ij}(s)))\mathrm{d}w_j(s)\Big\|_{\mathbb{O}}^2
\end{aligned}
$$

$$:= \sum_{i=1}^{17} F_i(t). \tag{3.17}$$

In view of the Cauchy-Schwarz inequality, we deduce that

$$\begin{aligned}F_1(t) \leq & 17 \max_{1\leq i\leq n} E\Big\| \int_{-\infty}^{t_q+t} e^{-\int_s^{t_q+t} a_i^{\phi}(u+s_k)\mathrm{d}u}\mathrm{d}s \int_{-\infty}^{t_q+t} e^{-\int_s^{t_q+t} a_i^{\phi}(u+s_k)\mathrm{d}u}\big(a_i^c(s+s_k)(y_i(s+s_k)-\tilde{y}_i(s))\big)^2\mathrm{d}s\Big\|_{\mathbb{O}} \\ \leq & \max_{1\leq i\leq n}\Big\{\frac{17}{\underline{a}_i^{\phi}}(\bar{a}_i^c)^2 \int_{-\infty}^{t_q+t} e^{-\underline{a}_i^{\phi}(t_q+t-s)} E\|y(s+s_k)-\tilde{y}(s)\|_0^2\mathrm{d}s\Big\} \\ \leq & \max_{1\leq i\leq n}\Big\{\frac{17}{\underline{a}_i^{\phi}}(\bar{a}_i^c)^2 \int_{-\infty}^{t} e^{-\underline{a}_i^{\phi}(t-s)} E\|y(t_q+s+s_k)-\tilde{y}(t_q+s)\|_0^2\mathrm{d}s\Big\}. \end{aligned} \tag{3.18}$$

Since $E\|\tilde{y}(s)\|_0^2 < (2K)^2$, $\lim_{k\to\infty}\|a_i^c(s+s_k)-\tilde{a}_i^c(s)\|_{\mathbb{O}}^2 = 0$, and by the Cauchy-Schwarz inequality and the Lebesgue dominated convergence theorem, we can get

$$F_2(t) \leq \max_{1\leq i\leq n}\Big\{\frac{17}{\underline{a}_i^{\phi}} \int_{-\infty}^{t_q+t} e^{-\int_s^{t_q+t} a_i^{\phi}(u+s_k)\mathrm{d}u}\|a_i^c(s+s_k)-\tilde{a}_i^c(s)\|_{\mathbb{O}}^2 E\|\tilde{y}(s)\|_0^2\mathrm{d}s\Big\} \to 0 \text{ as } k\to\infty. \tag{3.19}$$

Due to the fact that $E\|\tilde{y}(s)\|_0^2 < (2K)^2$, $\lim_{k\to\infty}|a_i^{\phi}(u+s_k)-\tilde{a}_i^{\phi}(u)|^2 = 0$, we can get

$$\begin{aligned}F_3(t) \leq & 17 \max_{1\leq i\leq n}\Big\{\Big(\int_{-\infty}^{t_q+t}\Big|e^{-\int_s^{t_q+t} a_i^{\phi}(u+s_k)\mathrm{d}u} - e^{-\int_s^{t_q+t} \tilde{a}_i^{\phi}(u)\mathrm{d}u}\Big|\mathrm{d}s\Big)^2(\bar{a}_i^c)^2 E\|\tilde{y}\|_0^2\Big\} \\ \leq & 17 \max_{1\leq i\leq n}\Big\{\Big(\int_{-\infty}^{t_q+t} e^{-\underline{a}_i^{\phi}(t_q+t-s)} \int_s^{t_q+t} |a_i^{\phi}(u+s_k)-\tilde{a}_i^{\phi}(u)|\mathrm{d}u\mathrm{d}s\Big)^2(\bar{a}_i^c)^2 E\|\tilde{y}(s)\|_0^2\Big\} \to 0 \text{ as } k\to\infty. \end{aligned} \tag{3.20}$$

Similarly, it is easy to see that

$$\begin{aligned}F_5(t) \leq & \max_{1\leq i\leq n}\Big\{\frac{17}{\underline{a}_i^{\phi}}\sum_{j=1}^n (L_j^f)^2(2K)^2 \int_{-\infty}^{t_q+t} e^{-\underline{a}_i^{\phi}(t_q+t-s)} \\ & \times \Big\|\sum_{j=1}^n (b_{ij}^1(s+s_k)-\tilde{b}_{ij}^1(s))\Big\|_{\mathbb{O}}^2\mathrm{d}s\Big\} \to 0 \text{ as } k\to\infty, \end{aligned} \tag{3.21}$$

$$\begin{aligned}F_6(t) \leq & \max_{1\leq i\leq n}\Big\{17\sum_{j=1}^n (\check{b}_{ij}^1)^2 \sum_{j=1}^n (L_j^f)^2(2K)^2\Big(\int_{-\infty}^{t_q+t} e^{-\underline{a}_i^{\phi}(t_q+t-s)} \\ & \times \int_s^{t_q+t} |a_i^{\phi}(u+s_k)-\tilde{a}_i^{\phi}(u)|\mathrm{d}u\mathrm{d}s\Big)^2\Big\} \to 0 \text{ as } k\to\infty, \end{aligned} \tag{3.22}$$

$$\begin{aligned}F_8(t) \leq & \max_{1\leq i\leq n}\Big\{\frac{17}{\underline{a}_i^{\phi}}\sum_{j=1}^n (L_j^g)^2(2K)^2 \int_{-\infty}^{t_q+t} e^{-\underline{a}_i^{\phi}(t_q+t-s)} \\ & \times \Big\|\sum_{j=1}^n (c_{ij}^1(s+s_k)-\tilde{c}_{ij}^1(s))\Big\|_{\mathbb{O}}^2\mathrm{d}s\Big\} \to 0 \text{ as } k\to\infty, \end{aligned} \tag{3.23}$$

$$F_9(t) \leq \max_{1\leq i\leq n}\Big\{\sum_{j=1}^n (\check{c}_{ij}^1)^2 \sum_{j=1}^n (L_j^g)^2(2K)^2\Big(\int_{-\infty}^{t_q+t} e^{-\underline{a}_i^{\phi}(t_q+t-s)}$$

$$\times\int_{s}^{t_q+t}|a_i^{\phi}(u+s_k)-\tilde{a}_i^{\phi}(u)|\mathrm{d}u\mathrm{d}s\Big)^2\Big\}\to 0 \text{ as } k\to\infty, \tag{3.24}$$

$$\begin{aligned}F_{11}(t)\le&\max_{1\le i\le n}\Big\{\frac{17}{\underline{a}_i^{\phi}}\sum_{j=1}^{n}(L_j^r)^2\sum_{l=1}^{n}(M_l^r)^2(2K)^2\int_{-\infty}^{t_q+t}e^{-\underline{a}_i^{\phi}(t_q+t-s)}\\&\times\Big\|\sum_{j=1}^{n}\sum_{l=1}^{n}(Q_{ijl}^1(s+s_k)-\tilde{Q}_{ijl}^1(s))\Big\|_{\mathbb{O}}^2\Big\}\to 0 \text{ as } k\to\infty,\end{aligned} \tag{3.25}$$

$$\begin{aligned}F_{12}(t)\le&\max_{1\le i\le n}\Big\{17\sum_{j=1}^{n}\Big[\sum_{l=1}^{n}(\check{Q}_{ijl}^1)^2\sum_{l=1}^{n}(L_l^r)^2\Big]\sum_{j=1}^{n}(M_j^r)^2(2K)^2\Big(\int_{-\infty}^{t_q+t}e^{-\underline{a}_i^{\phi}(t_q+t-s)}\\&\times\int_{s}^{t_q+t}|a_i^{\phi}(u+s_k)-\tilde{a}_i^{\phi}(u)|\mathrm{d}u\mathrm{d}s\Big)^2\Big\}\to 0 \text{ as } k\to\infty,\end{aligned} \tag{3.26}$$

$$F_{13}(t)\le\max_{1\le i\le n}\Big\{\frac{17}{\underline{a}_i^{\phi}}\int_{-\infty}^{t_q+t}e^{-\underline{a}_i^{\phi}(t_q+t-s)}\|I_i^1(s+s_k)-\tilde{I}_i^1(s)\|_{\mathbb{O}}^2\mathrm{d}s\Big\}\to 0 \text{ as } k\to\infty, \tag{3.27}$$

$$F_{14}(t)\le\max_{1\le i\le n}\Big\{\frac{17}{\underline{a}_i^{\phi}}(\check{I}_i^1)^2\Big(\int_{-\infty}^{t_q+t}e^{-\underline{a}_i^{\phi}(t_q+t-s)}\int_{s}^{t_q+t}|a_i^{\phi}(u+s_k)-\tilde{a}_i^{\phi}(u)|\mathrm{d}u\mathrm{d}s\Big)^2\Big\}\to 0 \text{ as } k\to\infty, \tag{3.28}$$

$$\begin{aligned}F_{16}(t)\le&\max_{1\le i\le n}\Big\{17\sum_{j=1}^{n}(L_{ij}^{\delta})^2(2K)^2\int_{-\infty}^{t_q+t}e^{-2\underline{a}_i^{\phi}(t_q+t-s)}\\&\times\Big\|\sum_{j=1}^{n}(h_{ij}^1(s+s_k)-\tilde{h}_{ij}^1(s))\Big\|_{\mathbb{O}}^2\mathrm{d}s\Big\}\to 0 \text{ as } k\to\infty,\end{aligned} \tag{3.29}$$

$$\begin{aligned}F_{17}(t)\le&\max_{1\le i\le n}\Big\{17\sum_{j=1}^{n}(\check{h}_{ij}^1)^2\sum_{j=1}^{n}(L_{ij}^{\delta})^2(2K)^2\int_{-\infty}^{t_q+t}e^{-2\underline{a}_i^{\phi}(t_q+t-s)}\\&\times\Big(\int_{s}^{t_q+t}|a_i^{\phi}(u+s_k)-\tilde{a}_i^{\phi}(u)|\mathrm{d}u\Big)^2\mathrm{d}s\Big\}\to 0 \text{ as } k\to\infty.\end{aligned} \tag{3.30}$$

In addition, we can get

$$F_4(t)\le\max_{1\le i\le n}\Big\{\frac{17}{\underline{a}_i^{\phi}}\sum_{j=1}^{n}(\check{b}_{ij}^1)^2\sum_{j=1}^{n}(L_j^f)^2\int_{-\infty}^{t}e^{-\underline{a}_i^{\phi}(t-s)}E\Big\|y(t_q+s+s_k)-\tilde{y}(t_q+s)\Big\|_0^2\mathrm{d}s\Big\}. \tag{3.31}$$

For $F_7(t)$, we have

$$\begin{aligned}F_7(t)\le&34\max_{1\le i\le n}E\Big\|\int_{-\infty}^{t_q+t}e^{-\int_s^{t_q+t}a_i^{\phi}(u+s_k)\mathrm{d}u}\sum_{j=1}^{n}c_{ij}^1(s+s_k)[g_j(y_j(s+s_k-\eta_{ij}(s+s_k)))\\&-g_j(\tilde{y}_j(s-\eta_{ij}(s+s_k)))]\mathrm{d}s\Big\|_{\mathbb{O}}^2+34\max_{1\le i\le n}E\Big\|\int_{-\infty}^{t_q+t}e^{-\int_s^{t_q+t}a_i^{\phi}(u+s_k)\mathrm{d}u}\sum_{j=1}^{n}c_{ij}^1(s+s_k)\\&\times[g_j(\tilde{y}_j(s-\eta_{ij}(s+s_k)))-g_j(\tilde{y}_j(s-\tilde{\eta}_{ij}(s)))]\mathrm{d}s\Big\|_{\mathbb{O}}^2\\:=&H_1(t)+H_2(t).\end{aligned} \tag{3.32}$$

By the Cauchy-Schwarz inequality and (3.13), we have

$$
\begin{aligned}
H_2(t) \le & \max_{1\le i\le n}\Bigg\{\frac{34}{\underline{a}_i^{\phi}}\sum_{j=1}^{n}(\check{c}_{ij}^{1})^2\sum_{j=1}^{n}(L_j^{g})^2\int_{-\infty}^{t_q+t}e^{-\underline{a}_i^{\phi}(t_q+t-s)} \\
& \times E\|\tilde{y}_j(s-\eta_{ij}(s+s_k))-\tilde{y}_j(s-\tilde{\eta}_{ij}(s))\|_{\mathbb{O}}^2\mathrm{d}s\Bigg\}\to 0 \text{ as } k\to\infty. \qquad (3.33)
\end{aligned}
$$

For $H_1(t)$, letting $u = s-\eta_{ij}(s+s_k)$, we infer that

$$
\begin{aligned}
H_1(t) \le & 34\max_{1\le i\le n}\Bigg\{\frac{1}{\underline{a}_i^{\phi}}\sum_{j=1}^{n}(\check{c}_{ij}^{1})^2\sum_{j=1}^{n}(L_j^{g})^2\int_{-\infty}^{t_q+t-\eta_{ij}(t+s_k)}\frac{e^{-\underline{a}_i^{\phi}(t_q+t-u-\eta_{ij}(s+s_k))}}{1-\dot{\eta}_{ij}^{+}}E\|y_j(u+s_k)-\tilde{y}_j(u)\|_{\mathbb{O}}^2\mathrm{d}u\Bigg\} \\
\le & \max_{1\le i\le n}\Bigg\{\frac{34}{\underline{a}_i^{\phi}}\sum_{j=1}^{n}(\check{c}_{ij}^{1})^2\sum_{j=1}^{n}(L_j^{g})^2\frac{e^{\underline{a}_i^{\phi}\eta_{ij}^{+}}}{1-\dot{\eta}_{ij}^{+}}\int_{-\infty}^{t}e^{-\underline{a}_i^{\phi}(t-s)}E\|y(t_q+s+s_k)-\tilde{y}(t_q+s)\|_0^2\mathrm{d}s\Bigg\}. \qquad (3.34)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
F_{10}(t) \le & \max_{1\le i\le n}\Bigg\{\frac{68n}{\underline{a}_i^{\phi}}\sum_{j=1}^{n}\Bigg[\sum_{l=1}^{n}(\check{Q}_{ijl}^{1})^2\int_{-\infty}^{t_q+t}e^{-\int_s^{t_q+t}a_i^{\phi}(u+s_k)\mathrm{d}u}\sum_{l=1}^{n}(M_j^{r}L_l^{r} \\
& \times E\|y_l(s+s_k-\xi_{ijl}(s+s_k))-\tilde{y}_l(s-\xi_{ijl}(s+s_k))\|_{\mathbb{O}}^2)\Bigg]\mathrm{d}s\Bigg\} \\
& +\max_{1\le i\le n}\Bigg\{\frac{68n}{\underline{a}_i^{\phi}}\sum_{j=1}^{n}\Bigg[\sum_{l=1}^{n}(\check{Q}_{ijl}^{1})^2\int_{-\infty}^{t_q+t}e^{-\int_s^{t_q+t}a_i^{\phi}(u+s_k)\mathrm{d}u}\sum_{l=1}^{n}(M_j^{r}L_l^{r} \\
& \times E\|\tilde{y}_l(s-\xi_{ijl}(s+s_k))-\tilde{y}_l(s-\tilde{\xi}_{ijl}(s))\|_{\mathbb{O}}^2)\Bigg]\mathrm{d}s\Bigg\} \\
& +\max_{1\le i\le n}\Bigg\{\frac{68n}{\underline{a}_i^{\phi}}\sum_{j=1}^{n}\Bigg[\sum_{l=1}^{n}(\check{Q}_{ijl}^{1})^2\int_{-\infty}^{t_q+t}e^{-\int_s^{t_q+t}a_i^{\phi}(u+s_k)\mathrm{d}u}\sum_{l=1}^{n}(M_l^{r}L_j^{r} \\
& \times E\|y_j(s+s_k-\theta_{ijl}(s+s_k))-\tilde{y}_j(s-\theta_{ijl}(s+s_k))\|_{\mathbb{O}}^2)\Bigg]\mathrm{d}s\Bigg\} \\
& +\max_{1\le i\le n}\Bigg\{\frac{68n}{\underline{a}_i^{\phi}}\sum_{j=1}^{n}\Bigg[\sum_{l=1}^{n}(\check{Q}_{ijl}^{1})^2\int_{-\infty}^{t_q+t}e^{-\int_s^{t_q+t}a_i^{\phi}(u+s_k)\mathrm{d}u}\sum_{l=1}^{n}(M_l^{r}L_j^{r} \\
& \times E\|\tilde{y}_j(s-\theta_{ijl}(s+s_k))-\tilde{y}_j(s-\tilde{\theta}_{ijl}(s))\|_{\mathbb{O}}^2)\Bigg]\mathrm{d}s\Bigg\} \\
= & B_1(t)+B_2(t)+B_3(t)+B_4(t). \qquad (3.35)
\end{aligned}
$$

By (3.14) and (3.15), we can infer that

$$
B_2(t)\to 0,\ \ B_4(t)\to 0 \text{ as } k\to\infty. \qquad (3.36)
$$

For $B_1(t)$, letting $u = s-\xi_{ijl}(s+s_k)$, we obtain

$$\begin{aligned} B_1(t) \leq \max_{1\leq i\leq n}\Big\{\frac{68n}{\underline{a}_i^{\phi}}\sum_{j=1}^{n}\Big[\sum_{l=1}^{n}(\check{Q}_{ijl}^{1})^2\sum_{l=1}^{n}(M_j^{r}L_l^{r})^2\Big]\frac{e^{\underline{a}_i^{\phi}\xi_{ijl}^{+}}}{1-\dot{\xi}_{ijl}^{+}} \\ \times\int_{-\infty}^{t}e^{-\underline{a}_i^{\phi}(t-s)}E\|y(t_q+s+s_k)-\tilde{y}(t_q+s)\|_0^2\mathrm{d}s\Big\}. \end{aligned} \tag{3.37}$$

Similarly, we have

$$\begin{aligned} B_3(t) \leq \max_{1\leq i\leq n}\Big\{\frac{68n}{\underline{a}_i^{\phi}}\sum_{j=1}^{n}\Big[\sum_{l=1}^{n}(\check{Q}_{ijl}^{1})^2\sum_{l=1}^{n}(M_l^{r}L_j^{r})^2\Big]\frac{e^{\underline{a}_i^{\phi}\theta_{ijl}^{+}}}{1-\dot{\theta}_{ijl}^{+}} \\ \times\int_{-\infty}^{t}e^{-\underline{a}_i^{\phi}(t-s)}E\|y(t_q+s+s_k)-\tilde{y}(t_q+s)\|_0^2\mathrm{d}s\Big\}. \end{aligned} \tag{3.38}$$

By the Itô isometry, we obtain

$$\begin{aligned} F_{15}(t) \leq& 34\max_{1\leq i\leq n}E\Big\|\int_{-\infty}^{t_q+t}e^{-\int_{-\infty}^{t_q+t}a_i^{\phi}(u+s_k)\mathrm{d}u}\sum_{j=1}^{n}h_{ij}^{1}(s+s_k)\delta_{ij}(y_j(s+s_k-v_{ij}(s+s_k))) \\ &-\delta_{ij}(\tilde{y}_j(s-v_{ij}(s+s_k)))\mathrm{d}w_j(s)\Big\|_{\mathbb{O}}^2 \\ &+34\max_{1\leq i\leq n}E\Big\|\int_{-\infty}^{t_q+t}e^{-\int_{-\infty}^{t_q+t}a_i^{\phi}(u+s_k)\mathrm{d}u}\sum_{j=1}^{n}h_{ij}^{1}(s+s_k)[\delta_{ij}(\tilde{y}_j(s-v_{ij}(s+s_k))) \\ &-\delta_{ij}(\tilde{y}_j(s-\tilde{v}_{ij}(s)))]\mathrm{d}w_j(s)\Big\|_{\mathbb{O}}^2 \\ \leq& 34\max_{1\leq i\leq n}\Big\{\sum_{j=1}^{n}(\check{h}_{ij}^{1})^2\sum_{j=1}^{n}(L_{ij}^{\delta})^2\int_{-\infty}^{t_q+t}e^{-2\int_{-\infty}^{t_q+t}a_i^{\phi}(u+s_k)\mathrm{d}u}E\|y_j(s+s_k-v_{ij}(s+s_k)) \\ &-\tilde{y}_j(s-v_{ij}(s+s_k))\|_{\mathbb{O}}^2\mathrm{d}s\Big\} \\ &+34\max_{1\leq i\leq n}\Big\{\sum_{j=1}^{n}(\check{h}_{ij}^{1})^2\sum_{j=1}^{n}(L_{ij}^{\delta})^2\int_{-\infty}^{t_q+t}e^{-2\int_{-\infty}^{t_q+t}a_i^{\phi}(u+s_k)\mathrm{d}u}E\|\tilde{y}_j(s-v_{ij}(s+s_k))-\tilde{y}_j(s-\tilde{v}_{ij}(s))\|_{\mathbb{O}}^2\mathrm{d}s\Big\} \\ =& C_1(t)+C_2(t). \end{aligned} \tag{3.39}$$

According to (3.16), we have

$$C_2(t)\to 0 \text{ as } k\to\infty. \tag{3.40}$$

For $C_1(t)$, letting $u = s - v_{ij}(s + s_k)$, we obtain

$$C_1(t) \leq \max_{1\leq i\leq n}\Big\{34\sum_{j=1}^{n}(\check{h}_{ij}^{1})^2\sum_{j=1}^{n}(L_{ij}^{\delta})^2\frac{e^{2\underline{a}_i^{\phi}v_{ij}^{+}}}{1-\dot{v}_{ij}^{+}}\int_{-\infty}^{t}e^{-2\underline{a}_i^{\phi}(t-s)}E\Big\|y(t_q+s+s_k)-\tilde{y}(t_q+s)\Big\|_0^2\mathrm{d}s\Big\}. \tag{3.41}$$

Therefore, from (3.17)–(3.41), it follows that

$$E\|y(t_q+t+s_k)-\tilde{y}(t_q+t)\|_0^2 \le J_k(t)+\Upsilon\int_{-\infty}^{t} e^{-a^-(t-s)}E\|y(t_q+s+s_k)-\tilde{y}(t_q+s)\|_0^2\mathrm{d}s,$$

where $a^- = \min_{1\le i\le n}\{\underline{a}_i^{\phi}\}$, Υ is given in (A_5), and

$$\begin{aligned}J_k(t) =&F_2(t)+F_3(t)+F_5(t)+F_6(t)+F_8(t)+F_9(t)+F_{11}(t)+F_{12}(t)+F_{13}(t)\\&+F_{14}(t)+F_{16}(t)+F_{17}(t)+H_2(t)+B_2(t)+B_4(t)+C_2(t)\to 0 \text{ as } k\to\infty.\end{aligned}$$

Thus, by Lemma 2.3, we can get $E\|y(t_q+t+s_k)-\tilde{y}(t_q+t)\|_0^2 \to 0$ as $k\to\infty$ for every $t, t_q\in\mathbb{R}$, which means that $y(t)$ is square-mean almost automorphic. By Lemma 2.1, we deduce that $y(t)$ is almost automorphic in finite-dimensional distributions.

Step 3. We will show that the x obtained in Step 1 is pseudo almost automorphic in finite-dimensional distributions.

Let $z = x-y$. Then, $x = y+z$. From Step 2, we have $y\in DAA(\mathbb{R},\mathcal{L}^2(\Omega,\mathbb{O}^n))$. So, to end the proof, we just need to prove that $z\in DPAA_0(\mathbb{R},\mathcal{L}^2(\Omega,\mathbb{O}^n))$.

From (3.2) and (3.10), it follows that

$$\begin{aligned}
&\frac{1}{2T}\int_{-T}^{T}E\|z(t_q+t)\|_0^2\mathrm{d}t\\
\le&10\max_{1\le i\le n}\left\{\frac{1}{2T}\int_{-T}^{T}E\left\|\int_{-\infty}^{t_q+t}e^{-\int_s^{t_q+t}a_i^{\phi}(u)\mathrm{d}u}\big[-a_i^c(s)(x_i(s)-y_i(s))\big]ds\right\|_{\mathbb{O}}^2\mathrm{d}t\right\}\\
&+10\max_{1\le i\le n}\left\{\frac{1}{2T}\int_{-T}^{T}E\left\|\int_{-\infty}^{t_q+t}e^{-\int_s^{t_q+t}a_i^{\phi}(u)\mathrm{d}u}\sum_{j=1}^{n}(b_{ij}(s)-b_{ij}^1(s))f_j(x_j(s))\mathrm{d}s\right\|_{\mathbb{O}}^2\mathrm{d}t\right\}\\
&+10\max_{1\le i\le n}\left\{\frac{1}{2T}\int_{-T}^{T}E\left\|\int_{-\infty}^{t_q+t}e^{-\int_s^{t_q+t}a_i^{\phi}(u)\mathrm{d}u}\sum_{j=1}^{n}b_{ij}^1(s)[f_j(x_j(s))-f_j(y_j(s))]\mathrm{d}s\right\|_{\mathbb{O}}^2\mathrm{d}t\right\}\\
&+10\max_{1\le i\le n}\left\{\frac{1}{2T}\int_{-T}^{T}E\left\|\int_{-\infty}^{t_q+t}e^{-\int_s^{t_q+t}a_i^{\phi}(u)\mathrm{d}u}\sum_{j=1}^{n}(c_{ij}(s)-c_{ij}^1(s))g_j(x_j(s-\eta_{ij}(s)))\mathrm{d}s\right\|_{\mathbb{O}}^2\mathrm{d}t\right\}\\
&+10\max_{1\le i\le n}\left\{\frac{1}{2T}\int_{-T}^{T}E\left\|\int_{-\infty}^{t_q+t}e^{-\int_s^{t_q+t}a_i^{\phi}(u)\mathrm{d}u}\sum_{j=1}^{n}c_{ij}^1(s)[g_j(x_j(s-\eta_{ij}(s)))-g_j(y_j(s-\eta_{ij}(s)))]\mathrm{d}s\right\|_{\mathbb{O}}^2\mathrm{d}t\right\}\\
&+10\max_{1\le i\le n}\left\{\frac{1}{2T}\int_{-T}^{T}E\left\|\int_{-\infty}^{t_q+t}e^{-\int_s^{t_q+t}a_i^{\phi}(u)\mathrm{d}u}\sum_{j=1}^{n}\left[\sum_{l=1}^{n}(Q_{ijl}(s)-Q_{ijl}^1(s))\right]\right.\right.\\
&\left.\left.\times r_j(x_j(s-\theta_{ijl}(s)))r_l(x_l(s-\xi_{ijl}(s)))\mathrm{d}s\right\|_{\mathbb{O}}^2\mathrm{d}t\right\}\\
&+10\max_{1\le i\le n}\left\{\frac{1}{2T}\int_{-T}^{T}E\left\|\int_{-\infty}^{t_q+t}e^{-\int_s^{t_q+t}a_i^{\phi}(u)\mathrm{d}u}\sum_{j=1}^{n}\sum_{l=1}^{n}Q_{ijl}^1(s)[r_j(x_j(s-\theta_{ijl}(s)))\right.\right.\\
&\left.\left.\times r_l(x_l(s-\xi_{ijl}(s)))-r_j(y_j(s-\theta_{ijl}(s)))r_l(y_l(s-\xi_{ijl}(s)))]\mathrm{d}s\right\|_{\mathbb{O}}^2\mathrm{d}t\right\}\\
&+10\max_{1\le i\le n}\left\{\frac{1}{2T}\int_{-T}^{T}E\left\|\int_{-\infty}^{t_q+t}e^{-\int_s^{t_q+t}a_i^{\phi}(u)\mathrm{d}u}[I_i(s)-I_i^1(s)]\mathrm{d}s\right\|_{\mathbb{O}}^2\mathrm{d}t\right\}
\end{aligned}$$

$$
\begin{aligned}
&+10\max_{1\le i\le n}\left\{\frac{1}{2T}\int_{-T}^{T}E\left\|\int_{-\infty}^{t_q+t}e^{-\int_s^{t_q+t}a_i^{\phi}(u)\mathrm{d}u}\sum_{j=1}^{n}(h_{ij}(s)-h_{ij}^{1}(s))\delta_{ij}(x_j(s-v_{ij}(s)))\mathrm{d}w_j(s)\right\|_{\mathbb{O}}^{2}\mathrm{d}t\right\}\\
&+10\max_{1\le i\le n}\left\{\frac{1}{2T}\int_{-T}^{T}E\left\|\int_{-\infty}^{t_q+t}e^{-\int_s^{t_q+t}a_i^{\phi}(u)\mathrm{d}u}\sum_{j=1}^{n}h_{ij}^{1}(s)[\delta_{ij}(x_j(s-v_{ij}(s)))\right.\right.\\
&\left.\left.-\delta_{ij}(y_j(s-v_{ij}(s)))]\mathrm{d}w_j(s)\right\|_{\mathbb{O}}^{2}\mathrm{d}t\right\}\\
:=&\sum_{i=1}^{10}D_i(t). \qquad (3.42)
\end{aligned}
$$

By a change of variables and Fubini's theorem, we can get

$$
\begin{aligned}
D_1(t)=&10\max_{1\le i\le n}\left\{\frac{1}{2T}\int_{-T}^{T}E\left\|\int_{-\infty}^{t_q+t}e^{-\int_s^{t_q+t}a_i^{\phi}(u)\mathrm{d}u}[-a_i^{c}(s)(x_i(s)-y_i(s))]\mathrm{d}s\right\|_{\mathbb{O}}^{2}\mathrm{d}t\right\}\\
\le&10\max_{1\le i\le n}\left\{\frac{1}{\underline{a}_i^{\phi}}(\bar{a}_i^{c})^2\frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{t_q+t}e^{-\underline{a}_i^{\phi}(t_q+t-s)}E\|z(s)\|_0^2\mathrm{d}s\mathrm{d}t\right\}\\
=&10\max_{1\le i\le n}\left\{\frac{1}{\underline{a}_i^{\phi}}(\bar{a}_i^{c})^2\int_{-\infty}^{0}\left(\frac{1}{2T}\int_{-T}^{T}e^{\underline{a}_i^{\phi}s}E\|z(s+t_q+t)\|_0^2\mathrm{d}t\right)\mathrm{d}s\right\}\\
=&10\max_{1\le i\le n}\left\{\frac{1}{\underline{a}_i^{\phi}}(\bar{a}_i^{c})^2\int_{-\infty}^{0}\left(\frac{1}{2T}\int_{-T+s}^{T+s}e^{\underline{a}_i^{\phi}s}E\|z(t_q+t)\|_0^2\mathrm{d}t\right)\mathrm{d}s\right\}\\
=&10\max_{1\le i\le n}\left\{\frac{1}{\underline{a}_i^{\phi}}(\bar{a}_i^{c})^2\int_{-\infty}^{T}e^{-\underline{a}_i^{\phi}(T-s)}\left(\frac{1}{2T}\int_{s-2T}^{s}E\|z(t_q+t)\|_0^2\mathrm{d}t\right)\mathrm{d}s\right\}. \qquad (3.43)
\end{aligned}
$$

Similarly, we can obtain that

$$
D_3(t)\le 10\max_{1\le i\le n}\left\{\frac{1}{\underline{a}_i^{\phi}}\sum_{j=1}^{n}(\check{b}_{ij}^{1})^2\sum_{j=1}^{n}(L_j^{f})^2\int_{-\infty}^{T}e^{-\underline{a}_i^{\phi}(T-s)}\left(\frac{1}{2T}\int_{s-2T}^{s}E\|z(t_q+t)\|_0^2\mathrm{d}t\right)\mathrm{d}s\right\}. \qquad (3.44)
$$

For $D_5(t)$, through variable substitutions and Fubini's theorem, one has

$$
\begin{aligned}
D_5(t)\le&10\max_{1\le i\le n}\left\{\frac{1}{\underline{a}_i^{\phi}}\sum_{j=1}^{n}(\check{c}_{ij}^{1})^2\sum_{j=1}^{n}(L_j^{g})^2\frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{t_q+t}e^{-\underline{a}_i^{\phi}(t_q+t-s)}E\|z(s-\eta_{ij}(s))\|_0^2\mathrm{d}s\mathrm{d}t\right\}\\
\le&10\max_{1\le i\le n}\left\{\frac{1}{\underline{a}_i^{\phi}}\sum_{j=1}^{n}(\check{c}_{ij}^{1})^2\sum_{j=1}^{n}(L_j^{g})^2\frac{e^{\underline{a}_i^{\phi}\eta_{ij}^{+}}}{1-\dot{\eta}_{ij}^{+}}\frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{t_q+t}e^{-\underline{a}_i^{\phi}(t_q+t-s)}E\|z(s)\|_0^2\mathrm{d}s\mathrm{d}t\right\}\\
=&10\max_{1\le i\le n}\left\{\frac{1}{\underline{a}_i^{\phi}}\sum_{j=1}^{n}(\check{c}_{ij}^{1})^2\sum_{j=1}^{n}(L_j^{g})^2\frac{e^{\underline{a}_i^{\phi}\eta_{ij}^{+}}}{1-\dot{\eta}_{ij}^{+}}\frac{1}{2T}\int_{-\infty}^{0}\int_{-T}^{T}e^{\underline{a}_i^{\phi}s}E\|z(s+t_q+t)\|_0^2\mathrm{d}t\mathrm{d}s\right\}\\
=&10\max_{1\le i\le n}\left\{\frac{1}{\underline{a}_i^{\phi}}\sum_{j=1}^{n}(\check{c}_{ij}^{1})^2\sum_{j=1}^{n}(L_j^{g})^2\frac{e^{\underline{a}_i^{\phi}\eta_{ij}^{+}}}{1-\dot{\eta}_{ij}^{+}}\int_{-\infty}^{0}\left(\frac{1}{2T}\int_{-T+s}^{T+s}e^{\underline{a}_i^{\phi}s}E\|z(t_q+t)\|_0^2\mathrm{d}t\right)\mathrm{d}s\right\}\\
\le&10\max_{1\le i\le n}\left\{\frac{1}{\underline{a}_i^{\phi}}\sum_{j=1}^{n}(\check{c}_{ij}^{1})^2\sum_{j=1}^{n}(L_j^{g})^2\frac{e^{\underline{a}_i^{\phi}\eta_{ij}^{+}}}{1-\dot{\eta}_{ij}^{+}}\int_{-\infty}^{T}e^{-\underline{a}_i(T-s)}\left(\frac{1}{2T}\int_{s-2T}^{s}E\|z(t_q+t)\|_0^2\mathrm{d}t\right)\mathrm{d}s\right\}. \qquad (3.45)
\end{aligned}
$$

For $D_7(t)$, it can be estimated in a similar way. That is, we have

$$\begin{aligned}
D_7(t) \leq & 20n \max_{1\leq i\leq n}\Big\{\sum_{j=1}^{n}\Big[\sum_{l=1}^{n}(\check{Q}_{ijl}^{1})^2\sum_{l=1}^{n}(M_j^r L_l^r)^2\int_{-\infty}^{t_q+t} e^{-\int_s^{t_q+t} a_i^{\phi}(u)\mathrm{d}u} E\|z(s-\xi_{ijl}(s))\|_0^2 \mathrm{d}s\mathrm{d}t\Big]\Big\} \\
& + 20n \max_{1\leq i\leq n}\Big\{\sum_{j=1}^{n}\Big[\sum_{l=1}^{n}(\check{Q}_{ijl}^{1})^2\sum_{l=1}^{n}(M_l^r L_j^r)^2\int_{-\infty}^{t_q+t} e^{-\int_s^{t_q+t} a_i^{\phi}(u)\mathrm{d}u} E\|z(s-\theta_{ijl}(s))\|_0^2 \mathrm{d}s\mathrm{d}t\Big]\Big\} \\
\leq & 20n \max_{1\leq i\leq n}\Big\{\sum_{j=1}^{n}\Big[\sum_{l=1}^{n}(\check{Q}_{ijl}^{1})^2\sum_{l=1}^{n}(M_j^r L_l^r)^2\Big]\frac{e^{\underline{a}_i^{\phi}\xi_{ijl}^{+}}}{1-\dot{\xi}_{ij}^{+}}\int_{-\infty}^{T} e^{-\underline{a}_i^{\phi}(T-s)}\Big(\frac{1}{2T}\int_{s-2T}^{s} E\|z(t_q+t)\|_0^2 \mathrm{d}t\Big)\mathrm{d}s\Big\} \\
& + 20n \max_{1\leq i\leq n}\Big\{\sum_{j=1}^{n}\Big[\sum_{l=1}^{n}(\check{Q}_{ijl}^{1})^2\sum_{l=1}^{n}(M_l^r L_j^r)^2\Big]\frac{e^{\underline{a}_i^{\phi}\theta_{ijl}^{+}}}{1-\dot{\theta}_{ijl}^{+}} \\
& \times \int_{-\infty}^{T} e^{-\underline{a}_i^{\phi}(T-s)}\Big(\frac{1}{2T}\int_{s-2T}^{s} E\|z(t_q+t)\|_0^2 \mathrm{d}t\Big)\mathrm{d}s\Big\}. \qquad (3.46)
\end{aligned}$$

Next, we will estimate $D_2(t)$. According to Fubini's theorem, we can get

$$\begin{aligned}
D_2(t) \leq & 10 \max_{1\leq i\leq n}\Big\{\frac{1}{\underline{a}_i^{\phi}}\sum_{j=1}^{n}(L_j^f)^2(2K)^2\frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{t_q+t} e^{-\underline{a}_i^{\phi}(t_q+t-s)}\Big\|\sum_{j=1}^{n} b_{ij}^{0}(s)\Big\|_{\mathbb{O}}^2 \mathrm{d}s\mathrm{d}t\Big\} \\
= & 10 \max_{1\leq i\leq n}\Big\{\frac{1}{\underline{a}_i^{\phi}}\sum_{j=1}^{n}(L_j^f)^2(2K)^2\int_{0}^{+\infty}\frac{1}{2T}\int_{-T}^{T} e^{-\underline{a}_i^{\phi}s}\Big\|\sum_{j=1}^{n} b_{ij}^{0}(t_q+t-s)\Big\|_{\mathbb{O}}^2 \mathrm{d}t\mathrm{d}s\Big\} \\
= & 10 \max_{1\leq i\leq n}\Big\{\frac{1}{\underline{a}_i^{\phi}}\sum_{j=1}^{n}(L_j^f)^2(2K)^2\int_{0}^{+\infty}\frac{1}{2T}\int_{t_q-T-s}^{t_q+T-s} e^{-\underline{a}_i^{\phi}s}\Big\|\sum_{j=1}^{n} b_{ij}^{0}(t)\Big\|_{\mathbb{O}}^2 \mathrm{d}t\mathrm{d}s\Big\} \\
\leq & 10 \max_{1\leq i\leq n}\Big\{\frac{1}{\underline{a}_i^{\phi}}\sum_{j=1}^{n}(L_j^f)^2(2K)^2\int_{0}^{+\infty}\Big(1+\frac{s}{T}\Big)\frac{1}{2(T+s)}\int_{t_q-T-s}^{t_q+T+s} e^{-\underline{a}_i^{\phi}s}\Big\|\sum_{j=1}^{n} b_{ij}^{0}(t)\Big\|_{\mathbb{O}}^2 \mathrm{d}t\mathrm{d}s\Big\} \\
\leq & 10 \max_{1\leq i\leq n}\Big\{\frac{1}{\underline{a}_i^{\phi}}\sum_{j=1}^{n}(L_j^f)^2(2K)^2\int_{0}^{+\infty} e^{-\underline{a}_i^{\phi}s}e^{\frac{s}{T}}\mathrm{d}s\frac{1}{2T}\int_{-T}^{T}\Big\|\sum_{j=1}^{n} b_{ij}^{0}(t)\Big\|_{\mathbb{O}}^2 \mathrm{d}t\Big\}. \qquad (3.47)
\end{aligned}$$

By the Lebesgue dominated convergence theorem, one has $D_2(t) \to 0$ as $T \to +\infty$. Similarly, we can get $D_4(t) \to 0$, $D_6(t) \to 0$, $D_8(t) \to 0$, and $D_9(t) \to 0$ as $T \to +\infty$.

Now, we estimate $D_{10}(t)$. By the Itô isometry and the Fubini theorem, we deduce that

$$\begin{aligned}
D_{10}(t) \leq & 10 \max_{1\leq i\leq n}\Big\{\sum_{j=1}^{n}(\check{h}_{ij}^{1})^2\sum_{j=1}^{n}(L_{ij}^{\delta})^2\frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{t_q+t} e^{-2\underline{a}_i^{\phi}(t_q+t-s)} E\|s-v_{ij}(s)\|_0^2 \mathrm{d}s\mathrm{d}t\Big\} \\
\leq & 10 \max_{1\leq i\leq n}\Big\{\sum_{j=1}^{n}(\check{h}_{ij}^{1})^2\sum_{j=1}^{n}(L_{ij}^{\delta})^2\frac{e^{2\underline{a}_i^{\phi}v_{ij}^{+}}}{1-\dot{v}_{ij}^{+}}\frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{t_q+t} e^{-2\underline{a}_i^{\phi}(t_q+t-s)} E\|z(s)\|_0^2 \mathrm{d}s\mathrm{d}t\Big\} \\
\leq & 10 \max_{1\leq i\leq n}\Big\{\sum_{j=1}^{n}(\check{h}_{ij}^{1})^2\sum_{j=1}^{n}(L_{ij}^{\delta})^2\frac{e^{2\underline{a}_i^{\phi}v_{ij}^{+}}}{1-\dot{v}_{ij}^{+}}\int_{-\infty}^{T} e^{-2\underline{a}_i^{\phi}(T-s)}\Big(\frac{1}{2T}\int_{s-2T}^{s} E\|z(t_q+t)\|_0^2 \mathrm{d}t\Big)\mathrm{d}s\Big\}. \qquad (3.48)
\end{aligned}$$

Therefore, by (3.32)–(3.48), we have

$$\frac{1}{2T}\int_{-T}^{T} E\|z(t_q+t)\|_0^2\mathrm{d}t \le \iota(t) + \Xi\int_{-\infty}^{T} e^{-a^-(T-s)}\Big(\frac{1}{2T}\int_{s-2T}^{s} E\|z(t_q+t)\|_0^2\mathrm{d}t\Big)\mathrm{d}s,$$

where $\iota(t) = D_2(t) + D_4(t) + D_6(t) + D_8(t) + D_9(t)$ with $D_h(t) = 0$ as $T \to \infty$, $h = 2, 4, 6, 8, 9$, and

$$\begin{aligned}\Xi = \max_{1\le i\le n}\Bigg\{&\frac{10}{\underline{a}_i^{\phi}}\Big[(\bar{a}_i^{c})^2 + \sum_{j=1}^{n}(\check{b}_{ij}^{1})^2\sum_{j=1}^{n}(L_j^{f})^2 + \sum_{j=1}^{n}(\check{c}_{ij}^{1})^2\sum_{j=1}^{n}(L_j^{g})^2\frac{e^{\underline{a}_i^{\phi}\eta_{ij}^{+}}}{1-\dot{\eta}_{ij}^{+}}\\ &+ 2n\sum_{j=1}^{n}\Big[\sum_{l=1}^{n}(\check{Q}_{ijl}^{1})^2\sum_{l=1}^{n}(M_j^{r}L_l^{r})^2\Big]\frac{e^{\underline{a}_i^{\phi}\xi_{ijl}^{+}}}{1-\dot{\xi}_{ijl}^{+}} + 2n\sum_{j=1}^{n}\Big[\sum_{l=1}^{n}(\check{Q}_{ijl}^{1})^2\sum_{l=1}^{n}(M_l^{r}L_j^{r})^2\Big]\\ &\times\frac{e^{\underline{a}_i^{\phi}\theta_{ijl}^{+}}}{1-\dot{\theta}_{ijl}^{+}} + \underline{a}_i^{\phi}\sum_{j=1}^{n}(\check{h}_{ij}^{1})^2\sum_{j=1}^{n}(L_{ij}^{\delta})^2\frac{e^{2\underline{a}_i^{\phi}v_{ij}^{+}}}{1-\dot{v}_{ij}^{+}}\Big]\Bigg\} < \min_{1\le i\le n}\{\underline{a}_i^{\phi}\}.\end{aligned}$$

By the definition of Υ in (A_5), we have $\Xi \le \Upsilon$. Hence, by (A_4), $\Xi < \min_{1\le i\le n}\{\underline{a}_i^{\phi}\}$.

Since $\iota(t) \to 0$ as $T \to +\infty$, by Lemma 2.3, we obtain that $\frac{1}{2T}\int_{-T}^{T} E\|z(t_q+t)\|_0^2\mathrm{d}t \to 0$ as $T \to \infty$ for any $t_q \in \mathbb{R}$. According to Lemma 2.2, we deduce that $z \in DPAA_0(\mathbb{R}, \mathcal{L}^2(\Omega, \mathbb{O}^n))$. The proof is completed.

Remark 3.1. *In the proof of Theorem 3.1, we take $\mathbb{H}$ to be the space of $\mathcal{L}^2$-bounded and $\mathcal{L}^2$-uniformly continuous functions in order to handle the case with time delay; if the system under consideration has no time delay, then it suffices to take $\mathbb{H}$ as the space of $\mathcal{L}^2$-bounded and $\mathcal{L}^2$-continuous functions.*

By adapting the method used in the proof of Theorem 3.2 in [34], one can easily obtain the following theorem.

Theorem 3.2. *Let conditions $(A_1)-(A_5)$ be fulfilled. Then, system (2.1) admits a unique pseudo almost automorphic solution in finite-dimensional distributions, which is global exponential stability, that is, for the pseudo almost automorphic solution in finite-dimensional distributions $x = (x_1, \cdots, x_n)$ of (2.1) agreeing with (2.2), there are positive constants $\lambda > 0$ and $M > 1$ such that each solution $x' = (x_1', \cdots, x_n')$ with the initial value $\phi = (\phi_1, \cdots, \phi_n)$ satisfies*

$$E\|x'(t) - x(t)\|_0^2 \le M(E\|\phi - \psi\|_0^2)e^{-\lambda t},\ t > 0,$$

where

$$\|\phi - \psi\|_0^2 = \sup_{s\in[-\rho,0]} E\|\phi_i(t) - \psi_i(t)\|_{\mathbb{O}}^2.$$

4. An example

In system (2.1), let $e_0 = 1, n = 2$, and for $i, j, l = 1, 2$, take

$$\begin{aligned}x_i =& x_i^0 + x_i^1e_1 + x_i^2e_2 + x_i^3e_3 + x_i^4e_4 + x_i^5e_5 + x_i^6e_6 + x_i^7e_7,\\ f_j(x_j) =& 0.031\sin(x_j^0) + 0.015e_1\sin(x_j^1) + 0.027e_2\tanh(x_j^2) + 0.024e_3\sin(x_j^3)\\ &+ 0.025e_4\sin(x_j^4) + 0.018e_5\sin(x_j^5) + 0.025e_6\tanh(x_j^6) + 0.029e_7\sin(x_j^7),\end{aligned}$$

$$\begin{aligned}
g_j(x_j) =&0.07\sin(x_j^0)+0.05e_1\sin(x_j^1)+0.04e_2\sin(x_j^2)+0.06e_3\sin(x_j^3)\\
&+0.05e_4\sin(x_j^4)+0.02e_5\sin(x_j^5)+0.04e_6\sin(x_j^6)+0.03e_7\sin(x_j^7),\\
r_j(x_j) =&0.04\sin(x_j^0)+0.02e_1\sin(x_j^1)+0.04e_2\sin(x_j^2)+0.03e_3\sin(x_j^3)\\
&+0.04e_4\sin(x_j^4)+0.03e_5\sin(x_j^5)+0.02e_6\sin(x_j^6)+0.01e_7\sin(x_j^7),\\
\delta_{ij}(x_j) =&0.037\sin(x_j^0)+0.015e_1\sin(x_j^1)+0.017e_2\tanh(x_j^2)+0.023e_3\sin(x_j^3)\\
&+0.01e_4\sin(x_j^4)+0.021e_5\sin(x_j^5)+0.031e_6\sin(x_j^6)+0.015e_7\sin(x_j^7),\\
a_1(t) =&(1.4+0.4\sin t)+(0.03+0.02\cos\sqrt{3}t)e_1\\
&+(0.03+0.02\sin\sqrt{2}t)e_2+(0.04+0.02\sin t)e_3\\
&+(0.04+0.01\cos t)e_4+(0.05+0.01\cos t)e_5\\
&+(0.04+0.01\cos\sqrt{3}t)e_6+(0.07+0.01\sin\sqrt{7}t)e_7,\\
a_2(t) =&(2.1-1.1\cos\sqrt{5}t)+(0.08+0.02\sin t)e_1\\
&+(0.05+0.04\cos\sqrt{3}t)e_2+(0.05+0.02\sin\sqrt{2}t)e_3\\
&+(0.05+0.01\sin\sqrt{3}t)e_4+(0.05+0.02\cos\sqrt{2}t)e_5\\
&+(0.04+0.02\sin t)e_6+(0.05+0.01\sin t)e_7,\\
b_{11}(t) =&0.07\sin t+0.03e_3\frac{\sin t}{1+t^2},\ b_{12}(t)=0.05e_1\cos t-0.02e_6\frac{1}{1+t^2},\\
b_{21}(t) =&0.05e_4\sin t+0.01e_7\sin\sqrt{2}t,\ b_{22}(t)=0.08e_2\cos t-0.02e_5\frac{\sin t}{1+t^2},\\
c_{11}(t) =&0.03e_4\sin\sqrt{5}t-0.01e_7\cos t,\ c_{12}(t)=0.07e_2\cos\sqrt{2}t-0.02e_5\sin t,\\
c_{21}(t) =&0.06\sin t-0.02e_3\frac{\sin t}{1+t^2},\ c_{22}(t)=0.07e_1\cos t-0.01e_6\sin\sqrt{3}t,\\
Q_{111}(t) =&Q_{112}(t)=0.08\sin t+0.01e_3\sin\sqrt{5}t,\ Q_{121}(t)=Q_{122}(t)=0.07e_1\cos\sqrt{5}t-0.02e_6\frac{1}{1+t^2},\\
Q_{211}(t) =&Q_{212}(t)=0.06e_4\cos t+0.01e_7\frac{\sin t}{1+t^2},\ Q_{221}(t)=Q_{222}(t)=0.09e_2\sin t-0.02e_5\cos t,\\
h_{11}(t) =&0.06\cos t+0.02e_3\frac{\sin t}{1+t^2},\ h_{12}(t)=0.04e_1\sin\sqrt{3}t-0.01e_6\cos t,\\
h_{21}(t) =&0.08e_4\sin\sqrt{2}t+0.01e_7\sin t,\ h_{22}(t)=0.05e_2\cos t-0.02e_5\sin\sqrt{3}t,\\
I_1(t) =&1.7\cos\sqrt{3}t+1.5e_1\sin t+1.4e_2\cos\left(\frac{1}{2+\sin t+\sin\sqrt{3}t}\right)+1.5e_3\sin\sqrt{2}t\\
&+1.7e_4\cos t+1.2e_5\frac{1}{1+t^2}+1.9e_6\cos\left(\frac{1}{2+\sin 2t+\sin\sqrt{2}t}\right)+1.8e_7\sin t,\\
I_2(t) =&1.6\sin\sqrt{5}t+1.7e_1\cos\sqrt{2}t+1.4e_2\sin\sqrt{3}t+1.5e_3\cos\left(\frac{1}{2+\sin t+\sin\sqrt{2}t}\right)\\
&+1.6e_4\sin t+0.8e_5\sin\sqrt{7}t+1.7e_6\cos t+1.8e_7\frac{1}{1+t^2},\\
\eta_{11}(t) =&\eta_{12}(t)=0.09+0.01\sin t,\ \eta_{21}(t)=\eta_{22}(t)=0.08+0.02\cos t,\\
\xi_{111}(t) =&\xi_{112}(t)=\xi_{121}(t)=\xi_{122}(t)=0.07+0.02\sin t,
\end{aligned}$$

$$\begin{aligned}
\xi_{211}(t) =&\xi_{212}(t) = \xi_{221}(t) = \xi_{222}(t) = 0.06 + 0.04\sin t,\\
\theta_{111}(t) =&\theta_{112}(t) = \theta_{121}(t) = \theta_{122}(t) = 0.07 + 0.03\cos t,\\
\theta_{211}(t) =&\theta_{212}(t) = \theta_{221}(t) = \theta_{222}(t) = 0.08 + 0.02\sin t,\\
v_{11}(t) =&v_{12}(t) = 0.08 + 0.01\cos t,\ \ v_{21}(t) = v_{22}(t) = 0.09 + 0.01\sin t.
\end{aligned}$$

Then, we have

$$L_j^f = 0.07, L_j^g \approx 0.134, L_j^r \approx 0.087, M_j^r = 0.04, L_{ij}^\delta \approx 0.064, \bar{a}_1^c \approx 0.154, \bar{a}_2^c \approx 0.197, \check{b}_{11}^1 = 0.07,$$
$$\check{b}_{11} \approx 0.076, \check{b}_{12}^1 = 0.05, \check{b}_{12} \approx 0.054, \check{b}_{21}^1 = \check{b}_{21} \approx 0.051, \check{b}_{22}^1 = 0.08, \check{b}_{22} \approx 0.082, \check{c}_{11}^1 = \check{c}_{11} = 0.1,$$
$$\check{c}_{12}^1 = \check{c}_{12} \approx 0.073, \check{c}_{21}^1 = 0.06, \check{c}_{21} \approx 0.063, \check{c}_{22}^1 = \check{c}_{22} \approx 0.071, \check{Q}_{111}^1 = \check{Q}_{112}^1 = \check{Q}_{111} = \check{Q}_{112} \approx$$
$$0.081, \check{Q}_{121}^1 = \check{Q}_{122}^1 = 0.07, \check{Q}_{121} = \check{Q}_{122} \approx 0.073, \check{Q}_{211}^1 = \check{Q}_{212}^1 = 0.06, \check{Q}_{211} = \check{Q}_{212} = 0.061,$$
$$\check{Q}_{211}^1 = \check{Q}_{222}^1 = \check{Q}_{221} = \check{Q}_{222}^1 \approx 0.092, \check{h}_{11}^1 = 0.06, \check{h}_{11} \approx 0.063, \check{h}_{12}^1 = \check{h}_{12} \approx 0.041, \check{h}_{21}^1 = \check{h}_{21} \approx$$
$$0.081, \check{h}_{22}^1 = \check{h}_{22} \approx 0.054, \underline{a}_1^\phi = \underline{a}_2^\phi = 1, R \approx 0.196, R' \approx 0.196, \Upsilon \approx 0.680.$$

So, $(A_1) - (A_5)$ are satisfied. Hence, we see that system (2.1) has a unique pseudo almost automorphic solution in finite-dimensional distribution that is globally exponentially stable (see Figures 1–6).

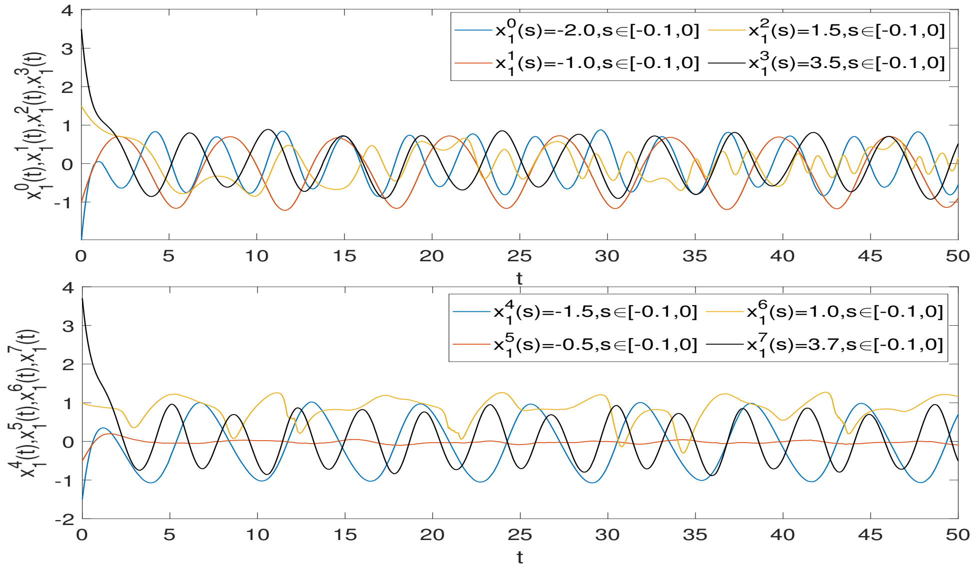

Figure 1. The states $x_1^0(t), x_1^1(t), x_1^2(t), x_1^3(t), x_1^4(t), x_1^5(t), x_1^6(t), x_1^7(t)$ of system (2.1) all exhibit almost automorphic oscillations.

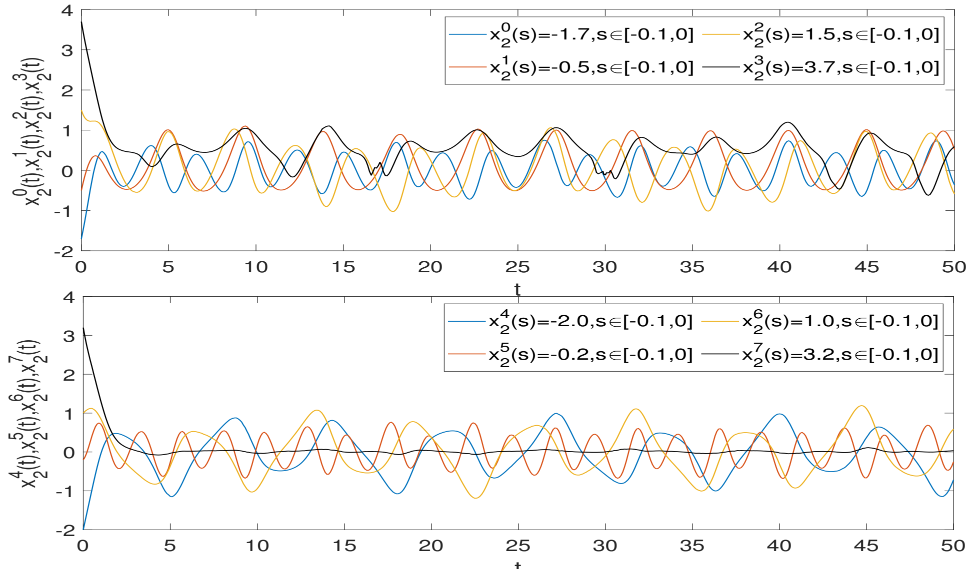

Figure 2. The states $x_2^0(t), x_2^1(t), x_2^2(t), x_2^3(t), x_2^4(t), x_2^5(t), x_2^6(t), x_2^7(t)$ of (2.1) all exhibit almost automorphic oscillations.

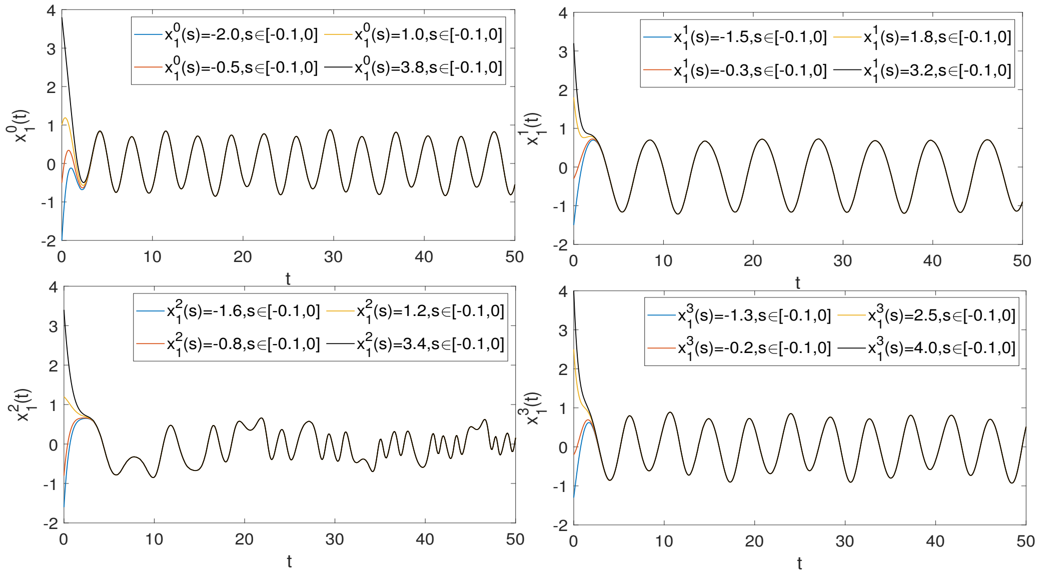

Figure 3. Global exponential stability of states $x_1^0(t), x_1^1(t), x_1^2(t), x_1^3(t)$ of (2.1) with different initial values.

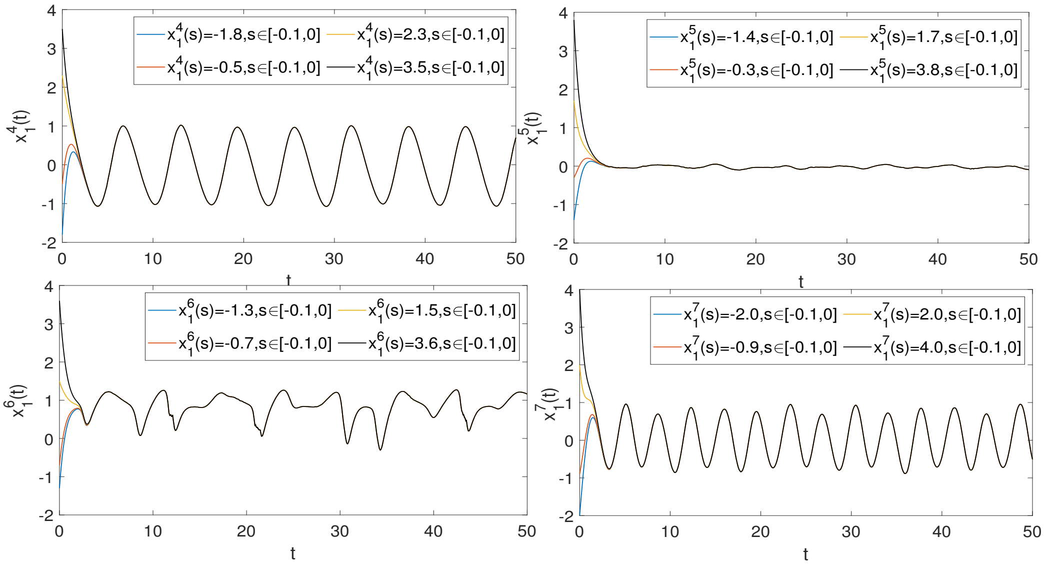

Figure 4. Global exponential stability of states $x_1^4(t), x_1^5(t), x_1^6(t), x_1^7(t)$ of (2.1) with different initial values.

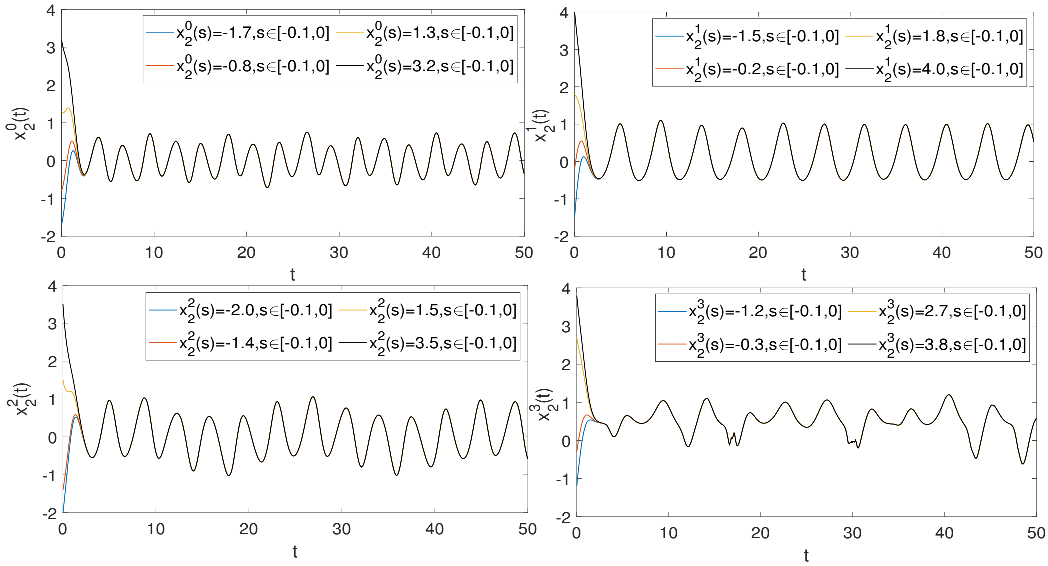

Figure 5. Global exponential stability of states $x_2^0(t), x_2^1(t), x_2^2(t), x_2^3(t)$ of (2.1) with different initial values.

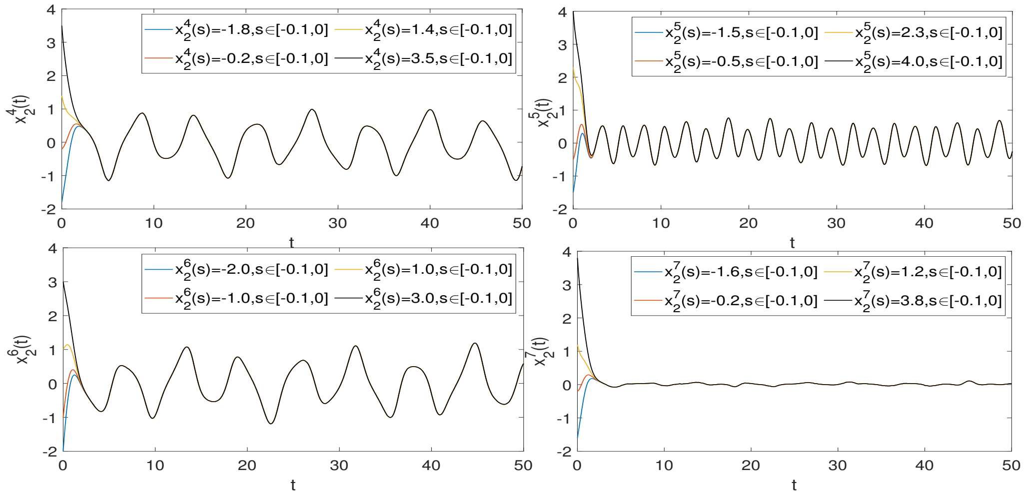

Figure 6. Global exponential stability of states $x_2^4(t), x_2^5(t), x_2^6(t), x_2^7(t)$ of (2.1) with different initial values.

Figures 1 and 2 show that each component of the state variables of system (2.1) exhibits a certain recurrent oscillation phenomenon over time, indicating that system (2.1) has a pseudo almost automorphic solution.

Figures 3–6 show that the same component of the state variables of system (2.1) with different initial values rapidly coincide over time, indicating that the solution of system (2.1) is exponentially stable.

5. Conclusions

This paper has investigated the pseudo almost automorphic dynamics for a class of octonion-valued stochastic higher-order Hopfield neural networks with time-varying delays in the finite-dimensional distribution sense. The main work and contributions are summarized as follows.

First, to achieve a higher accuracy in capturing the oscillatory characteristics of stochastic systems, a new definition of pseudo almost automorphic stochastic processes based on finite-dimensional distributions was introduced, which provides a significantly more thorough description than the conventional one-dimensional distribution sense.

Second, by effectively handling the complexities arising from the non-commutative and non-associative nature of octonions, stochastic noise, time delays, and higher-order interactions, five easily verifiable sufficient conditions were derived. These conditions guarantee the existence and global exponential stability of the finite-dimensional distribution pseudo almost automorphic solution for the proposed network model. The results significantly extend the dynamical analysis of into the octonion domain with stochastic perturbations.

Third, the theoretical findings presented in this paper are fundamentally new. Their novelty persists even when the octonion-valued system is reduced to a quaternion-valued, complex-valued, or real-valued system, thereby enriching the dynamical theory for across various number systems.

Finally, the research framework and analytical techniques employed, including the fixed-point theorem and Lyapunov method, are not limited to the specific model studied. They offer a viable and general approach for exploring pseudo almost automorphic dynamics and other collective behaviors in a broader class of stochastic with high-dimensional algebra or complex topology.

Future work may focus on exploring the practical applications of these theoretical guarantees in areas such as high-dimensional associative memory and secure communication. Additionally, extending the current analysis to include other complex dynamics, such as multistability or synchronization in finite-time/fixed-time, and considering a wider class of generalized noise (e.g., Lévy noise) or impulsive effects, present important and challenging research directions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

References

1. C. A. Popa, Octonion-valued neural networks, in *Artificial Neural Networks and Machine Learning – ICANN 2016*, **9886** (2016), 435–443. https://doi.org/10.1007/978-3-319-44778-0_51
2. M. Amdouni, μ-Stability of (μ_1, μ_2)-pseudo almost periodic solution for octonion-valued fuzzy BAM cellular neural networks with mixed delays, *Chaos Solitons Fractals*, **203** (2026), 117665. https://doi.org/10.1016/j.chaos.2025.117665
3. N. Zhao, Y. Qiao, J. Miao, L. Duan, Fixed-time synchronization of impulsive octonion-valued fuzzy inertial neural networks via improving fixed-time stability, *IEEE Trans. Fuzzy Syst.*, **32** (2024), 1978–1990. https://doi.org/10.1109/TFUZZ.2023.3339162
4. K. Takahashi, M. Fujita, M. Hashimoto, Remarks on octonion–valued neural networks with application to robot manipulator control, in *IEEE International Conference on Mechatronics (ICM)*, (2021), 1–6. https://doi.org/10.1109/ICM46511.2021.9385617
5. U. Kandasamy, R. Rajan, Hopf bifurcation of a fractional-order octonion-valued neural networks with time delays, *Discrete Contin. Dyn. Syst. Ser. S*, **13** (2020), 2537–2559. https://doi.org/10.3934/dcdss.2020137
6. V. Agrawal, S. Singh, V. K. Singh, S. Das, Fixed-time synchronization of octonion-valued neural networks with mixed delays: A non-separation norm approach, *Neurocomputing*, **652** (2025), 130995. https://doi.org/10.1016/j.neucom.2025.130995
7. B. Li, Y. Li, H. Xu, Almost automorphic dynamics to stochastic octonion-valued fuzzy neural networks with delays, *Fuzzy Sets Syst.*, **521** (2025), 109592. https://doi.org/10.1016/j.fss.2025.109592

8. Y. Li, R. Xu, Global exponential Weyl almost periodic synchronization for Hypercomplex-valued high-order Hopfield neural networks with delays, *Qual. Theory Dyn. Syst.*, **25** (2026), 33. https://doi.org/10.1007/s12346-026-01459-3

9. C. Aouiti, F. Dridi, (μ, ν)-Pseudo-almost automorphic solutions for high-order Hopfield bidirectional associative memory neural networks, *Neural Comput. Appl.*, **32** (2020), 1435–1456. https://doi.org/10.1007/s00521-018-3651-6

10. S. Dhama, S. Abbas, Existence and stability of weighted pseudo almost automorphic solution of dynamic equation on time scales with weighted Stepanov-like (S^p) pseudo almost automorphic coefficients, *Qual. Theory Dyn. Syst.*, **19** (2020), 46. https://doi.org/10.1007/s12346-020-00385-2

11. C. Aouiti, H. Jallouli, M. Miraoui, Global exponential stability of pseudo almost automorphic solutions for delayed Cohen-Grosberg neural networks with measure, *Appl. Math.*, **67** (2022), 393-–418. https://doi.org/10.21136/AM.2022.0201-20

12. A. M. Alimi, S. Khelifi, M. Miraoui, The measure pseudo almost-periodicty and automorphy in HOHNNs with time-varying delays, *Int. J. Comput. Math.*, **100** (2023), 1284–1302. https://doi.org/10.1080/00207160.2023.2178275

13. F. X. Zheng, H. X. Li, Pseudo compact almost automorphic solutions to a family of delay differential equations, *Demonstr. Math.*, **57** (2024), 20240074. https://doi.org/10.1515/dema-2024-0074

14. M. Dieye, A. Diop, M. M. Mbaye, M. A. McKibben, On weighted pseudo almost automorphic mild solutions for some mean field stochastic evolution equations, *Stochastics*, **96** (2024), 1388–1427. https://doi.org/10.1080/17442508.2023.2283554

15. M. Abdelaziz, F. Chérif, Piecewise pseudo almost automorphic solution for impulsive Lasota-Wazewska model with mixed delays, *Appl. Math. J. Chin. Univ.*, **40** (2025), 102–121. https://doi.org/10.1007/s11766-025-4356-0

16. T. Su, Y. Kao, D. Jiang, Analysis of a generalized stochastic population diffusion epidemic model perturbed by Black–Karasinski process, *Nonlinear Dyn.*, **114** (2026), 121. https://doi.org/10.1007/s11071-025-11990-8

17. T. Su, Y. Kao, D. Jiang, Dynamical behaviors of a stochastic SIR epidemic model with reaction–diffusion and spatially heterogeneous transmission rate, *Chaos Solitons Fractals*, **195** (2025), 116283. https://doi.org/10.1016/j.chaos.2025.116283

18. M. Shi, D. Tong, Q. Chen, W. Zhou, Pth moment exponential synchronization for delayed multi-agent systems with Lévy noise and Markov switching, *IEEE Trans. Circuits Syst. II Express Briefs*, **71** (2024), 697–701. https://doi.org/10.1109/TCSII.2023.3304635

19. W. Wang, Mean-square exponential input-to-state stability of stochastic fuzzy delayed Cohen-Grossberg neural networks, *J. Exp. Theor. Artif. Intell.*, **36** (2024), 1823–1836. https://doi.org/10.1080/0952813X.2023.2165725

20. Q. Yao, T. Wei, P. Lin, L. Wang, Finite-time boundedness of impulsive delayed reaction–diffusion stochastic neural networks, *IEEE Trans. Neural Netw. Learn. Syst.*, **36** (2024), 4794–4804. https://doi.org/10.1109/TNNLS.2024.3360711

21. M. Luo, Z. Jiang, M. Tan, Mean-square stability of stochastic Clifford-valued Cohen-Grossberg neural networks with variable coefficients and time-varying delays, *Neurocomputing*, **625** (2025), 129500. https://doi.org/10.1016/j.neucom.2025.129500
22. M. Wang, S. Zhu, W. Luo, Z. Zhang, Finite-time and fixed-time self-triggered synchronization of stochastic memristive neural networks and applications in secure communication, *Neural Netw.*, **193** (2026), 108033. https://doi.org/10.1016/j.neunet.2025.108033
23. B. Li, Y. Cao, Y. Li, Almost periodic oscillation in distribution for octonion-valued neutral-type stochastic recurrent neural networks with D operator, *Nonlinear Dyn.*, **111** (2023), 11371–11388. https://doi.org/10.1007/s11071-023-08411-z
24. O. Mellah, P. Fitte, Counterexamples to mean square almost periodicity of the solutions of some SDEs with almost periodic coefficients, preprint, arXiv:1208.6384.
25. J. Baez, The octonions, preprint, arXiv:math/0105155.
26. G. M. N'Guérékata, *Almost Automorphic and Almost Periodic Functions in Abstract Spaces*, Springer, New York, 2001. https://doi.org/10.1007/978-3-030-73718-4
27. J. Liang, G. M. N'Guérékata, T. Xiao, J. Zhang, Some properties of pseudo almost automorphic functions and applications to abstract differential equations, *Nonlinear Anal.*, **70** (2009), 2731–2735. https://doi.org/10.1016/j.na.2008.03.061
28. P. H. Bezandry, T. Diagana, *Almost Periodic Stochastic Processes*, Springer, New York, 2011. https://doi.org/10.1007/978-1-4419-9476-9
29. P. H. Bezandry, T. Diagana, p-th Mean pseudo almost automorphic mild solutions to some nonautonomous stochastic differential equations, *Afr. Diaspora J. Math. (N.S.)*, **12** (2011), 60–79.
30. A. Klenke, *Probability Theory: A Comprehensive Course*, Springer, London, 2008. https://doi.org/10.1007/978-1-84800-048-3
31. M. Kamenskii, O. Mellah, P. Fitte, Weak averaging of semilinear stochastic differential equations with almost periodic coefficients, *J. Math. Anal. Appl.*, **427** (2015), 336–364. https://doi.org/10.1016/j.jmaa.2015.02.036
32. Y. Li, Z. Bai, Besicovitch almost automorphic solutions in finite-dimensional distributions to stochastic semilinear differential equations driven by both Brownian and fractional Brownian motions, *Math. Methods Appl. Sci.*, **48** (2025), 1685–1700. https://doi.org/10.1002/mma.10403
33. Y. Li, B. Li, Pseudo almost periodic solutions in distribution for stochastic differential equations driven by Brownian and fractional Brownian motions, *Stochastics*, **2025** (2025), 1–22, https://doi.org/10.1080/17442508.2025.2593356
34. X. Huang, Y. Li, Weyl almost periodic solutions of octonion-valued high-order fuzzy neural networks with delays, *Comput. Appl. Math.*, **42** (2023), 155. https://doi.org/10.1007/s40314-023-02294-x

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