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Research article

Existence results for hybrid Caputo fractional models for a thermostat with hybrid boundary conditions

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Abstract: In this paper, we study a generalized hybrid fractional thermostat model involving the Caputo fractional derivative and the Riemann–Liouville fractional integral under nonlocal hybrid boundary conditions with feedback control. The problem describes a class of heat regulation systems in which the controller's response depends on both the process's thermal memory and sensor measurements taken at interior points of the domain. Working in the Banach space of integral-type Hölder functions $\mathcal{J}_{\alpha,\beta}$, we establish sufficient conditions for the existence and uniqueness of solutions to the associated boundary value problem. This functional framework is more suitable than the classical spaces $C(E_0)$ and $C^{\alpha}(E_0)$, since it provides stronger regularity, better stability under fractional integral operators, and improved continuity properties for the nonlinear superposition terms involved in the hybrid structure. Our analysis is based on fractional calculus techniques, an appropriate measure of noncompactness, and Darbo-type fixed-point theorem. The obtained results extend several existing thermostat and hybrid fractional models in the literature. Finally, a numerical example is presented to illustrate and verify the theoretical findings for different values of α and β.

Keywords: integral-form Hölder space; fixed-point theorem; models of thermostats; measure of noncompactness; hybrid fractional BVP

1. Introduction

Mathematical models involving nonlocal functional integro-differential equations, whether of integer or fractional order, play an important role in describing systems whose future evolution depends not only on the current state and its instantaneous variation but also on their past history and distributed spatial interactions. Such models naturally arise in heat transfer, elasticity, viscoelasticity, thermodynamics, control theory, and biological dynamics, where memory effects and nonlocal interactions cannot be neglected. Their importance lies in the combination of differential operators, which describe local variations, and integral operators, which capture hereditary effects and distributed memory phenomena [1–5].

Nonlocal conditions describe not only boundary interactions but also intermediate processes inside the domain. They appear naturally in wave propagation, elasticity, and thermal regulation problems, where a controller located at the boundary adjusts the system's response according to information collected from sensors placed at interior points of the domain [6–9]. Such mechanisms lead to boundary conditions involving both endpoint and interior-point contributions.

The boundary value problem (BVP) considered in this work can be interpreted as a generalized fractional thermostat model describing the stationary thermal regulation of a heated rod equipped with an interior sensor and a boundary controller. In this setting, the temperature distribution depends not only on the instantaneous thermal state but also on accumulated thermal history and delayed feedback generated by the interaction between the sensor and the controller.

Fractional derivatives provide an effective framework for modeling memory and hereditary effects, since the present state depends on the entire past evolution of the process (cf. [10]). In the present model, the Caputo fractional derivative represents the delayed thermal response and memory effect of the controller, while the Riemann–Liouville (RL)-fractional integral describes the accumulated distributed influence of past temperature variations across the domain. Moreover, the quotient-type nonlinear structure reflects the nonlinear interaction between the state variable and the control mechanism, which frequently appears in thermostat regulation processes. Moreover, the existence and uniqueness results obtained in this work ensure that the thermostat system admits a well-defined and predictable thermal profile, which is essential for the reliability of such control processes.

Motivated by these considerations, we study the following nonlocal hybrid fractional BVP involving the Caputo fractional derivative and the RL-fractional integral:

$$ {}^{\mathrm{C}}\mathcal{D}^{\tau_1}\left(\frac{\mathbb{w}(r)-g(r)-K_1\left(r,{}^{\mathrm{RL}}\mathcal{I}^{\tau_2}\mathbb{w}(r)\right)}{K_2(r,\mathbb{w}(r))}\right)+K_3(r,\mathbb{w}(r))=0, \qquad r\in\mathrm{E}_0:=[0,1], \tag{1.1} $$

for $1<\tau_1<2$ and $\tau_2\in\mathrm{E}_0\setminus\{0\}$, subject to the following nonlocal hybrid boundary value conditions:

$$ \begin{cases} \mathbb{D}\left(\frac{\mathbb{w}(r)-g(r)-K_1\left(r,{}^{\mathrm{RL}}\mathcal{I}^{\tau_2}\mathbb{w}(r)\right)}{K_2(r,\mathbb{w}(r))}\right)\Bigg|_{r=0}=0, \qquad \mathbb{D}=\frac{\mathrm{d}}{\mathrm{d}r}, \\ \lambda\,{}^{\mathrm{C}}\mathcal{D}^{\tau_1-1}\left(\frac{\mathbb{w}(r)-g(r)-K_1\left(r,{}^{\mathrm{RL}}\mathcal{I}^{\tau_2}\mathbb{w}(r)\right)}{K_2(r,\mathbb{w}(r))}\right)\Bigg|_{r=1} \\ \qquad +\left(\frac{\mathbb{w}(r)-g(r)-K_1\left(r,{}^{\mathrm{RL}}\mathcal{I}^{\tau_2}\mathbb{w}(r)\right)}{K_2(r,\mathbb{w}(r))}\right)\Bigg|_{r=\eta}=0, \quad \lambda>0,\ \eta\in\mathrm{E}_0. \end{cases} \tag{1.2} $$

The analysis of Problems (1.1) and (1.2) is carried out in the Hölder-type space with the integral modulus of continuity $J_{\alpha,\beta}$, where $0<\alpha<1$ and $\beta>\tau_1>\alpha$. This choice is essential because the problem involves RL-fractional operators together with nonlinear superposition operators acting on the unknown function.

Classical spaces such as $C(\mathrm{E}_0)$ and $C^{\alpha}(\mathrm{E}_0)$ are not sufficient for this purpose. The space $C(\mathrm{E}_0)$ guarantees only continuity and does not provide enough regularity to control fractional integral terms. On the other hand, in the classical Hölder space $C^{\alpha}(\mathrm{E}_0)$, the continuity of Nemytskii-type superposition operators is not always preserved under the present nonlinear structure (see [11, Theorems 3.8, 3.9, 3.13]), which limits the applicability of standard fixed-point arguments (cf. [12]).

The Banach algebra $J_{\alpha,\beta}$ resolves these difficulties. It is stable under RL-fractional integral operators, preserves the continuity of nonlinear superposition operators, and is fully compatible with the quotient-type structure appearing in the model. Furthermore, since $C^{\alpha}(\mathrm{E}_0) \subset J_{\alpha,\beta}(\mathrm{E}_0)$, this framework extends the classical Hölder setting while providing stronger regularity of the solutions and better continuity properties for the associated operators [13, 14].

The presence of nonlinear superposition terms and fractional integral components, such as $K_1\left(r, {}^{\mathrm{RL}}\mathcal{I}^{\tau_2}\mathbb{w}(r)\right)$, prevents the associated operator from preserving relative compactness in general. Consequently, Schauder's fixed-point theorem (FPT) is not directly applicable. To overcome this difficulty, we use a Darbo-type FPT based on measures of noncompactness (MNC), particularly the product-form result due to Banaś and Lecko [15]. This approach is more suitable because the operator associated with Problems (1.1) and (1.2) can be naturally decomposed into the product of two nonlinear operators.

Hybrid differential equations (DEs) with a variety of disturbances were implicitly considered by Krasnoselskii [16] and extensively discussed in many papers; see [17–23].

The classical thermostat models studied in [24–26] mainly concern integer-order BVPs of the form

$$\mathbb{w}''(r) + K(r, \mathbb{w}(r)) = 0, \qquad r \in \mathrm{E}_0 \setminus 0, 1,$$

under the conditions $\mathbb{w}'(0) = 0$ and $\lambda\mathbb{w}'(1) + \mathbb{w}(\eta) = 0$, analyzed in the classical space $C(\mathrm{E}_0)$.

The fractional extension considered in [27] involves Caputo-type derivatives in continuous spaces

$${}^{\mathrm{C}}\mathcal{D}^{\tau}\mathbb{w}(r) + K(r, \mathbb{w}(r)) = 0,$$

with analogous nonlocal boundary conditions. Later, hybrid fractional thermostat models such as [28] introduced quotient-type nonlinear structures of the form

$${}^{\mathrm{C}}\mathcal{D}^{\tau}\left(\frac{\mathbb{w}(r)}{K(r, \mathbb{w}(r))}\right) + \phi(r, \mathbb{w}(r)) = 0, \qquad \tau \in (1, 2),$$

subject to hybrid boundary conditions.

In parallel, several works investigated fractional equations in the Banach algebra $J_{\alpha,\beta}$. Appell et al. [13] showed that this space contains classical Hölder spaces and is suitable for RL-fractional BVPs, while the authors of [14, 29] established existence results for quadratic RL-type and Hadamard fractional integral equations using the structural properties of $J_{\alpha,\beta}$.

In contrast to the abovementioned works, the present problem combines the Caputo fractional derivative, the RL-fractional integral, and nonlocal hybrid boundary conditions involving both boundary and interior-point contributions in a single framework. Moreover, the analysis is performed in the integral-form Hölder space $J_{\alpha,\beta}$ rather than in the classical spaces $C(\mathrm{E}_0)$ or $C^{\alpha}(\mathrm{E}_0)$. This yields stronger regularity of solutions, and better continuity properties of the nonlinear operators and allows the use of a Darbo-type FPT based on MNC. Therefore, the present work extends previous thermostat and hybrid fractional models by establishing the existence, uniqueness, and numerical verification for a more general and analytically richer class of hybrid fractional BVPs.

The main contributions of this paper are summarized as follows.

- We study a new extension of fractional thermostat models involving Caputo derivatives, RL-fractional integrals, and hybrid nonlocal boundary conditions with contributions from both boundary and interior points.

- We establish existence and uniqueness results for Problems (1.1) and (1.2) in the Banach algebra $J_{\alpha,\beta}$.
- We apply a Darbo-type FPT based on MNC, which is more appropriate than Schauder's theorem for the present nonlinear product structure.
- By working in $J_{\alpha,\beta}$, we obtain stronger regularity results than those derived in the classical spaces $C(\mathrm{E}_0)$ and $C^{\alpha}(\mathrm{E}_0)$.
- The fractional formulation is physically motivated by thermal memory effects and delayed feedback mechanisms inherent in thermostat control systems.
- We conclude with a numerical example that verifies the assumptions of the main theorems, illustrates the admissibility of the parameter conditions for different values of α and β, and supports the practical relevance of the theoretical results.

2. Preliminaries

Let $C = C(\mathrm{E}_0)$, $\mathrm{E}_0 = [0,1]$ with the supremum norm $\|\cdot\|_C$. For $\alpha \in \mathrm{E}_0 \setminus \{0\}$, $\alpha < \beta < \infty$, we represent the integral-form Hölder space $\mathcal{J}_{\alpha,\beta} = \mathcal{J}_{\alpha,\beta}(\mathrm{E}_0)$ using the norm [30]

$$\left\|\left|\mathbb{w}\right|\right\|_{\alpha,\beta} = \|\mathbb{w}\|_C + \mathcal{J}_{\alpha,\beta}(\mathbb{w},\mathrm{E}_0)^{\alpha/\beta},$$

where

$$\mathcal{J}_{\alpha,\beta}(\mathbb{w},\mathrm{E}_0) = \int_0^1 \varpi\,(\mathbb{w},\xi)^{\beta/\alpha}\ \xi^{-(\beta+1)}\,\mathrm{d}\xi,$$

and where $\varpi(\mathbb{w},\xi)$ is known as the modulus of continuity of a function $\mathbb{w} \in C$ and given by

$$\varpi(\mathbb{w},\xi) = \sup\left\{|\mathbb{w}(r) - \mathbb{w}(s)| \ :\ r,t \in \mathrm{E}_0,\ |r-s| \le \xi\right\}.$$

Theorem 2.1 ([14]). *The space* $(\mathcal{J}_{\alpha,\beta}, \left\|\left|\mathbb{w}\right|\right\|_{\alpha,\beta})$ *is a commutative Banach algebra with*

$$\left\|\left|\mathbb{w}\,\grave{\mathbb{w}}\right|\right\|_{\alpha,\beta} \le \left(1 + 2^{2-\alpha/\beta}\right)\cdot \left\|\left|\mathbb{w}\right|\right\|_{\alpha,\beta}\cdot \left\|\left|\grave{\mathbb{w}}\right|\right\|_{\alpha,\beta}, \qquad \forall \mathbb{w},\grave{\mathbb{w}} \in \mathcal{J}_{\alpha,\beta}.$$

We also recall the following operator. The superposition operator of Nemytskii type was formulated by [30].

$$\begin{cases} \mathcal{N}_K = K(r,\mathbb{w}) : \mathrm{E}_0 \times \mathbb{R} \to \mathbb{R}, \\ \mathcal{N}_K(\mathbb{w})(r) = K(r,\mathbb{w}(r)), \qquad \forall r \in \mathrm{E}_0. \end{cases}$$

For a function of two variables $K(r,\mathbb{w}) : \mathrm{E}_0 \times \mathbb{R} \to \mathbb{R}$, the modulus of continuity was formulated by [30]

$$\varpi(K,\xi,\mu) = \sup\left\{|K(s,u) - K(r,v)| \ :\ s,r \in \mathrm{E}_0,\ |s-r| \le \xi,\ u,v \in \mathbb{R}, |u-v| \le \mu\right\}.$$

If $\beta \to \infty$, we get the classical Hölder space $C^{\alpha}(\mathrm{E}_0)$ i.e., $\lim_{\beta\to\infty} \mathcal{J}_{\alpha,\beta}(\mathrm{E}_0) = C^{\alpha}(\mathrm{E}_0)$.

Theorem 2.2 ([30]). *Consider a function* $K : \mathrm{E}_0 \times \mathbb{R} \to \mathbb{R}$ *with the following assumptions.*

(a) *For any* $\rho > 0$*, there is* $\tilde{c}_i^* > 0$ *such that for each* $s, r \in \mathrm{E}_0$ *and* $u \in [-\rho,\rho]$*, we have*

$$\left|K(s,u) - K(r,u)\right| \le \tilde{c}_i^*|s-r|;$$

(b) *The function* $K(r,\cdot)$ *is differentiable for every* $r \in \mathrm{E}_0$, *and* $B_\rho > 0$ *exists such that for any* $u, v \in \mathbb{R}$

$$\left|\partial_2 K(\cdot,u) - \partial_2 K(\cdot,v)\right| \leq B_\rho \cdot |u-v|.$$

Then the operator $\mathcal{N}_{K} : \mathcal{J}_{\alpha,\beta}(\mathrm{E}_0) \to \mathcal{J}_{\alpha,\beta}(\mathrm{E}_0)$ *is continuous.*

Moreover, in relation to $\mathcal{N}_{K}$ in the space $\mathcal{J}_{\alpha,\beta}$, the following results for the acting conditions hold.

Theorem 2.3 ([30]). *Let* $\mathrm{b}_\rho > 0$, $\mathrm{a}_\rho \in L_1(\mathrm{E}_0)$, $\alpha \in \mathrm{E}_0 \setminus \{0,1\}$, *and* $\beta > \alpha$ *attached to a* $K : \mathrm{E}_0 \times \mathbb{R} \to \mathbb{R}$ *with the features*

$$\varpi(K,\xi,\varpi(\mathbb{w},\xi))^{\beta/\alpha} \leq \mathrm{a}_\rho(\xi)\xi^{\beta+1} + \mathrm{b}_\rho \varpi(\mathbb{w},\xi)^{\beta/\alpha}, \qquad \forall \xi \geq 0.$$

Then the ball $B_\rho(\mathcal{J}_{\alpha,\beta})$ *maps into* $B_R(\mathcal{J}_{\alpha,\beta})$ *by operator* $\mathcal{N}_{K}$, *where*

$$R = \max\left\{|f(r,u)| \,:\, r \in \mathrm{E}_0,\ |u| \leq \rho\right\} + \left[\|\mathrm{a}_\rho\|_{L_1} + \mathrm{b}_\rho \rho^{\beta/\alpha}\right]^{\alpha/\beta},$$

$\rho > 0$ *and* $B_R = \{\mathbb{z} \in \mathcal{J}_{\alpha,\beta} \,:\, \left\||\mathbb{z}|\right\|_{\alpha,\beta} \leq R\}$.

For a bounded set $\mathbb{G}$ belongs to $\mathcal{J}_{\alpha,\beta}(\mathrm{E}_0)$, the Hausdorff MNC $\chi_{\mathrm{H}}(X)$, is given as [31]

$$\chi_{\mathrm{H}}(X) = \inf\left\{\varepsilon > 0 \,:\, X \text{ admits a finite } \varepsilon\text{-net in } \mathbb{G}\right\}.$$

The MNC in the space $\mathcal{J}_{\alpha,\beta}$ is given as follows.

Proposition 2.4 ([30]). *The* χ_{H} *in the space* $\mathcal{J}_{\alpha,\beta}$ *is known as*

$$c(X) = \limsup_{r\to 0}\ \sup_{\mathbb{w}\in X} \mathcal{J}_{\alpha,\beta}(\mathbb{w},[0,r]),$$

and verifies $2^{-\beta/\alpha}\chi_{\mathrm{H}}(X) \leq c(X) \leq 2^{\beta/\alpha}\chi_{\mathrm{H}}(X)$.

Lemma 2.5 ([14]). *The c-MNC satisfies the property* $(\mathfrak{m})$*:* $c(AB) \leq \Upsilon_A \cdot c(B) + \Upsilon_B \cdot c(A)$ *on* $\mathcal{J}_{\alpha,\beta}$ *where* $\Upsilon_A = 2^{\beta/\alpha-1}\|A\|_C^{\beta/\alpha}$ *and* $\Upsilon_B = 2^{\beta/\alpha-1}\|B\|_C^{\beta/\alpha}$.

The following Darbo-type FPT for product operators, established by Banaś and Lecko [15], plays a central role in our analysis.

Lemma 2.6. *Assume that* $\varnothing \neq \mathbb{G} \subseteq \mathcal{J}_{\alpha,\beta}$ *is a bounded, closed, and convex and consider the continuous operator* $\Theta_i : \mathbb{G} \to \mathcal{J}_{\alpha,\beta}$, $i = 1,2$, *with* $\Theta_i(\mathbb{G})$ *being a bounded operator. Moreover, assume that* $\Theta = \Theta_1 \cdot \Theta_2$ *transforms* $\mathbb{G}$ *into itself.*

(i) *If for arbitrary bounded subset* $X \subseteq \mathbb{G}$, *there is a constant* k_i *such that* Θ_i *satisfies an inequality* $c(\Theta_i(X)) \leq k_i \cdot c(X)$, $i = 1,2$*;*

(ii) *If the inequality* $2^{\beta/\alpha-1}\|\Theta_1\|_C^{\beta/\alpha} \cdot k_2 + 2^{\beta/\alpha-1}\|\Theta_2\|_C^{\beta/\alpha} \cdot k_1 < 1$ *holds true, then* Θ *has at least one fixed point in the set* $\mathbb{G}$.

Next, we introduce some concepts related to fractional calculus.

The fractional RL-type integral and Caputo derivative of order $\tau > 0$ of a well-defined function $\mathbb{w}$ on E_0 are given for $r > 0$ by

$${}^{\mathrm{RL}}\mathcal{I}^{\tau}\mathbb{w}(r) = \int_0^r \frac{\mathbb{w}(s)}{\Gamma(\tau)(r-s)^{1-\tau}}\,\mathrm{d}s,$$

and

$$^{\mathrm{C}}\mathcal{D}^{\tau}\mathbb{w}(r)=\int_0^r \tfrac{\mathbb{w}^{(n)}(s)}{\Gamma(n-\tau)(r-s)^{1+\tau-n}}\,\mathrm{d}s,\qquad \mathbb{w}\in C^{(n)}(\mathrm{E}_0,\mathbb{R}),$$

where $\Gamma(\tau)=\int_0^{\infty}e^{-r}r^{\tau-1}\,\mathrm{d}r$ and $n–1<\tau<n$ with $\tau=[n]+1$ [32].

Theorem 2.7 ([32])**.** *If we let* $n-1<\tau<n$, $\tau>0$, *then*

(i) *The FDE* $^{\mathrm{C}}\mathcal{D}^{\tau}\mathbb{w}(r)=0$ *is verified if and only if*

$$\mathbb{w}(r)=\tilde{b}_0+\tilde{b}_1r+b_2\tilde{b}^2+\cdots+\tilde{b}_{n-1}r^{n-1},\qquad \tilde{b}_i\in\mathbb{R}.$$

Particularly, if $1<\tau<2$, *the relation* $^{\mathrm{C}}\mathcal{D}^{\tau}\mathbb{w}(r)=0$ *is verified if and only if* $\mathbb{w}(r)=\tilde{b}_0+\tilde{b}_1r$, $\tilde{b}_0,\tilde{b}_1\in\mathbb{R}$.

(ii) $^{\mathrm{C}}\mathcal{D}^{\tau}\,{}^{\mathrm{RL}}\mathcal{I}^{\tau}\mathbb{w}(r)=\mathbb{w}(r)$ *is verified for every* $\mathbb{w}(r)\in L^p(\mathrm{E}_0)$.

(iii) *Let* $\mathbb{w}\in C(\mathbb{R}^+)\cap L^p(\mathbb{R}^+)$. *The following formula is verified:*

$$^{\mathrm{RL}}\mathcal{I}^{\tau}\;{}^{\mathrm{C}}\mathcal{D}^{\tau}\mathbb{w}(r)=\mathbb{w}(r)+\sum_{i=0}^{n-1}\tilde{b}_ir^i=\mathbb{w}(r)+\tilde{b}_0+\tilde{b}_1r+\tilde{b}_2r^2+\cdots+\tilde{b}_{n-1}r^{n-1}.$$

Theorem 2.8 ([30])**.** *Let* $0<\alpha,\tau<1$ *and* $\beta>\tau>\alpha>0$. *Then the operator* $^{\mathrm{RL}}\mathcal{I}^{\tau}$ *continuously maps the space* $\mathcal{J}_{\alpha,\beta}$ *into itself with*

$$\varpi\left({}^{\mathrm{RL}}\mathcal{I}^{\tau}\mathbb{w},\xi\right)\le\tfrac{\varpi(\mathbb{w},\xi)+\xi^{\tau}\|\mathbb{w}\|_C}{\Gamma(\tau+1)},\qquad \xi\in\mathrm{E}_0,$$

and the norm

$$\left\|\left|{}^{\mathrm{RL}}\mathcal{I}^{\tau}\mathbb{w}\right|\right\|_{\alpha,\beta}\le\max\left\{\left(\tfrac{2^{1-\alpha/\beta}}{\Gamma(\tau+1)}\right)^{\alpha/\beta},\ \tfrac{1}{\tau}+\left(\tfrac{(2^{1-\alpha/\beta})\cdot\alpha}{\Gamma(\tau+1)\cdot\beta\cdot(\tau-\alpha)}\right)^{\alpha/\beta}\right\}\cdot\left\|\left|\mathbb{w}\right|\right\|_{\alpha,\beta}.$$

Corollary 2.9. *Let* $0<\alpha<1$, $1<\tau<2$, *and* $\beta>\tau>\alpha>0$. *Then the operator* $^{\mathrm{RL}}\mathcal{I}^{\tau}$ *continuously maps the space* $\mathcal{J}_{\alpha,\beta}$ *into itself with*

$$\varpi\left({}^{\mathrm{RL}}\mathcal{I}^{\tau}\mathbb{w},\xi\right)\le\frac{\varpi(\mathbb{w},\xi)+\tau\xi\|\mathbb{w}\|_C}{\Gamma(\tau+1)}$$

and the norm

$$\left\|\left|{}^{\mathrm{RL}}\mathcal{I}^{\tau}\mathbb{w}\right|\right\|_{\alpha,\beta}\le C_{\alpha,\beta,\tau}\left\|\left|\mathbb{w}\right|\right\|_{\alpha,\beta},$$

where

$$C_{\alpha,\beta,\tau}=\max\left\{\left(\frac{2^{1-\alpha/\beta}}{\Gamma(\tau+1)}\right)^{\alpha/\beta},\ \frac{1}{\tau}+\left(\frac{\left(\tau^{\frac{\beta}{\alpha}}2^{1-\alpha/\beta}\right)\alpha}{\Gamma(\tau+1)\beta(1-\alpha)}\right)^{\alpha/\beta}\right\}.$$

Proof. The proof follows the same general lines as in [30, Lemma 3.4.], with suitable modifications for the case $1<\tau<2$.

For $\mathbb{w}\in\mathcal{J}_{\alpha,\beta}$ and $\xi\in\mathrm{E}_0$, we estimate the modulus of continuity of the operator $^{\mathrm{RL}}\mathcal{I}^{\tau}\mathbb{w}$.

Since $1<\tau<2$, we apply the mean value theorem to the function

$$|r^{\tau}-s^{\tau}|\le\tau\xi,$$

for a suitable $r, s \in \mathrm{E}_0$. Consequently, we obtain

$$\varpi\left({}^{\mathrm{RL}}\mathcal{I}^{\tau}\mathbb{w}, \xi\right) \leq \frac{\varpi(\mathbb{w}, \xi) + \tau\xi\|\mathbb{w}\|_C}{\Gamma(\tau+1)}.$$

Similarly, using the norm definition of $\mathcal{J}_{\alpha,\beta}$ and the same decomposition technique as in [30, Lemma 3.4.], together with the estimate above, we derive

$$\left\|\left|{}^{\mathrm{RL}}\mathcal{I}^{\tau}\mathbb{w}\right|\right\|_{\alpha,\beta} \leq C_{\alpha,\beta,\tau}\left\|\left|\mathbb{w}\right|\right\|_{\alpha,\beta}.$$

3. Main results

Next, we study the "nonlocal" three-point BVP in (1.1) and (1.2) by stating and proving the first lemma.

Lemma 3.1. *For $\zeta \in \mathcal{J}_{\alpha,\beta}(\mathrm{E}_0)$, a function $\mathbb{w}$ is a solution for the following hybrid $\mathbb{FDE}$:*

$$ {}^{\mathrm{C}}\mathscr{D}^{\tau_1}\left(\frac{\mathbb{w}(r)-g(r)-K_1\left(r, {}^{\mathrm{RL}}\mathcal{I}^{\tau_2}\mathbb{w}(r)\right)}{K_2(r,\mathbb{w}(r))}\right) + \zeta(r) = 0, \qquad r \in \mathrm{E}_0,\ 1 < \tau_1 < 2, \tag{3.1}$$

under the three-point hybrid boundary value conditions

$$\begin{cases} \mathbb{D}\left(\frac{\mathbb{w}(r)-g(r)-K_1\left(r, {}^{\mathrm{RL}}\mathcal{I}^{\tau_2}\mathbb{w}(r)\right)}{K_2(r,\mathbb{w}(r))}\right)\Bigg|_{r=0} = 0, \\ \lambda\, {}^{\mathrm{C}}\mathscr{D}^{\tau_1-1}\left(\frac{\mathbb{w}(r)-g(r)-K_1\left(r, {}^{\mathrm{RL}}\mathcal{I}^{\tau_2}\mathbb{w}(r)\right)}{K_2(r,\mathbb{w}(r))}\right)\Bigg|_{r=1} \\ \qquad + \left(\frac{\mathbb{w}(r)-g(r)-K_1\left(r, {}^{\mathrm{RL}}\mathcal{I}^{\tau_2}\mathbb{w}(r)\right)}{K_2(r,\mathbb{w}(r))}\right)\Bigg|_{r=\eta} = 0, \end{cases} \tag{3.2}$$

where $\eta \in \mathrm{E}_0$, $\lambda > 0$, $\tau_2 \in \mathrm{E}_0 \setminus \{0\}$ if and only if $\mathbb{w}$ is a solution for the following integral equation IE:

$$\begin{aligned} \mathbb{w}(r) = g(r) + K_1\left(r, {}^{\mathrm{RL}}\mathcal{I}^{\tau_2}\mathbb{w}(r)\right) + K_2(r, \mathbb{w}(r))\Bigg[&- \int_0^r \frac{\zeta(s)}{\Gamma(\tau_1)(r-s)^{1-\tau_1}}\,\mathrm{d}s \\ &+ \int_0^{\eta} \frac{\zeta(s)}{\Gamma(\tau_1)(\eta-s)^{1-\tau_1}}\,\mathrm{d}s + \lambda\int_{\mathrm{E}_0}\zeta(s)\,\mathrm{d}s\Bigg]. \end{aligned} \tag{3.3}$$

Proof. Let $\mathbb{w}$ be a solution of Eq (3.1). By applying the properties of the Caputo fractional derivative of order $1 < \tau_1 < 2$, the constants $\tilde{b}_0, \tilde{b}_1 \in \mathbb{R}$ exists such that

$$\begin{aligned} \mathbb{w}(r) &= g(r) + K_1\left(r, {}^{\mathrm{RL}}\mathcal{I}^{\tau_2}\mathbb{w}(r)\right) \\ &\quad + K_2(r, \mathbb{w}(r))\left[-\int_0^r \frac{\zeta(s)}{\Gamma(\tau_1)(r-s)^{1-\tau_1}}\,ds + \tilde{b}_0 + \tilde{b}_1 r\right]. \end{aligned} \tag{3.4}$$

Applying the operator $\mathbb{D}$ to the normalized expression yields

$$\mathbb{D}\left(\frac{\mathbb{w}(r)-g(r)-K_1\left(r, {}^{\mathrm{RL}}\mathcal{I}^{\tau_2}\mathbb{w}(r)\right)}{K_2(r,\mathbb{w}(r))}\right) = -\int_0^r \frac{\zeta(s)}{\Gamma(\tau_1-1)(r-s)^{2-\tau_1}}\,ds + \tilde{b}_1.$$

Using the first boundary condition in (3.2) at $r = 0$, we obtain

$$\tilde{b}_1 = 0.$$

Hence

$$ {}^{\mathrm{C}}\mathcal{D}^{\tau_1-1}\left(\frac{\mathbb{w}(r) - g(r) - K_1\Big(r, {}^{\mathrm{RL}}\mathcal{I}^{\tau_2}\mathbb{w}(r)\Big)}{K_2(r, \mathbb{w}(r))}\right) = -\int_0^r \zeta(s)\,ds + \tilde{b}_0. $$

Now, applying the second boundary condition in (3.2), we evaluate at $r = 1$ and $r = \eta$, which gives

$$\tilde{b}_0 = \lambda \int_0^1 \zeta(s)\,ds + \int_0^\eta \frac{\zeta(s)}{\Gamma(\tau_1)(\eta - s)^{1-\tau_1}}\,ds.$$

Substituting $\tilde{b}_0$ and $\tilde{b}_1 = 0$ into (3.4), we obtain

$$\begin{aligned} \mathbb{w}(r) &= g(r) + K_1\left(r, {}^{\mathrm{RL}}\mathcal{I}^{\tau_2}\mathbb{w}(r)\right) \\ &\quad + K_2(r, \mathbb{w}(r))\Big[-\int_0^r \frac{\zeta(s)}{\Gamma(\tau_1)(r - s)^{1-\tau_1}}\,ds + \int_0^\eta \frac{\zeta(s)}{\Gamma(\tau_1)(\eta - s)^{1-\tau_1}}\,ds + \lambda \int_{\mathrm{E}_0} \zeta(s)\,ds \Big]. \end{aligned}$$

This shows that $\mathbb{w}$ satisfies the integral equation (3.3).

Conversely, applying the Caputo derivative and using standard fractional calculus properties, one verifies directly that (3.3) satisfies the hybrid fractional differential equation (3.1) together with the boundary conditions (3.2). This completes the proof.

3.1. Existence results in $\mathcal{J}_{\alpha,\beta}$

Regarding Lemma 3.1, the solution to the BVP in (1.1) and (1.2) can be formed by

$$\begin{aligned} \mathbb{w}(r) = g(r) + K_1\left(r, {}^{\mathrm{RL}}\mathcal{I}^{\tau_2}\mathbb{w}(r)\right) + K_2(r, \mathbb{w}(r))\Big[&-\int_0^r \tfrac{K_3(s,\mathbb{w}(s))}{\Gamma(\tau_1)(r-s)^{1-\tau_1}}\,\mathrm{d}s \\ &+ \int_0^\eta \tfrac{K_3(s,\mathbb{w}(s))}{\Gamma(\tau_1)(\eta-s)^{1-\tau_1}}\,\mathrm{d}s + \lambda \int_{\mathrm{E}_0} K_3(s, \mathbb{w}(s))\,\mathrm{d}s \Big]. \end{aligned} \tag{3.5}$$

Let us put Eq (3.5) in the following operator form:

$$\mathbb{w}(r) = \Theta(\mathbb{w})(r) = \Theta_1^{(1)}(\mathbb{w})(r) + \Theta_1^{(2)}(\mathbb{w})(r) \cdot \Theta_2(\mathbb{w})(r),$$

with

$$\Theta_1^{(1)}(\mathbb{w})(r) = g(r) + \mathcal{N}_{K_1} \circ {}^{\mathrm{RL}}\mathcal{I}^{\tau_2}(\mathbb{w})(r), \qquad \Theta_1^{(2)}(\mathbb{w})(r) = \mathcal{N}_{K_2}(\mathbb{w})(r),$$

and

$$\Theta_2(\mathbb{w})(r) = -\,{}^{\mathrm{RL}}\mathcal{I}^{\tau_1} \circ \mathcal{N}_{K_3}(\mathbb{w})(r) + {}^{\mathrm{RL}}\mathcal{I}^{\tau_1} \circ \mathcal{N}_{K_3}(\mathbb{w})(\eta) + \lambda\mathbb{I} \circ \mathcal{N}_{K_3}(\mathbb{w})(1),$$

where

$$\mathbb{I} \circ \mathcal{N}_{K_3}(\mathbb{w})(1) = \int_{\mathrm{E}_0} K_3(s, \mathbb{w}(s))\,\mathrm{d}s,$$

and $\mathcal{N}_{K_i}$, $i = 1, 2, 3$, are the superposition operator.

Let $0 < \alpha, \tau_2 < 1,\ 1 < \tau_1 < 2, \beta > \tau_1$, and for $i = 1, 2, 3$, consider the following list of hypotheses.

(i) $g \in \mathcal{J}_{\alpha,\beta}(\mathrm{E}_0)$;

(ii) For any $\rho > 0$, the functions $K_1, K_3 : \mathrm{E}_0 \times \mathbb{R} \to \mathbb{R}$ and $K_2 : \mathrm{E}_0 \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ satisfy:

- We have $\mathrm{b}_{\rho_i} > 0$ and a function $\mathrm{a}_{\rho_i} \in L_1(\mathrm{E}_0)$ such that
$$\varpi(K_i, \xi, \varpi(\mathrm{w}, \xi))^{\beta/\alpha} \leq \mathrm{a}_{\rho_i}(\xi)\, \xi^{\beta+1} + \mathrm{b}_{\rho_i} \varpi(\mathrm{w}, \xi)^{\beta/\alpha}, \qquad \xi \geq 0. \tag{3.6}$$
- For every $s, r \in \mathrm{E}_0$ and $u, v \in [-\rho, \rho]$, $\tilde{c}_i^* > 0$ and $\tilde{c}_i > 0$ exist such that
$$\begin{aligned} \left|K_i(s,u) - K_i(r,u)\right| &\leq \tilde{c}_i^* |s - r|, \\ \left|K_i(s,u) - K_i(s,v)\right| &\leq \tilde{c}_i |u - v|; \end{aligned}$$
- The function $K_i(r, \cdot)$ are differentiable for all $r \in \mathrm{E}_0$, and for any $\rho > 0$, there is a $B_i > 0$ such that
$$\left|\partial_2 K_i(r,u) - \partial_2 K_i(r,v)\right| \leq B_i |u - v|, \qquad \forall\, u, v \in [-\rho, \rho].$$

(iii) Assume that there is $\ell > 0$ satisfying
$$\begin{aligned} \left\|\left\|g\right\|\right\|_{\alpha,\beta} &+ \widehat{K}_1 + 2^{1-\alpha/\beta} \|\mathrm{a}_{\rho_1}\|_{L_1}^{\alpha/\beta} + +\hat{c}_{11} \cdot \ell \\ &+ \left(1 + 2^{2-\alpha/\beta}\right) \left[\widehat{K}_2 + 2^{1-\alpha/\beta} \|\mathrm{a}_{\rho_2}\|_{L_1}^{\alpha/\beta} + \hat{c}_{22} \cdot \ell\right] \\ &\times \left[\tfrac{2^{1-\alpha/\beta}}{\Gamma(\tau_1+1)} \|\mathrm{a}_{\rho_3}\|_{L_1}^{\alpha/\beta} + \hat{c}_{33} \widehat{K}_3 + \hat{c}_{33} \cdot \ell\right] \leq \ell, \end{aligned}$$

where $\widehat{K}_i = \left\|K_i(r,0)\right\|_C$, and
$$\begin{aligned} \hat{c}_{11} &= \max\left\{ \tfrac{\mathrm{b}_{\rho_1}^{\alpha/\beta} 2^{1-\alpha/\beta}}{\Gamma(\tau_2+1)},\ \tfrac{\tilde{c}_1}{\Gamma(\tau_2+1)} + \left[\tfrac{\alpha \cdot \mathrm{b}_{\rho_1} 2^{\beta/\alpha - 1}}{\beta(\tau_2 - \alpha) \Gamma^{\beta/\alpha}(\tau_2+1)} \right]^{\alpha/\beta} \right\}, \\ \hat{c}_{22} &= \max\left\{ \tilde{c}_2,\ 2^{1-\alpha/\beta} \mathrm{b}_{\rho_2}^{\alpha/\beta} \right\}, \\ \hat{c}_{33} &= \max\left\{ \tfrac{\mathrm{b}_{\rho_3}^{\alpha/\beta} 2^{1-\alpha/\beta}}{\Gamma(\tau_1+1)},\ \tfrac{\tilde{c}_3}{\Gamma(\tau_1+1)} \left[2 + \lambda \Gamma(\tau_1 + 1) + \left[\tfrac{\alpha \tau_1^{\beta/\alpha} \cdot 2^{\beta/\alpha - 1}}{\beta(1-\alpha)} \right]^{\alpha/\beta} \right] \right\}. \end{aligned} \tag{3.7}$$

(iv) The constant below Σ is less than 1:
$$\begin{aligned} \Sigma = \tfrac{\mathrm{b}_{\rho_1} 2^{3\beta/\alpha - 2}}{\Gamma^{\beta/\alpha}(\tau_2+1)} &+ \left[\tfrac{\tilde{c}_3 \cdot \ell + \widehat{K}_3}{\Gamma(\tau_1+1)} [2 + \lambda \Gamma(\tau_1 + 1)] \right]^{\beta/\alpha} \mathrm{b}_{\rho_2} 2^{\beta/\alpha} \\ &+ \left[\tilde{c}_2 \cdot \ell + \widehat{K}_2 \right]^{\beta/\alpha} \tfrac{\mathrm{b}_{\rho_3} 2^{2\beta/\alpha - 1}}{\Gamma^{\beta/\alpha}(\tau_1+1)} < 1. \end{aligned} \tag{3.8}$$

Theorem 3.2. *Let assumptions* (i) *and* (iv) *be satisfied. Then the hybrid BVP in* (1.1) *and* (1.2) *has at least one solution* $\mathrm{w} \in \mathcal{J}_{\alpha,\beta}$ *defined on* E_0.

Proof. In view of Lemma 2.6's conditions, the proof is divided into a few steps.

Step I. First, by virtue of Theorem 2.8 and Corollary 2.9, the RL operators ${}^{\mathrm{RL}}\mathcal{I}^{\tau_i}$, $i = 1, 2$, map $\mathcal{J}_{\alpha,\beta}$ continuously into themselves. Thanks to Hypothesis (ii) along with Theorem 2.2, we find that the operators $\mathcal{N}_{K_i} : \mathcal{J}_{\alpha,\beta} \to \mathcal{J}_{\alpha,\beta}$, $i = 1, 2, 3$, are continuous consequently, the operators $\Theta_1^{(1)}$, $\Theta_1^{(2)}$, and $\Theta_2 : \mathcal{J}_{\alpha,\beta} \to \mathcal{J}_{\alpha,\beta}$ are also continuous. Employing Lemma 2.1, we find that $\Theta = \Theta_1^{(1)} + \Theta_1^{(2)} \cdot \Theta_2$ continuously maps $\mathcal{J}_{\alpha,\beta}$ into $\mathcal{J}_{\alpha,\beta}$.

Step II. We will show that the operator Θ is bounded on the ball $B_\ell(\mathcal{J}_{\alpha,\beta}) = \{\mathbb{w} \in \mathcal{J}_{\alpha,\beta} : \left\||\mathbb{w}|\right\|_{\alpha,\beta} \leq \ell\}$, where ℓ is given by Hypothesis (iii). If we let $\mathbb{w} \in \mathcal{J}_{\alpha,\beta}$, then

$$\begin{aligned}\left\||\Theta_1^{(1)}(\mathbb{w})|\right\|_{\alpha,\beta} &= \left\||g + \mathcal{N}_{K_1} \circ {}^{\mathrm{RL}}\mathcal{I}^{\tau_2}(\mathbb{w})|\right\|_{\alpha,\beta}\\ &\leq \left\||g|\right\|_{\alpha,\beta} + \left\||\mathcal{N}_{K_1} \circ {}^{\mathrm{RL}}\mathcal{I}^{\tau_2}(\mathbb{w})|\right\|_{\alpha,\beta}\\ &= \left\||g|\right\|_{\alpha,\beta} + \left\|\mathcal{N}_{K_1} \circ {}^{\mathrm{RL}}\mathcal{I}^{\tau_2}(\mathbb{w})\right\|_{C}\\ &\quad + \mathcal{J}_{\alpha,\beta}(\mathcal{N}_{K_1} \circ {}^{\mathrm{RL}}\mathcal{I}^{\tau_2}(\mathbb{w}), \mathrm{E}_0)^{\alpha/\beta},\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}\|\Theta_1^{(1)}(\mathbb{w})\|_C &= \left\|g + \mathcal{N}_{K_1} \circ {}^{\mathrm{RL}}\mathcal{I}^{\tau_2}(\mathbb{w})\right\|_C\\ &= \|g\|_C + \left\|K_1\left(r, {}^{\mathrm{RL}}\mathcal{I}^{\tau_2}(\mathbb{w})\right) - K_1(r,0) + K_1(r,0)\right\|_C\\ &\leq \|g\|_C + \tilde{c}_1 \left\|{}^{\mathrm{RL}}\mathcal{I}^{\tau_2}(\mathbb{w})\right\|_C + \widehat{K}_1\\ &\leq \|g\|_C + \widehat{K}_1 + \tilde{c}_1\|\mathbb{w}\|_C \int_0^r \tfrac{(r-s)^{\tau_2-1}}{\Gamma(\tau_2)}\,\mathrm{d}s\\ &= \|g\|_C + \widehat{K}_1 + \tilde{c}_1\|\mathbb{w}\|_C \tfrac{r^{\tau_2}}{\tau_2\Gamma(\tau_2)}\\ &\leq \|g\|_C + \widehat{K}_1 + \tfrac{\tilde{c}_1}{\Gamma(\tau_2+1)}\|\mathbb{w}\|_C.\end{aligned} \tag{3.10}$$

Moreover, applying Theorem 2.8, we obtain

$$\begin{aligned}&\mathcal{J}_{\alpha,\beta}\left(\mathcal{N}_{K_1} \circ {}^{\mathrm{RL}}\mathcal{I}^{\tau_2}(\mathbb{w})(r), \mathrm{E}_0\right)\\ &= \int_{\mathrm{E}_0} \xi^{-(\beta+1)}\varpi\left(\mathcal{N}_{K_1} \circ {}^{\mathrm{RL}}\mathcal{I}^{\tau_2}(\mathbb{w})(r), \xi\right)^{\beta/\alpha}\,\mathrm{d}\xi\\ &\leq \int_{\mathrm{E}_0} \xi^{-(\beta+1)}\left[\mathrm{a}_{\rho_1}(\xi)\xi^{\beta+1} + \mathrm{b}_{\rho_1}\varpi\left({}^{\mathrm{RL}}\mathcal{I}^{\tau_2}(\mathbb{w})(r), \xi\right)^{\beta/\alpha}\right]\mathrm{d}\xi\\ &\leq \int_{\mathrm{E}_0} \mathrm{a}_{\rho_1}(\xi)\,\mathrm{d}\xi + \mathrm{b}_{\rho_1}\int_{\mathrm{E}_0} \xi^{-(\beta+1)}\varpi\left({}^{\mathrm{RL}}\mathcal{I}^{\tau_2}(\mathbb{w})(r), \xi\right)^{\beta/\alpha}\,\mathrm{d}\xi\\ &\leq \|\mathrm{a}_{\rho_1}\|_{L_1} + \mathrm{b}_{\rho_1}2^{\beta/\alpha-1}\int_{\mathrm{E}_0} \xi^{-(\beta+1)}\left[\tfrac{\varpi(\mathbb{w},\xi)^{\beta/\alpha} + \xi^{\beta\tau_2/\alpha}\|\mathbb{w}\|_C^{\beta/\alpha}}{\Gamma^{\beta/\alpha}(\tau_2+1)}\,\mathrm{d}\xi\right]\\ &\leq \|\mathrm{a}_{\rho_1}\|_{L_1} + \tfrac{\mathrm{b}_{\rho_1}2^{\beta/\alpha-1}}{\Gamma^{\beta/\alpha}(\tau_2+1)}\Big[\int_{\mathrm{E}_0} \xi^{-(\beta+1)}\varpi(\mathbb{w}, \xi)^{\beta/\alpha}\,\mathrm{d}\xi\\ &\quad + \int_{\mathrm{E}_0} \xi^{-(\beta+1)}\xi^{\beta\tau_2/\alpha}\|\mathbb{w}\|_C^{\beta/\alpha}\,\mathrm{d}\xi\Big]\\ &\leq \|\mathrm{a}_{\rho_1}\|_{L_1} + \tfrac{\mathrm{b}_{\rho_1}2^{\beta/\alpha-1}}{\Gamma^{\beta/\alpha}(\tau_2+1)}\mathcal{J}_{\alpha,\beta}(\mathbb{w}, \mathrm{E}_0)\\ &\quad + \tfrac{\mathrm{b}_{\rho_1}2^{\beta/\alpha-1}\cdot\|\mathbb{w}\|_C^{\beta/\alpha}}{\Gamma^{\beta/\alpha}(\tau_2+1)}\left[\tfrac{\alpha}{\beta(\tau_2-\alpha)}\right].\end{aligned} \tag{3.11}$$

By substituting from (3.10) and (3.11) in (3.9), we have

$$\begin{aligned}\left\|\left|\Theta_1^{(1)}(\mathbb{w})\right|\right\|_{\alpha,\beta} &\leq \left\||g|\right\|_{\alpha,\beta} + \tfrac{\tilde{c}_1\|\mathbb{w}\|_C}{\Gamma(\tau_2+1)} + \widehat{K}_1 \\ &\quad + \left[\|\mathrm{a}_{\rho_1}\|_{L_1} + \tfrac{\mathrm{b}_{\rho_1}2^{\beta/\alpha-1}}{\Gamma^{\beta/\alpha}(\tau_2+1)}\mathcal{J}_{\alpha,\beta}(\mathbb{w},\mathrm{E}_0)^{\alpha/\beta}\right. \\ &\quad \left. + \tfrac{\mathrm{b}_{\rho_1}2^{\beta/\alpha-1}\cdot\|\mathbb{w}\|_C^{\beta/\alpha}}{\Gamma^{\beta/\alpha}(\tau_2+1)}\left[\tfrac{\alpha}{\beta(\tau_2-\alpha)}\right]\right]^{\alpha/\beta} \\ &\leq \left\||g|\right\|_{\alpha,\beta} + \tfrac{\tilde{c}_1\|\mathbb{w}\|_C}{\Gamma(\tau_2+1)} + \widehat{K}_1 + 2^{1-\alpha/\beta}\|\mathrm{a}_{\rho_1}\|_C^{\alpha/\beta} \\ &\quad + \tfrac{\mathrm{b}_{\rho_1}^{\alpha/\beta}2^{1-\alpha/\beta}}{\Gamma(\tau_2+1)}\left[\mathcal{J}_{\alpha,\beta}(\mathbb{w},\mathrm{E}_0)^{\alpha/\beta} + \left[\tfrac{\alpha}{\beta(\tau_2-\alpha)}\right]^{\alpha/\beta}\|\mathbb{w}\|_C\right] \\ &\leq \left\||g|\right\|_{\alpha,\beta} + \widehat{K}_1 + 2^{1-\alpha/\beta}\|\mathrm{a}_{\rho_1}\|_{L_1}^{\alpha/\beta} + +\hat{c}_{11}\left\||\mathbb{w}|\right\|_{\alpha,\beta}. \end{aligned} \tag{3.12}$$

For the operator $\Theta_1^{(2)}\mathbb{w}$, we have

$$\begin{aligned}\left\|\left|\Theta_1^{(2)}(\mathbb{w})\right|\right\|_{\alpha,\beta} &\leq \tilde{c}_2\|\mathbb{w}\|_C + \widehat{K}_2 + 2^{1-\alpha/\beta}\left[\left\|\mathrm{a}_{\rho_2}\right\|_{L_1}^{\alpha/\beta} + \mathrm{b}_{\rho_2}^{\alpha/\beta}\cdot\mathcal{J}_{\alpha,\beta}(\mathbb{w},\mathrm{E}_0)^{\alpha/\beta}\right] \\ &\leq \widehat{K}_2 + 2^{1-\alpha/\beta}\left\|\mathrm{a}_{\rho_2}\right\|_{L_1}^{\alpha/\beta} + \hat{c}_{22}\left\||\mathbb{w}|\right\|_{\alpha,\beta}. \end{aligned} \tag{3.13}$$

Estimations (3.12) and (3.13) yield that the operators $\Theta_1^{(1)}(\mathbb{w})$, $\Theta_1^{(2)}(\mathbb{w})$ are bounded on E_0. Combining these facts with the continuity of $\Theta_1^{(1)}(\mathbb{w})$, $\Theta_1^{(2)}(\mathbb{w})$ on E_0, we find that the operators $\Theta_1^{(1)}, \Theta_1^{(2)} : \mathcal{J}_{\alpha,\beta} \to \mathcal{J}_{\alpha,\beta}$. Similarly, we can show that $\Theta_2(\mathbb{w})$ is bounded on E_0 as follows:

$$\begin{aligned}\left\|\left|\Theta_2(\mathbb{w})\right|\right\|_{\alpha,\beta} &= \left\|\left| -{}^{\mathrm{RL}}\mathcal{I}^{\tau_1}\circ\mathcal{N}_{K_3}(\mathbb{w}) + {}^{\mathrm{RL}}\mathcal{I}^{\tau_1}\circ\mathcal{N}_{K_3}(\mathbb{w})(\eta) + \lambda\mathbb{I}\circ\mathcal{N}_{K_3}(\mathbb{w})(1)\right|\right\|_{\alpha,\beta} \\ &\leq \left\|\left|{}^{\mathrm{RL}}\mathcal{I}^{\tau_1}\circ\mathcal{N}_{K_3}(\mathbb{w})\right|\right\|_{\alpha,\beta} + \left\|\left|{}^{\mathrm{RL}}\mathcal{I}^{\tau_1}\circ\mathcal{N}_{K_3}(\mathbb{w})(\eta)\right|\right\|_{\alpha,\beta} \\ &\quad + \lambda\left\|\left|\mathbb{I}\circ\mathcal{N}_{K_3}(\mathbb{w})(1)\right|\right\|_{\alpha,\beta} \\ &= \left\|{}^{\mathrm{RL}}\mathcal{I}^{\tau_1}\circ\mathcal{N}_{K_3}(\mathbb{w})\right\|_C + \left\|{}^{\mathrm{RL}}\mathcal{I}^{\tau_1}\circ\mathcal{N}_{K_3}(\mathbb{w})(\eta)\right\|_C \\ &\quad + \lambda\left\|\mathbb{I}\circ\mathcal{N}_{K_3}(\mathbb{w})(1)\right\|_C + \mathcal{J}_{\alpha,\beta}\left({}^{\mathrm{RL}}\mathcal{I}^{\tau_1}\circ\mathcal{N}_{K_3}(\mathbb{w}),\mathrm{E}_0\right)^{\alpha/\beta} \\ &\quad + \mathcal{J}_{\alpha,\beta}\left({}^{\mathrm{RL}}\mathcal{I}^{\tau_1}\circ\mathcal{N}_{K_3}(\mathbb{w})(\eta),\mathrm{E}_0\right)^{\alpha/\beta} + \lambda\mathcal{J}_{\alpha,\beta}(\mathbb{I}\circ\mathcal{N}_{K_3}(\mathbb{w})(1),\mathrm{E}_0)^{\alpha/\beta}, \end{aligned} \tag{3.14}$$

where

$$\begin{aligned}\left\|\Theta_2(\mathbb{w})\right\|_C &= \left\|\int_0^r \tfrac{K_3(s,\mathbb{w}(s))}{\left(r-s\right)^{1-\tau_1}}\,\mathrm{d}s\right\|_C + \left\|\int_0^\eta \tfrac{K_3(s,\mathbb{w}(s))}{\Gamma(\tau_1)(\eta-s)^{1-\tau_1}}\,\mathrm{d}s\right. \\ &\quad + \lambda\left\|\int_{\mathrm{E}_0} K_3(s,\mathbb{w}(s))\,\mathrm{d}s\right\|_C \\ &\leq \tfrac{\left\|\mathcal{N}_{K_3}(\mathbb{w})\right\|_C}{\Gamma(\tau_1)}\left[\int_0^r \tfrac{1}{(r-s)^{1-\tau_1}}\,\mathrm{d}s\right. \\ &\quad \left. + \int_0^\eta \tfrac{1}{(\eta-s)^{1-\tau_1}}\,\mathrm{d}s + \lambda\Gamma(\tau_1)\int_{\mathrm{E}_0}\mathrm{d}s\right] \\ &\leq \tfrac{\|K_3(r,\mathbb{w})-K_3(r,0)+K_3(r,0)\|_C}{\Gamma(\tau_1)}\left[\tfrac{r^{\tau_1}}{\tau_1} + \tfrac{\eta^{\tau_1}}{\tau_1} + \lambda\Gamma(\tau_1)\right] \\ &\leq \tfrac{\tilde{c}_3\|\mathbb{w}\|_C+\widehat{K}_3}{\Gamma(\tau_1+1)}[2 + \lambda\Gamma(\tau_1+1)]. \end{aligned} \tag{3.15}$$

Moreover, applying Corollary 2.9, we obtain

$$\begin{aligned}
\mathcal{J}_{\alpha,\beta}(\Theta_2(\mathbbm{w}),\mathrm{E}_0) &\leq \mathcal{J}_{\alpha,\beta}\left({}^{\mathrm{RL}}\mathcal{I}^{\tau_1}\circ\mathcal{N}_{K_3}(\mathbbm{w}),\mathrm{E}_0\right)+\mathcal{J}_{\alpha,\beta}\left({}^{\mathrm{RL}}\mathcal{I}^{\tau_1}\circ\mathcal{N}_{K_3}(\mathbbm{w}),\mathrm{E}_0\right)\\
&\quad+\lambda\mathcal{J}_{\alpha,\beta}(\mathbb{I}\circ\mathcal{N}_{K_3}(\mathbbm{w})(1),\mathrm{E}_0)\\
&=\int_{\mathrm{E}_0}\xi^{-(\beta+1)}\varpi\left({}^{\mathrm{RL}}\mathcal{I}^{\tau_1}\circ\mathcal{N}_{K_3}(\mathbbm{w}),\xi\right)^{\beta/\alpha}\,\mathrm{d}\xi\\
&\leq 2^{\beta/\alpha-1}\int_{\mathrm{E}_0}\xi^{-(\beta+1)}\frac{\varpi\left(\mathcal{N}_{K_3}(\mathbbm{w}),\xi\right)^{\beta/\alpha}+(\tau_1\xi)^{\beta/\alpha}\left\|\mathcal{N}_{K_3}(\mathbbm{w})\right\|_C^{\beta/\alpha}}{\Gamma^{\beta/\alpha}(\tau_1+1)}\,\mathrm{d}\xi\\
&\leq\frac{2^{\beta/\alpha-1}}{\Gamma^{\beta/\alpha}(\tau_1+1)}\Bigg[\int_{\mathrm{E}_0}\xi^{-(\beta+1)}\varpi(\mathcal{N}_{K_3}(\mathbbm{w}),\xi)^{\beta/\alpha}\,\mathrm{d}\xi\\
&\quad+\tau_1^{\beta/\alpha}\int_{\mathrm{E}_0}\xi^{-(\beta+1)}\xi^{\beta/\alpha}\left\|\mathcal{N}_{K_3}(\mathbbm{w})\right\|_C^{\beta/\alpha}\,\mathrm{d}\xi\Bigg]\\
&\leq\frac{2^{\beta/\alpha-1}}{\Gamma^{\beta/\alpha}(\tau_1+1)}\Bigg[\int_{\mathrm{E}_0}\xi^{-(\beta+1)}\left[\mathrm{a}_{\rho_3}(\xi)\xi^{\beta+1}+\mathrm{b}_{\rho_3}\varpi(\mathbbm{w},\xi)^{\beta/\alpha}\right]\mathrm{d}\xi\\
&\quad+\frac{\alpha\ \tau_1^{\beta/\alpha}}{\beta(1-\alpha)}[\tilde{c}_3\|\mathbbm{w}\|_C+\widehat{K}_3]^{\beta/\alpha}\Bigg]\\
&\leq\frac{2^{\beta/\alpha-1}}{\Gamma^{\beta/\alpha}(\tau_1+1)}\Bigg[\int_{\mathrm{E}_0}\mathrm{a}_{\rho_3}(\xi)\,\mathrm{d}\xi+\mathrm{b}_{\rho_3}\int_{\mathrm{E}_0}\xi^{-(\beta+1)}\varpi(\mathbbm{w},\xi)^{\beta/\alpha}\,\mathrm{d}\xi\\
&\quad+\frac{\alpha\ \tau_1^{\beta/\alpha}}{\beta(1-\alpha)}[\tilde{c}_3\|\mathbbm{w}\|_C+\widehat{K}_3]^{\beta/\alpha}\Bigg]\\
&\leq\frac{2^{\beta/\alpha-1}}{\Gamma^{\beta/\alpha}(\tau_1+1)}\left[\left\|\mathrm{a}_{\rho_3}\right\|_{L_1}+\mathrm{b}_{\rho_3}\mathcal{J}_{\alpha,\beta}[\mathbbm{w},\mathrm{E}_0]+\frac{\alpha\ \tau_1^{\beta/\alpha}}{\beta(1-\alpha)}\left[\tilde{c}_3\|\mathbbm{w}\|_C+\widehat{K}_3\right]^{\beta/\alpha}\right],
\end{aligned}\tag{3.16}$$

where

$$\begin{aligned}
&\varpi\left({}^{\mathrm{RL}}\mathcal{I}^{\tau_1}\circ\mathcal{N}_{K_3}(\mathbbm{w})(\eta),\xi\right)=0 &&\Longrightarrow\mathcal{J}_{\alpha,\beta}\left({}^{\mathrm{RL}}\mathcal{I}^{\tau_1}\circ\mathcal{N}_{K_3}(\mathbbm{w})(\eta),\ \mathrm{E}_0\right)=0,\\
&\varpi\left(\mathbb{I}\circ\mathcal{N}_{K_3}(\mathbbm{w})(1),\xi\right)=0 &&\Longrightarrow\mathcal{J}_{\alpha,\beta}\left(\mathbb{I}\circ\mathcal{N}_{K_3}(\mathbbm{w})(1),\ \mathrm{E}_0\right)=0.
\end{aligned}$$

By substituting from (3.15) and (3.16) in (3.14), we have

$$\begin{aligned}
\left\|\left|\Theta_2(\mathbbm{w})\right|\right\|_{\alpha,\beta}&\leq\frac{\tilde{c}_3\|\mathbbm{w}\|_C+\widehat{K}_3}{\Gamma(\tau_1+1)}[2+\lambda\Gamma(\tau_1+1)]\\
&\quad+\frac{2^{1-\alpha/\beta}}{\Gamma(\tau_1+1)}\Bigg[\left\|\mathrm{a}_{\rho_3}\right\|_{L_1}^{\alpha/\beta}+\mathrm{b}_{\rho_3}^{\alpha/\beta}\mathcal{J}_{\alpha,\beta}(\mathbbm{w},\mathrm{E}_0)^{\alpha/\beta}\\
&\quad+\left[\frac{\alpha\ \tau_1^{\beta/\alpha}}{\beta(1-\alpha)}\right]^{\alpha/\beta}[\tilde{c}_3\|\mathbbm{w}\|_C+\widehat{K}_3]\Bigg]\\
&=\frac{2^{1-\alpha/\beta}}{\Gamma(\tau_1+1)}\left\|\mathrm{a}_{\rho_3}\right\|_{L_1}^{\alpha/\beta}+\frac{\widehat{K}_3}{\Gamma(\tau_1+1)}\left[2+\lambda\Gamma(\tau_1+1)+\left[\frac{\alpha\ \tau_1^{\beta/\alpha}2^{1-\alpha/\beta}}{\beta(1-\alpha)}\right]^{\alpha/\beta}\right]\\
&\quad+\frac{\tilde{c}_3\|\mathbbm{w}\|_C}{\Gamma(\tau_1+1)}\left[2+\lambda\Gamma(\tau_1+1)+\left[\frac{\alpha\ \tau_1^{\beta/\alpha}2^{1-\alpha/\beta}}{\beta(1-\alpha)}\right]^{\alpha/\beta}\right]\\
&\quad+\frac{\mathrm{b}_{\rho_3}^{\alpha/\beta}2^{1-\alpha/\beta}}{\Gamma(\tau_1+1)}\mathcal{J}_{\alpha,\beta}(\mathbbm{w},\mathrm{E}_0)^{\alpha/\beta}\\
&\leq\frac{2^{1-\alpha/\beta}}{\Gamma(\tau_1+1)}\left\|\mathrm{a}_{\rho_3}\right\|_{L_1}^{\alpha/\beta}+\hat{c}_{33}\widehat{K}_3+\hat{c}_{33}\left\|\left|\mathbbm{w}\right|\right\|_{\alpha,\beta}.
\end{aligned}\tag{3.17}$$

The estimate above shows that the operator $\Theta_2:\mathcal{J}_{\alpha,\beta}\to\mathcal{J}_{\alpha,\beta}$. Now, combining (3.12), (3.13), and (3.17), and recalling Lemma 2.1, we have

$$\begin{aligned}\Big\|\|\Theta(\mathbb{w})\|\Big\|_{\alpha,\beta} &\leq \Big\|\|\Theta_1^{(1)}(\mathbb{w})\|\Big\|_{\alpha,\beta} + \Big[1 + 2^{2-\alpha/\beta}\Big]\Big\|\|\Theta_1^{(2)}(\mathbb{w})\|\Big\|_{\alpha,\beta}\Big\|\|\Theta_2(\mathbb{w})\|\Big\|_{\alpha,\beta} \\ &\leq \Big\|\|g\|\Big\|_{\alpha,\beta} + \widehat{K}_1 + 2^{1-\alpha/\beta}\|\mathrm{a}_{\rho_1}\|_{L_1}^{\alpha/\beta} + \hat{c}_{11}\Big\|\|\mathbb{w}\|\Big\|_{\alpha,\beta} \\ &\quad + \Big[1 + 2^{2-\alpha/\beta}\Big]\Big[\widehat{K}_2 + 2^{1-\alpha/\beta}\,\Big\|\mathrm{a}_{\rho_2}\Big\|_{L_1}^{\alpha/\beta} + \hat{c}_{22}\Big\|\|\mathbb{w}\|\Big\|_{\alpha,\beta}\Big] \\ &\quad \times\Big[\tfrac{2^{1-\alpha/\beta}}{\Gamma(\tau_1+1)}\,\Big\|\mathrm{a}_{\rho_3}\Big\|_{L_1}^{\alpha/\beta} + \hat{c}_{33}\widehat{K}_3 + \hat{c}_{33}\Big\|\|\mathbb{w}\|\Big\|_{\alpha,\beta}\Big].\end{aligned}$$

Consequently, $\Theta : \mathcal{J}_{\alpha,\beta} \to \mathcal{J}_{\alpha,\beta}$. For $\mathbb{w} \in B_\ell(\mathcal{J}_{\alpha,\beta})$, we obtain

$$\begin{aligned}\Big\|\|\Theta(\mathbb{w})\|\Big\|_{\alpha,\beta} &\leq \Big\|\|g\|\Big\|_{\alpha,\beta} + \widehat{K}_1 + 2^{1-\alpha/\beta}\|\mathrm{a}_{\rho_1}\|_{L_1}^{\alpha/\beta} + \hat{c}_{11}\cdot\ell \\ &\quad + \Big[1 + 2^{2-\alpha/\beta}\Big]\Big[\widehat{K}_2 + 2^{1-\alpha/\beta}\,\Big\|\mathrm{a}_{\rho_2}\Big\|_{L_1}^{\alpha/\beta} + \hat{c}_{22}\cdot\ell\Big] \\ &\quad \times\Big[\tfrac{2^{1-\alpha/\beta}}{\Gamma(\tau_1+1)}\,\Big\|\mathrm{a}_{\rho_3}\Big\|_{L_1}^{\alpha/\beta} + \hat{c}_{33}\widehat{K}_3 + \hat{c}_{33}\cdot\ell\Big] \leq \ell.\end{aligned}$$

Then $\Theta : B_\ell(\mathcal{J}_{\alpha,\beta}) \to B_\ell(\mathcal{J}_{\alpha,\beta})$ and is continuous. Moreover, by virtue of (3.12), (3.13), and (3.17), the following estimates are deduced:

$$\begin{aligned}\Big\|\|\Theta_1^{(1)}(B_\ell)\|\Big\|_{\alpha,\beta} &\leq \Big\|\|g\|\Big\|_{\alpha,\beta} + \widehat{K}_1 + 2^{1-\alpha/\beta}\|\mathrm{a}_{\rho_1}\|_{L_1}^{\alpha/\beta} + \hat{c}_{11}\cdot\ell, \\ \Big\|\|\Theta_1^{(2)}(B_\ell)\|\Big\|_{\alpha,\beta} &\leq \widehat{K}_2 + 2^{1-\alpha/\beta}\,\Big\|\mathrm{a}_{\rho_2}\Big\|_{L_1}^{\alpha/\beta} + \hat{c}_{22}\cdot\ell, \\ \Big\|\|\Theta_2(B_\ell)\|\Big\|_{\alpha,\beta} &\leq \tfrac{2^{1-\alpha/\beta}}{\Gamma(\tau_1+1)}\,\Big\|\mathrm{a}_{\rho_3}\Big\|_{L_1}^{\alpha/\beta} + \hat{c}_{33}\widehat{K}_3 + \hat{c}_{33}\cdot\ell.\end{aligned}$$

Step III. Clearly, as a direct calculation from the definition, the ball $\varnothing \neq B_\ell(\mathcal{J}_{\alpha,\beta})$ is bounded, convex, and also closed.

Step IV. To show that Θ satisfies Condition (ii) in Lemma 2.6, first, we need to check that the operators $\Theta_1^{(1)}$, $\Theta_1^{(2)}$, and Θ_2 satisfy Hypothesis (i) of Lemma 2.6. Let $\varnothing \neq \mathbb{G} \subset B_\ell(\mathcal{J}_{\alpha,\beta})$ and for $\mathbb{w} \in \mathbb{G}$, $r \in \mathrm{E}_0 \setminus \{0, 1\}$, we have

$$\begin{aligned}\mathcal{J}_{\alpha,\beta}\Big(\Theta_1^{(1)}(\mathbb{w}), [0,r]\Big) &= \int_0^r \xi^{-(\beta+1)}\varpi\Big(\Theta_1^{(1)}(\mathbb{w}), \xi\Big)^{\beta/\alpha}\,\mathrm{d}\xi, \\ &\leq 2^{\beta/\alpha-1}\int_0^r \xi^{-(\beta+1)}\varpi(g,\xi)^{\beta/\alpha}\,\mathrm{d}\xi \\ &\quad + 2^{\beta/\alpha-1}\int_0^r \xi^{-(\beta+1)}\varpi\Big(\mathcal{N}_{K_1}\circ{}^{\mathrm{RL}}\mathcal{I}^{\tau_2}(\mathbb{w}), \xi\Big)^{\beta/\alpha}\,\mathrm{d}\xi \\ &\leq 2^{\beta/\alpha-1}\int_0^r \xi^{-(\beta+1)}\varpi(g,\xi)^{\beta/\alpha}\,\mathrm{d}\xi \\ &\quad + 2^{\beta/\alpha-1}\int_0^r \xi^{-(\beta+1)}\Big[\mathrm{a}_{\rho_1}(\xi)\xi^{\beta+1} + \mathrm{b}_{\rho_1}\varpi\Big({}^{\mathrm{RL}}\mathcal{I}^{\tau_2}(\mathbb{w})(\xi), \xi\Big)^{\beta/\alpha}\Big]\,\mathrm{d}\xi \\ &\leq 2^{\beta/\alpha-1}\Big[\int_0^r \xi^{-(\beta+1)}\varpi(g,\xi)^{\beta/\alpha}\,\mathrm{d}\xi + \int_0^r \mathrm{a}_{\rho_1}(\xi)\,\mathrm{d}\xi\Big] \\ &\quad + \mathrm{b}_{\rho_1}2^{2\beta/\alpha-2}\int_0^r \xi^{-(\beta+1)}\Big[\tfrac{\varpi(\mathbb{w},\xi)^{\beta/\alpha} + \xi^{\beta\tau_2/\alpha}\|\mathbb{w}\|_C^{\beta/\alpha}}{\Gamma^{\beta/\alpha}(\tau_2+1)}\Big]\,\mathrm{d}\xi \\ &\leq 2^{\beta/\alpha-1}\Big[\int_0^r \xi^{-(\beta+1)}\varpi(g,\xi)^{\beta/\alpha}\,\mathrm{d}\xi + \int_0^r \mathrm{a}_{\rho_1}(\xi)\,\mathrm{d}\xi\Big] \\ &\quad + \tfrac{\mathrm{b}_{\rho_1}2^{2\beta/\alpha-2}}{\Gamma^{\beta/\alpha}(\tau_2+1)}\Big[\int_0^r \xi^{-(\beta+1)}\varpi(\mathbb{w},\xi)^{\beta/\alpha}\,\mathrm{d}\xi \\ &\quad + \|\mathbb{w}\|_C^{\beta/\alpha}\int_0^r \xi^{-(\beta+1)+\beta\tau_2/\alpha}\,\mathrm{d}\xi\Big].\end{aligned}$$

Taking the limit with $r \to 0$, we get

$$c(\Theta_1^{(1)}(\mathbb{G})) \leq \frac{\mathrm{b}_{\rho_1} 2^{2\beta/\alpha - 2}}{\Gamma^{\beta/\alpha}(\tau_2+1)} c(\mathbb{G}).$$

Then, applying Proposition 2.4, we obtain

$$\chi_{\mathrm{H}}(\Theta_1^{(1)}(\mathbb{G})) \leq \frac{\mathrm{b}_{\rho_1} 2^{3\beta/\alpha - 2}}{\Gamma^{\beta/\alpha}(\tau_2+1)} \chi_{\mathrm{H}}(\mathbb{G}). \tag{3.18}$$

Similarly, we have

$$\chi_{\mathrm{H}}\left(\Theta_1^{(2)}(\mathbb{G})\right) \leq 2^{\beta/\alpha} \mathrm{b}_{\rho_2} \cdot \chi_{\mathrm{H}}(\mathbb{G}). \tag{3.19}$$

Moreover, for the operator $\Theta_2(\mathtt{w})$, we have

$$\begin{aligned}
\mathcal{J}_{\alpha,\beta}(\Theta_2(\mathtt{w}), [0,r]) &= \int_0^r \xi^{-(\beta+1)} \varpi(\Theta_2(\mathtt{w}), \xi)^{\beta/\alpha} \, \mathrm{d}\xi \\
&\leq \int_0^r \xi^{-(\beta+1)} \varpi\left({}^{\mathrm{RL}}\mathcal{I}^{\tau_1} \circ \mathcal{N}_{\kappa_3}(\mathtt{w}), \xi\right)^{\beta/\alpha} \, \mathrm{d}\xi \\
&\quad + \int_0^r \xi^{-(\beta+1)} \varpi\left({}^{\mathrm{RL}}\mathcal{I}^{\tau_1} \circ \mathcal{N}_{\kappa_3}(\mathtt{w})(\eta), \xi\right)^{\beta/\alpha} \, \mathrm{d}\xi \\
&\quad + \int_0^r \xi^{-(\beta+1)} \varpi(\mathbb{I} \circ \mathcal{N}_{\kappa_3}(\mathtt{w})(1), \xi)^{\beta/\alpha} \, \mathrm{d}\xi \\
&\leq \frac{2^{\beta/\alpha - 1}}{\Gamma^{\beta/\alpha}(\tau_1+1)} \Bigg[\int_0^r \xi^{-(\beta+1)} \varpi(\mathcal{N}_{\kappa_3}(\mathtt{w}), \xi)^{\beta/\alpha} \, \mathrm{d}\xi \\
&\quad + \int_0^r \xi^{-(\beta+1)} (\xi \tau_1)^{\beta/\alpha} \left\| \mathcal{N}_{\kappa_3}(\mathtt{w}) \right\|_C^{\beta/\alpha} \, \mathrm{d}\xi \Bigg] \\
&\leq \frac{2^{\beta/\alpha - 1}}{\Gamma^{\beta/\alpha}(\tau_1+1)} \Bigg[\int_0^r \mathrm{a}_{\rho_3}(\xi) \, \mathrm{d}\xi + \mathrm{b}_{\rho_3} \int_0^r \xi^{-(\beta+1)} \varpi(\mathtt{w}, \xi)^{\beta/\alpha} \, \mathrm{d}\xi \\
&\quad + \left\| \mathcal{N}_{\kappa_3}(\mathtt{w}) \right\|_C^{\beta/\alpha} \tau_1^{\beta/\alpha} \int_0^r \xi^{-(\beta+1) + \beta/\alpha} \, \mathrm{d}\xi \Bigg],
\end{aligned}$$

where $\varpi({}^{\mathrm{RL}}\mathcal{I}^{\tau_1} \circ \mathcal{N}_{\kappa_3}(\mathtt{w})(\eta), \xi) = 0$ and $\varpi(\mathbb{I} \circ \mathcal{N}_{\kappa_3}(\mathtt{w})(1), \xi) = 0$. Taking the limit as $r \to 0$, we get

$$c(\Theta_2(\mathbb{G})) \leq \frac{\mathrm{b}_{\rho_3} 2^{\beta/\alpha - 1}}{\Gamma^{\beta/\alpha}(\tau_1+1)} c(\mathbb{G}).$$

Then, applying Proposition 2.4, we obtain

$$\chi_{\mathrm{H}}(\Theta_2(\mathtt{w})) \leq \frac{\mathrm{b}_{\rho_3} 2^{2\beta/\alpha - 1}}{\Gamma^{\beta/\alpha}(\tau_1+1)} \chi_{\mathrm{H}}(\mathbb{G}). \tag{3.20}$$

The operators $\Theta_1^{(1)}$, $\Theta_1^{(2)}$, and Θ_2 satisfy Assumption (i) of Lemma 2.6 with the constants

$$k_1 = \frac{\mathrm{b}_{\rho_1} 2^{3\beta/\alpha - 2}}{\Gamma^{\beta/\alpha}(\tau_2+1)}, \quad k_2 = 2^{\beta/\alpha} \mathrm{b}_{\rho_2}, \quad k_3 = \frac{\mathrm{b}_{\rho_3} 2^{2\beta/\alpha - 1}}{\Gamma^{\beta/\alpha}(\tau_1+1)}. \tag{3.21}$$

Now, Lemma 2.5 confirms the operator $\Theta = \Theta_1^{(1)} + \Theta_1^{(2)} \cdot \Theta_2$ satisfies Condition (ii) of Lemma 2.5, which has the contraction constant $\Sigma < 1$, where is given by (3.8) in Hypothesis (iv). At the end, by considering all the abovementioned properties, we can use Lemma 2.5 to complete the proof.

3.2. Uniqueness results

Now, we show that the BVP in (1.1) and (1.2) has exactly one solution.

Theorem 3.3. *Let assumptions of Theorem 3.2 be verified by replacing the inequality* (3.6) *with*

$$\varpi(\mathcal{N}_{K_i}(\mathbb{w}) - \mathcal{N}_{K_i}(\grave{\mathbb{w}}), \xi) \leq \mathrm{b}^*_{\rho_i} \cdot \varpi\left(\mathbb{w} - \grave{\mathbb{w}}, \xi\right), \qquad \mathrm{b}^*_{\rho_i} > 0,\ i = 1, 2, 3, \tag{3.22}$$

and assume that

$$\begin{aligned}\hat{c}_{44} = \Big[\hat{c}^*_{11} + \hat{c}^*_{22}\big[1 + 2^{2-\alpha/\beta}\big]\Big[\tfrac{2^{1-\alpha/\beta}}{\Gamma(\tau_1+1)}\left\|\mathrm{a}_{\rho_3}\right\|^{\alpha/\beta}_{L_1} + \hat{c}_{33}\widehat{K}_3 + \hat{c}_{33}\cdot\ell\Big]\\ + \hat{c}^*_{33}\big[1 + 2^{2-\alpha/\beta}\big]\Big[\widehat{K}_2 + 2^{1-\alpha/\beta}\left\|\mathrm{a}_{\rho_2}\right\|^{\alpha/\beta}_{L_1} + \hat{c}_{22}\cdot\ell\Big]\Big] < 1,\end{aligned}$$

where $\hat{c}_{22}, \hat{c}_{33}$, *and* ℓ *are defined in Theorem 3.2, and*

$$\begin{aligned}\hat{c}^*_{11} &= \max\left\{\tfrac{\left(\mathrm{b}^*_{\rho_1}\right)\cdot 2^{1-\alpha/\beta}}{\Gamma(\tau_2+1)},\ \tfrac{1}{\Gamma(\tau_2+1)}\left[\tilde{c}_1 + \left[\tfrac{\alpha\cdot\left(\mathrm{b}^*_{\rho_1}\right)^{\beta/\alpha}\cdot 2^{\beta/\alpha-1}}{\beta(\tau_2-\alpha)}\right]^{\alpha/\beta}\right]\right\},\\ \hat{c}^*_{22} &= \max\left\{\tilde{c}_2,\ \mathrm{b}^*_{\rho_2}\right\},\\ \hat{c}^*_{33} &= \max\left\{\tfrac{\mathrm{b}^*_{\rho_3}2^{1-\alpha/\beta}}{\Gamma(\tau_1+1)},\ \tfrac{\tilde{c}_3}{\Gamma(\tau_1+1)}\left[2 + \lambda\Gamma(\tau_1+1) + \left[\tfrac{\alpha\ \tau_1^{\beta/\alpha}\cdot 2^{\beta/\alpha-1}}{\beta(1-\alpha)}\right]^{\alpha/\beta}\right]\right\}.\end{aligned} \tag{3.23}$$

Then the BVP in (1.1) *and* (1.2) *has a unique solution* $\mathbb{w} \in B_\ell(\mathcal{J}_{\alpha,\beta})$.

Proof. In what follows, let $i = 1, 2, 3$. By using the inequality (3.22), we have

$$\varpi(\mathcal{N}_{K_i}(\mathbb{w}) - \mathcal{N}_{K_i}(0),\ \xi) \leq \mathrm{b}^*_{\rho_i}\cdot\varpi\left(\mathbb{w}, \xi\right). \tag{3.24}$$

Moreover

$$\begin{aligned}\varpi(\mathcal{N}_{K_i}(\mathbb{w}) - \mathcal{N}_{K_i}(0), \xi, \mu) &= \sup_{s,r\in\mathrm{E}_0}\Big\{\big|K_i(s,u) - K_i(r,v) - K_i(s,0) + K_i(r,0)\big|\\ &\qquad : |s-r| \leq \xi,\ u, v \in \mathbb{R},\ |u-v| \leq \mu\Big\}\\ &\geq \sup_{s,r\in\mathrm{E}_0}\Big\{\big|K_i(s,u) - K_i(r,v)\big|\ :\ |s-r| \leq \xi,\\ &\qquad u, v \in \mathbb{R},\ |u-v| \leq \mu\Big\}\\ &\qquad - \sup_{s,r\in\mathrm{E}_0}\Big\{\big|K_i(s,0) - K_i(r,0)\big|\ :\ |s-r| \leq \xi\Big\}\\ &\geq \varpi\left(\mathcal{N}_{K_i}(\mathbb{w}), \xi\right) - \tilde{c}^*_i\cdot\sup_{s,r\in\mathrm{E}_0}\Big\{|s-r|\ :\ |s-r| \leq \xi\Big\}\\ &= \varpi\left(\mathcal{N}_{K_i}(\mathbb{w}), \xi\right) - \tilde{c}^*_i\cdot\xi.\end{aligned} \tag{3.25}$$

Combining (3.24) and (3.25), we get

$$\varpi\left(\mathcal{N}_{K_i}(\mathbb{w}), \xi\right) - \tilde{c}^*_i\cdot\xi \leq \varpi(\mathcal{N}_{K_i}(\mathbb{w}) - \mathcal{N}_{K_i}(0), \xi) \leq \mathrm{b}^*_{\rho_i}\cdot\varpi\left(\mathbb{w}, \xi\right),$$

and so $\varpi(\mathcal{N}_{\kappa_i}(\mathbb{w}), \xi) \leq \tilde{c}_i^* \cdot \xi + \mathrm{b}_{\rho_i}^* \cdot \varpi(\mathbb{w}, \xi)$. Thus

$$\varpi(\mathcal{N}_{\kappa_i}(\mathbb{w}), \xi)^{\beta/\alpha} \leq 2^{\beta/\alpha-1}\left((\tilde{c}_i^*)^{\beta/\alpha} \cdot \xi^{\beta/\alpha} + \left(\mathrm{b}_{\rho_i}^*\right)^{\beta/\alpha} \cdot \varpi(\mathbb{w}, \xi)^{\beta/\alpha}\right),$$

which implies that the inequality (3.6) is verified by

$$\mathrm{a}_{\rho_i} = 2^{\beta/\alpha-1}(\tilde{c}_i^*)^{\beta/\alpha} \cdot \xi^{\beta/\alpha-\beta-1}, \qquad \mathrm{b}_{\rho_i} = 2^{\beta/\alpha-1}(\mathrm{b}_{\rho_i}^*)^{\beta/\alpha}.$$

Then Theorem 3.2 implies that the BVP in (1.1) and (1.2) has at least one solution $\mathbb{w} \in B_\ell(\mathcal{J}_{\alpha,\beta})$. Now, let $\mathbb{w}$ and $\grave{\mathbb{w}}$ be any two different solutions of Eq (3.5), we get

$$\begin{aligned}
\left|\left|\left|\Theta(\mathbb{w}) - \Theta(\grave{\mathbb{w}})\right|\right|\right|_{\alpha,\beta} &= \Big|\Big|\Big|\Theta_1^{(1)}(\mathbb{w}) + \Theta_1^{(2)}(\mathbb{w}) \cdot \Theta_2(\mathbb{w}) \\
&\quad - \Theta_1^{(1)}(\grave{\mathbb{w}}) - \Theta_1^{(2)}(\grave{\mathbb{w}}) \cdot \Theta_2(\grave{\mathbb{w}})\Big|\Big|\Big|_{\alpha,\beta} \\
&\leq \left|\left|\left|\Theta_1^{(1)}(\mathbb{w}) - \Theta_1^{(1)}(\grave{\mathbb{w}})\right|\right|\right|_{\alpha,\beta} \\
&\quad + \left|\left|\left|(\Theta_1^{(2)}(\mathbb{w}) - \Theta_1^{(2)}(\grave{\mathbb{w}})) \cdot \Theta_2(\mathbb{w})\right|\right|\right|_{\alpha,\beta} \\
&\quad + \left|\left|\left|\Theta_1^{(2)}(\grave{\mathbb{w}}) \cdot (\Theta_2(\mathbb{w}) - \Theta_2(\grave{\mathbb{w}}))\right|\right|\right|_{\alpha,\beta} \\
&\leq \left|\left|\left|\Theta_1^{(1)}(\mathbb{w}) - \Theta_1^{(1)}(\grave{\mathbb{w}})\right|\right|\right|_{\alpha,\beta} \\
&\quad + \left[1 + 2^{2-\alpha/\beta}\right] \left|\left|\left|\Theta_1^{(2)}(\mathbb{w}) - \Theta_1^{(2)}(\grave{\mathbb{w}})\right|\right|\right|_{\alpha,\beta} \cdot \left|\left|\left|\Theta_2(\mathbb{w})\right|\right|\right|_{\alpha,\beta} \\
&\quad + \left[1 + 2^{2-\alpha/\beta}\right] \left|\left|\left|\Theta_1^{(2)}(\grave{\mathbb{w}})\right|\right|\right|_{\alpha,\beta} \cdot \left|\left|\left|\Theta_2(\mathbb{w}) - \Theta_2(\grave{\mathbb{w}})\right|\right|\right|_{\alpha,\beta}.
\end{aligned} \tag{3.26}$$

To estimate the inequality above, we have the following estimations:

$$\begin{aligned}
\left|\left|\left|\Theta_1^{(1)}(x) - \Theta_1^{(1)}(z)\right|\right|\right|_{\alpha,\beta} &\leq \left|\left|\left|\mathcal{N}_{\kappa_1} \circ {}^{\mathrm{RL}}\mathcal{I}^{\tau_2}(\mathbb{w}) - \mathcal{N}_{\kappa_1} \circ {}^{\mathrm{RL}}\mathcal{I}^{\tau_2}(\grave{\mathbb{w}})\right|\right|\right|_{\alpha,\beta} \\
&= \left\|\mathcal{N}_{\kappa_1} \circ {}^{\mathrm{RL}}\mathcal{I}^{\tau_2}(\mathbb{w}) - \mathcal{N}_{\kappa_1} \circ {}^{\mathrm{RL}}\mathcal{I}^{\tau_2}(\grave{\mathbb{w}})\right\|_C \\
&\quad + \mathcal{J}_{\alpha,\beta}\left(\mathcal{N}_{\kappa_1} \circ {}^{\mathrm{RL}}\mathcal{I}^{\tau_2}(\mathbb{w}) - \mathcal{N}_{\kappa_1} \circ {}^{\mathrm{RL}}\mathcal{I}^{\tau_2}(\grave{\mathbb{w}}), \mathrm{E}_0\right)^{\alpha/\beta}.
\end{aligned} \tag{3.27}$$

Therefore

$$\left\|\mathcal{N}_{\kappa_1} \circ {}^{\mathrm{RL}}\mathcal{I}^{\tau_2}(\mathbb{w}) - \mathcal{N}_{\kappa_1} \circ {}^{\mathrm{RL}}\mathcal{I}^{\tau_2}(\grave{\mathbb{w}})\right\|_C \leq \tfrac{\tilde{c}_1}{\Gamma(\tau_2)} \int_0^r \tfrac{\|\mathbb{w}-\grave{\mathbb{w}}\|_C}{(r-s)^{1-\tau_2}}\,\mathrm{d}s = \tfrac{\tilde{c}_1\|\mathbb{w}-\grave{\mathbb{w}}\|_C}{\Gamma(\tau_2+1)}. \tag{3.28}$$

Moreover, recalling the inequality 3.22, we obtain

$$\begin{aligned}
&\mathcal{J}_{\alpha,\beta}(\mathcal{N}_{\kappa_1} \circ {}^{\mathrm{RL}}\mathcal{I}^{\tau_2}(\mathbb{w}) - \mathcal{N}_{\kappa_1} \circ {}^{\mathrm{RL}}\mathcal{I}^{\tau_2}(\grave{\mathbb{w}}), \mathrm{E}_0) \\
&\qquad = \int_{\mathrm{E}_0} \xi^{-(\beta+1)}\varpi\Big((\mathcal{N}_{\kappa_1} \circ {}^{\mathrm{RL}}\mathcal{I}^{\tau_2}(\mathbb{w}) - \mathcal{N}_{\kappa_1} \circ {}^{\mathrm{RL}}\mathcal{I}^{\tau_2}(\grave{\mathbb{w}})), \xi\Big)^{\beta/\alpha}\,\mathrm{d}\xi \\
&\qquad \leq \int_{\mathrm{E}_0} \xi^{-(\beta+1)}\left[\left(\mathrm{b}_{\rho_1}^*\right)^{\beta/\alpha} \cdot \varpi\left({}^{\mathrm{RL}}\mathcal{I}^{\tau_2}(\mathbb{w} - \grave{\mathbb{w}}), \xi\right)^{\beta/\alpha}\right]\mathrm{d}\xi \\
&\qquad \leq \left(\mathrm{b}_{\rho_1}^*\right)^{\beta/\alpha} \cdot 2^{\beta/\alpha-1} \int_{\mathrm{E}_0} \xi^{-(\beta+1)}\left[\tfrac{\varpi(\mathbb{w}-\grave{\mathbb{w}},\xi)^{\beta/\alpha}+\xi^{\beta\tau_2/\alpha}\|\mathbb{w}-\grave{\mathbb{w}}\|_C^{\beta/\alpha}}{\Gamma^{\beta/\alpha}(\tau_2+1)}\right]\mathrm{d}\xi \\
&\qquad \leq \tfrac{\left(\mathrm{b}_{\rho_1}^*\right)^{\beta/\alpha}\cdot 2^{\beta/\alpha-1}}{\Gamma^{\beta/\alpha}(\tau_2+1)}\left[\mathcal{J}_{\alpha,\beta}(\mathbb{w} - \grave{\mathbb{w}}, \mathrm{E}_0) + \|\mathbb{w} - \grave{\mathbb{w}}\|_C^{\beta/\alpha}\left[\tfrac{\alpha}{\beta(\tau_2-\alpha)}\right]\right].
\end{aligned} \tag{3.29}$$

By substituting from (3.28) and (3.29) in (3.27), we have

$$\begin{aligned}\left\|\left|\Theta_1^{(1)}(\mathbb{w}) - \Theta_1^{(1)}(\grave{\mathbb{w}})\right|\right\|_{\alpha,\beta} \leq \tfrac{\tilde{c}_1\|\mathbb{w}-\grave{\mathbb{w}}\|_C}{\Gamma(\tau_2+1)} + \tfrac{\left(\mathrm{b}^*_{\rho_1}\right)\cdot 2^{1-\alpha/\beta}}{\Gamma(\tau_2+1)}\Big[&\mathcal{J}_{\alpha,\beta}\left(\mathbb{w} - \grave{\mathbb{w}}, \mathrm{E}_0\right)^{\alpha/\beta} \\ &+ \|\mathbb{w} - \grave{\mathbb{w}}\|_C\left[\tfrac{\alpha}{\beta(\tau_2-\alpha)}\right]^{\alpha/\beta}\Big] \leq \hat{c}^*_{11}\left\||\mathbb{w} - \grave{\mathbb{w}}|\right\|_{\alpha,\beta}.\end{aligned} \tag{3.30}$$

Similarly, we have

$$\begin{aligned}\left\||\Theta_1^{(2)}(\mathbb{w}) - \Theta_1^{(2)}(\grave{\mathbb{w}})|\right\|_{\alpha,\beta} &\leq \tilde{c}_2\|\mathbb{w} - \grave{\mathbb{w}}\|_C + \left(\mathrm{b}^*_{\rho_2}\right)\cdot\mathcal{J}_{\alpha,\beta}(\mathbb{w} - \grave{\mathbb{w}}, \mathrm{E}_0)^{\alpha/\beta} \\ &\leq \hat{c}^*_{22}\left\||\mathbb{w} - \grave{\mathbb{w}}|\right\|_{\alpha,\beta}.\end{aligned} \tag{3.31}$$

Moreover, applying Corollary 2.9, we obtain

$$\begin{aligned}\left\||\Theta_2(\mathbb{w}) - \Theta_2(\grave{\mathbb{w}})|\right\|_{\alpha,\beta} &\leq \left\|\left|{}^{\mathrm{RL}}\mathcal{I}^{\tau_1}\left(\mathcal{N}_{K_3}(\mathbb{w}) - \mathcal{N}_{K_3}(\grave{\mathbb{w}})\right)\right|\right\|_{\alpha,\beta} \\ &\quad+ \left\|\left|{}^{\mathrm{RL}}\mathcal{I}^{\tau_1}\left(\mathcal{N}_{K_3}(\mathbb{w}) - \mathcal{N}_{K_3}(\grave{\mathbb{w}})\right)(\eta)\right|\right\|_{\alpha,\beta} \\ &\quad+ \lambda\left\|\left|\mathbb{I}\left(\mathcal{N}_{K_3}(\mathbb{w}) - \mathcal{N}_{K_3}(\grave{\mathbb{w}})\right)(1)\right|\right\|_{\alpha,\beta} \\ &= \left\|{}^{\mathrm{RL}}\mathcal{I}^{\tau_1}\left(\mathcal{N}_{K_3}(\mathbb{w}) - \mathcal{N}_{K_3}(\grave{\mathbb{w}})\right)\right\|_C \\ &\quad+ \left\|{}^{\mathrm{RL}}\mathcal{I}^{\tau_1}\left(\mathcal{N}_{K_3}(\mathbb{w}) - \mathcal{N}_{K_3}(\grave{\mathbb{w}})\right)(\eta)\right\|_C \\ &\quad+ \lambda\left\|\mathbb{I}\left(\mathcal{N}_{K_3}(\mathbb{w}) - \mathcal{N}_{K_3}(\grave{\mathbb{w}})\right)(1)\right\|_C \\ &\quad+ \mathcal{J}_{\alpha,\beta}\left({}^{\mathrm{RL}}\mathcal{I}^{\tau_1}\left(\mathcal{N}_{K_3}(\mathbb{w}) - \mathcal{N}_{K_3}(\grave{\mathbb{w}})\right), \mathrm{E}_0\right)^{\alpha/\beta} \\ &\quad+ \mathcal{J}_{\alpha,\beta}\left({}^{\mathrm{RL}}\mathcal{I}^{\tau_1}\left(\mathcal{N}_{K_3}(\mathbb{w}) - \mathcal{N}_{K_3}(\grave{\mathbb{w}})\right)(\eta), \mathrm{E}_0\right)^{\alpha/\beta} \\ &\quad+ \lambda\mathcal{J}_{\alpha,\beta}\left(\mathbb{I}\left(\mathcal{N}_{K_3}(\mathbb{w}) - \mathcal{N}_{K_3}(\grave{\mathbb{w}})\right)(1), \mathrm{E}_0\right)^{\alpha/\beta},\end{aligned} \tag{3.32}$$

where

$$\begin{aligned}\left\|\Theta_2(\mathbb{w}) - \Theta_2(\grave{\mathbb{w}})\right\|_C &\leq \left\|{}^{\mathrm{RL}}\mathcal{I}^{\tau_1}(\mathcal{N}_{K_3}(\mathbb{w}) - \mathcal{N}_{K_3}(\grave{\mathbb{w}}))\right\|_C \\ &\quad+ \left\|{}^{\mathrm{RL}}\mathcal{I}^{\tau_1}(\mathcal{N}_{K_3}(\mathbb{w}) - \mathcal{N}_{K_3}(\grave{\mathbb{w}}))(\eta)\right\|_C \\ &\quad+ \lambda\left\|\mathbb{I}\circ(\mathcal{N}_{K_3}(\mathbb{w}) - \mathcal{N}_{K_3}(\grave{\mathbb{w}}))(1)\right\|_C \\ &= \left\|\int_0^r \tfrac{K_3(s,\mathbb{w}(s)) - K_3(s,\grave{\mathbb{w}}(s))}{\Gamma(\tau_1)(r-s)^{1-\tau_1}}\,\mathrm{d}\right\|_C \\ &\quad+ \left\|\int_0^\eta \tfrac{K_3(s,\mathbb{w}(s)) - K_3(s,\grave{\mathbb{w}}(s))}{\Gamma(\tau_1)(\eta-s)^{1-\tau_1}}\,\mathrm{d}s\right\|_C \\ &\quad+ \lambda\left\|\int_{\mathrm{E}_0} |K_3(s, \mathbb{w}(s)) - K_3(s, \grave{\mathbb{w}}(s))|\,\mathrm{d}s\right\|_C \\ &\leq \tfrac{\tilde{c}_3\left\|\mathbb{w}-\grave{\mathbb{w}}\right\|_C}{\Gamma(\tau_1)}\Big[\int_0^r \tfrac{1}{(r-s)^{1-\tau_1}}\,\mathrm{d}s \\ &\quad+ \int_0^\eta \tfrac{1}{(\eta-s)^{1-\tau_1}}\,\mathrm{d}s + \lambda\Gamma(\tau_1)\int_{\mathrm{E}_0}\mathrm{d}s\Big] \\ &\leq \tfrac{\tilde{c}_3\|\mathbb{w}-\grave{\mathbb{w}}\|_C}{\Gamma(\tau_1+1)}[2 + \lambda\Gamma(\tau_1 + 1)].\end{aligned} \tag{3.33}$$

Moreover

$$\begin{aligned}
\mathcal{J}_{\alpha,\beta}(\Theta_2(\mathbb{w})-\Theta_2(\grave{\mathbb{w}}),\mathrm{E}_0) &\leq \mathcal{J}_{\alpha,\beta}\big({}^{\mathrm{RL}}\mathcal{I}^{\tau_1}\left(\mathcal{N}_{K_3}(\mathbb{w})-\mathcal{N}_{K_3}(\grave{\mathbb{w}})\right),\mathrm{E}_0\big)\\
&\quad+\mathcal{J}_{\alpha,\beta}\big({}^{\mathrm{RL}}\mathcal{I}^{\tau_1}(\mathcal{N}_{K_3}(\mathbb{w})-\mathcal{N}_{K_3}(\grave{\mathbb{w}}))(\eta),\mathrm{E}_0\big)\\
&\quad+\lambda\mathcal{J}_{\alpha,\beta}\left(\mathbb{I}\circ\left(\mathcal{N}_{K_3}(\mathbb{w})-\mathcal{N}_{K_3}(\grave{\mathbb{w}})\right)(1),\mathrm{E}_0\right)\\
&=\int_{\mathrm{E}_0}\xi^{-(\beta+1)}\varpi\left({}^{\mathrm{RL}}\mathcal{I}^{\tau_1}\left(\mathcal{N}_{K_3}(\mathbb{w})-\mathcal{N}_{K_3}(\grave{\mathbb{w}})\right),\xi\right)^{\beta/\alpha}\,\mathrm{d}\xi\\
&\leq 2^{\beta/\alpha-1}\int_{\mathrm{E}_0}\tfrac{\xi^{-(\beta+1)}}{\Gamma^{\beta/\alpha}(\tau_1+1)}\Big[\varpi\left(\mathcal{N}_{K_3}(\mathbb{w})-\mathcal{N}_{K_3}(\grave{\mathbb{w}}),\xi\right)^{\beta/\alpha}\\
&\quad+(\tau_1\xi)^{\beta/\alpha}\|\mathcal{N}_{K_3}(\mathbb{w})-\mathcal{N}_{K_3}(\grave{\mathbb{w}})\|_C^{\beta/\alpha}\Big]\mathrm{d}\xi\\
&\leq\tfrac{2^{\beta/\alpha-1}}{\Gamma^{\beta/\alpha}(\tau_1+1)}\Big[\int_{\mathrm{E}_0}\xi^{-(\beta+1)}\left(\mathrm{b}^*_{\rho_3}\right)^{\beta/\alpha}\varpi(\mathbb{w}-\grave{\mathbb{w}},\xi)^{\beta/\alpha}\,\mathrm{d}\xi\\
&\quad+(\tau_1\tilde{c}_3)^{\beta/\alpha}\|\mathbb{w}-\grave{\mathbb{w}}\|_C^{\beta/\alpha}\int_{\mathrm{E}_0}\xi^{-(\beta+1)}\xi^{\beta/\alpha}\,\mathrm{d}\xi\Big]\\
&\leq\tfrac{2^{\beta/\alpha-1}}{\Gamma^{\beta/\alpha}(\tau_1+1)}\Big[\left(b^*_{\rho_3}\right)^{\beta/\alpha}\mathcal{J}_{\alpha,\beta}(\mathbb{w}-\grave{\mathbb{w}},\mathrm{E}_0)\\
&\quad+\tfrac{\alpha\cdot(\tau_1\tilde{c}_3)^{\beta/\alpha}}{\beta(1-\alpha)}\|\mathbb{w}-\grave{\mathbb{w}}\|_C^{\beta/\alpha}\Big],
\end{aligned}\tag{3.34}$$

where

$$\begin{aligned}
&\varpi\Big({}^{\mathrm{RL}}\mathcal{I}^{\tau_1}(\mathcal{N}_{K_3}(\mathbb{w})-\mathcal{N}_{K_3}(\grave{\mathbb{w}}))(\eta),\xi\Big)=0,\\
&\qquad\Longrightarrow\mathcal{J}_{\alpha,\beta}\Big({}^{\mathrm{RL}}\mathcal{I}^{\tau_1}(\mathcal{N}_{K_3}(\mathbb{w})-\mathcal{N}_{K_3}(\grave{\mathbb{w}}))(\eta),\mathrm{E}_0\Big)=0,\\
&\varpi\Big(\mathbb{I}\circ(\mathcal{N}_{K_3}(\mathbb{w})-\mathcal{N}_{K_3}(\grave{\mathbb{w}}))(1),\xi\Big)=0,\\
&\qquad\Longrightarrow\mathcal{J}_{\alpha,\beta}\Big(\mathbb{I}\circ(\mathcal{N}_{K_3}(\mathbb{w})-\mathcal{N}_{K_3}(\grave{\mathbb{w}}))(1),\mathrm{E}_0\Big)=0.
\end{aligned}$$

By substituting from (3.33) and (3.34) in (3.32), we have

$$\begin{aligned}
\left\|\left|\Theta_2(\mathbb{w})-\Theta_2(\grave{\mathbb{w}})\right|\right\|_{\alpha,\beta} &\leq\tfrac{\tilde{c}_3\|\mathbb{w}-\grave{\mathbb{w}}\|_C}{\Gamma(\tau_1+1)}[2+\lambda\Gamma(\tau_1+1)]\\
&\quad+\tfrac{2^{1-\alpha/\beta}}{\Gamma(\tau_1+1)}\Big[\left(\mathrm{b}^*_{\rho_3}\right)\cdot\mathcal{J}_{\alpha,\beta}(\mathbb{w}-\grave{\mathbb{w}},\mathrm{E}_0)^{\alpha/\beta}\\
&\quad+\left[\tfrac{\alpha}{\beta(1-\alpha)}\right]^{\alpha/\beta}\tau_1\tilde{c}_3\cdot\|\mathbb{w}-\grave{\mathbb{w}}\|_C\Big]\\
&\leq\hat{c}^*_{33}\left\|\left|\mathbb{w}-\grave{\mathbb{w}}\right|\right\|_{\alpha,\beta}.
\end{aligned}\tag{3.35}$$

Now, recalling (3.13), and (3.17) and by substituting from (3.30), (3.31), and (3.35) in (3.26), we get

$$\begin{aligned}
\left\|\left|\Theta(\mathbb{w})-\Theta(\grave{\mathbb{w}})\right|\right\|_{\alpha,\beta} &\leq\hat{c}^*_{11}\left\|\left|\mathbb{w}-\grave{\mathbb{w}}\right|\right\|_{\alpha,\beta}+\left[1+2^{2-\alpha/\beta}\right]\hat{c}^*_{22}\left\|\left|\mathbb{w}-\grave{\mathbb{w}}\right|\right\|_{\alpha,\beta}\\
&\quad\times\left[\tfrac{2^{1-\alpha/\beta}}{\Gamma(\tau_1+1)}\left\|\mathrm{a}_{\rho_3}\right\|_{L_1}^{\alpha/\beta}+\hat{c}_{33}\widehat{K}_3+\hat{c}_{33}\cdot\ell\right]\\
&\quad+\left[1+2^{2-\alpha/\beta}\right]\Big[\widehat{K}_2+2^{1-\alpha/\beta}\left\|\mathrm{a}_{\rho_2}\right\|_{L_1}^{\alpha/\beta}\\
&\quad+\hat{c}_{22}\cdot\ell\Big]\hat{c}^*_{33}\left\|\left|\mathbb{w}-\grave{\mathbb{w}}\right|\right\|_{\alpha,\beta}=\hat{c}_{44}\left\|\left|\mathbb{w}-\grave{\mathbb{w}}\right|\right\|_{\alpha,\beta},
\end{aligned}$$

where

$$\begin{aligned}\hat{c}_{44} = \Big[\hat{c}^*_{11} + \hat{c}^*_{22}\big[1 + 2^{2-\alpha/\beta}\big]\Big[\tfrac{2^{1-\alpha/\beta}}{\Gamma(\tau_1+1)}\left\|\mathrm{a}_{\rho_3}\right\|^{\alpha/\beta}_{L_1} + \hat{c}_{33}\widehat{K}_3 + \hat{c}_{33}\cdot\ell\Big] \\ + \hat{c}^*_{33}\big[1 + 2^{2-\alpha/\beta}\big]\Big[\widehat{K}_2 + 2^{1-\alpha/\beta}\left\|\mathrm{a}_{\rho_2}\right\|^{\alpha/\beta}_{L_1} + \hat{c}_{22}\cdot\ell\Big]\Big] < 1.\end{aligned}$$

This wraps up the proof.

4. Application

Now, we conclude with an example that verifies the proposed results.

Remark 4.1. *The chosen parameters are selected to reflect a practical thermostat system in which a heated rod is monitored by an internal sensor located at position η, while the controller's response is regulated by the feedback parameter λ. The fractional orders τ_1 and τ_2 represent memory effects associated with heat diffusion and the accumulated thermal response, respectively. Such choices allow the example to illustrate not only the validity of the theoretical assumptions but also the practical relevance of the proposed model.*

Example 1. *Consider the following hybrid BVP:*

$$ {}^{\mathrm{C}}\mathcal{D}^{\tau_1}\left(\frac{\mathbb{w}(r)-\frac{r}{50}-\sin\left(\frac{1}{100}\left(r+{}^{\mathrm{RL}}\mathcal{I}^{\tau_2}\mathbb{w}(r)\right)\right)}{\cos\left(\frac{t+\mathbb{w}(r)}{100}\right)}\right) + \cos\left(\frac{1}{100}\left(r+\mathbb{w}(r)\right)\right) = 0, \tag{4.1}$$

for $r \in \mathrm{E}_0 = [0, 1]$, with the three-point hybrid boundary value conditions

$$\begin{cases} \mathbb{D}\left(\frac{\mathbb{w}(r)-\frac{r}{50}-\sin\left(\frac{1}{100}\left(r+{}^{\mathrm{RL}}\mathcal{I}^{\tau_2}\mathbb{w}(r)\right)\right)}{\cos\left(\frac{1}{100}(r+\mathbb{w}(r))\right)}\right)\Bigg|_{r=0} = 0, \qquad \tau_2 \in \mathrm{E}_0 \setminus \{0\},\ \mathbb{D} = \frac{\mathrm{d}}{\mathrm{d}r} \\ \lambda\,{}^{\mathrm{C}}\mathcal{D}^{\tau_1-1}\left(\frac{\mathbb{w}(r)-\frac{r}{50}-\sin\left(\frac{1}{100}\left(r+{}^{\mathrm{RL}}\mathcal{I}^{\tau_2}\mathbb{w}(r)\right)\right)}{\cos\left(\frac{1}{100}(r+\mathbb{w}(r))\right)}\right)\Bigg|_{r=1} \\ \qquad +\left(\frac{\mathbb{w}(r)-\frac{r}{50}-\sin\left(\frac{1}{100}\left(r+{}^{\mathrm{RL}}\mathcal{I}^{\tau_2}\mathbb{w}(r)\right)\right)}{\cos\left(\frac{1}{100}(r+\mathbb{w}(r))\right)}\right)\Bigg|_{r=\eta} = 0, \end{cases} \tag{4.2}$$

and three cases as follows:

$$\begin{aligned} &\mathit{Case}\,1: \quad 1 < \tau_1 < 2,\ \tau_2 = \tfrac{11}{12} \in (0,1],\ \alpha = \tfrac{10}{11} \in (0,1),\ \beta = \tau_1 + 0.1 > \tau_1 > \alpha; \\ &\mathit{Case}\,2: \quad 0 < \tau_2 \le 1,\ \alpha = \tfrac{10}{11} \in (0,1),\ \tau_1 = \tfrac{9}{7} \in (1,2),\ \beta = \tfrac{97}{70} > \tau_1 > \alpha; \\ &\mathit{Case}\,3: \quad 0.3 < \alpha < 0.95,\ \tau_1 = \tfrac{9}{7} \in (1,2),\ \tau_2 = \tfrac{1}{4} \in (0,1],\ \beta = \tfrac{97}{70} > \tau_1 > \alpha. \end{aligned}$$

We can note that $g(r) = \frac{r}{50}$ with $\left\|\|g\|\right\|_{\alpha,\beta} = \frac{1}{50} + \frac{1}{50}\left(\frac{\alpha}{\beta(1-\alpha)}\right)^{\alpha/\beta}$, $K_1(r,\mathbb{w}) = \sin\left(\frac{r+\mathbb{w}}{100}\right)$ with $\tilde{c}_1 = \tilde{c}^*_1 = \frac{1}{100}$ and $B_1 = \frac{1}{10^4}$, where

$$\begin{aligned} &\left|K_1(r,\mathbb{w}) - K_1(\grave{r},\mathbb{w})\right| = \left|\sin\left(\tfrac{r+\mathbb{w}}{100}\right) - \sin\left(\tfrac{\grave{r}+\mathbb{w}}{100}\right)\right| \le \left|\tfrac{r+\mathbb{w}}{100} - \tfrac{\grave{r}+\mathbb{w}}{100}\right| = \tfrac{1}{100}|r-\grave{r}|, \\ &\left|K_1(r,\mathbb{w}) - K_1(r,\grave{\mathbb{w}})\right| = \left|\sin\left(\tfrac{r+\mathbb{w}}{100}\right) - \sin\left(\tfrac{r+\grave{\mathbb{w}}}{100}\right)\right| \le \left|\tfrac{r+\mathbb{w}}{100} - \tfrac{r+\grave{\mathbb{w}}}{100}\right| = \tfrac{1}{100}|\mathbb{w}-\grave{\mathbb{w}}|, \\ &\left|\partial_2 K_1(r,\mathbb{w}) - \partial_2 K_1(r,\grave{\mathbb{w}})\right| \le \tfrac{1}{10^4}|\mathbb{w}-\grave{\mathbb{w}}|, \end{aligned}$$

and

$$\begin{aligned}\varpi(K_1, \xi, \varpi(\mathtt{w}, \xi))^{\beta/\alpha} &= \left|\sin\left(\tfrac{r+\mathtt{w}(r)}{100}\right) - \sin\left(\tfrac{s+\mathtt{w}(s)}{100}\right)\right|^{\beta/\alpha} \\ &\leq \left|\tfrac{r+\mathtt{w}(r)}{100} - \tfrac{s+\mathtt{w}(s)}{100}\right|^{\beta/\alpha} \\ &\leq \tfrac{100^{\beta/\alpha-1}}{100^{\beta/\alpha}}|r-s|^{\beta/\alpha} + \tfrac{100^{\beta/\alpha-1}}{100^{\beta/\alpha}}|\mathtt{w}(r) - \mathtt{w}(s)|^{\beta/\alpha} \\ &\leq \tfrac{1}{100}\xi^{\beta/\alpha} + \tfrac{1}{100}\varpi(\mathtt{w}, \xi)^{\beta/\alpha},\end{aligned}$$

such that $\mathrm{a}_{\rho_1}(\xi) = \frac{1}{100}\xi^{\beta/\alpha-\beta-1}$, *and* $\mathrm{b}_{\rho_1} = \frac{1}{100}$. $K_2(r, \mathtt{w}) = K_3(r, \mathtt{w}) = \cos\left(\frac{r+\mathtt{w}}{100}\right)$ *with* $\tilde{c}_2 = \tilde{c}_3 = \tilde{c}_2^* = \tilde{c}_3^* = \frac{1}{100}$ *and* $B_2 = B_3 = \frac{1}{10^4}$, *such that*

$$\varpi(K_2, \xi, \varpi(\mathtt{w}, \xi))^{\beta/\alpha} = \varpi(K_3, \xi, \varpi(\mathtt{w}, \xi))^{\beta/\alpha} \leq \frac{1}{100}\xi^{\beta/\alpha} + \frac{1}{100}\varpi(\mathtt{w}, \xi)^{\beta/\alpha},$$

with $\mathrm{a}_{\rho_2}(\xi) = \mathrm{a}_{\rho_3}(\xi) = \frac{1}{100}\xi^{\beta/\alpha-\beta-1}$, *and* $\mathrm{b}_{\rho_2} = \mathrm{b}_{\rho_3} = \frac{1}{100}$. *Next, for different predicted cases, we review the challenges in order to confirm the existence of a solution. Therefore, Assumptions* (i) *and* (ii) *are satisfied. The remaining assumptions are examined in the following three cases.*

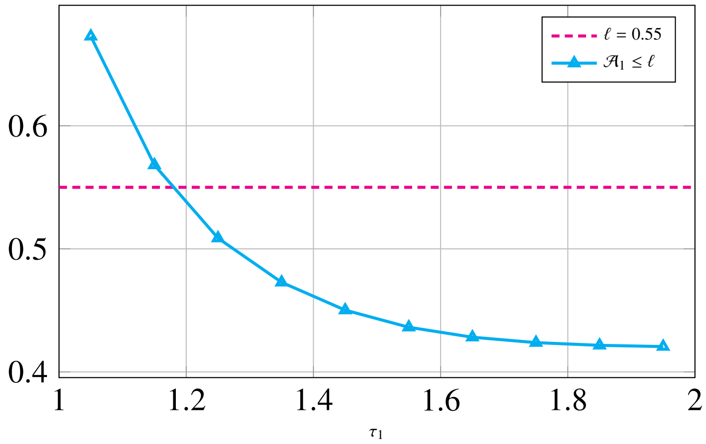

Figure 1. Two-dimensional plots of $\mathcal{A}_1$ for BVP (4.1) in Case 1.

Case 1: *From the given data in this case of BVP* (4.1), *for* $\lambda = 0.15$, *and* $\ell = 0.55$, *we have*

$$\begin{aligned}\mathcal{A}_1 = \big\||g|\big\|_{10/11,\beta} &+ \widehat{K}_1 + 2^{1-10/11\beta}\|\mathrm{a}_{\rho_1}\|_{L_1}^{10/11\beta} + \hat{c}_{11}\cdot\ell \\ &+ [1 + 2^{2-10/11\beta}]\Big[\widehat{K}_2 + 2^{1-10/11\beta}\left\|\mathrm{a}_{\rho_2}\right\|_{L_1}^{10/11\beta} + \hat{c}_{22}\cdot\ell\Big] \\ &\times\left[\tfrac{2^{1-10/11\beta}}{\Gamma(\tau_1+1)}\left\|\mathrm{a}_{\rho_3}\right\|_{L_1}^{10/11\beta} + \hat{c}_{33}\widehat{K}_3 + 0.55\hat{c}_{33}\right] \simeq 0.5087 \leq \ell,\end{aligned}$$

where from the relations in (3.7), *we have*

$$\begin{aligned}\hat{c}_{11} &= \max\left\{\frac{\left(\frac{1}{100}\right)^{10/11\beta}2^{1-10/11\beta}}{\Gamma(\tau_2+1)}, \frac{\frac{1}{100}}{\Gamma\left(\frac{23}{12}\right)} + \left[\frac{\frac{1}{110}2^{11\beta/10-1}}{\beta\left(\frac{1}{120}\right)\Gamma^{10/11\beta}\left(\frac{23}{12}\right)}\right]^{10/11\beta}\right\} \simeq 1.2462, \\ \hat{c}_{22} &= \max\left\{\frac{1}{100}, 2^{1-10/11\beta}\left(\frac{1}{100}\right)^{10/11\beta}\right\} \simeq 0.1908,\end{aligned}$$

$$\hat{c}_{33} = \max\left\{\frac{\left(\frac{1}{100}\right)^{10/11\beta}2^{1-10/11\beta}}{\Gamma(\tau_1+1)}, \frac{\frac{1}{100}}{\Gamma(\tau_1+1)}\left[2 + 0.15\Gamma(\tau_1 + 1) + \left[\frac{\frac{10}{11}\tau_1^{\frac{11\beta}{10}}2^{11\beta/10-1}}{\beta\left(1-\frac{10}{11}\right)}\right]^{10/11\beta}\right]\right\} \simeq 0.0999.$$

We plot the changes in $\mathcal{A}_1$ *in Figure 1 for the range of* $1 < \tau_1 < 2$*. These estimated results are shown in Table 1. Therefore, Assumption* (iii) *is verified.*

Further, the constant Σ *which is defined by* (3.8) *is less than* 1 *as follows:*

$$\begin{aligned}\Sigma = {} & \frac{\frac{1}{100}2^{33\beta/10-2}}{\Gamma^{10\beta/11}\left(\frac{23}{12}\right)} + \left[\frac{\frac{11}{2000}+\widehat{K}_3}{\Gamma(\tau_1+1)}[2 + 0.15\Gamma(\tau_1 + 1)]\right]^{11\beta/10} \frac{1}{100}2^{11\beta/10} \\ & + \left[\frac{11}{2000} + \widehat{K}_2\right]^{11\beta/10} \frac{\frac{1}{100}2^{11\beta/5-1}}{\Gamma^{11\beta/10}(\tau_1+1)} \simeq 0.3921 < 1.\end{aligned}$$

Therefore, Assumption (iv) *is verified. The results thus obtained are presented in Table 1, and we also plot the changes in* Σ *in Figure 2 for the range of* $1 < \tau_1 < 2$*.*

Table 1. Estimated results of Theorem 3.2 for BVP (4.1) for $1 < \tau_1 < 2$ in Case 1.

τ_1	$1 \xrightarrow{\tau_1} 2$				
	$\hat{c}_{11}$	$\hat{c}_{22}$	$\hat{c}_{33}$	$\mathcal{A}_1 < \ell_0$	$\Sigma < 1$
1.05	$\underline{1.2462}$	0.0303	0.0653	0.6727	0.1266
1.15	1.2224	0.0424	0.0453	0.5681	0.1448
1.25	1.2076	0.0564	0.0498	$\underline{0.5087}$	0.1645
1.35	1.1991	0.0722	0.0600	0.4728	0.1856
1.45	1.1950	0.0894	0.0696	0.4502	0.2086
1.55	1.1942	0.1080	0.0784	0.4363	0.2341
1.65	1.1957	0.1276	0.0859	0.4282	0.2633
1.75	1.1989	0.1480	0.0920	0.4238	0.2977
1.85	1.2035	0.1692	0.0967	0.4216	0.3397
1.95	1.2090	$\underline{0.1908}$	$\underline{0.0999}$	0.4206	$\underline{0.3921}$

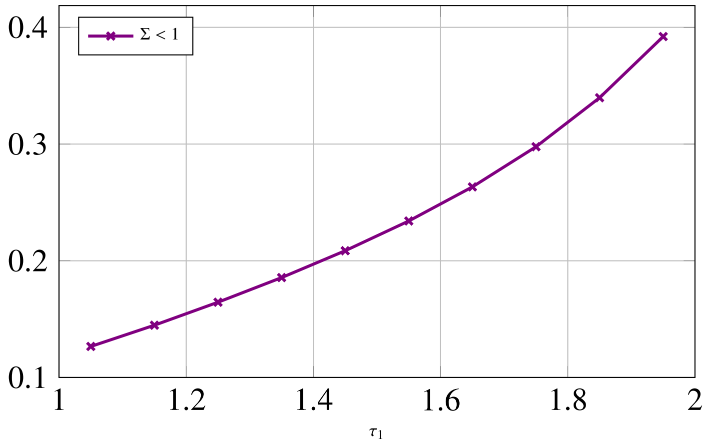

Figure 2. Two-dimensional plots of Σ for BVP (4.1) in Case 1.

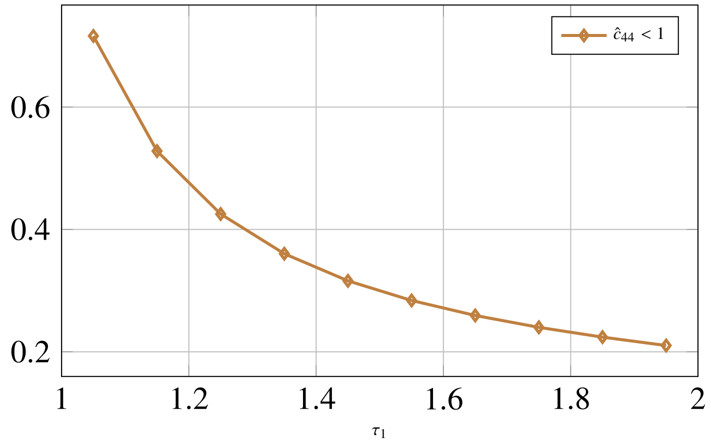

Figure 3. Two-dimensional plots of $\hat{c}_{44}$ for BVP (4.1) in Case 1.

Table 2. Estimated results of Theorem 3.3 for BVP (4.1) for $1 < \tau_1 < 2$ in Case 1.

τ_1	$1 \xrightarrow{\tau_1} 2$			
	$\hat{c}^*_{11}$	$\hat{c}^*_{22}$	$\hat{c}^*_{33}$	$\hat{c}_{44} < 1$
1.05	$\underline{0.4813}$	0.0100	$\underline{0.0653}$	$\underline{0.7162}$
1.15	0.3555	0.0100	0.0453	0.5281
1.25	0.2764	0.0100	0.0367	0.4252
1.35	0.2236	0.0100	0.0315	0.3603
1.45	0.1868	0.0100	0.0279	0.3160
1.55	0.1600	0.0100	0.0251	0.2839
1.65	0.1400	0.0100	0.0228	0.2594
1.75	0.1246	0.0100	0.0207	0.2400
1.85	0.1125	0.0100	0.0189	0.2240
1.95	0.1027	0.0100	0.0172	0.2103

Hence, Assumptions (i) *and* (iv) *hold. Therefore, thanks to Theorem 3.2, BVP* (4.1) *has at least one solution* $\mathbb{w} \in \mathcal{J}_{\alpha,\beta}$ *for Case 1.*

Additionally, using (3.23) *and setting* $\mathrm{b}^*_{\rho_i} = \frac{1}{100}$, $i = 1, 2, 3$, *we obtain*

$$\hat{c}^*_{11} = \max\left\{\frac{\left(\frac{1}{100}\right)2^{1-10/11\beta}}{\Gamma\left(\frac{23}{12}\right)}, \frac{1}{\Gamma\left(\frac{23}{12}\right)}\left[\frac{1}{100} + \left[\frac{\frac{10}{11}\left(\frac{1}{100}\right)^{11\beta/10} 2^{11\beta/10-1}}{\frac{1}{120}\beta}\right]^{10/11\beta}\right]\right\} \simeq 0.4813,$$

$$\hat{c}^*_{22} = \max\left\{\frac{1}{100}, \frac{1}{100}\right\} = 0.01,$$

$$\hat{c}^*_{33} = \max\left\{\frac{\frac{1}{100}2^{1-10/11\beta}}{\Gamma(\tau_1+1)}, \frac{\frac{1}{100}}{\Gamma(\tau_1+1)}\left[2 + 0.55\Gamma(\tau_1+1) + \left[\frac{\frac{10}{11}\tau_1^{\frac{11\beta}{\alpha}} 2^{11\beta/10-1}}{\beta\left(1-\frac{10}{11}\right)}\right]^{10/11\beta}\right]\right\} \simeq 0.0653.$$

For the contraction constant $\hat{c}_{44}$, *we have*

$$\begin{aligned}\hat{c}_{44} = \hat{c}^*_{11} + \hat{c}^*_{22}\left[1 + 2^{2-10/11\beta}\right]\left[\frac{4^{1-10/11\beta}}{\Gamma(\tau_1+1)}\left\|\mathrm{a}_{\rho_3}\right\|^{10/11\beta}_{L_1} + \hat{c}_{33}\widehat{K}_3 + 0.55\hat{c}_{33}\left\|\left|\mathbb{w}\right|\right\|_{10/11,\beta}\right] \\ + \hat{c}^*_{33}\left[1 + 2^{2-10/11\beta}\right]\left[\widehat{K}_2 + 2^{1-10/11\beta}\left\|\mathrm{a}_{\rho_2}\right\|^{10/11\beta}_{L_1} + 0.55\hat{c}_{22}\right] \simeq 0.7162 < 1.\end{aligned}$$

We plot the changes in $\hat{c}_{44}$ *in Figure 3 for the range* $1 < \tau_1 < 2$. *All the results obtained for* $\hat{c}_{11}^*$, $\hat{c}_{22}^*$, $\hat{c}_{33}^*$, *and* $\hat{c}_{44}$ *are given in Table 2.*

Therefore, these numerical results confirm that all assumptions of Theorem 3.3 hold and, consequently, BVP (4.1) *has a unique solution* $\mathbb{w} \in B_\ell(\mathcal{J}_{\alpha,\beta})$ *for Case 1.*

Case 2: *From the given data in the case of BVP* (4.1), *for* $\lambda = 0.15$, *and* $\ell = 0.55$, *we have*

$$\begin{aligned}\mathcal{A}_2 = {} & \left\|\|g\|\right\|_{10/11,97/70} + \widehat{K}_1 + 2^{367/1067}\|\mathrm{a}_{\rho_1}\|_{L_1}^{700/1067} + 0.55\hat{c}_{11} \\ & + [1 + 2^{1434/1067}]\left[\widehat{K}_2 + 2^{367/1067}\left\|\mathrm{a}_{\rho_2}\right\|_{L_1}^{700/1067} + 0.55\hat{c}_{22}\right] \\ & \times\left[\frac{2^{367/1067}}{\Gamma(\frac{17}{7})}\left\|\mathrm{a}_{\rho_3}\right\|_{L_1}^{700/1067} + \hat{c}_{33}\widehat{K}_3 + 0.55\hat{c}_{33}\right] \simeq 0.4498 \leq \ell,\end{aligned}$$

where, from the relations in (3.7), *we have*

$$\begin{aligned}\hat{c}_{11} &= \max\left\{\frac{(\frac{1}{100})^{700/1067}2^{367/1067}}{\Gamma(\tau_2+1)}, \frac{\frac{1}{100}}{\Gamma(\tau_2+1)} + \left[\frac{\frac{1}{110}2^{367/700}}{\frac{97}{70}(\tau_2-\frac{10}{11})\Gamma^{1067/700}(\tau_2+1)}\right]^{700/1067}\right\} \simeq 0.4000, \\ \hat{c}_{22} &= \max\left\{\frac{1}{100}, 2^{367/1067}\left(\frac{1}{100}\right)^{700/1067}\right\} \simeq 0.0619, \\ \hat{c}_{33} &= \max\left\{\frac{(\frac{1}{100})^{700/1067}2^{367/1067}}{\Gamma(\frac{16}{7})}, \frac{\frac{1}{100}}{\Gamma(\frac{16}{7})}\left[2 + 0.15\Gamma\left(\frac{16}{7}\right) + \left[\frac{(\frac{10}{11})(\frac{9}{7})^{1067/700}2^{367/700}}{\frac{97}{770}}\right]^{700/1067}\right]\right\} \simeq 0.0535.\end{aligned}$$

We plot the changes in $\mathcal{A}_2$ *in Figure 4 for the range* $\tau_2 \in (0, 1]$. *These estimated results are shown in Table 3. Therefore, Assumption* (iii) *is verified.*

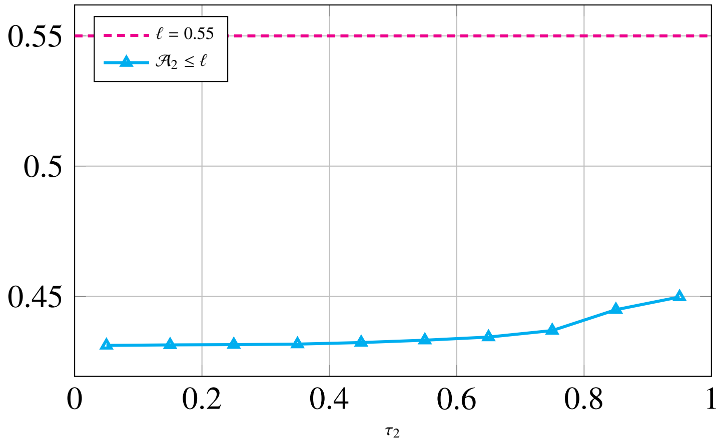

Figure 4. Two-dimensional plots of $\mathcal{A}_2$ for BVP (4.1) in Case 2.

Furthermore, the constant Σ *which is defined by* (3.8) *is less than* 1 *as follows:*

$$\begin{aligned}\Sigma = {} & \frac{\frac{1}{100}2^{157/70}}{\Gamma^{1067/700}(\tau_2+1)} + \left[\frac{\frac{11}{2000}+\widehat{K}_3}{\Gamma(\frac{16}{7})}\left[2 + 0.15\Gamma\left(\frac{16}{7}\right)\right]\right]^{1067/700}\frac{1}{100}2^{1067/700} \\ & + \left[\frac{11}{2000} + \widehat{K}_2\right]^{1067/700}\frac{\frac{1}{100}2^{62/35}}{\Gamma^{1067/700}(\frac{16}{7})} \simeq 0.1706 < 1.\end{aligned}$$

The obtained results are presented in Table 3, and we plot the changes in Σ *in Figure 5 for the range* $\tau_2 \in (0, 1]$. *Hence, Assumptions* (i) *and* (iv) *hold. Therefore, thanks to Theorem 3.2, the BVP* (4.1) *has at least one solution* $\mathbb{w} \in B_\ell(\mathcal{J}_{\alpha,\beta})$ *for Case 2.*

Table 3. Estimated results of Theorem 3.2 for BVP (4.1) for $\tau_2 \in (0, 1]$ in Case 2.

τ_2	$\mathbf{0} \xrightarrow{\tau_2} \mathbf{1}$				
	$\hat{c}_{11}$	$\hat{c}_{22}$	$\hat{c}_{33}$	$\mathscr{A}_2 < \ell_0$	$\Sigma < 1$
0.05	0.0636	0.0619	0.0535	0.4312	$\underline{0.1713}$
0.15	0.0663	0.0619	0.0535	0.4314	0.1754
0.25	0.0683	0.0619	0.0535	0.4315	0.1784
0.35	0.0725	0.0619	0.0535	0.4317	0.1802
0.45	0.0836	0.0619	0.0535	0.4323	0.1809
0.55	0.0986	0.0619	0.0535	0.4332	0.1805
0.65	0.1216	0.0619	0.0535	0.4344	0.1791
0.75	0.1657	0.0619	0.0535	0.4369	0.1769
0.85	0.3125	0.0619	0.0535	0.4449	0.1741
0.95	$\underline{0.4000}$	0.0619	0.0535	$\underline{0.4498}$	0.1706

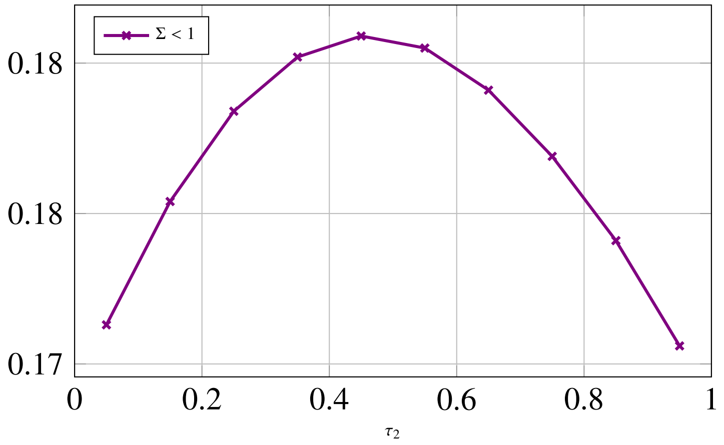

Figure 5. Two-dimensional plots of Σ for BVP (4.1) in Case 2.

Additionally, using (3.23) *and setting* $\mathrm{b}^*_{\rho_i} = \frac{1}{100}$, $i = 1, 2, 3$, *we obtain*

$$\hat{c}^*_{11} = \max\left\{\frac{\frac{1}{100}2^{367/1067}}{\Gamma(\tau_2+1)}, \frac{1}{\Gamma(\tau_2+1)}\left[\frac{1}{100} + \left[\frac{\frac{10}{11}\left(\frac{1}{100}\right)^{1067/700}2^{367/700}}{\frac{97}{70}\left(\tau_2 - \frac{10}{11}\right)}\right]^{700/1067}\right]\right\} \simeq 0.0902,$$

$$\hat{c}^*_{22} = \max\left\{\frac{1}{100}, \frac{1}{100}\right\} \simeq 0.01,$$

$$\hat{c}^*_{33} = \max\left\{\frac{\frac{1}{100}2^{367/1067}}{\Gamma\left(\frac{16}{7}\right)}, \frac{\frac{1}{100}}{\Gamma\left(\frac{16}{7}\right)}\left[2 + 0.15\Gamma\left(\frac{16}{7}\right) + \left[\frac{\left(\frac{10}{11}\right)\left(\frac{9}{7}\right)^{1067/700}2^{367/700}}{\frac{97}{770}}\right]^{700/1067}\right]\right\} \simeq 0.0346,$$

and for the contraction constant $\hat{c}_{44}$, *we have*

$$\hat{c}_{44} = \left[\hat{c}^*_{11} + \hat{c}^*_{22}\left[1 + 2^{1434/1067}\right]\left[\frac{4^{1-367/1067}}{\Gamma\left(\frac{16}{7}\right)}\left\|\mathrm{a}_{\rho_3}\right\|^{700/1067}_{L_1} + \hat{c}_{33}\widehat{K}_3 + \hat{c}_{33}\cdot\ell\right]\right.$$
$$\left. + \hat{c}^*_{33}\left[1 + 2^{1434/1067}\right]\left[\widehat{K}_2 + 2^{367/1067}\left\|\mathrm{a}_{\rho_2}\right\|^{\alpha/\beta}_{C} + \hat{c}_{22}\cdot\ell\right]\right] \simeq 0.2339 < 1.$$

We plot the changes in $\hat{c}_{44}$ *in Figure 6 for the range* $\tau_2 \in (0, 1]$. *All results obtained for* $\hat{c}^*_{11}$, $\hat{c}^*_{22}$, $\hat{c}^*_{33}$, *and* $\hat{c}_{44}$ *are given in Table 4.*

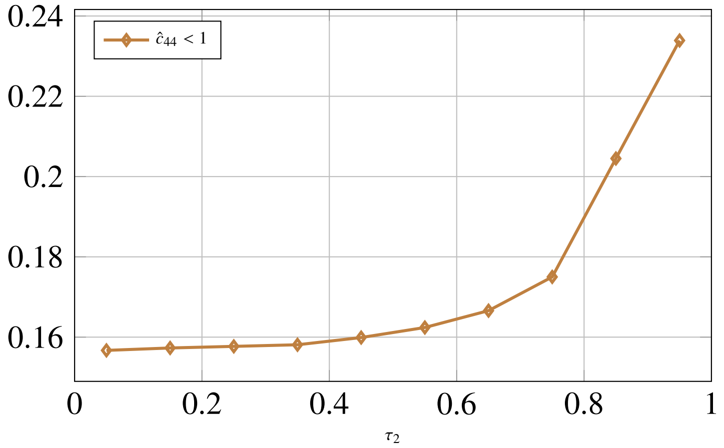

Figure 6. Two-dimensional plots of $\hat{c}_{44}$ for BVP (4.1) in Case 2.

Table 4. Estimated results of Theorem 3.3 for BVP (4.1) for $\tau_2 \in (0, 1]$ in Case 2.

τ_2	$\mathbf{0} \xrightarrow{\tau_2} \mathbf{1}$			
	$\hat{c}^*_{11}$	$\hat{c}^*_{22}$	$\hat{c}^*_{33}$	$\hat{c}_{44} < 1$
0.05	0.0130	0.0100	0.0346	0.1567
0.15	0.0136	0.0100	0.0346	0.1573
0.25	0.0140	0.0100	0.0346	0.1577
0.35	0.0145	0.0100	0.0346	0.1581
0.45	0.0162	0.0100	0.0346	0.1599
0.55	0.0187	0.0100	0.0346	0.1624
0.65	0.0229	0.0100	0.0346	0.1666
0.75	0.0314	0.0100	0.0346	0.1750
0.85	0.0609	0.0100	0.0346	0.2045
0.95	0.0902	0.0100	0.0346	$\underline{0.2339}$

These numerical results confirms that all assumptions of Theorem 3.3 hold and, consequently, BVP (4.1) *has a unique solution* $\mathbb{w} \in B_\ell(\mathcal{J}_{\alpha,\beta})$ *for Case 2.*

Case 3: *From the given data in this case of BVP* (4.1), *for* $\lambda = 0.15$ *and* $\ell = 0.55$, *we have*

$$\begin{aligned}\mathcal{A}_3 = \left|\left|\left|g\right|\right|\right|_{\alpha,97/70} &+ \widehat{K}_1 + 2^{70\alpha/97}\|\mathrm{a}_{\rho_1}\|_{L_1}^{70\alpha/97} + 0.55\hat{c}_{11}\\ &+ [1 + 2^{2-70\alpha/97}]\left[\widehat{K}_2 + 2^{1-70\alpha/97}\left\|\mathrm{a}_{\rho_2}\right\|_{L_1}^{70\alpha/97} + 0.55\hat{c}_{22}\right]\\ &\times\left[\frac{2^{1-70\alpha/97\beta}}{\Gamma(\frac{16}{7})}\left\|\mathrm{a}_{\rho_3}\right\|_{L_1}^{70\alpha/97} + \hat{c}_{33}\widehat{K}_3 + 0.55\hat{c}_{33}\right] \simeq 0.5332 \le \ell,\end{aligned}$$

where, from the relations in (3.7), *we have*

$$\begin{aligned}\hat{c}_{11} &= \max\left\{\frac{\mathrm{b}_{\rho_1}^{70\alpha/97}2^{1-70\alpha/97}}{\Gamma(\tau_2+1)}, \frac{\tilde{c}_1}{\Gamma(\frac{5}{4})} + \left[\frac{\frac{1}{100}\alpha 2^{97/70\alpha-1}}{\frac{97}{70}(\frac{1}{4}-\alpha)\Gamma^{97/70\alpha}(\frac{1}{4}-\alpha)}\right]^{70\alpha/97}\right\} \simeq 0.9710,\\ \hat{c}_{22} &= \max\left\{\frac{1}{100}, 2^{1-70\alpha/97}\left(\frac{1}{100}\right)^{70\alpha/97}\right\} \simeq 0.6351,\end{aligned}$$

$$\hat{c}_{33} = \max\left\{\frac{\left(\frac{1}{100}\right)^{70\alpha/97} 2^{1-70\alpha/97}}{\Gamma\left(\frac{16}{7}\right)}, \frac{\frac{1}{100}}{\Gamma\left(\frac{16}{7}\right)}\left[2 + 0.15\Gamma\left(\frac{16}{7}\right) + \left[\frac{\alpha\left(\frac{9}{7}\right)^{97/70\alpha} 2^{97/70\alpha-1}}{\frac{97}{70}(1-\alpha)}\right]^{\alpha/\beta}\right]\right\} \simeq 0.5490.$$

We plot the changes in $\mathcal{A}_3$ *in Figure 7 for the range* $\alpha \in (0.3, 0.95] \subset (0, 1]$. *These estimated results are shown in Table 5. Therefore, Assumption* (iii) *is verified.*

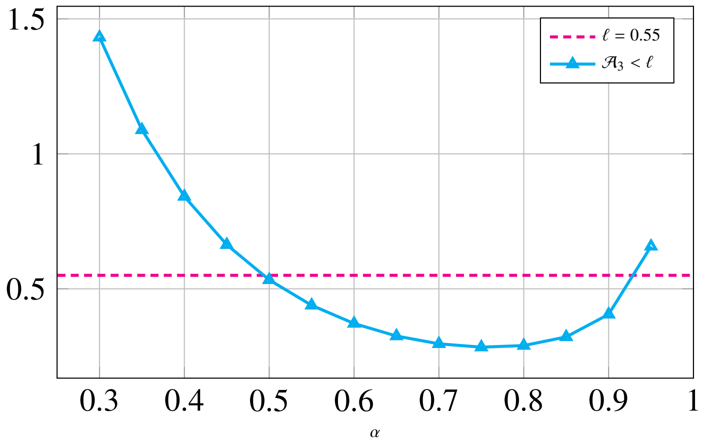

Figure 7. Two-dimensional plots of $\mathcal{A}_3$ for BVP (4.1) in Case 3.

Furthermore, the constant Σ *which is defined by* (3.8) *is less than* 1 *as follows:*

$$\Sigma = \frac{\frac{1}{100} 2^{291/70\alpha-2}}{\Gamma^{97/70\alpha}\left(\frac{5}{4}\right)} + \left[\frac{\frac{11}{2000} + \widehat{K}_3}{\Gamma\left(\frac{16}{7}\right)}\left[2 + 0.15\left(\frac{16}{7}\right)\right]\right]^{97/70\alpha} \frac{1}{100} 2^{97/70\alpha}$$
$$+ \left[\frac{11}{2000} + \widehat{K}_2\right]^{97/70\alpha} \frac{\frac{1}{100} 2^{194/70\alpha-1}}{\Gamma^{97/70\alpha}\left(\frac{16}{7}\right)} \simeq 0.6865.$$

Table 5. Estimated results of Theorem 3.2 for BVP (4.1) for $\alpha \in (0, 1]$ in Case 3.

α	$\mathbf{0} \xrightarrow{\alpha} \mathbf{1}$				
	$\hat{c}_{11}$	$\hat{c}_{22}$	$\hat{c}_{33}$	$\mathcal{A}_3 < \ell$	$\Sigma < 1$
0.30	$\underline{0.9710}$	$\underline{0.6351}$	$\underline{0.5490}$	1.4318	64.6319
0.35	0.7392	0.5246	0.4535	1.0886	16.5057
0.40	0.5844	0.4333	0.3746	0.8420	6.0992
0.45	0.4681	0.3579	0.3094	0.6633	2.8746
0.50	0.3771	0.2956	0.2556	$\underline{0.5332}$	1.6010
0.55	0.3048	0.2442	0.2111	0.4389	1.0039
0.60	0.2465	0.2017	0.1744	0.3715	$\underline{0.6865}$
0.65	0.1994	0.1666	0.1440	0.3251	0.5009
0.70	0.1612	0.1376	0.1190	0.2963	0.3842
0.75	0.1299	0.1137	0.0983	0.2839	0.3063

The results obtained are presented in Table 5, and we plot the changes in Σ *in Figure 8 for the range* $\alpha \in (0.30, 0.95] \subset (0, 1]$. *Hence, Assumptions* (i) *and* (iv) *hold. Therefore, thanks to Theorem 3.2, BVP* (4.1) *has at least one solution in* E_0 *for Case 3.*

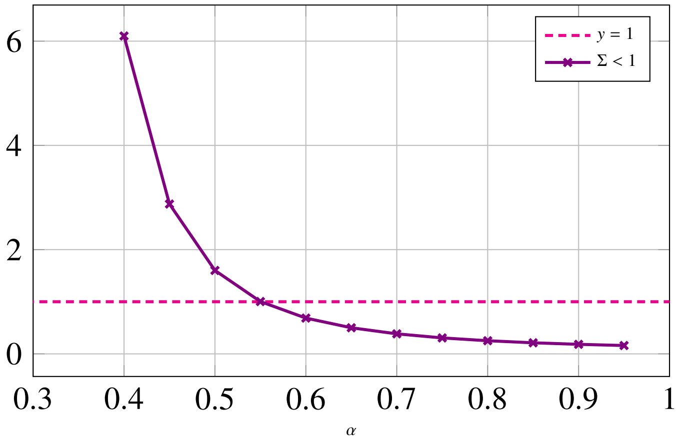

Figure 8. Two-dimensional plots of Σ for BVP (4.1) in Case 3.

Additionally, using (3.23) *and setting* $\mathrm{b}^*_{\rho_i} = \frac{1}{100}$, $i = 1, 2, 3$, *we obtain*

$$\begin{aligned}
\hat{c}^*_{11} &= \max\left\{ \frac{\frac{1}{100} 2^{1-70\alpha/97}}{\Gamma(\frac{5}{4})}, \frac{1}{\Gamma(\frac{5}{4})}\left[\frac{1}{100\,1} + \left[\frac{\alpha\left(\frac{1}{100}\right)^{97/70\alpha} 2^{97/70\alpha-1}}{\frac{97}{70}\left(\frac{1}{4}-\alpha\right)} \right]^{70\alpha/97} \right] \right\} \simeq 0.0353,\\
\hat{c}^*_{22} &= \max\left\{ \frac{1}{100}, \frac{1}{100} \right\} = 0.01,\\
\hat{c}^*_{33} &= \max\left\{ \frac{\frac{1}{100} 2^{1-70\alpha/97}}{\Gamma(\frac{16}{7})}, \frac{\frac{1}{100}}{\Gamma(\frac{16}{7})}\left[2 + 0.15\Gamma\left(\frac{16}{7}\right) + \left[\frac{\alpha(\frac{9}{7})^{97/70\alpha} 2^{97/70\alpha-1}}{\frac{97}{70}(1-\alpha)} \right]^{70\alpha/97} \right] \right\} \simeq 0.0363,
\end{aligned}$$

and for the contraction constant $\hat{c}_{44}$, *we have*

$$\begin{aligned}
\hat{c}_{44} = \Bigg[&\hat{c}^*_{11} + \hat{c}^*_{22}\left[1 + 2^{2-70\alpha/97}\right]\left[\frac{4^{1-70\alpha/97}}{\Gamma(\frac{16}{7})} \left\|\mathrm{a}_{\rho_3}\right\|_{L_1}^{70\alpha/97} + \hat{c}_{33}\widehat{K}_3 + \hat{c}_{33}\cdot\ell \right]\\
&+ \hat{c}^*_{33}\left[1 + 2^{2-70\alpha/97}\right]\left[\widehat{K}_2 + 2^{1-70\alpha/97} \left\|\mathrm{a}_{\rho_2}\right\|_{L_1}^{70\alpha/97} + \hat{c}_{22}\cdot\ell \right] \Bigg] \simeq 0.4237 < 1.
\end{aligned}$$

We plot the changes in $\hat{c}_{44}$ *in Figure 9 for the range* $\alpha \in (0, 1]$. *All the results obtained for* $\hat{c}^*_{11}$, $\hat{c}^*_{22}$, $\hat{c}^*_{33}$, *and* $\hat{c}_{44}$ *are given in Table 6.*

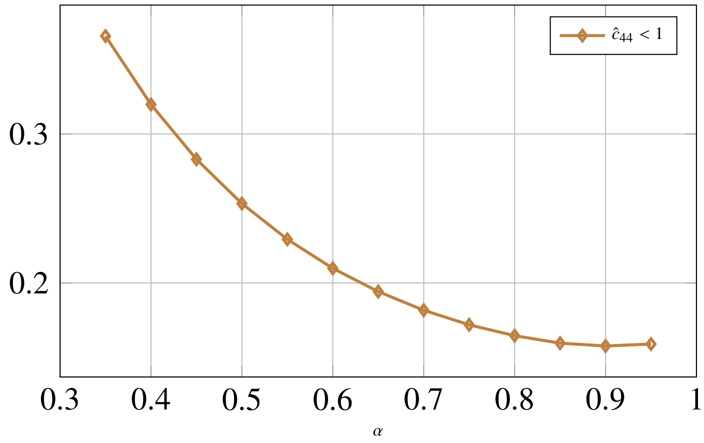

Figure 9. Two-dimensional plots of $\hat{c}_{44}$ for BVP (4.1) in Case 3.

These numerical results confirm that all assumptions of Theorem 3.3 hold and, consequently, BVP (4.1) *has a unique solution* $\mathbb{w} \in B_\ell(\mathcal{J}_{\alpha,\beta})$ *for* Case 3.

Table 6. Estimated results of Theorem 3.3 for BVP (4.1) for $\alpha \in (0,1]$ in Case 3.

α	$\mathbf{0 \overset{\alpha}{\longrightarrow} 1}$			
	$\hat{c}^*_{11}$	$\hat{c}^*_{22}$	$\hat{c}^*_{33}$	$\hat{c}_{44} < 1$
0.30	$\underline{0.0353}$	0.0100	0.0295	$\underline{0.4237}$
0.35	0.0321	0.0100	0.0292	0.3657
0.40	0.0299	0.0100	0.0290	0.3198
0.45	0.0280	0.0100	0.0289	0.2830
0.50	0.0263	0.0100	0.0290	0.2533
0.55	0.0246	0.0100	0.0291	0.2293
0.60	0.0229	0.0100	0.0293	0.2098
0.65	0.0213	0.0100	0.0296	0.1942
0.70	0.0197	0.0100	0.0301	0.1817
0.75	0.0181	0.0100	0.0307	0.1719
0.80	0.0165	0.0100	0.0316	0.1646
0.85	0.0149	0.0100	0.0327	0.1596
0.90	0.0141	0.0100	0.0343	0.1577
0.95	0.0137	0.0100	$\underline{0.0363}$	0.1590

Remark 4.2. *The numerical verification shows that all assumptions required for the existence and uniqueness theorems are satisfied under the chosen parameter regime, which guarantees that the considered thermostat model admits a unique and stable temperature profile. This confirms that the heat regulation process remains mathematically well posed and physically predictable, even in the presence of a delayed thermal response and nonlocal interactions. Moreover, varying the parameters* α *and* β *illustrates how the regularity of the solutions depends on the functional framework* $J_{\alpha,\beta}$ *and confirms that this space is more suitable than the classical spaces* $C(E_0)$ *and* $C^{\alpha}(E_0)$ *for the analysis of hybrid fractional thermostat models. Therefore, the numerical example serves not only as a verification of the theoretical results but also as an illustration of the practical applicability and reliability of the proposed fractional thermostat model.*

5. Conclusions

In this article, an extension and generalization of the hybrid Caputo BVP related to models of thermostats in the Banach space of the integral-type Hölder space $\mathcal{J}_{\alpha,\beta}$ is studied. We derived sufficient conditions guaranteeing the existence and uniqueness of solutions to the hybrid Caputo BVP for heat controllers, satisfying certain nonlocal hybrid conditions with valued feedback control. The existence of the results and the unique solution to the nonlocal BVP, which contains contributions at both the domain's boundary and the intermediate processes, is proved. We used the techniques related to Darbo-type FPT with MNC. Because we studied our problem in the space $J_{\alpha,\beta}$, our results are more regular than the prior ones, which is a suitable choice for solving the BVP in (1.1) and (1.2). Finally, a numerical example was presented to verify the assumptions of the main theorems for different values of α and β. The results obtained confirmed that the theoretical conditions are satisfied under admissible parameter choices and demonstrated that the proposed fractional thermostat model admits a unique and stable temperature profile.

Author contributions

MMAM: Methodology, validation, formal analysis, actualization, investigation, initial draft, and was a major contributor to writing the manuscript. FSA: Methodology, validation, formal analysis, review, and editing. MES: Actualization, methodology, formal analysis, validation, investigation, software, simulation, initial draft, and was a major contributor to writing the manuscript. All authors read and approved the final manuscript.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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