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Theory article

Positive solutions for a Kirchhoff-Schrödinger-Poisson system with singular term

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Abstract: This work is concerned with a Kirchhoff-Schrödinger-Poisson (KSP) system posed in a bounded domain of $\mathbb{R}^3$. The model features a singular nonlinearity $\alpha v^{-\tau}$ with $0 < \tau < 1$, together with a coupling term of the form $\varphi|v|^{q-2}v$, where $2 < q < 3$. The singular term destroys differentiability of the energy functional while the nonlocal potential φ_v causes compactness issues. Using nonsmooth critical point theory, we establish a key estimate linking the weak slope with the derivative of the regular part, prove the Palais-Smale (PS) condition, and characterize critical points as weak solutions. By means of Ekeland's variational principle and the mountain pass theorem, we establish the existence of a constant $\Gamma > 0$ with the property that the system admits two distinct positive solutions whenever $\alpha \in (0, \Gamma)$.

Keywords: weak slope; Palais-Smale condition; nonlocal potential

1. Introduction and main result

The Kirchhoff-Schrödinger-Poisson (KSP) system originates from the description of charged quantum particles interacting with their self-generated electrostatic field [1,2]. Its general form [3] is

$$\begin{cases} -(a + b\int_\Omega |\nabla v|^2 dx)\Delta v + \varphi v = f(v), & \text{in } \Omega, \\ -\Delta \varphi = v^2, & \text{in } \Omega, \\ v = \varphi = 0, & \text{on } \partial\Omega, \end{cases}$$

where the coupling term φv appears on the left-hand side of the equation as a linear potential. The corresponding energy functional contains a positive definite term $\int_\Omega \varphi_v v^2 dx$, which helps maintain the coercivity and compactness properties of the functional, thus this structure is widely adopted (see [4–8]). When the nonlinearity $f(v)$ is smooth, classical variational methods can be effectively applied.

When a singular term $v^{-\tau}$ $(0 < \tau < 1)$ is introduced, the problem becomes more delicate. Zhang [8] obtained two positive solutions using a variational method combined with a perturbation technique under a general weak singularity assumption. Subsequently, Zhang [7] obtained one positive solution in the strong singularity case $(\tau > 1)$ by constructing a special constraint set and applying Ekeland's variational principle. Mu and Lu [5] employed the Nehari manifold method to obtain multiple positive solutions for the (KSP) system in the weak singularity setting. In these works, the coupling term remains on the left-hand side (i.e., φv as a linear term), keeping the nonlocal term positive definite. They also rely on the Nehari manifold or perturbation methods, which require the functional to be C^1 on the relevant space and impose monotonicity conditions on the nonlinearity to ensure uniqueness of the critical point of the fibering map.

In recent years, some works have focused on the critical growth case. Liang et al. [6] studied a p-Laplacian KSP-type system involving both singular and critical Sobolev nonlinearities, obtaining one positive solution via the Nehari manifold method. Feng et al. [9] and Ghosh [10] investigated fractional KSP systems and critical logarithmic nonlinearities, respectively, also relying on differentiability of the functional. Meanwhile, sign-changing solutions have been extensively studied; for instance, Chen et al. [11] and Wang and Zhang [12] obtained sign-changing solutions using perturbation methods and invariant set techniques. These studies have greatly enriched the theory of KSP systems, but they share the common premise that the coupling term appears on the left-hand side and the functional is differentiable.

In this paper, we consider a KSP system with a different structure:

$$\begin{cases} -(a + b\int_\Omega |\nabla v|^2 dx)\Delta v = \varphi |v|^{q-2} v + \alpha v^{-\tau}, & \text{in } \Omega, \\ -\Delta \varphi = |v|^q, & \text{in } \Omega, \\ v = \varphi = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $2 < q < 3$. This range of q is critical: $q > 2$ ensures the coupling term grows superlinearly, which is essential for establishing the mountain pass geometry; on the other hand, $q < 3$ keeps the nonlinearity subcritical, so the integral $\int \varphi_v |v|^q dx$ is well defined without requiring concentration-compactness arguments. Compared with the classical form, the coupling term $\varphi |v|^{q-2} v$ appears on the right-hand side as a nonlinear source. This change leads to the emergence of a negative term $-\frac{1}{2q}\int_\Omega \varphi_v |v|^q dx$ in the energy functional

$$I(v) = \frac{a}{2}\|v\|^2 + \frac{b}{4}\|v\|^4 - \frac{1}{2q}\int_\Omega \varphi_v |v|^q dx - \frac{\alpha}{1-\tau}\int_\Omega |v|^{1-\tau} dx,$$

while the singular term $\alpha v^{-\tau}$ makes the functional nondifferentiable. These two difficulties render the Nehari manifold method, perturbation techniques, and the classical mountain pass lemma inapplicable. Thus, under the combined difficulties of right-hand side coupling, weak singularity, and nonlocal interactions, the existence of solutions becomes a nontrivial problem.

To overcome these difficulties, we adopt the theory of weak slope introduced by Degiovanni and Marzocchi [13] (further developed by Canino and Degiovanni [14] and Liu et al. [15]). For a continuous functional defined on a complete metric space, this theory introduces an abstract notion of gradient, denoted $|df|(v)$, which is obtained by testing the existence of a descent flow. When f is C^1, one has $|df|(v) = \|f'(v)\|$; when f is nondifferentiable, the weak slope still allows one to define critical points [16] and (PS) sequences and to establish deformation lemmas and mountain pass theorems. Therefore, this nonsmooth critical point theory is a natural tool for handling KSP systems with right-hand side coupling and weak singularity.

Our main result is the following.

Theorem 1.1 *Assume* $0 < \tau < 1$ *and* $2 < q < 3$*. Then there exists* $\Gamma > 0$ *such that for every* $\alpha \in (0, \Gamma)$*, system (1.1) admits at least two distinct positive solutions* v_α *and* v^* *satisfying* $I(v_\alpha) < 0 < I(v^*)$.

Compared with existing results, our method does not require differentiability of the functional and applies to a broader range of parameters. It extends the weak slope theory to KSP systems with nonlocal coupling for the first time and successfully handles the interaction between the Kirchhoff term and the Poisson term. Under the weak singularity assumption ($0 < \tau < 1$), we establish the existence of two positive solutions. In contrast, only one solution is obtained when the singularity is strong.

Our analysis shows that even when the coupling term appears in a way that destroys the positive definiteness, nonsmooth critical point theory, together with appropriate estimates and limit arguments, can directly yield existence and multiplicity results within the framework of continuous functionals. This approach may be extended to other nonlocal singular systems with similar structures, providing a possible direction for further studies on critical exponents and sign-changing solutions.

The proof is organized as follows. Section 2 recalls the definition of weak slope and properties of φ_v. Section 3 establishes a key inequality (Lemma 3.1), proves the PS condition (Lemma 3.2), and shows that points with zero weak slope are weak solutions (Lemma 3.3). Section 4 verifies the mountain pass geometry and obtains two solutions: a local minimizer with negative energy via Ekeland's principle and a second solution with positive energy via the nonsmooth mountain pass theorem. Section 5 contains concluding remarks.

2. Preliminaries

2.1. Function spaces and notation

$H_0^1(\Omega)$ is equipped with the norm $\|v\|^2 = \int_\Omega |\nabla v|^2 dx$. $\|\cdot\|_p$ denotes the L^p-norm. S is the best Sobolev constant for $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$. The positive cone is $P = \{v \in H_0^1(\Omega) : v \geq 0 \text{ a.e. in } \Omega\}$.

2.2. The auxiliary potential φ_v

For any $v \in H_0^1(\Omega)$, the unique solution of $-\Delta\varphi = |v|^q$ with Dirichlet condition is denoted φ_v. Standard results give the following properties (see [17]).

Proposition 2.1.

1) $\varphi_v \geq 0$ *a.e. in* Ω *and* $\varphi_{tv} = t^q \varphi_v$ *for* $t > 0$.
2) *There exists* $C_0 > 0$ *depending on* Ω, q, S *such that* $\|\varphi_v\| \leq C_0\|v\|^q$ *and* $\int_\Omega \varphi_v |v|^q dx \leq C_0\|v\|^{2q}$.
3) *If* $v_n \to v$ *strongly in* $H_0^1(\Omega)$*, then* $\varphi_{v_n} \to \varphi_v$ *strongly in* $H_0^1(\Omega)$.

2.3. Weak slope and nonsmooth critical point theory

Suppose (X, d) is a complete metric space and $f : X \to \mathbb{R}$ is continuous. For a point $v \in X$, the weak slope $|df|(v)$ is defined to be the supremum of all $\delta \geq 0$ for which one can find a radius $\rho > 0$ and a continuous map $\sigma : B_\rho(v) \times [0, \rho] \to X$ satisfying

$$f(\sigma(w, t)) \leq f(w) - \delta t, \quad d(\sigma(w, t), w) \leq t,$$

for every $w \in B_\rho(v)$ and $t \in [0, \rho]$.

If no such δ exists, set $|df|(v) = 0$. For C^1 functionals, $|df|(v) = \|f'(v)\|$. A point v is critical if $|df|(v) = 0$. A sequence v_n is called a PS sequence at level c whenever $|df|(v_n) \to 0$ and $f(v_n) \to c$. The functional f is said to fulfill the PS condition provided every such PS sequence possesses a convergent subsequence. Moreover, the weak slope is lower semicontinuous: $v_n \to v$ in $H_0^1(\Omega)$ implies

$$|df|(v) \leq \liminf_{n\to\infty} |df|(v_n).$$

We work on P, which is closed in $H_0^1(\Omega)$ and hence complete. The functional I is continuous on P. Consequently, the nonsmooth formulations of Ekeland's variational principle and the mountain pass theorem are directly applicable.

3. Nonsmooth analysis of the energy functional

3.1. A key inequality

Lemma 3.1. *Assume $v \in P$ satisfies $|dI|(v) < +\infty$. Then for every $\eta \in P$,*

$$\begin{aligned}\alpha \int_\Omega \frac{\eta - v}{v^\tau} dx \leq (a + b\|v\|^2) \int_\Omega \nabla v \cdot \nabla(\eta - v) dx \\ - \int_\Omega \varphi_v |v|^{q-2} v(\eta - v) dx + |dI|(v)\|\eta - v\|.\end{aligned} \tag{3.1}$$

Proof. Fix $c > |dI|(v)$. We first prove (3.1) with $|dI|(v)$ replaced by c, then let $c \to |dI|(v)$. Assume $\eta \neq v$ (otherwise trivial). Choose $\rho > 0$ sufficiently small such that $\|\eta - z\| \geq \frac{1}{2}\|\eta - v\| > 0$ for all $z \in B_\rho(v)$. Define $\sigma(z, t) = z + t\frac{\eta - z}{\|\eta - z\|}$ on $B_\rho(v) \times [0, \rho]$. Then $\|\sigma(z, t) - z\| = t$ and $\sigma(z, t) \in P$ by convexity of P.

If $I(\sigma(z, t)) \leq I(z) - ct$ for all $(z, t) \in B_\rho(v) \times [0, \rho]$, then by the definition of weak slope, we have $|dI|(v) \geq c$, contradicting $c > |dI|(v)$. Hence, for each ρ, there exists $(z, t) \in B_\rho(v) \times [0, \rho]$ with $I(\sigma(z, t)) > I(z) - ct$. Taking $\rho_n \to 0^+$, we obtain sequences $\{v_n\} \subset P$ and $\{t_n\} \subset [0, \rho_n]$ such that $v_n \to v$, $t_n \to 0^+$ as $n \to \infty$, and

$$I(v_n + t_n\tfrac{\eta - v_n}{\|\eta - v_n\|}) \geq I(v_n) - ct_n.$$

Set $\xi_n = t_n/\|\eta - v_n\| \to 0^+$ as $n \to \infty$, then

$$I(v_n + \xi_n(\eta - v_n)) \geq I(v_n) - c\xi_n\|\eta - v_n\|. \tag{3.2}$$

Now substitute the expression of I into (3.2), subtract $I(v_n)$, and divide by $\xi_n > 0$, and we have

$$\begin{aligned}
&\frac{\alpha}{1-\tau}\int_\Omega \frac{[v_n+\xi_n(\eta-v_n)]^{1-\tau}-v_n^{1-\tau}}{\xi_n}dx \\
&\le \frac{a}{2}\frac{\|v_n+\xi_n(\eta-v_n)\|^2-\|v_n\|^2}{\xi_n}+\frac{b}{4}\frac{\|v_n+\xi_n(\eta-v_n)\|^4-\|v_n\|^4}{\xi_n} \\
&-\frac{1}{2q}\int_\Omega \frac{\varphi_{v_n+\xi_n(\eta-v_n)}|v_n+\xi_n(\eta-v_n)|^q-\varphi_{v_n}|v_n|^q}{\xi_n}dx \\
&+c\|\eta-v_n\|.
\end{aligned} \tag{3.3}$$

We analyze each term as $n\to\infty$.

Singular term. Let

$$L_n=\frac{[v_n+\xi_n(\eta-v_n)]^{1-\tau}-v_n^{1-\tau}}{\xi_n(1-\tau)}.$$

Decompose the numerator as

$$([v_n+\xi_n(\eta-v_n)]^{1-\tau}-[(1-\xi_n)v_n]^{1-\tau})+([(1-\xi_n)v_n]^{1-\tau}-v_n^{1-\tau}).$$

Applying the mean value theorem, we find κ_n between $(1-\xi_n)v_n$ and $v_n+\xi_n(\eta-v_n)$ such that the first difference equals $(1-\tau)\kappa_n^{-\tau}\xi_n\eta$. Hence,

$$\frac{[v_n+\xi_n(\eta-v_n)]^{1-\tau}-[(1-\xi_n)v_n]^{1-\tau}}{\xi_n(1-\tau)}=\frac{\eta}{\kappa_n^\tau}.$$

Since $v_n\to v$ a.e. in Ω and $\xi_n\to 0$, $\kappa_n\to v$ a.e. in Ω as $n\to\infty$. Also, $\eta\ge 0$, so $\eta/\kappa_n^\tau\ge 0$. Fatou's lemma [18] gives

$$\liminf_{n\to\infty}\int_\Omega \frac{\eta}{\kappa_n^\tau}dx\ge\int_\Omega\frac{\eta}{v^\tau}dx. \tag{3.4}$$

For the second part,

$$\frac{[(1-\xi_n)v_n]^{1-\tau}-v_n^{1-\tau}}{\xi_n(1-\tau)}=v_n^{1-\tau}\frac{(1-\xi_n)^{1-\tau}-1}{\xi_n(1-\tau)}.$$

As $\xi_n\to 0$, $\frac{(1-\xi_n)^{1-\tau}-1}{\xi_n(1-\tau)}\to -1$ and $v_n^{1-\tau}\to v^{1-\tau}$ strongly in L^1. By dominated convergence [19],

$$\lim_{n\to\infty}\int_\Omega v_n^{1-\tau}\frac{(1-\xi_n)^{1-\tau}-1}{\xi_n(1-\tau)}dx=-\int_\Omega v^{1-\tau}dx. \tag{3.5}$$

Combining with (3.4) and (3.5), we have

$$\liminf_{n\to\infty}\alpha\int_\Omega L_n dx\ge\alpha\Big(\int_\Omega\frac{\eta}{v^\tau}dx-\int_\Omega v^{1-\tau}dx\Big)=\alpha\int_\Omega\frac{\eta-v}{v^\tau}dx. \tag{3.6}$$

Gradient terms. Direct calculations yield

$$\begin{aligned}
&\frac{\|v_n+\xi_n(\eta-v_n)\|^2-\|v_n\|^2}{\xi_n} \\
&=2\int_\Omega\nabla v_n\cdot\nabla(\eta-v_n)dx+\xi_n\|\eta-v_n\|^2 \\
&\to 2\int_\Omega\nabla v\cdot\nabla(\eta-v)dx,\ \text{as } n\to\infty,
\end{aligned}$$

and

$$\begin{aligned}&\frac{\|v_n+\xi_n(\eta-v_n)\|^4-\|v_n\|^4}{\xi_n}\\&=(2\int_\Omega \nabla v_n\cdot\nabla(\eta-v_n)dx+\xi_n\|\eta-v_n\|^2)(\|v_n+\xi_n(\eta-v_n)\|^2+\|v_n\|^2)\\&\to 4\|v\|^2\int_\Omega \nabla v\cdot\nabla(\eta-v)dx,\ \text{as } n\to\infty.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&\frac{a}{2}\frac{\|v_n+\xi_n(\eta-v_n)\|^2-\|v_n\|^2}{\xi_n}+\frac{b}{4}\frac{\|v_n+\xi_n(\eta-v_n)\|^4-\|v_n\|^4}{\xi_n}\\&\to(a+b\|v\|^2)\int_\Omega \nabla v\cdot\nabla(\eta-v)dx,\ \text{as } n\to\infty.\end{aligned}\tag{3.7}$$

Nonlocal term. Write

$$J_n=\int_\Omega \frac{\varphi_{v_n+\xi_n(\eta-v_n)}|v_n+\xi_n(\eta-v_n)|^q-\varphi_{v_n}|v_n|^q}{\xi_n}dx.$$

Split $J_n=J_n^1+J_n^2$ with

$$J_n^1=\int_\Omega \varphi_{v_n+\xi_n(\eta-v_n)}\frac{|v_n+\xi_n(\eta-v_n)|^q-|v_n|^q}{\xi_n}dx,$$

and

$$J_n^2=\int_\Omega \frac{\varphi_{v_n+\xi_n(\eta-v_n)}-\varphi_{v_n}}{\xi_n}|v_n|^qdx.$$

Since $t\mapsto|t|^q$ is C^1, the difference quotient in J_n^1 converges a.e. to $q|v|^{q-2}v(\eta-v)$. Using the strong convergence $\varphi_{v_n+\xi_n(\eta-v_n)}\to\varphi_v$ (Proposition 2.1 (3)) and dominated convergence,

$$J_n^1\to\int_\Omega \varphi_v q|v|^{q-2}v(\eta-v)dx.\tag{3.8}$$

For J_n^2, the map $v\mapsto\varphi_v$ is C^1 (see [17]), thus the difference quotient converges strongly in H_0^1 to $\varphi_v'(\eta-v)$, which satisfies $-\Delta\varphi_v'(\eta-v)=q|v|^{q-2}v(\eta-v)$. Hence,

$$J_n^2\to\int_\Omega \varphi_v'(\eta-v)|v|^qdx.$$

Integrating by parts and using the equation for $\varphi_v'(\eta-v)$,

$$\int_\Omega \varphi_v'(\eta-v)|v|^qdx=\int_\Omega(-\Delta\varphi_v'(\eta-v))\varphi_vdx=\int_\Omega q|v|^{q-2}v(\eta-v)\varphi_vdx.\tag{3.9}$$

Adding (3.8) and (3.9) gives $J_n\to 2q\int_\Omega \varphi_v|v|^{q-2}v(\eta-v)dx$, so

$$-\frac{1}{2q}J_n\to-\int_\Omega \varphi_v|v|^{q-2}v(\eta-v)dx.\tag{3.10}$$

Remaining term. Clearly, $c\|\eta-v_n\|\to c\|\eta-v\|$.

Now take the limit inferior on the left of (3.3) and the limit on the right. Using (3.6), (3.7), (3.10), and the limit of the last term, we obtain

$$\alpha\int_\Omega \frac{\eta - v}{v^\tau}dx \le (a + b\|v\|^2)\int_\Omega \nabla v\cdot\nabla(\eta - v)dx \\ - \int_\Omega \varphi_v|v|^{q-2}v(\eta - v)dx + c\|\eta - v\|.$$

Letting $c \to |dI|(v)$ yields (3.1). □

3.2. Compactness

The following gives a compactness result.

Lemma 3.2. *If $\{v_n\} \subset P$ satisfies $|dI|(v_n) \to 0$ and $I(v_n) \to c$, then $\{v_n\}$ has a strongly convergent subsequence in $H_0^1(\Omega)$.*

Proof. *Boundedness.* Applying Lemma 3.1 with $\eta = 2v_n$ gives

$$\alpha\int_\Omega v_n^{1-\tau}dx \le (a + b\|v_n\|^2)\|v_n\|^2 - \int_\Omega \varphi_{v_n}|v_n|^q dx + o(1)\|v_n\|. \tag{3.11}$$

From the expression of $I(v_n)$,

$$I(v_n) = \frac{a}{2}\|v_n\|^2 + \frac{b}{4}\|v_n\|^4 - \frac{1}{2q}\int_\Omega \varphi_{v_n}|v_n|^q dx - \frac{\alpha}{1-\tau}\int v_n^{1-\tau}dx.$$

Using (3.11) to bound the nonlocal term from above,

$$\int_\Omega \varphi_{v_n}|v_n|^q \le (a + b\|v_n\|^2)\|v_n\|^2 - \alpha\int_\Omega v_n^{1-\tau}dx + o(1)\|v_n\|.$$

Substituting into $I(v_n)$, we have

$$I(v_n) \ge \frac{a}{2}\|v_n\|^2 + \frac{b}{4}\|v_n\|^4 - \frac{1}{2q}[(a + b\|v_n\|^2)\|v_n\|^2 - \alpha\int_\Omega v_n^{1-\tau}dx] \\ - \frac{\alpha}{1-\tau}\int v_n^{1-\tau}dx + o(1)\|v_n\|.$$

Rearranging,

$$I(v_n) \ge \frac{a(q-1)}{2q}\|v_n\|^2 + \frac{b(q-2)}{4q}\|v_n\|^4 \\ - \alpha\Big(\frac{1}{1-\tau} - \frac{1}{2q}\Big)\int_\Omega v_n^{1-\tau}dx + o(1)\|v_n\|. \tag{3.12}$$

By using Young's inequality, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that $\int_\Omega v_n^{1-\tau}dx \le \frac{\varepsilon(1-\tau)}{2}\|v_n\|^2 + C_\varepsilon$. Choose ε so that

$$\alpha\Big(\frac{1}{1-\tau} - \frac{1}{2q}\Big)\frac{\varepsilon(1-\tau)}{2} = \frac{a(q-1)}{4q},$$

then

$$\alpha\Big(\frac{1}{1-\tau}-\frac{1}{2q}\Big)\int_\Omega v_n^{1-\tau}dx \le \frac{a(q-1)}{4q}\|v_n\|^2 + C_1,$$

where $C_1 := \alpha(\frac{1}{1-\tau}-\frac{1}{2q})C_\varepsilon > 0$ (independent of n). Inserting this into (3.12) yields

$$I(v_n) \ge \frac{a(q-1)}{4q}\|v_n\|^2 + \frac{b(q-2)}{4q}\|v_n\|^4 - C_1 + o(1)\|v_n\|.$$

Since $I(v_n) \to c$, the right-hand side is bounded, and the quartic term forces $\|v_n\|$ to be bounded. Hence, $\{v_n\}$ is bounded in $H_0^1(\Omega)$.

Strong convergence. Extract a subsequence $v_n \rightharpoonup v$ in $H_0^1(\Omega)$. Then $v_n \to v$ strongly in $L^p(\Omega)$ for all $2 \le p < 6$ and a.e. in Ω. By Proposition 2.1 (3), $\varphi_{v_n} \to \varphi_v$ strongly in $H_0^1(\Omega)$. By the lower semicontinuity of the weak slope, we have

$$|dI|(v) \le \liminf_{n\to\infty} |dI|(v_n) = 0,$$

hence $|dI|(v) = 0 < +\infty$. Applying Lemma 3.1 with $\eta = v$ and $\eta = v_n$ (base point v),

$$\alpha\int_\Omega \frac{v-v_n}{v_n^\tau}dx \le (a+b\|v_n\|^2)\int \nabla v_n\cdot\nabla(v-v_n)dx - \int_\Omega \varphi_{v_n}|v_n|^{q-2}v_n(v-v_n)dx \\ + o(1)\|v-v_n\|,$$

and

$$\alpha\int_\Omega \frac{v_n-v}{v^\tau}dx \le (a+b\|v\|^2)\int_\Omega \nabla v\cdot\nabla(v_n-v)dx - \int_\Omega \varphi_v|v|^{q-2}v(v_n-v)dx.$$

Adding these, we get

$$\begin{aligned}&\alpha\int_\Omega\Big(\frac{v-v_n}{v_n^\tau}+\frac{v_n-v}{v^\tau}\Big)dx\\ &\le (a+b\|v_n\|^2)\int_\Omega \nabla v_n\cdot\nabla(v-v_n)dx + (a+b\|v\|^2)\int_\Omega \nabla v\cdot\nabla(v_n-v)dx\\ &\quad - \int_\Omega[\varphi_{v_n}|v_n|^{q-2}v_n(v-v_n)+\varphi_v|v|^{q-2}v(v_n-v)]dx + o(1)\|v-v_n\|.\end{aligned}$$

The sum of the first two integrals equals

$$-(a+b\|v_n\|^2)\|v-v_n\|^2 + b(\|v_n\|^2-\|v\|^2)\int_\Omega \nabla v\cdot\nabla(v-v_n)dx.$$

Since $v_n \rightharpoonup v$ in $H_0^1(\Omega)$, $\int_\Omega \nabla v\cdot\nabla(v-v_n)dx \to 0$ as $n\to\infty$, and $\|v_n\|^2$ is bounded, the extra term tends to zero. The nonlocal integral converges to zero because of strong convergence. Moreover, because $t \mapsto t^{-\tau}$ is decreasing, the left-hand side is nonpositive. Hence,

$$0 \le \liminf_{n\to\infty}[-(a+b\|v_n\|^2)\|v-v_n\|^2 + o(1)].$$

Since $a+b\|v_n\|^2 \ge a > 0$, we obtain $\|v-v_n\| \to 0$ as $n\to\infty$. □

3.3. Characterization of critical points

Lemma 3.3. *If $v \in P$ satisfies $|dI|(v) = 0$, then v is a weak solution of* (1.1). *For every $\psi \in H_0^1(\Omega)$,*

$$(a + b\|v\|^2) \int_\Omega \nabla v \cdot \nabla \psi dx = \int_\Omega \varphi_v |v|^{q-2} v \psi dx + \alpha \int_\Omega \frac{\psi}{v^\tau} dx.$$

Proof. From Lemma 3.1 with $|dI|(v) = 0$, for any $\eta \in P$,

$$\alpha \int_\Omega \frac{\eta - v}{v^\tau} dx \le (a + b\|v\|^2) \int_\Omega \nabla v \cdot \nabla(\eta - v) dx - \int_\Omega \varphi_v |v|^{q-2} v(\eta - v) dx. \tag{3.13}$$

We first assume $\psi \ge 0$. Fix such a $\psi \in H_0^1(\Omega)$ with $\psi \ge 0$, and for small $\varepsilon > 0$, define $\eta_\varepsilon = (v + \varepsilon\psi)^+ \in P$. Substituting $\eta = \eta_\varepsilon$ into (3.13) and using

$$\eta_\varepsilon - v = \begin{cases} \varepsilon\psi & \text{on } A_\varepsilon := \{x : v(x) + \varepsilon\psi(x) \ge 0\}, \\ -v & \text{on } B_\varepsilon := \{x : v(x) + \varepsilon\psi(x) < 0\}, \end{cases}$$

we obtain

$$\begin{aligned} \alpha \int_{A_\varepsilon} \frac{\varepsilon\psi}{v^\tau} dx - \alpha \int_{B_\varepsilon} v^{1-\tau} dx \le{}& (a + b\|v\|^2)\Big(\int_{A_\varepsilon} \nabla v \cdot \nabla(\varepsilon\psi) dx - \int_{B_\varepsilon} |\nabla v|^2 dx\Big) \\ & - \Big(\int_{A_\varepsilon} \varphi_v |v|^{q-2} v(\varepsilon\psi) dx - \int_{B_\varepsilon} \varphi_v |v|^q dx\Big). \end{aligned} \tag{3.14}$$

As $\varepsilon \to 0^+$, B_ε shrinks to a subset of $\{v = 0\}$. By a property of Sobolev functions, $\nabla v = 0$ a.e. on $\{v = 0\}$ (Stampacchia's lemma [20]). Hence, the integrals over B_ε tend to zero. Moreover, $A_\varepsilon \to \Omega$ in measure. Using dominated convergence,

$$\int_{A_\varepsilon} \nabla v \cdot \nabla \psi dx \to \int_\Omega \nabla v \cdot \nabla \psi dx, \qquad \int_{A_\varepsilon} \varphi_v |v|^{q-2} v \psi dx \to \int_\Omega \varphi_v |v|^{q-2} v \psi dx.$$

For the singular term, since $\psi \ge 0$, the integrand $\frac{\varepsilon\psi}{v^\tau}$ is nonnegative. By Fatou's lemma [18], we have

$$\liminf_{\varepsilon \to 0^+} \alpha \int_{A_\varepsilon} \frac{\varepsilon\psi}{v^\tau} dx \ge \alpha \int_\Omega \frac{\psi}{v^\tau} dx.$$

Thus, dividing (3.14) by $\varepsilon > 0$ and taking $\varepsilon \to 0^+$, we obtain

$$\alpha \int_\Omega \frac{\psi}{v^\tau} dx \le (a + b\|v\|^2) \int_\Omega \nabla v \cdot \nabla \psi dx - \int_\Omega \varphi_v |v|^{q-2} v \psi dx. \tag{3.15}$$

Now we prove that (3.15) holds for every $\psi \in H_0^1(\Omega)$ without any sign restriction. Let $\psi \in H_0^1(\Omega)$ be arbitrary and set $\eta_\varepsilon = (v + \varepsilon\psi)^+$ as before. The same computation that led to (3.14) is valid, and after dividing by ε, we obtain

$$\alpha \int_{A_\varepsilon} \frac{\psi}{v^\tau} dx \le (a + b\|v\|^2) \int_{A_\varepsilon} \nabla v \cdot \nabla \psi dx - \int_{A_\varepsilon} \varphi_v |v|^{q-2} v \psi dx + \frac{1}{\varepsilon} R_\varepsilon. \tag{3.16}$$

Decompose $\psi = \psi^+ - \psi^-$ with $\psi^+ = \max\{\psi, 0\}$, $\psi^- = \max\{-\psi, 0\}$. Both are nonnegative and belong to $H_0^1(\Omega)$. Because $\psi^+/v^\tau \geq 0$ and $A_\varepsilon \to \Omega$ in measure as $\varepsilon \to 0^+$, Fatou's lemma gives

$$\liminf_{\varepsilon\to 0^+} \int_{A_\varepsilon} \frac{\psi^+}{v^\tau} dx \geq \int_\Omega \frac{\psi^+}{v^\tau} dx.$$

For the negative part, $\psi^-/v^\tau \geq 0$ and the sets A_ε are monotone increasing as $\varepsilon \to 0^+$ (the condition $v + \varepsilon\psi \geq 0$ becomes less restrictive for smaller ε). Hence, we may apply the monotone convergence theorem to obtain

$$\lim_{\varepsilon\to 0^+} \int_{A_\varepsilon} \frac{\psi^-}{v^\tau} dx = \int_\Omega \frac{\psi^-}{v^\tau} dx.$$

Now observe that

$$\int_{A_\varepsilon} \frac{\psi}{v^\tau} dx = \int_{A_\varepsilon} \frac{\psi^+}{v^\tau} dx - \int_{A_\varepsilon} \frac{\psi^-}{v^\tau} dx.$$

Taking liminf on both sides and using the fact that $\liminf(X_\varepsilon - Y_\varepsilon) \geq \liminf X_\varepsilon - \lim Y_\varepsilon$ whenever the limit of Y_ε exists, we deduce

$$\liminf_{\varepsilon\to 0^+} \int_{A_\varepsilon} \frac{\psi}{v^\tau} dx \geq \int_\Omega \frac{\psi^+}{v^\tau} dx - \int_\Omega \frac{\psi^-}{v^\tau} dx = \int_\Omega \frac{\psi}{v^\tau} dx.$$

On the other hand, taking liminf in (3.16) and using $\liminf u_\varepsilon \leq \liminf v_\varepsilon$ when $u_\varepsilon \leq v_\varepsilon$, together with the convergence of the right-hand side, we obtain

$$\alpha \liminf_{\varepsilon\to 0^+} \int_{A_\varepsilon} \frac{\psi}{v^\tau} dx \leq (a + b\|v\|^2) \int_\Omega \nabla v \cdot \nabla\psi dx - \int_\Omega \varphi_v |v|^{q-2} v\psi dx.$$

Combining this with the lower bound above yields precisely (3.15) for the general ψ.

Finally, replacing ψ by $-\psi$ in (3.15) gives the reverse inequality. Hence, equality holds for every $\psi \in H_0^1(\Omega)$. The integrability of ψ/v^τ follows from the finiteness of the right-hand side. □

4. Existence of two positive solutions

4.1. Mountain pass geometry

Lemma 4.1. *There exist constants $\gamma, \delta, \Gamma > 0$ such that for every $\alpha \in (0, \Gamma)$:*
(i) $I(v) \geq \delta$ for all v with $\|v\| = \gamma$;
(ii) $\inf\limits_{\|v\|<\gamma} I(v) < 0$;
(iii) there exists $\rho \in H_0^1(\Omega)$ with $\|\rho\| > \gamma$ and $I(\rho) < 0$.

Proof. (*i*) Using Proposition 2.1 (2) and Hölder's inequality, we have for any v,

$$I(v) \geq \frac{a}{2}\|v\|^2 + \frac{b}{4}\|v\|^4 - \frac{C_0}{2q}\|v\|^{2q} - \frac{\alpha C_3}{1-\tau}\|v\|^{1-\tau}$$

with $C_0 > 0$ depending on Ω, q, S (the constant from Proposition 2.1 (2)), and $C_3 = C_2^{1-\tau}|\Omega|^{\frac{1+\tau}{2}} > 0$, where C_2 is the Sobolev embedding constant. Set $t = \|v\|$ and define $F(t) = \frac{a}{2}t^{1+\tau} - \frac{C_0}{2q}t^{2q-1+\tau}$. Multiplying by $t^{-(1-\tau)}$ gives $t^{-(1-\tau)}I(v) \geq F(t) - \frac{\alpha C_3}{1-\tau}$. The function F attains a positive maximum at some

$\gamma > 0$; let $M = F(\gamma)$. Choose $\Gamma = \frac{1-\tau}{C_3}M$. For $\alpha \in (0,\Gamma)$ and $\|v\| = \gamma$, we have $I(v) \geq \gamma^{1-\tau}(M - \frac{\alpha C_3}{1-\tau}) > 0$. Thus, for any fixed $\alpha \in (0,\Gamma)$, the choice $\delta = \gamma^{1-\tau}(M - \frac{\alpha C_3}{1-\tau}) > 0$ satisfies condition (i).

(ii) Take a nonzero nonnegative function $\psi \in H_0^1(\Omega)$ with $\|\psi\| = 1$. For $t > 0$, the energy functional satisfies

$$I(t\psi) = \frac{a}{2}t^2\|\psi\|^2 + \frac{b}{4}t^4\|\psi\|^4 - \frac{t^{2q}}{2q}\int_\Omega \varphi_\psi|\psi|^q dx - \frac{\alpha t^{1-\tau}}{1-\tau}\int_\Omega |\psi|^{1-\tau}dx.$$

Since $1 - \tau < 2 < 2q$, we have

$$\lim_{t\to 0^+}\frac{I(t\psi)}{t^{1-\tau}} = -\frac{\alpha}{1-\tau}\int_\Omega |\psi|^{1-\tau}dx < 0.$$

Thus, there exists $t_0 > 0$ sufficiently small such that $I(t_0\psi) < 0$ and $t_0\|\psi\| = t_0 < \gamma$, which gives $\inf_{\|v\|<\gamma} I(v) < 0$.

(iii) For large t, the term $-\frac{t^{2q}}{2q}\int_\Omega \varphi_\psi|\psi|^q dx$ dominates because $2q > 4 > 2 > 1-\tau$. Hence,

$$\lim_{t\to\infty}\frac{I(t\psi)}{t^{2q}} = -\frac{1}{2q}\int_\Omega \varphi_\psi|\psi|^q dx < 0.$$

Therefore, we can choose t_1 large enough so that $I(t_1\psi) < 0$ and $t_1 > \gamma$. Setting $\rho = t_1\psi$ yields $\|\rho\| > \gamma$ and $I(\rho) < 0$, completing the proof of (iii). □

4.2. A local minimizer with negative energy

Theorem 4.2 *For $\alpha \in (0,\Gamma)$, system* (1.1) *has a positive solution v_α with $I(v_\alpha) < 0$.*

Proof. Let $m = \inf\{I(v) : v \in P, \|v\| \leq \gamma\}$. By Lemma 4.1$(ii)$, $m < 0$. Take a minimizing sequence $\{v_n\} \subset P$ with $\|v_n\| \leq \gamma$. Since $\overline{B_\gamma(0)} \cap P$ is a complete metric space, Ekeland's variational principle (see [21]) gives a new sequence (still denoted $\{v_n\}$) such that

$$I(v_n) \leq m + 1/n, \qquad I(w) \geq I(v_n) - \frac{1}{n}\|w - v_n\|, \quad \forall w \in \overline{B_\gamma(0)} \cap P.$$

Because $I(v_n) \to m < 0$ and $I(v) \geq \delta > 0$ on $\|v\| = \gamma$, the v_n lie in the interior of $B_\gamma(0)$ for large n. Hence, there exists $r_n > 0$ with $\overline{B_{r_n}(v_n)} \subset B_\gamma(0) \cap P$. Fix $\eta \in P$ with $\|\eta\| \leq \gamma$. For small $t > 0$, $v_n + t(\eta - v_n) \in \overline{B_\gamma(0)} \cap P$. From the Ekeland inequality,

$$I(v_n + t(\eta - v_n)) \geq I(v_n) - \frac{t}{n}\|\eta - v_n\|.$$

Assume for contradiction that $|dI|(v_n) > 1/n$. Then there exist $\delta > 1/n$, $\rho > 0$ and a continuous map $\sigma : B_\rho(v_n) \times [0,\rho] \to P$ with $I(\sigma(z,t)) \leq I(z) - \delta t$ and $\|\sigma(z,t) - z\| \leq t$. Choose $t < \min\{\rho, r_n\}$ and set $w = \sigma(v_n, t)$. Then $w \in B_\gamma(0) \cap P$ and

$$I(w) \geq I(v_n) - \frac{1}{n}\|w - v_n\| \geq I(v_n) - \frac{t}{n}.$$

But $I(w) \leq I(v_n) - \delta t$, so $I(v_n) - \delta t \geq I(v_n) - t/n$, i.e., $\delta \leq 1/n$, which is a contradiction. Hence, $|dI|(v_n) \leq 1/n$. Thus, $\{v_n\}$ is a PS sequence at level m. By Lemma 3.2, a subsequence converges strongly to some $v_\alpha \in P$ with $I(v_\alpha) = m < 0$. Lower semicontinuity gives $|dI|(v_\alpha) = 0$, and Lemma 3.3 shows v_α is a weak solution. Positivity follows from the strong maximum principle. Indeed, from the weak formulation, we have $-\Delta v_\alpha \geq 0$ in the distributional sense in Ω because $\varphi_{v_\alpha} \geq 0$, $|v_\alpha|^{q-2}v_\alpha \geq 0$ and $\alpha v_\alpha^{-\tau} > 0$ a.e. in Ω. Since $v_\alpha \not\equiv 0$, the strong maximum principle (see [22]) implies $v_\alpha > 0$ in Ω. Thus, v_α is exactly the negative energy solution stated in Theorem 1.1. □

4.3. A second solution with positive energy

Theorem 4.3 *For $\alpha \in (0, \Gamma)$, system* (1.1) *has a positive solution v^* with $I(v^*) > 0$.*

Proof. Take ρ from Lemma 4.1(*iii*). Define $\Phi = \{\zeta \in C([0, 1], P) : \zeta(0) = 0, \zeta(1) = \rho\}$ and the mountain pass value

$$c = \inf_{\zeta \in \Phi} \max_{t \in [0,1]} I(\zeta(t)).$$

Every path must cross the sphere $\|v\| = \gamma$ (since $\|0\| = 0 < \gamma < \|\rho\|$), so by Lemma 4.1(*i*), $c \geq \delta > 0$. The nonsmooth mountain pass theorem [13, Theorem 1.3.1] guarantees a PS sequence $\{v_n\} \subset P$ with $|dI|(v_n) \to 0$ and $I(v_n) \to c$. By Lemma 3.2, a subsequence converges strongly to some $v^* \in P$ with $I(v^*) = c > 0$ and $|dI|(v^*) = 0$. Lemma 3.3 then implies v^* is a weak solution. As before, the right-hand side of the equation is non-negative, so $-\Delta v^* \geq 0$ in the distributional sense. The strong maximum principle (see [22]) then gives $v^* > 0$ in Ω. The v^* is exactly the positive energy solution stated in Theorem 1.1. Combining Theorems 4.2 and 4.3, we obtain Theorem 1.1.

5. Discussion and concluding remarks

We have proved the existence of two positive solutions for a singular KSP system with coupling exponent $2 < q < 3$. The main novelty is the systematic use of nonsmooth critical point theory to handle the mild singularity and the nonlocal coupling, extending our previous work [24]. Compared with existing results [5,7,8,23], our method does not require C^1 regularity and yields multiplicity under a smallness condition on the singular parameter. Future work may consider the critical case $q = 3$ or sign-changing solutions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest in this paper.

References

1. V. Benci, D. Fortunato, An eigenvalue problem for the Schrödinger-Maxwell equations, *Topol. Methods Nonlinear Anal.*, **1998** (1998), 283–293. https://doi.org/10.12775/TMNA.1998.019

2. V. Benci, D. Fortunato, Solitons in Schrödinger-Maxwell equations, *J. Fixed Point Theory Appl.* **15** (2014), 101–132. https://doi.org/10.1007/s11784-014-0184-1
3. D. F. Lü, Positive solutions for Kirchhoff-Schrödinger-Poisson systems with general nonlinearity, *Commun. Pure Appl. Anal.* **17** (2018), 605–626. https://doi.org/10.3934/cpaa.2018033
4. C. J. Batkam, J. R. S. Júnior, Schrödinger-Kirchhoff-Poisson type systems, *Commun. Pure Appl. Anal.* **15** (2016), 429–444. https://doi.org/10.3934/cpaa.2016.15.429
5. M. J. Mu, H. Q. Lu, Existence and multiplicity of positive solutions for Schrödinger-Kirchhoff-Poisson system with singularity, *J. Funct. Spaces* **2017** (2017), 5985962. https://doi.org/10.1155/2017/5985962
6. B. L. Yang, D. L. Zhang, S. H. Liang, On Kirchhoff-Schrödinger-Poisson-type systems with singular and critical nonlinearity, *Adv. Nonlinear Anal.*, **13**, (2024), 20240050. https://doi.org/10.1515/anona-2024-0050
7. Q. Zhang, Existence of positive solution to Kirchhoff-Schrödinger-Poisson system with strong singular term, *J. Math. Phys.*, **60** (2019), 041504. https://doi.org/10.1063/1.5065521
8. Q. Zhang, Multiple positive solutions for Kirchhoff-Schrödinger-Poisson system with general singularity, *Bound. Value Probl.*, **17** (2017), 127. https://doi.org/10.1186/s13661-017-0858-x
9. S. H. Feng, L. Wang, L. Huang, Least energy sign-changing solutions of fractional Kirchhoff-Schrödinger-Poisson system with critical and logarithmic nonlinearity, *Complex Var. Elliptic Equations*, **68** (2023), 81–106. https://doi.org/10.1080/17476933.2021.1975116
10. S. Ghosh, An existence result for singular nonlocal fractional Kirchhoff-Schrödinger-Poisson system, *Complex Var. Elliptic Equations*, **67** (2022), 1817–1846. https://doi.org/10.1080/17476933.2021.1900137
11. J. H. Chen, X. H. Tang, Z. Gao, Existence of multiple solutions for modified Schrödinger-Kirchhoff-Poisson type systems via perturbation method with sign-changing potential, *Comput. Math. Appl.*, **73** (2017), 505–519. https://doi.org/10.1016/j.camwa.2016.12.006
12. Y. Wang, Z. H. Zhang, Ground state solutions for Kirchhoff-Schrödinger-Poisson system with sign-changing potentials, *Bull. Malays. Math. Sci. Soc.*, **44** (2021), 2319–2333. https://doi.org/10.1007/s40840-020-01061-z
13. M. Degiovanni, M. Marzocchi, A critical point theory for nonsmooth functionals, *Ann. Mat. Pura Appl.*, **167** (1994), 73–100. https://doi.org/10.1007/BF01760329
14. A. Canino, M. Degiovanni, Nonsmooth critical point theory and quasilinear elliptic equations, in *Topological Methods in Differential Equations and Inclusions* (eds. A. Granas, M. Frigon and G. Sabidussi), Springer Netherlands, (1995), 1–50. https://doi.org/10.1007/978-94-011-0339-8_1
15. X. Liu, Y. Guo, J. Liu, Solutions for singular p-Laplacian equation in $\mathbb{R}^n$, *J. Syst. Sci. Complexity*, **22** (2009), 597–613. https://doi.org/10.1007/s11424-009-9190-6
16. A. Ambrosetti, P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.*, **14** (1973), 349–381. https://doi.org/10.1016/0022-1236(73)90051-7
17. D. Ruiz, The Schrödinger-Poisson equation under the effect of a nonlinear local term, *J. Funct. Anal.*, **237** (2006), 655–674. https://doi.org/10.1016/j.jfa.2006.04.005

18. D. Schmeidler, Fatou's lemma in several dimensions, *Proc. Amer. Math. Soc.*, **24** (1970), 300–306. https://doi.org/10.1090/S0002-9939-1970-0248316-7

19. H. Lebesgue, Sur l'intégration des fonctions discontinues, *Ann. Sci. École Norm. Sup.*, **27** (1910), 361–450. https://doi.org/10.24033/asens.624

20. G. Stampacchia, *Équations Elliptiques du Second Ordre à Coefficients Discontinus*, Les Presses de l'Université de Montréal, Montréal, Canada, 1966.

21. I. Ekeland, On the variational principle, *J. Math. Anal. Appl.*, **47** (1974), 324–353. https://doi.org/10.1016/0022-247X(74)90025-0

22. D. Gilbarg, N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin-Heidelberg-New York, 1977. https://doi.org/10.1007/978-3-642-96379-7

23. C. Y. Lei, J. F. Liao, C. M. Chu, H. M. Suo, A nonsmooth theory for a logarithmic elliptic equation with singular nonlinearity, *Acta Math. Sci. Ser. B*, **42** (2022), 502–510. https://doi.org/10.1007/s10473-022-0205-x

24. Y. Zhou, W. Wei, Y. Wang, J. Lei, Existence of optimal control for a class of Kirchhoff-Poisson system, *Qual.Theory Dyn. Syst.*, **23** (2024), 156. https://doi.org/10.1007/s12346-024-01019-7

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