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Research article

Noncanonical second-order differential equations with several time-delay arguments: Enhancing oscillation criteria

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Abstract: This study investigates the oscillatory properties of solutions for a general class of neutral differential equations with multiple delays. Using Riccati and comparison techniques, we establish five distinct oscillation theorems that address the limitations of previous results in this topic. Our criteria not only extend and generalize earlier findings but also reduce the required constraints. Notably, they provide sharper results when applied to special cases like Euler's equation. The novelty and effectiveness of the proposed oscillation criteria are illustrated through a detailed analysis of a given example, supported by tables and figures.

Keywords: differential equations; oscillatory properties; Riccati and comparison techniques; noncanonical second-order; several time-delay arguments

1. Introduction

The theory of oscillation for differential equations (DEs) has a long and rich history, dating back to the pioneering work by Poincaré and Lyapunov in the late 19th and early 20th centuries. Their contributions were followed by the first published paper [1] establishing the first oscillation criteria for a particular class of DEs. These findings were later expanded through generalizations based on the well-known comparison theorems of Kneser and Fite [2,3], which expanded oscillation theory to encompass functional differential equations (FDEs). This theory has subsequently shown significant contributions to the modeling and analysis of many dynamic systems across multiple disciplines, including biology and economics [4].

The evolution of mathematical analysis has led to the development of new techniques for studying the oscillation of DEs. Beyond the traditional comparison theorem, which reduces the order of the DE to utilize criteria for lower-order equations, alternative approaches have emerged. These include the Riccati technique, the interval technique, the iterative method, and fixed-point theorems, as discussed in [5].

Recent research has focused on enhancing oscillation criteria and applying novel ideas and tools for analyzing various types of differential equations. This development has extended from ordinary differential equations (ODEs) to more sophisticated functional forms, where the dependent variable depends on multiple arguments of the independent variable, often representing time.

Differential equations can be classified based on the presence of delays in their terms. If a delay appears in the lower derivatives of the DE but not in the highest derivative, the equation is called a delay differential equation (DDE). If the delay affects the highest derivative, it is classified as an advanced differential equation (ADE). When the highest derivative appears both with and without delay, the equation is classified as a neutral delay differential equation (NDDE). In some cases, these classifications overlap, resulting in mixed-type equations. For a more comprehensive discussion on oscillation theory and its application to various classifications and orders of DEs, see [6].

It is remarkable to observe the interplay between deducing oscillation criteria for second-order DEs from higher-order equations, and vice versa. This relationship often relies on order-reduction techniques, as previously mentioned, and reflects the frequent appearance of these equations in the modeling of natural phenomena [7]. The theoretical significance and practical applications of these equations have motivated extensive studies into their oscillatory behavior, as evidenced in [8].

The primary goal of this study is to obtain sufficient conditions to test the oscillation of solutions to second-order NDDEs with multiple delayed arguments:

$$\left(a(t)\,\psi(x(t))\,[z'(t)]^{\delta}\right)' + \sum_{\ell=1}^{n} c_{\ell}(t)\, x^{\delta}(\theta_{\ell}(t)) = 0, \tag{1.1}$$

over the interval $\mathbb{I}_1 = [t_0, \infty)$, where the function $z(t)$ is defined as

$$z(t) = x(t) + b(t)\, x(\omega(t)).$$

To ensure the validity of our results, the following conditions are assumed:

(A_1) δ is a quotient of odd positive integers;

(A_2) $a \in C^1(\mathbb{I}_1, \mathbb{R}^+)$, and there exists a positive function

$$\eta(t) := \int_t^{\infty} \frac{1}{\sqrt[\delta]{a(v)}} \mathrm{d}v,$$

with $\eta(t_0) < \infty$;

(A_3) $b \in C^2(\mathbb{I}_1, \mathbb{I}_2)$, $c_\ell \in C(\mathbb{I}_1, \mathbb{R}^+)$ for $\ell = 1, 2, ..., n,\ n \in \mathbb{N},\ \mathbb{I}_2 = [0, b_0],\ b_0 \in \mathbb{R}^+ \cup \{0\}$, and $\sum_{\ell=1}^{n} c_\ell(t)$ does not vanish eventually;

(A$_4$) $\psi \in C^1(\mathbb{R}, \mathbb{I}_3)$, and $\mathbb{I}_3 = [n_1, n_2]$, where $n_1, n_2 \in \mathbb{R}^+$. We will denote

$$L = \sqrt[\delta]{\frac{n_2}{n_1}};$$

(A$_5$) $\theta_\ell \in C(\mathbb{I}_1, \mathbb{R}^+)$, $\omega \in C^2(\mathbb{I}_1, \mathbb{R}^+)$, with

$$\theta(t) = \max_{n\in\mathbb{N}} \{\theta_\ell(t) : \ell = 1, 2, ..., n\}, \tag{1.2}$$

satisfying $\theta(t) \le t$, $\omega(t) \le t$, $\theta'(t) \in \mathbb{R}^+ \cup \{0\}$, and $\lim_{t\to\infty} \theta(t) = \infty$, $\lim_{t\to\infty} \omega(t) = \infty$, for all $\ell = 1, 2, ..., n$, $n \in \mathbb{N}$.

A function $x \in C^2(\mathbb{I}_1, \mathbb{R})$ is a nontrivial solution of (1.1), if x satisfies (1.1) on $\mathbb{I}_1$ with $a\psi(x)[z']^\delta \in C^1(\mathbb{I}_1, \mathbb{R})$, and $\sup\{|x(t)| : t \ge T\} > 0$ holds for all $T \in \mathbb{I}_1$. Such a solution is called oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is classified as nonoscillatory. The equation itself is said to oscillate if all of its solutions oscillate [7].

This study begins with a review of the most significant prior research, identifying gaps that motivate our investigation. Based on this, in Section 3, we define the notations, classify the positive solutions, and present some preliminary results. This section is further divided into two subsections, each focusing on different classes of positive solutions, improving the relationships between variables, their derivatives, and associated functions. The core section of this paper presents five oscillation theorems, each derived using different methods. Finally, we analyze and compare our results with previous findings, identifying the most effective oscillation criteria. These comparisons are illustrated through a detailed analysis of a given example, supported by tables and graphs for clarity.

2. Review of previous literature

In this section, we provide a brief overview of the key results related to NDDEs, compare them, and highlight their shortcomings, illustrating the motivation behind this work.

In 1985, Ladas et al. [9] studied the oscillation of a very simple form of (1.1),

$$(x(t) + b(t)x(t-\omega))'' + c(t)x(t-\theta) = 0, \tag{2.1}$$

and derived an oscillation criterion that states that the solutions of (2.1) are oscillatory if

$$\int_{t_0}^{t} c(\nu)(1 - b(\nu-\omega))\,\mathrm{d}\nu \to \infty \quad \text{as} \quad t \to \infty.$$

Subsequently, Grace and Lalli [10] generalized this result by incorporating a coefficient a, studying the equation

$$(a(t)(x(t) + b(t)x(t-\omega)))' + c(t)x(t-\theta) = 0.$$

They established that the equation oscillates if

$$\int_{t_0}^{t} \left[\beta(\nu)c(\nu)(1 - b(\nu-\theta)) - \frac{(\beta'(\nu))^2 a(\nu-\theta)}{4L\beta(\nu)}\right]\mathrm{d}\nu \to \infty \quad \text{as} \quad t \to \infty$$

for some differentiable function $\beta \in C^1(\mathbb{I}_1, \mathbb{R}^+)$.

These findings inspired subsequent studies that followed various directions; see [11]. In 2010, Han et al. [12], investigated the oscillation of the second-order half-linear NDDE

$$\left(a(t)\left[z'(t)\right]^{\delta}\right)' + c(t)\, x^{\delta}(\theta(t)) = 0, \tag{2.2}$$

and established two key theorems:

Theorem 2.1. *Assume that $b' \geq 0$, $\theta(t) \leq t - \omega(t)$ for $t \in \mathbb{I}_1$, and there exists a differentiable function $\beta \in C^1(\mathbb{I}_1, \mathbb{R}^+)$ on $\mathbb{I}_1$, such that*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[\beta(\nu)\, c(\nu)\,(1 - b(\theta(\nu)))^{\delta} - \left(\frac{\beta_+(\nu)}{\delta+1}\right)^{\delta+1} \frac{a(\theta(\nu))}{(\theta'(\nu))^{\delta}\beta^{\delta}(\nu)}\right] \mathrm{d}\nu = \infty \tag{2.3}$$

and

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[c(\nu)\,\eta^{\delta}(\nu)\left(\frac{1}{1+b(\nu)}\right)^{\delta} - \left(\frac{\delta}{\delta+1}\right)^{\delta+1} \frac{1}{\eta(\nu)\sqrt[\delta]{a(\nu)}}\right] \mathrm{d}\nu = \infty \tag{2.4}$$

for $\beta_+ = \max\{0, \beta'\}$. Then, Eq (2.2) is oscillatory.

Theorem 2.2. *Assume that $b' \geq 0$, $\theta(t) \leq t - \omega(t)$ for $t \in \mathbb{I}_1$, and there exists a differentiable function $\beta \in C^1(\mathbb{I}_1, \mathbb{R}^+)$ on $\mathbb{I}_1$, such that (2.3) holds and*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \frac{1}{\sqrt[\delta]{a(s)}} \left(\int_{t_1}^{s} c(\nu)\left(\frac{\eta(\nu)}{1+b(\nu)}\right)^{\delta} \mathrm{d}\nu\right)^{1/\delta} \mathrm{d}s = \infty, \tag{2.5}$$

where $\beta_+ = \max\{0, \beta'\}$. Then, Eq (2.2) is oscillatory.

These criteria were later extended by Agarwal et al. [13], who introduced the following theorem:

Theorem 2.3. *Assume that $\delta \geq 1$, $b' \geq 0$, and $\theta(t) \leq t - \omega(t)$, on $\mathbb{I}_1$. If there exists a function $\beta \in C^1(\mathbb{I}_1, \mathbb{R}^+)$ such that (2.3) holds and*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[\widehat{\varsigma}(\nu) - \left(\frac{\widehat{\beta}_+(\nu)}{\delta+1}\right)^{\delta+1} a(\nu)\beta(\nu)\right] \mathrm{d}\nu = \infty, \tag{2.6}$$

for $\hat{\beta}^+ = \max\left\{0, \hat{\beta}\right\}$, where

$$\widehat{\varsigma}(t) = \beta(t)\left(\widetilde{\varsigma}(t) + \frac{1-\delta}{\sqrt[\delta]{a(t)}\eta^{\delta+1}(t)}\right),$$

$$\widetilde{\varsigma}(t) = c(t)\left(1 - \frac{b(\theta(t))\,\eta(\omega(\theta(t)))}{\eta(\theta(t))}\right)^{\delta},$$

and

$$\widehat{\beta}(t) = \frac{\beta'(t)}{\beta(t)} + \frac{1+\delta}{\sqrt[\delta]{a(t)}\eta(t)},$$

then every solution of (2.2) oscillates.

Later, Bohner et al. [14] improved these results by deriving the following simpler one-condition criteria:

Theorem 2.4. *If*

$$\int_{t_1}^{\infty} \frac{1}{\sqrt[\delta]{a(s)}} \left(\int_{t_1}^{t} \widetilde{\varsigma}(v)\, \eta^{\delta}(\theta(v))\, \mathrm{d}v \right)^{1/\delta} \mathrm{d}s = \infty,$$

then every solution of (2.2) oscillates.

Theorem 2.5. *If*

$$\limsup_{t\to\infty} \left(\eta^{\delta}(t) \int_{t_1}^{t} \widetilde{\varsigma}(v)\, \mathrm{d}v \right) > 1,$$

then every solution of (2.2) oscillates.

Theorem 2.6. *If*

$$\liminf_{t\to\infty} \int_{\omega(t)}^{t} \left(\frac{1}{a(s)} \int_{t_1}^{s} \widetilde{\varsigma}(v)\, \mathrm{d}v \right)^{1/\delta} \mathrm{d}s > \frac{1}{\mathrm{e}},$$

then every solution of (2.2) oscillates.

Remark 2.1. *It is obvious that these results do not account for the presence of the function $\psi(x)$ and delays θ_{ℓ}. Furthermore, Theorem 2.3 fails when $\delta < 1$, $b' < 0$, and $b < 1$. Additionally, Theorems 2.4–2.6, are not applicable when $b < 1$.*

On the other hand, there are very few papers dealing with the special case of (1.1) involving the function $\psi(x)$:

$$\left(a(t)\, \psi(x(t)) \left[z'(t) \right]^{\delta} \right)' + c(t)\, x^{\delta}(\theta(t)) = 0. \tag{2.7}$$

In 2009, Luhong and Zhiting [15] addressed this gap and introduced oscillation criteria for this case, proving several theorems.

Theorem 2.7. *If*

$$\int_{t_1}^{\infty} \left[\eta^{\delta}(\theta(v))\, \varsigma(v) - \frac{1}{M} \left(\frac{\delta}{\delta+1} \right)^{\delta+1} \frac{\theta'(v)}{\sqrt[\delta]{a(\theta(v))}\eta^{\delta}(\theta(v))} \right] \mathrm{d}v = \infty \tag{2.8}$$

and

$$\int_{t_1}^{\infty} \left[\eta^{\delta}(v)\, \varsigma(v) - \frac{1}{M} \left(\frac{\delta}{\delta+1} \right)^{\delta+1} \frac{a(\theta(v))}{a^{1+1/\delta}(v)\, (\theta'(v))^{\delta}\, \eta(v)} \right] \mathrm{d}v = \infty, \tag{2.9}$$

for $\psi(x) \le 1/M$ and

$$\varsigma(t) = c(t)\left(1 - b(\theta(t))\right)^{\delta},$$

then every solution of (2.7) oscillates.

Theorem 2.8. *If (2.8) holds and*

$$\int_{t_1}^{\infty} \eta^{\delta+1}(v)\, \varsigma(v)\, \mathrm{d}v = \infty,$$

then every solution of (2.7) oscillates.

Theorem 2.9. *If (2.9) holds and*

$$\int_{t_1}^{\infty}\left[\eta(\theta(\nu))\varsigma(\nu)-\frac{1}{M(\delta+1)^{\delta+1}}\frac{\theta'(\nu)}{\sqrt[\delta]{a(\theta(\nu))}\eta^{\delta}(\theta(\nu))}\right]\mathrm{d}\nu=\infty,$$

then every solution of (2.7) oscillates.

Remark 2.2. *The last criteria of Ye and Xu [15] requires that any positive solution of (1.1) on $\mathbb{I}_1$ satisfies the inequality*

$$\left(a(t)\psi(x(t))\left[z'(t)\right]^{\delta}\right)'+\varsigma(t)z^{\delta}(\theta(t))\leq 0.$$

However, this inequality may not always hold, particularly for decreasing positive solutions ($z'<0$); see Han et al. [12] for a discussion of this issue and the corresponding alternative conditions.

The following theorem is based on the conditions outlined above.

Theorem 2.10. *Assume that $b'\geq 0$, $\theta(t)\leq t-\omega(t)$, and there exists a differentiable function $\beta\in C^1(\mathbb{I}_1,\mathbb{R}^+)$ on $\mathbb{I}_1$, such that*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[\beta(\nu)\varsigma(\nu)-\frac{1}{M(\delta+1)^{\delta+1}}\frac{(\beta_+(\nu))^{\delta+1}a(\theta(\nu))}{(\theta'(\nu))^{\delta}\beta^{\delta}(\nu)}\right]\mathrm{d}\nu=\infty$$

and

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[c(\nu)\eta^{\delta}(\nu)\left(\frac{1}{1+b(\nu)}\right)^{\delta}-\frac{1}{M}\left(\frac{\delta}{\delta+1}\right)^{\delta+1}\frac{1}{\eta(\nu)\sqrt[\delta]{a(\nu)}}\right]\mathrm{d}\nu=\infty,$$

where $\beta_+=\max\{0,\beta'\}$. Then, Eq (2.7) is oscillatory.

Remark 2.3. *The derivation of the above conditions and (2.3)–(2.6) relies on establishing a new relationship between the dependent variable x and its associated function z as:*

$$x(\theta(t))\geq(1+b(t))^{-1}z(t)$$

under the assumptions $b\geq 0$ and $\lim_{t\to\infty}b(t)=0$. This relationship is completely different from the known relationships in the literature, which were also called into question after the study published in 2019 by Chatzarakis et al. [16].

As a result, considering the analytical shortcomings outlined in Remarks 2.1–2.3, and the relatively limited research on the oscillation of the general form of the NDDEs that involve the function $\psi(x)$, it is clear that many existing results require refinement or further scrutiny. This has led us to develop more precise and sufficient criteria to ensure the oscillation of this type of equations.

3. Properties of positive solutions

In this section, we will first introduce some auxiliary lemmas and notations. Then, we establish some lemmas describing the monotonic and asymptotic properties of positive solutions to (1.1). Let us define the following notations:

$$\widehat{c}(t)=\sum_{\ell=1}^{n}c_{\ell}(t)\left(1-\frac{b(\theta_{\ell}(t))\eta^{L}(\omega(\theta_{\ell}(t)))}{\eta^{L}(\theta_{\ell}(t))}\right)^{\delta},$$

$$\alpha_0(t) = \frac{1}{\sqrt[\delta]{n_2}} \limsup_{t\to\infty} \eta(t) \left(\int_{t_1}^{t} \widehat{c}(\nu)\, \mathrm{d}\nu \right)^{1/\delta},$$

and

$$\overline{c}(t) = \frac{\left(\alpha \sqrt[\delta]{n_1}\right)^{1-\delta}}{\delta} \widehat{c}(t)\, \eta^{\delta-1}(\theta(t)),$$

for every $\ell = 1, 2, ..., n,\ n \in \mathbb{N}$. Furthermore, we present the following auxiliary lemma:

Lemma 3.1. *[17] Let $\xi \in \mathbb{R}^+$ with*

$$G(\xi) = \left(q_1 - \sqrt[\delta]{q_2^{\delta+1}}\right)\xi,$$

where q_1, q_2 are arbitrary constants, $q_2 > 0$, and $\delta \in \mathbb{Q}^+_{odd}$ as in (A_1)*. Then, the following properties hold:*
(P_{0-1}) $G(\xi^*) = \max\{G(\xi) : \xi \in \mathbb{R}\} = \frac{\delta^\delta}{(\delta+1)^{\delta+1}} q_1^{\delta+1} q_2^{-\delta}$ *for* $\xi^* = \left(\frac{q_1\delta}{q_2(\delta+1)}\right)^\delta$*, which is the critical point of* G;
(P_{0-2}) $\left(q_1 - \sqrt[\delta]{q_2^{\delta+1}}\right)\xi \leq \frac{\delta^\delta}{(\delta+1)^{\delta+1}} q_1^{\delta+1} q_2^{-\delta}$.

According to Remark 2.3, we obtain that for any positive solution $x(t)$ of (1.1), the associated function $z(t)$ satisfies one of the following two cases:

$$\begin{aligned} C_1 &: \quad z^{(i)}(t) > 0 \text{ for } i = 0, 1, \text{ and } \left(a(t)\psi(x(t))(z'(t))^\delta\right)' \leq 0; \\ C_2 &: \quad z(t) > 0,\ z'(t) < 0, \text{ and } \left(a(t)\psi(x(t))(z'(t))^\delta\right)' \leq 0, \end{aligned}$$

on $\mathbb{I}_1$.

3.1. Preliminary lemmas for case C_1

Here, we establish the monotonic properties for Case C_1.

Lemma 3.2. *Assume that x is a positive solution of (1.1) on $\mathbb{I}_1$ and that Case C_1 holds eventually. Then, the following properties hold:*
(P_1) $x(t) \geq \left(1 - \frac{b(t)\eta^L(\omega(t))}{\eta^L(t)}\right) z(t)$;
(P_2) $\left(a(t)\psi(x(t))[z'(t)]^\delta\right)' + k_1\widehat{c}(t) \leq 0,$
for $k_1 \in \mathbb{R}^+ \cup \{0\}$.

Proof. Assume that x is a positive solution of (1.1) and C_1 holds eventually for all $t \in (t_1, \infty)$. It is obvious from the definition of the corresponding variable z and (A_2) that (P_1) holds. Now, from (1.1), we obtain

$$\begin{aligned} \left(a\psi(x)[z']^\delta\right)' &= -\sum_{\ell=1}^{n} \left(c_\ell x^\delta(\theta_\ell)\right) \\ &\leq -\sum_{\ell=1}^{n} c_\ell \left(1 - \frac{b(\theta_\ell)\eta^L(\omega(\theta_\ell))}{\eta^L(\theta_\ell)}\right)^\delta z^\delta(\theta_\ell). \end{aligned}$$

Since z is a positive increasing function, then there exists a positive constant k_1 such that $z \geq k_1$, and so,

$$\begin{aligned}\left(a\psi(x)[z']^{\delta}\right)' &\leq -k_1 \sum_{\ell=1}^{n} c_\ell \left(1 - \frac{b(\theta_\ell)\eta^L(\omega(\theta_\ell))}{\eta^L(\theta_\ell)}\right)^{\delta} \\ &= -k_1 \widehat{c},\end{aligned}$$

which gives (P$_2$). This completes the proof.

3.2. Preliminary lemmas for case C$_2$

Lemma 3.3. *Assume that x is a positive solution of (1.1) on* $\mathbb{I}_1$ *and that Case* C$_2$ *holds eventually. Then, the following properties hold:*

(P$_3$) $\sqrt[\delta]{a(t)}\eta(t)z'(t) + Lz(t) \geq 0$;

(P$_4$) $z(t)/\eta^L(t)$ *is nondecreasing;*

(P$_5$) $x(t) \geq \left(1 - \frac{b(t)\eta^L(\omega(t))}{\eta^L(t)}\right)z(t)$;

(P$_6$) $\left(a(t)\psi(x(t))[z'(t)]^{\delta}\right)' + \widehat{c}(t)z^{\delta}(\theta(t)) \leq 0$.

Proof. Assume that x is a positive solution of (1.1) and C$_2$ holds eventually for all $t \in [t_1, \infty)$. The nonincreasing monotonicity of $a\psi(x)(z')^{\delta}$ yields

$$\begin{aligned}-z(t) &\leq \int_t^{\infty} \frac{\sqrt[\delta]{a(v)\psi(x(v))}z'(v)}{\sqrt[\delta]{a(v)\psi(x(v))}}\mathrm{d}v \\ &\leq \sqrt[\delta]{a(t)\psi(x(t))}z'(t)\int_t^{\infty}\frac{1}{\sqrt[\delta]{a(v)\psi(x(v))}}\mathrm{d}v.\end{aligned} \tag{3.1}$$

But (A$_2$) and (A$_4$) imply that

$$\begin{aligned}\sqrt[\delta]{\psi(x(t))}\int_t^{\infty}\frac{1}{\sqrt[\delta]{a(v)\psi(x(v))}}\mathrm{d}v &\geq \sqrt[\delta]{n_1}\int_t^{\infty}\frac{1}{\sqrt[\delta]{n_2 a(v)}}\mathrm{d}v \\ &= \sqrt[\delta]{\frac{n_1}{n_2}}\int_t^{\infty}\frac{1}{\sqrt[\delta]{a(v)}}\mathrm{d}v \\ &= \frac{\eta(t)}{L}.\end{aligned}$$

Substituting into (3.1), we have

$$\sqrt[\delta]{a(t)}\eta(t)z'(t) + Lz(t) \geq 0.$$

Therefore,

$$\begin{aligned}\left(\frac{z(t)}{\eta^L(t)}\right)' &= \frac{\eta^L(t)z'(t) + L\eta^{L-1}(t)\sqrt[\delta]{1/a(t)}z(t)}{\eta^{2L}(t)} \\ &= \frac{\eta^{L-1}(t)}{\eta^{2L}(t)\sqrt[\delta]{a(t)}}\left(\sqrt[\delta]{a(t)}\eta(t)z'(t) + Lz(t)\right) \geq 0.\end{aligned}$$

Now, by using this result and the definition of the corresponding variable z, we can prove (P_5). First, based on the increasing nature of z/η^L, for $\omega(t) \leq t$, we find that

$$z(t) - b(t) z(\omega(t)) \geq \left(1 - \frac{b(t) \eta^L(\omega(t))}{\eta^L(t)}\right) z(t),$$

and therefore, we have

$$\begin{aligned} x(t) &= z(t) - b(t) x(\omega(t)) \\ &\geq z(t) - b(t) z(\omega(t)) \\ &\geq \left(1 - \frac{b(t) \eta^L(\omega(t))}{\eta^L(t)}\right) z(t). \end{aligned}$$

Substituting into (1.1) and using (A_5), we obtain

$$\begin{aligned} \left(a\psi(x)[z']^\delta\right)' &= -\sum_{\ell=1}^{n} c_\ell x^\delta(\theta_\ell) \\ &\leq -\sum_{\ell=1}^{n} c_\ell \left(1 - \frac{b(\theta_\ell) \eta^L(\omega(\theta_\ell))}{\eta^L(\theta_\ell)}\right)^\delta z^\delta(\theta_\ell) \\ &\leq -z^\delta(\theta) \sum_{\ell=1}^{n} c_\ell \left(1 - \frac{b(\theta_\ell) \eta^L(\omega(\theta_\ell))}{\eta^L(\theta_\ell)}\right)^\delta \\ &= -\widehat{c} z^\delta(\theta). \end{aligned}$$

This completes the proof.

Lemma 3.4. *Assume that x is a positive solution of (1.1) on $\mathbb{I}_1$. If*

$$\int_{t_1}^{\infty} \widehat{c}(v)\,\mathrm{d}v = \infty, \tag{3.2}$$

then x satisfies Case C_2.

Proof. Assume, on the contrary, that there is an eventually positive solution x to (1.1) that satisfies case C_1 for all $t \geq t_1 \geq t_0$. For convenience, let us define the positive function:

$$\pi = \frac{a\psi(x)[z']^\delta}{k_1}.$$

By differentiating the above function and using (P_2) of Lemma 3.2, we get

$$\pi' = \frac{\left(a\psi(x)[z']^\delta\right)'}{k_1} \leq -\widehat{c}.$$

Integrating the above inequality over the interval $[t_1, t]$ yields

$$\pi(t) - \pi(t_1) \leq -\int_{t_1}^{t} \widehat{c}(v)\,\mathrm{d}v.$$

Taking the limit on both sides as t approaches to infinity and using (3.2) implies a contradiction. Therefore, we conclude that x must satisfy Case C_2. This completes the proof.

Lemma 3.5. *Assume that x is a positive solution of (1.1). If*

$$\int_{t_1}^{\infty}\left[\frac{1}{a(s)}\int_{t_1}^{s}\widehat{c}(v)\,\mathrm{d}v\right]^{1/\delta}\mathrm{d}s=\infty, \tag{3.3}$$

then $\lim_{t\to\infty} z(t)=0$.

Proof. Assume that x is a positive solution of (1.1) for $t\geq t_1\geq t_0$. It is obvious that condition (3.3), together with (3.2), implies that x satisfies Case C_2. Since the corresponding z is positive and decreasing, then there exists a nonnegative constant $k_2\geq 0$ such that $\lim_{t\to\infty} z(t)=k_2$. Assume, on the contrary, that $z\geq k_2>0$, and so, $z^{\delta}(\theta)\geq k_2^{\delta}>0$ for sufficiently large t. From property (P_6) in Lemma 3.3, it follows that

$$\begin{aligned}-\left(a(t)\psi(x(t))[z'(t)]^{\delta}\right)' &\geq \widehat{c}(t)z^{\delta}(\theta(t))\\ &\geq \widehat{c}(t)k_2^{\delta}.\end{aligned}$$

Integrating this inequality from t_1 to t, we obtain

$$-a(t)\psi(x(t))[z'(t)]^{\delta}\geq k_2^{\delta}\int_{t_1}^{t}\widehat{c}(v)\,\mathrm{d}v.$$

Now, integrating again from t_1 to ∞ gives

$$z(t_1)\geq\frac{k_2}{\sqrt[\delta]{n_2}}\int_{t_1}^{\infty}\left[\frac{1}{a(s)}\int_{t_1}^{s}\widehat{c}(v)\,\mathrm{d}v\right]^{1/\delta}\mathrm{d}s.$$

Applying condition (3.3) yields

$$z(t_1)\geq\infty,$$

which is a contradiction. Therefore, $k_2=0$, and we conclude that $\lim_{t\to\infty} z(t)=0$. This completes the proof.

Lemma 3.6. *Assume that x is an eventually positive solution of (1.1) on $\mathbb{I}_1$ and $L=1$. Then z/η is a nondecreasing function.*

Proof. Let x be an eventually positive solution of (1.1) on $\mathbb{I}_1$. From the nonincreasing monotonicity of $a\psi(x)(z')^{\delta}$ and condition (A_4), we have, for $t\leq v$ and $L=1$, that

$$\begin{aligned}a(t)(z'(t))^{\delta} &\geq a(t)\psi(x(t))(z'(t))^{\delta}\\ &\geq a(v)\psi(x(v))(z'(v))^{\delta}\geq a(v)(z'(v))^{\delta}.\end{aligned} \tag{3.4}$$

Next, using (3.4), we arrive at

$$\begin{aligned}-z(t) &\leq \int_{t}^{\infty}\frac{\sqrt[\delta]{a(v)}z'(v)}{\sqrt[\delta]{a(v)}}\mathrm{d}v\\ &\leq \sqrt[\delta]{a(t)}z'(t)\int_{t}^{\infty}\frac{1}{\sqrt[\delta]{a(v)}}\mathrm{d}v\\ &= \sqrt[\delta]{a(t)}\eta(t)z'(t),\end{aligned}$$

which yields

$$\sqrt[\delta]{a(t)}z'(t)\eta(t)+z(t)\geq 0,$$

and this completes the proof.

Lemma 3.7. *Assume that x is an eventually positive solution of (1.1) on $\mathbb{I}_1$ with $L = 1$ and condition (3.2) holding. If*

$$\int_{t_1}^{\infty} \widehat{c}(v)\,\eta^{\delta}(\theta(v))\,\mathrm{d}v = \infty, \tag{3.5}$$

then

$$\lim_{t\to\infty} \frac{z(t)}{\eta(t)} = \infty.$$

Proof. Assume that x is an eventually positive solution of (1.1) on $\mathbb{I}_1$. Similarly, as mentioned earlier, it follows from (3.5) and (A_2) that $\int_{t_1}^{\infty} \widehat{c}(v)\,\eta^{\delta}(\theta(v))\,\mathrm{d}v$ must be unbounded, and η is a positive decreasing function. Hence, condition (3.5) implies condition (3.2), guaranteeing that x satisfies Case C_2. Integrating the inequality (P_6) of Lemma 3.3 from t_1 to t, we get

$$\begin{aligned} -a(t)\,\psi(x(t))\,[z'(t)]^{\delta} &\geq \int_{t_1}^{t} \widehat{c}(v)\,z^{\delta}(\theta(v))\,\mathrm{d}v \\ &= \int_{t_1}^{t} \widehat{c}(v)\,\eta^{\delta}(\theta(v))\,\frac{z^{\delta}(\theta(v))}{\eta^{\delta}(\theta(v))}\mathrm{d}v. \end{aligned}$$

Using the monotonicity of $z(t)/\eta$ (from Lemma (3.6)), we get

$$-a(t)\,\psi(x(t))\,[z'(t)]^{\delta} \geq \frac{z^{\delta}(\theta(t_1))}{\eta^{\delta}(\theta(t_1))} \int_{t_1}^{t} \widehat{c}(v)\,\eta^{\delta}(\theta(v))\,\mathrm{d}v. \tag{3.6}$$

Now, suppose for contradiction that the positive increasing function $-a\psi(x)[z']^{\delta}$ is bounded above. This means that there exists a positive constant $k_3 \in \mathbb{R}^+$, where $k_3 \geq -a\psi(x)[z']^{\delta}$ and $\lim_{t\to\infty} -a(t)\,\psi(x(t))\,[z'(t)]^{\delta} = k_3 < \infty$. Substituting into (3.6), we have

$$k_3 \geq \left(\frac{z(\theta(t_1))}{\eta(\theta(t_1))}\right)^{\delta} \int_{t_1}^{t} \widehat{c}(v)\,\eta^{\delta}(\theta(v))\,\mathrm{d}v.$$

Taking the limit on both sides as $t \to \infty$ yields

$$k_3 \geq \left(\frac{z(\theta(t_1))}{\eta(\theta(t_1))}\right)^{\delta} \int_{t_1}^{\infty} \widehat{c}(v)\,\eta^{\delta}(\theta(v))\,\mathrm{d}v = \infty,$$

a contradiction, and we conclude that $\lim_{t\to\infty} -a(t)\,\psi(x(t))\,[z'(t)]^{\delta} = \infty$. Finally, since

$$\lim_{t\to\infty} \frac{z(t)}{\eta(t)} \overset{L'H}{=} -\lim_{t\to\infty} a^{1/\delta}(t)\,z'(t),$$

we conclude the result.

Lemma 3.8. *Assume that x is a positive solution of (1.1) on $\mathbb{I}_1$ and that Case C_2 holds eventually. Then,*

$$\left(\frac{z(t)}{\eta^{\alpha}(t)}\right)' < 0 \tag{3.7}$$

for $\alpha = \alpha_0 - \epsilon$, where $\epsilon > 0$.

Proof. Assume that x is a positive solution of (1.1) and C_2 holds eventually for all $t \in [t_1, \infty)$. From (P_3) in Lemma 3.3, we have

$$z(\theta(t)) \geq z(t) \geq -\frac{1}{L}\sqrt[\delta]{a(t)}\eta(t)\, z'(t). \tag{3.8}$$

Integrating (P_6) of the same Lemma from t_1 to t, yields

$$-a(t)\psi(x(t))[z'(t)]^{\delta} \geq z^{\delta}(\theta(t)) \int_{t_1}^{t} \widehat{c}(\nu)\,\mathrm{d}\nu,$$

i.e.,

$$\begin{aligned} -z'(t) &\geq \sqrt[\delta]{\frac{1}{a(t)\psi(x(t))}}\, z(\theta(t)) \left(\int_{t_1}^{t} \widehat{c}(\nu)\,\mathrm{d}\nu\right)^{1/\delta} \\ &\geq \sqrt[\delta]{\frac{1}{a(t)\psi(x(t))}}\, z(t) \left(\int_{t_1}^{t} \widehat{c}(\nu)\,\mathrm{d}\nu\right)^{1/\delta}. \end{aligned} \tag{3.9}$$

On the other hand,

$$\begin{aligned} \left(\frac{z(t)}{\eta^{\alpha}(t)}\right)' &= \frac{\eta^{\alpha}(t)\, z'(t)}{\eta^{2\alpha}(t)} + \frac{\alpha z(t)\, \eta^{\alpha-1}(t)}{\eta^{2\alpha}(t)\sqrt[\delta]{a(t)}} \\ &= \frac{z(t)}{\sqrt[\delta]{a(t)}\eta^{\alpha+1}(t)} \left(\sqrt[\delta]{a(t)}\eta(t)\frac{z'(t)}{z(t)} + \alpha\right). \end{aligned}$$

From (3.9), we have

$$\sqrt[\delta]{a(t)\psi(x(t))}\frac{z'(t)}{z(t)} \leq -\left(\int_{t_1}^{t} \widehat{c}(\nu)\,\mathrm{d}\nu\right)^{1/\delta},$$

i.e.,

$$\sqrt[\delta]{a(t)}\frac{z'(t)}{z(t)} \leq -\frac{1}{\sqrt[\delta]{n_2}}\left(\int_{t_1}^{t} \widehat{c}(\nu)\,\mathrm{d}\nu\right)^{1/\delta}.$$

Thus,

$$\alpha + \sqrt[\delta]{a(t)}\eta(t)\frac{z'(t)}{z(t)} \leq \alpha - \frac{\eta(t)}{\sqrt[\delta]{n_2}}\left(\int_{t_1}^{t} \widehat{c}(\nu)\,\mathrm{d}\nu\right)^{1/\delta} < 0.$$

This completes the proof.

Lemma 3.9. *Assume that x is a positive solution of (1.1) satisfying Case C_2 on $\mathbb{I}_1$ and $\delta \geq 1$. Then,*

$$\left(\sqrt[\delta]{a(t)\psi(x(t))}z'(t)\right)' + \overline{c}(t)\, z(\theta(t)) \leq 0, \tag{3.10}$$

eventually.

Proof. Let x be a positive solution of (1.1) and suppose that C_2 holds eventually for all $t \in [t_1, \infty)$. From (3.7) in Lemma 3.8, we have

$$\sqrt[\delta]{a(\theta)\psi(x(\theta))}z'(\theta) < -\alpha\sqrt[\delta]{n_1}\frac{z(\theta)}{\eta(\theta)}.$$

Since $a\psi(x)(z')^{\delta}$ is noninceasing, we have

$$\sqrt[\delta]{a\psi(x)}z' \leq \sqrt[\delta]{a(\theta)\psi(x(\theta))}z'(\theta),$$

and hence

$$\sqrt[\delta]{a\psi(x)}z' < -\alpha\sqrt[\delta]{n_1}\frac{z(\theta)}{\eta(\theta)}.$$

Raising both sides to the power $\delta - 1$ yields:

$$\left(\sqrt[\delta]{a\psi(x)}z'\right)^{\delta-1} \leq \left(\alpha\sqrt[\delta]{n_1}\frac{z(\theta)}{\eta(\theta)}\right)^{\delta-1}.$$

Now,

$$\begin{aligned}\left(a\psi(x)[z']^{\delta}\right)' &= \left(\left(\sqrt[\delta]{a\psi(x)}z'\right)^{\delta}\right)' \\ &= \delta\left(\sqrt[\delta]{a\psi(x)}z'\right)^{\delta-1}\left(\sqrt[\delta]{a\psi(x)}z'\right)' \\ &\geq \delta\left(\alpha\sqrt[\delta]{n_1}\frac{z(\theta)}{\eta(\theta)}\right)^{\delta-1}\left(\sqrt[\delta]{a\psi(x)}z'\right)',\end{aligned}$$

and from (P_6) in Lemma 3.3, we have

$$-\widehat{c}z^{\delta}(\theta) \geq \delta\left(\alpha\sqrt[\delta]{n_1}\frac{z(\theta)}{\eta(\theta)}\right)^{\delta-1}\left(\sqrt[\delta]{a\psi(x)}z'\right)',$$

that is,

$$\left(\sqrt[\delta]{a\psi(x)}z'\right)' + \frac{\left(\alpha\sqrt[\delta]{n_1}\right)^{1-\delta}}{\delta}\widehat{c}\eta^{\delta-1}(\theta)z(\theta) \leq 0.$$

And the proof is complete.

4. Main oscillation criteria

Below, we derive our main results and introduce some oscillation theorems for (1.1) by using different techniques.

Theorem 4.1. *Assume that (3.2) holds. If*

$$\limsup_{t\to\infty}\eta^{\delta}(t)\int_{t_1}^{t}\widehat{c}(\nu)\,\mathrm{d}\nu > \frac{n_2^2}{n_1}, \tag{4.1}$$

then (1.1) is oscillatory.

Proof. Assume, to the contrary, that (1.1) has an eventually positive solution x on $t \geq t_1 \geq t_0$. As discussed previously in Lemma 3.4, condition (3.2) guarantees that $x(t)$ satisfies Case C_2. Substituting (3.8) into (3.9) implies that

$$-z' \geq -\frac{1}{L}\sqrt[\delta]{\frac{1}{a\psi(x)}}\sqrt[\delta]{a}\eta z'\left(\int_{t_1}^{t}\widehat{c}(\nu)\,\mathrm{d}\nu\right)^{1/\delta},$$

which gives

$$\begin{aligned} L &\geq \sqrt[\delta]{\frac{1}{\psi(x)}}\eta\left(\int_{t_1}^{t}\widehat{c}(v)\,\mathrm{d}v\right)^{1/\delta} \\ &\geq \sqrt[\delta]{\frac{1}{n_2}}\eta\left(\int_{t_1}^{t}\widehat{c}(v)\,\mathrm{d}v\right)^{1/\delta}. \end{aligned}$$

And so, (A_4) implies

$$\frac{n_2^2}{n_1} \geq \eta^{\delta}(t)\int_{t_1}^{t}\widehat{c}(v)\,\mathrm{d}v.$$

Taking the limit superior of both sides as $t \to \infty$, we get a contradiction with (4.1). Thus, the proof is complete.

Theorem 4.2. *If*

$$\int_{t_1}^{\infty}\frac{1}{\sqrt[\delta]{a(s)}}\left(\int_{t_1}^{s}\widehat{c}(v)\,\eta^{\delta}(\theta(v))\,\mathrm{d}v\right)^{1/\delta}\mathrm{d}s = \infty, \tag{4.2}$$

then (1.1) is oscillatory.

Proof. Assume, to the contrary, that (1.1) has an eventually positive solution x on $t \geq t_1 \geq t_0$. It is not difficult to see that (4.2) implies (3.2), which guarantees that x satisfies Case C_2 and there are no increasing positive solutions for (1.1). Since $a\psi(x)(z')^{\delta}$ is a decreasing function, then there exist a constant $k_4 \in \mathbb{R}^+$ such that

$$-\sqrt[\delta]{a}z' \geq \sqrt[\delta]{\frac{k_4}{\psi(x)}}.$$

Therefore, from part (P_3) of Lemma 3.3, we have

$$\begin{aligned} z &\geq -\frac{1}{L}\sqrt[\delta]{a}\eta z' \\ &\geq \frac{\eta}{L}\sqrt[\delta]{\frac{k_4}{\psi(x)}} \\ &\geq \frac{\eta}{L}\sqrt[\delta]{\frac{k_4}{n_2}} \\ &= \sqrt[\delta]{\frac{k_4 n_1}{n_2^2}}\eta. \end{aligned}$$

Substituting this estimate into part (P_6) of Lemma 3.3,

$$\begin{aligned} \left(a\psi(x)[z']^{\delta}\right)' &\leq -\widehat{c}z^{\delta}(\theta) \\ &\leq -\frac{k_4 n_1}{n_2^2}\widehat{c}\eta^{\delta}(\theta). \end{aligned}$$

Integrating the above inequality from t_1 to t, yields

$$a\psi(x)[z']^{\delta} \leq -\frac{k_4 n_1}{n_2^2}\int_{t_1}^{t}\widehat{c}(v)\,\eta^{\delta}(\theta(v))\,\mathrm{d}v.$$

Integrating once more from t_1 to t, we obtain

$$\begin{aligned} z(t) &\leq z(t_1) - \frac{k_4 n_1}{n_2^2} \int_{t_1}^{t} \frac{1}{\sqrt[\delta]{a(s)\psi(x(s))}} \left(\int_{t_1}^{t} \widehat{c}(v)\, \eta^{\delta}(\theta(v))\, \mathrm{d}v \right)^{1/\delta} \mathrm{d}s \\ &\leq z(t_1) - \frac{k_4 n_1}{n_2^{2+1/\delta}} \int_{t_1}^{t} \frac{1}{\sqrt[\delta]{a(s)}} \left(\int_{t_1}^{t} \widehat{c}(v)\, \eta^{\delta}(\theta(v))\, \mathrm{d}v \right)^{1/\delta} \mathrm{d}s. \end{aligned}$$

Taking the limit on both sides as t approaches to infinity leads to a contradiction with (4.2), and this completes the proof.

Theorem 4.3. *Assume that* $\theta(t) < t$ *and*

$$\liminf_{t \to \infty} \int_{\theta(t)}^{t} \left(\frac{1}{a(s)} \int_{t_1}^{s} \widehat{c}(v)\, \mathrm{d}v \right)^{1/\delta} \mathrm{d}s > \frac{n_2}{\mathrm{e}}. \tag{4.3}$$

Then, (1.1) is oscillatory.

Proof. Assume, to the contrary, that (1.1) has an eventually positive solution x on $t \geq t_1 \geq t_0$. Since x has two possible cases, let us firstly consider that x satisfies Case C_1. Under (A_2), it is obvious that condition (4.3) implies (3.2), which in turn, ensures that there are no positive increasing solutions of (1.1) under (4.3).
Now, consider that x satisfies Case C_2. And so, from (3.9), it is not difficult to see that

$$\begin{aligned} 0 &\geq z' + \sqrt[\delta]{\frac{1}{a\psi(x)}} z(\theta) \left(\int_{t_1}^{t} \widehat{c}(v)\, \mathrm{d}v \right)^{1/\delta} \\ &\geq z' + \sqrt[\delta]{\frac{1}{a n_2}} z(\theta) \left(\int_{t_1}^{t} \widehat{c}(v)\, \mathrm{d}v \right)^{1/\delta}; \end{aligned}$$

this inequality has no positive solution under (4.3) according to [18] and [19]. A contradiction completes the proof.

Theorem 4.4. *Assume that (3.2) and* $\delta \geq 1$ *hold. If*

$$\liminf_{t \to \infty} \int_{\theta(t)}^{t} \eta(v)\, \overline{c}(v)\, \mathrm{d}v > \frac{\sqrt[\delta]{n_2} - \alpha \sqrt[\delta]{n_1}}{\mathrm{e}}, \tag{4.4}$$

for $\alpha = \alpha_0 - \epsilon, \epsilon > 0$, *then (1.1) is oscillatory.*

Proof. Assume, to the contrary, that (1.1) has an eventually positive solution x on $t \geq t_1 \geq t_0$. As discussed previously in Lemma 3.4, condition (3.2) guarantees that x satisfies Case C_2. For simplicity, let us define the function

$$\phi = \sqrt[\delta]{a\psi(x)}\eta z' + \sqrt[\delta]{n_2} z.$$

Inequality (3.7) in Lemma 3.8 indicates that

$$\sqrt[\delta]{a\psi(x)}\eta z' < -\alpha \sqrt[\delta]{n_1} z.$$

Substituting this inequality into the definition of ϕ, we get

$$\begin{aligned} \phi &< \sqrt[\delta]{n_2} z - \alpha \sqrt[\delta]{n_1} z \\ &= \left(\sqrt[\delta]{n_2} - \alpha \sqrt[\delta]{n_1} \right) z. \end{aligned} \tag{4.5}$$

Now, differentiating ϕ gives us

$$\begin{aligned} \phi' &\leq \sqrt[\delta]{a\psi(x)} \eta z'' + \left(\sqrt[\delta]{a\psi(x)} \right)' \eta z' \\ &= \left(\sqrt[\delta]{a\psi(x)} z' \right)' \eta. \end{aligned}$$

By using (3.10) in Lemma 3.9, we obtain

$$\phi' \leq -\bar{c} z(\theta) \eta.$$

But (4.5) implies that

$$\phi' + \frac{1}{\left(\sqrt[\delta]{n_2} - \alpha \sqrt[\delta]{n_1} \right)} \bar{c} \eta \phi(\theta) \leq 0.$$

This inequality has no positive solutions under (4.4) according to [20], so we get a contradiction that completes the proof.

Theorem 4.5. *Assume that (3.2) holds and there exists a differentiable function $\beta(t) \in C^1(\mathbb{I}_1, \mathbb{R}^+)$ such that*

$$\limsup_{t\to\infty} \frac{\eta^\delta(t)}{\beta(t)} \int_{t_2}^{t} \left[\beta(v) \widehat{c}(v) - n_2 \left(\frac{\beta'(v)}{\delta+1} \right)^{\delta+1} \frac{a(v)}{\beta^\delta(v)} \right] \mathrm{d}v > \frac{n_2^2}{n_1}. \tag{4.6}$$

Then (1.1) is oscillatory.

Proof. Assume, to the contrary, that (1.1) has an eventually positive solution x on $t \geq t_1 \geq t_0$. As discussed previously in Lemma 3.4, condition (3.2) guarantees that x satisfies Case C_2.
For simplicity, let us define the function φ as

$$\varphi = \beta \left(a\psi(x) [z']^\delta z^{-\delta} + \eta^{-\delta} \right).$$

Differentiating φ, we arrive at

$$\varphi' = \beta \left(\frac{\left(a\psi(x) [z']^\delta \right)'}{z^\delta} - \delta (a\psi(x)) \left(\frac{z'}{z} \right)^{\delta+1} + \frac{\delta}{\sqrt[\delta]{a} \eta^{\delta+1}} + \frac{\beta'}{\beta} \varphi \right), \tag{4.7}$$

but

$$\varphi - \frac{\beta}{\eta^\delta} = \beta \frac{a\psi(x) [z']^\delta}{z^\delta}.$$

By raising both sides to the power of $1 + 1/\delta$, we get

$$\left(\varphi - \frac{\beta}{\eta^\delta} \right)^{1+1/\delta} = \sqrt[\delta]{\beta [a\psi(x)]} \beta (a\psi(x)) \left(\frac{z'}{z} \right)^{\delta+1}.$$

Substituting into (4.7), we get

$$\begin{aligned}\varphi' &= \beta\frac{\left(a\psi(x)[z']^{\delta}\right)'}{z^{\delta}} - \frac{\delta}{\sqrt[\delta]{\beta[a\psi(x)]}}\left(\varphi - \frac{\beta}{\eta^{\delta}}\right)^{1+1/\delta} \\ &\quad +\beta\frac{\delta}{\sqrt[\delta]{a}\eta^{\delta+1}} + \frac{\beta'}{\beta}\varphi.\end{aligned}$$

Using part (P_6) of Lemma 3.3 in the above inequality, yields

$$\varphi' \le -\beta\widehat{c} - \frac{\delta}{\sqrt[\delta]{\beta[a\psi(x)]}}\left(\varphi - \frac{\beta}{\eta^{\delta}}\right)^{1+1/\delta} + \beta\frac{\delta}{\sqrt[\delta]{a}\eta^{\delta+1}} + \frac{\beta'}{\beta}\varphi.$$

Now, from part (P_{0-2}) of Lemma 3.1 with

$$q_1 = \beta'\beta^{-1} \quad \text{and} \quad q_2 = \frac{\delta}{\sqrt[\delta]{\beta[a\psi(x)]}},$$

we get

$$\varphi' \le -\beta\widehat{c} + \left(\frac{\beta}{\eta^{\delta}}\right)' + \frac{(\beta')^{\delta+1}}{(\delta+1)^{\delta+1}}\frac{a\psi(x)}{\beta^{\delta}}.$$

Integrating the last inequality from t_2 to t, we have

$$\begin{aligned}\varphi(t_2) - \varphi(t) &\ge \int_{t_2}^{t}\left[\beta(v)\widehat{c}(v) - \left(\frac{\beta'(v)}{\delta+1}\right)^{\delta+1}\frac{a(v)\psi(x(v))}{\beta^{\delta}(v)}\right]\mathrm{d}v \\ &\quad -\frac{\beta}{\eta^{\delta}}(t) + \left(\frac{\beta}{\eta^{\delta}}\right)(t_2).\end{aligned}$$

However, the definition of φ and part (P_3) of Lemma 3.3 indicates that

$$\frac{n_2^2}{n_1}\beta\eta^{-\delta} \ge \int_{t_2}^{t}\left[\beta(v)\widehat{c}(v) - \left(\frac{\beta'(v)}{\delta+1}\right)^{\delta+1}\frac{a(v)\psi(x(v))}{\beta^{\delta}(v)}\right]\mathrm{d}v,$$

i.e.,

$$\begin{aligned}\frac{n_2^2}{n_1} &\ge \frac{\eta^{\delta}}{\beta}\int_{t_2}^{t}\left[\beta(v)\widehat{c}(v) - \left(\frac{\beta'(v)}{\delta+1}\right)^{\delta+1}\frac{a(v)\psi(x(v))}{\beta^{\delta}(v)}\right]\mathrm{d}v \\ &\ge \frac{\eta^{\delta}}{\beta}\int_{t_2}^{t}\left[\beta(v)\widehat{c}(v) - n_2\left(\frac{\beta'(v)}{\delta+1}\right)^{\delta+1}\frac{a(v)}{\beta^{\delta}(v)}\right]\mathrm{d}v.\end{aligned}$$

Taking the limit superior of both sides, we get a contradiction with (4.6). Thus, the proof is complete.

5. Conclusions and illustrative examples

In this paper, we generalize the study of NDDEs of the second order with multiple delays by including the existence of the function $\psi(x)$. This type of equation, as we mentioned before, has received little attention from researchers and the results available in previous studies have been questioned, with some found to be less accurate, while others continue to be a subject of discussion and investigation. This served as the motivation for our study. In addition, we aimed to improve some previous works and provide better oscillation criteria with fewer constraints compared to the previous ones by using more than one approach for the analysis.

First, we classified the positive solutions of (1.1) into two cases (C_1 and C_2) and established certain monotonic and asymptotic properties of these solutions. We also identified relationships between the dependent variable x, its derivatives, and the associated function z. These relationships form the foundation of our theorems, which include five oscillation theorems. Theorems 4.1 and 4.2 were derived from the relationships presented in Lemmas 3.3 and 3.8. Theorems 4.3 and 4.4 were developed using the comparison method with first-order DEs. Finally, Theorem 4.5 is based on the Riccati technique. Below, we presented some results based on this theorem.

Remark 5.1. *Putting $\psi(x) = 1$ and $n = 1$ in Theorem 4.2 yields Theorem 2.4 of Bohner et al. [14].*

Remark 5.2. *Putting $\beta = 1$ in Theorem 4.5 yields Theorem 4.1.*

Remark 5.3. *Theorem 4.5 is notable for its extreme flexibility, as its results are determined by the function β, which has an infinite number of possibilities and choices. This gives the theory the possibility of improving its results over time.*

Below we will present some results from applying Theorem 4.5 to different values of β to clarify the previous remark and use these results later to compare them and derive the best oscillation results.

Corollary 5.1. *Assume that (3.2) holds. If*

$$\limsup_{t\to\infty} \eta^{\delta-1}(t) \int_{t_2}^{t} \left[\eta(v)\widehat{c}(v) - \frac{n_2}{(\delta+1)^{\delta+1}} \frac{1}{\sqrt[\delta]{a(v)}\eta^{\delta}(v)} \right] \mathrm{d}v > \frac{n_2^2}{n_1}, \tag{5.1}$$

then (1.1) is oscillatory.

Proof. Conversely, let us assume that for (1.1), there exists an eventually positive solution x on $t \geq t_1 \geq t_0$. Lemma (3.4) states that (3.2) guarantees that x satisfies Case C_2. The proof is finished by taking $\beta = \eta$ in Theorem 4.5, which leads to (5.1).

Corollary 5.2. *Assume that (3.2) holds. If*

$$\limsup_{t\to\infty} \int_{t_2}^{t} \left[\eta^{\delta}(v)\widehat{c}(v) - n_2 \left(\frac{\delta}{(\delta+1)} \right)^{\delta+1} \frac{1}{\sqrt[\delta]{a(v)}\eta(v)} \right] \mathrm{d}v > \frac{n_2^2}{n_1}, \tag{5.2}$$

then (1.1) is oscillatory.

Proof. In contrast, suppose that on $t \geq t_1 \geq t_0$, there exists an eventually positive solution x for (1.1). As mentioned in Lemma 3.4, (3.2) ensures that x satisfies Case C_2. By taking $\beta = \eta^{\delta}$ in Theorem 4.5, we arrive at (5.2), and this completes the proof.

Example 5.1. *Consider the following second-order NDE:*

$$\left(t^2\left([x(t) + b_0 x(\lambda t)]'\right)^{\delta}\right)' + c_0 t^{1-\delta} \sum_{\ell=1}^{n} x^{\delta}(\mu_\ell t) = 0, \tag{$E1$}$$

for $t \in [0,\infty)$, $\delta \in \mathbb{Q}^{+}_{odd}$, $\lambda \in \left(b_0^{\frac{\delta}{2-\delta}}, 1\right]$, $b_0 \in \left[0, \lambda^{2/\delta-1}\right]$, $c_\ell(t) = c_0 t^{1-\delta}$, $c_0 \in (0,\infty)$, *and* $\mu_\ell \in (0,1)$, *for all* $\ell = 1,2,...,n$, $n \in \mathbb{N}$. *It is obvious that conditions* (A_1)*–*(A_4) *hold eventually for* $\delta < 2$ *with* $\psi(t) = 1$ *(which means that* $L = n_1 = n_2 = 1$*) and* $\eta(t) = t^{1-2/\delta}/2/\delta - 1$. *Furthermore,* (A_5) *yields*

$$\theta(t) = \mu t = \max_{\ell\in\mathbb{N}} \{\mu_\ell t\} \quad \forall \quad \mu = \max_{\ell\in\mathbb{N}} \{\mu_\ell\},$$

and after some calculations, we get

$$\widehat{c}(t) = c_0 \left(1 - b_0\lambda^{1-2/\delta}\right)^{\delta} t^{1-\delta},$$

$$\alpha_0(t) = \frac{\delta c_0^{1/\delta}}{(2-\delta)^{1+1/\delta}} \left(1 - b_0\lambda^{1-2/\delta}\right),$$

and

$$\overline{c}(t) = c_0 \frac{(\alpha(2/\delta - 1))^{1-\delta}}{\delta} \left(1 - b_0\lambda^{1-2/\delta}\right)^{\delta} \mu^{\delta+2/\delta-3} t^{2/\delta-2}$$

with $\alpha = \alpha_0 - \epsilon$, $\epsilon > 0$. *Thus, (3.2) easily holds for any* $\delta < 2$. *And so, we guarantee the oscillation of (E*1*) under the following conditions obtained by applying Theorems 4.1–4.5, respectively:*

$\mathrm{O}_{\mathrm{E}11}$- *Theorem 4.1:*

$$c_0 > \frac{(2-\delta)^{\delta+1}}{(\delta(1 - b_0\lambda^{1-2/\delta}))^{\delta}}.$$

$\mathrm{O}_{\mathrm{E}12}$- *Theorem 4.3:*

$$c_0 > \frac{2-\delta}{(\mathrm{e}(1 - b_0\lambda^{1-2/\delta})\ln(\mu^{-1}))^{\delta}}.$$

$\mathrm{O}_{\mathrm{E}13}$- *Theorem 4.4:*

$$c_0 > \frac{(2/\delta - 1)^{\delta}}{\delta} \frac{(1-\alpha)}{\alpha^{1-\delta}} \frac{\mu^{3-\delta-2/\delta}}{\mathrm{e}(1 - b_0\lambda^{1-2/\delta})^{\delta}\ln(\mu^{-1})},$$

for $\alpha = \alpha_0 - \epsilon, \epsilon > 0$.

$\mathrm{O}_{\mathrm{E}14}$- *Theorem 4.5:*

$$c_0 > \left(\frac{2/\delta - 1}{\delta + 1}\right)^{\delta+1} \frac{1}{(1 - b_0\lambda^{1-2/\delta})^{\delta}},$$

for $\beta(t) = \eta(t)$.

Remark 5.4. *By setting* $b_0 = 0$, $\mu = 1$, *and* $n = 1$, *then (E*1*) reduces to the famous Euler second order DE:*

$$\left(t^2 x'(t)\right)' + c_0 x(t) = 0.$$

Condition $\mathrm{O}_{\mathrm{E}14}$ *gives the sufficient and necessary oscillation condition for this equation, namely* $c_0 > 0.25$. *This confirms the importance and effectiveness of our results, where our criterion yields the famous sharp condition of a special case.*

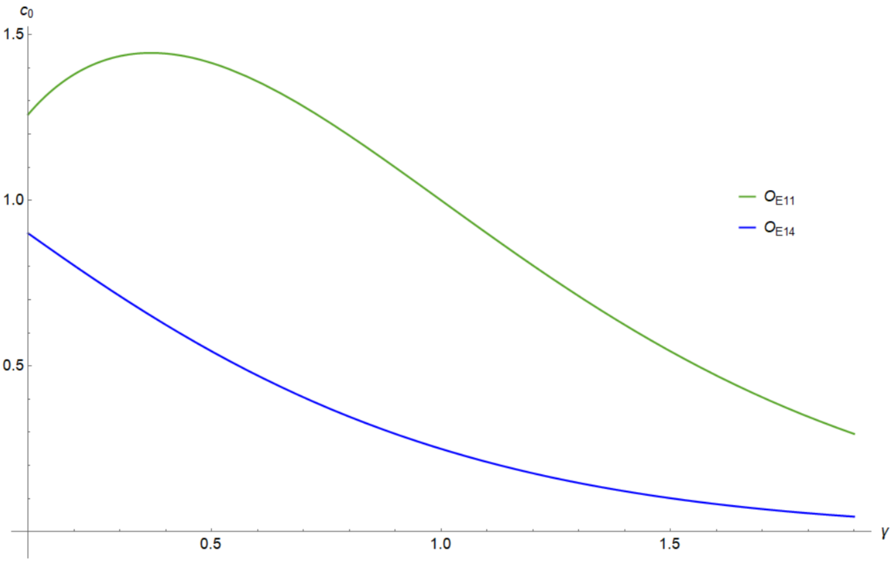

Figure 1. Range of Criteria O_{E11} and O_{E14} .

Table 1. Lower bounds of coefficient c_0 for different special cases of $(E1)$.

	δ	b_0	λ	μ	O_{E12}	O_{E13}	O_{E14}
Special Case 1	1	0.2	0.7	0.5	0.7430	0.6559	0.3500
Special Case 2	1	0.4	0.7	0.1	0.3711	0.3569	0.5833
Special Case 3	11/9	0.5	0.6	0.4	1.0756	0.38244	0.2620
Special Case 4	11/9	0.5	0.6	0.2	0.5403	0.2021	0.2620

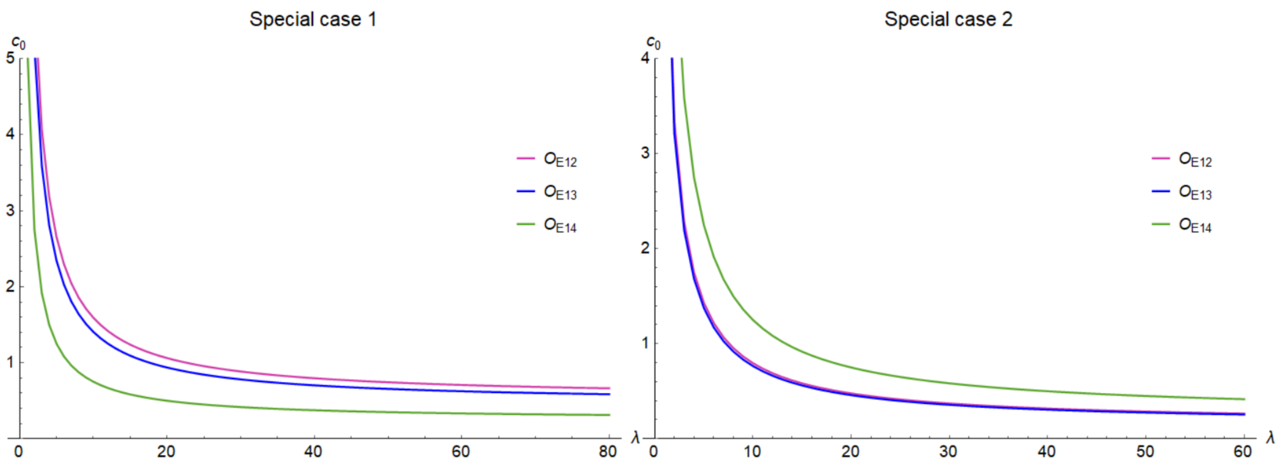

Figure 2. Large scale comparison of criteria O_{E12} –O_{E14} .

Remark 5.5. *It is obvious that criterion O_{E14} from Theorem 4.5 improves upon O_{E11} from Theorem 4.1 by covering a wider range of scenarios. This superiority is illustrated in Figure 2, which shows that O_{E14} is more effective than O_{E11} for all values of δ.*

Remark 5.6. *Criteria* O_{E12} *and* O_{E13} *are distinct from criterion* O_{E14} *in that they explicitly account for the effects of the delay function* μ*. This differentiation allows for varying dominance of the criteria under specific conditions. In certain cases related to* $(E1)$*, one criterion may prove to be more effective, while in other scenarios, the alternative criterion may take precedence. To illustrate these distinctions, we will provide a comprehensive overview of various cases in Table 1. From Table 1, we see that, in special cases 1 and 3, criterion* O_{E14} *overcomes criteria* O_{E12} *and* O_{E13}*, while, in special cases 2 and 4, the opposite occurs. The following figure illustrates this comparison on a large scale of the delay argument* λ*.*

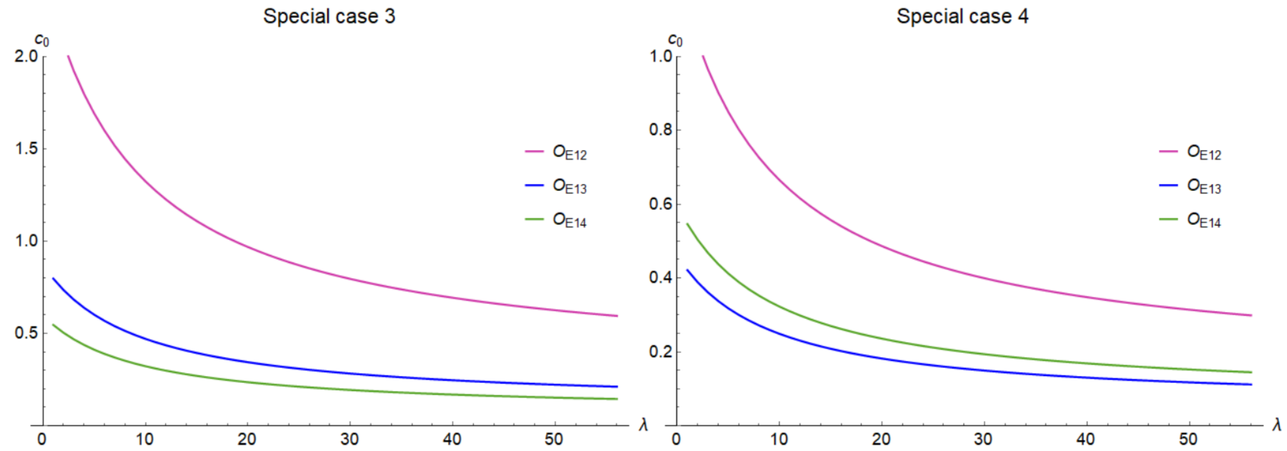

Figure 3. Impact of delay function on application scope.

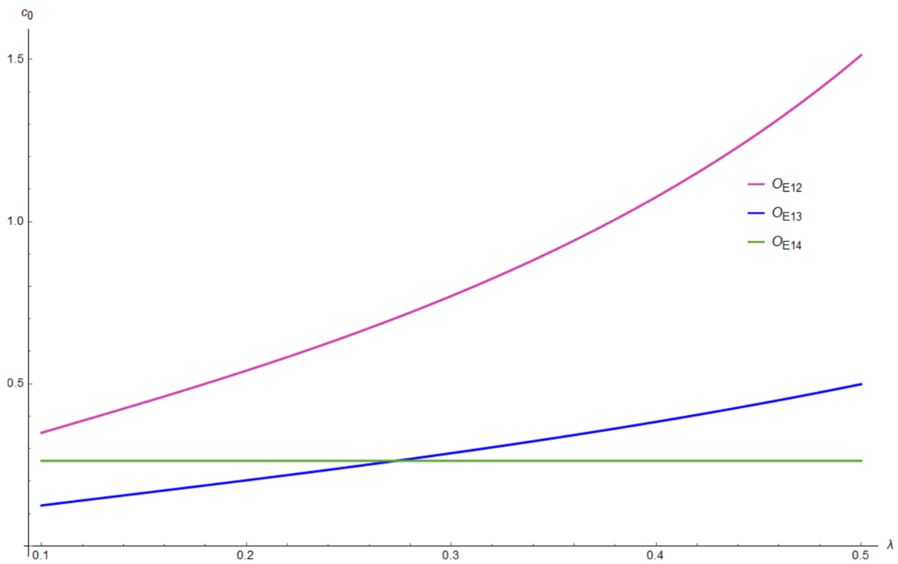

Figure 4. The effect of μ on criteria O_{E12} –O_{E14} .

Figure 3 shows the impact of the delay function μ *on expanding the scope of the application. By comparing special cases 3 and 4 concerning the values of the coefficient* λ*, we observe that the values of* O_{E14} *remain unchanged and are unaffected by variations in* μ*. In contrast,* O_{E12} *and* O_{E13} *exhibit* *significant differences, with* O_{E13} *even surpassing* O_{E14} *in special case 4 due to the incorporation of the effects of* μ.

Remark 5.7. *By applying Eq* $(E1)$ *in the last example to earlier works of Han et al. [12] and Agarwal et al. [13], with* $n = 1$, $\mu \leq (1-\lambda)$, *and* $\beta = 1$, *it becomes evident that the oscillation criteria in Theorems 2.1–2.3 fail to apply. This arises because condition (2.3) is unmet, as the* $\limsup(\cdot)$ *on the right-hand side does not approach infinity under the given functions and coefficients in Example 5.1.*

Remark 5.8. *For* $n = 1$, *our criteria* O_{E11} *and* O_{E12} *yield oscillation results consistent with those of Theorems 2.5 and 2.6 established by Bohner et al. [14]. However, in cases involving multiple delays* (θ_ℓ, $\ell = 1, 2, ..., n$, $n \in \mathbb{N}$)*, Bohner's theorems fail to apply, highlighting the limitations of their approach. Our findings not only address these limitations but also introduce robust criteria adaptable to more complex scenarios, showcasing the originality and practical relevance of our contributions to the field.*

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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