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Research article

Raising operators and a parametric polynomial continued fraction for π^2

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Abstract: We study a one-parameter family of polynomial J-fractions whose coefficients depend polynomially on the index and on an integer parameter $u \geq 0$. After a factorial normalization of the denominator sequence, the difference of two consecutive convergents factors into the fixed central-binomial Apéry-type term $1/(n^2\binom{2n}{n})$ and a rational factor determined by a normalized polynomial family P_u. We construct P_u by an explicit parameter-raising operator. We then prove a parameter-shift telescoping identity, which allows induction on u and gives

$$X(u) = \binom{2u}{u}^3 \frac{\pi^2}{18} + \rho_u, \qquad \rho_u \in \mathbb{Q}.$$

Thus, the operator identity and the telescoping identity provide the algebraic mechanism behind the evaluation of the whole family.

Keywords: continued fractions; polynomial continued fractions; J-fractions; parameter-raising operators; central binomial coefficients; telescoping; π^2

1. Introduction

Polynomial continued fractions may contain a simple kernel behind complicated coefficients. In this paper, the kernel is the central-binomial Apéry-type series for π^2. Our aim is to explain why the same kernel appears for a whole family of J-fractions.

We study the polynomial J-fraction

$$X(u) = \cfrac{1}{a_1(u) + \cfrac{b_1(u)}{a_2(u) + \cfrac{b_2(u)}{a_3(u) + \ddots}}}, \qquad u \in \mathbb{N}_0, \tag{1.1}$$

with

$$a_n(u) = 5n^2 + (14u - 4)n + (3u - 1)^2, \qquad b_n(u) = -2n(2n - 1)(n + u)^2. \tag{1.2}$$

The family fits naturally into the theory of polynomial J-fractions, where convergents satisfy three-term recurrences with polynomial coefficients; see [1–8].

The underlying summation kernel of (1.1) is the classical central binomial identity

$$\sum_{n\geq 1} \frac{1}{n^2\binom{2n}{n}} = \frac{\pi^2}{18}, \tag{1.3}$$

which is a standard Apéry-type series evaluation; see for instance Apéry, Beukers, van der Poorten, and Borwein–Girgensohn [9–12]. The point of the present construction is that the consecutive convergent differences of (1.1) retain exactly this kernel after normalization. The parameter dependence is concentrated in a normalized denominator polynomial family P_u.

This family is closely related to Cohen's parametric continued fractions for π^2, $\zeta(3)$, and other constants [13]. Cohen's work provides important motivating and structural background for such families. Here, we give a proof mechanism for (1.1). We build a difference operator $\mathcal{L}_u$ and a parameter-raising operator Ψ_u; the latter sends solutions at level u to solutions at level $u + 1$.

We now explain the main idea of the proof. The continued fraction gives a three-term recurrence for its denominator sequence. We write this recurrence as an operator equation and denote the operator by $\mathcal{L}_u$. This notation is useful because it lets us compare the recurrence for the parameter u with the recurrence for the next parameter $u + 1$. We then construct a parameter-raising operator. This operator sends a solution at level u to a solution at level $u + 1$. In this way, we obtain the normalized polynomial family P_u.

The role of the polynomials P_u is to separate two parts of the problem. One part is universal and contains the central-binomial Apéry-type kernel. The other part depends on the parameter u. After this separation, we prove a parameter-shift telescoping identity. This identity compares the summand at level $u + 1$ with the summand at level u. It allows induction on u and therefore proves the evaluation for every integer $u \geq 0$:

$$X(u) = \binom{2u}{u}^3 \frac{\pi^2}{18} + \rho_u, \qquad \rho_u \in \mathbb{Q}.$$

Thus, the polynomial identities are not merely formal identities. They are the algebraic mechanism, which makes the induction possible.

The surrounding literature has several related but distinct strands. Classical continued-fraction theory provides the analytic and recurrence background [1–6]. Flajolet's theory explains many continued fractions through combinatorial models [7]. Apéry's proof and its later interpretations show how binomial kernels can encode special values such as $\zeta(2)$ and $\zeta(3)$ [9–12]. Symbolic summation and holonomic methods often reduce identities to finite algebraic checks [14–17]. The present paper uses ideas close to these methods, but its role is more specific. The distinctive feature of the approach is that the parameter shift is realized by an explicit first-order parameter-raising operator. This operator is then combined with a telescoping identity for the summand itself, which turns the construction into an induction on the parameter u. To the best of our knowledge, this is the first use of an explicit first-order parameter-raising operator as the main proof mechanism for evaluating a complete parametric family of polynomial J-fractions for π^2.

Our first theorem records the structural decomposition that drives the rest of the paper. Its significance is that the u-independent π^2-producing factor and the parameter-dependent algebraic factor are separated in a precise form: central-binomial term is fixed, while all dependence on u is concentrated in two consecutive values of one polynomial family generated by the parameter-raising operator.

The polynomials P_u in the following theorem are constructed in Section 3; they are normalized by $P_u(0) = 1$.

Theorem 1.1. *For each fixed $u \in \mathbb{N}_0$ and every $n \geq 1$, the convergents of* (1.1) *satisfy*

$$X_n(u) - X_{n-1}(u) = \frac{\binom{n+u-1}{u}^2}{n^2\binom{2n}{n}} \cdot \frac{1}{P_u(n)P_u(n-1)},$$

where P_u is the normalized denominator polynomial defined in Section 3.

Remark 1.2. Theorem 1.1 separates the problem into a u-independent central-binomial factor and a parameter-dependent algebraic factor. The factor

$$\frac{1}{n^2\binom{2n}{n}}$$

is the same factor that appears in (1.3), while the factor

$$\frac{\binom{n+u-1}{u}^2}{P_u(n)P_u(n-1)}$$

contains the full dependence on u. This separation is the main structural point of the paper.

The paper is organized as follows: Section 2 derives the convergent recurrence and the central-binomial increment formula. Section 3 constructs the parameter-raising operator and the normalized denominator polynomials P_u. Section 4 proves the parameter-shift telescoping identity and evaluates the full family.

2. Convergents and the central-binomial kernel

Definition 2.1. Set $b_0(u) := 1$. Define sequences $(p_n(u))_{n\geq-1}$ and $(q_n(u))_{n\geq-1}$ by

$$p_{-1} = 1, \qquad p_0 = 0, \qquad q_{-1} = 0, \qquad q_0 = 1,$$

and for $n \geq 1$,

$$p_n = a_n(u)p_{n-1} + b_{n-1}(u)p_{n-2}, \qquad q_n = a_n(u)q_{n-1} + b_{n-1}(u)q_{n-2}. \tag{2.1}$$

Whenever $q_n(u) \neq 0$, the nth convergent of (1.1) is

$$X_n(u) = \frac{p_n(u)}{q_n(u)}.$$

Proposition 2.2. *For each fixed $u \in \mathbb{N}_0$, the denominator sequence satisfies $q_n(u) > 0$ for all $n \geq 0$. Consequently, every truncation $X_n(u)$ is well defined. If one sets*

$$\widetilde{P}_u(n) := \frac{q_n(u)}{(2n)!}, \qquad n \geq 0, \tag{2.2}$$

then the consecutive convergent differences satisfy

$$X_n(u) - X_{n-1}(u) = \frac{\binom{n+u-1}{u}^2}{n^2\binom{2n}{n}} \cdot \frac{1}{\widetilde{P}_u(n)\widetilde{P}_u(n-1)}, \qquad n \geq 1, \tag{2.3}$$

and the normalized sequence obeys

$$(2n)(2n-1)\widetilde{P}_u(n) = a_n(u)\widetilde{P}_u(n-1) - (n+u-1)^2\widetilde{P}_u(n-2), \qquad n \geq 2, \tag{2.4}$$

with initial values

$$\widetilde{P}_u(0) = 1, \qquad \widetilde{P}_u(1) = \frac{a_1(u)}{2}.$$

Proof. We begin by proving positivity of the denominator sequence. Write

$$c_n(u) := -b_n(u) = 2n(2n-1)(n+u)^2 > 0.$$

For $N \geq 1$, consider the symmetric tridiagonal matrix

$$T_N(u) = \begin{pmatrix} a_1(u) & \sqrt{c_1(u)} & & \\ \sqrt{c_1(u)} & a_2(u) & \sqrt{c_2(u)} & \\ & \ddots & \ddots & \ddots \\ & & \sqrt{c_{N-1}(u)} & a_N(u) \end{pmatrix}.$$

Its leading principal minors satisfy the same recurrence as $q_N(u)$ with the same initial values, so $q_N(u) = \det T_N(u)$. We show that $T_N(u)$ is strictly diagonally dominant with positive diagonal. First, $a_n(u) > 0$ for all $n \geq 1$ and $u \geq 0$. For $n \geq 2$,

$$\sqrt{2n(2n-1)} < 2n, \qquad \sqrt{2(n-1)(2n-3)} < 2(n-1),$$

hence

$$\sqrt{c_n(u)} < 2n(n+u), \qquad \sqrt{c_{n-1}(u)} < 2(n-1)(n+u-1).$$

So, it is enough to prove

$$a_n(u) \geq 2n(n+u) + 2(n-1)(n+u-1), \qquad n \geq 2.$$

A direct simplification gives

$$a_n(u) - (2n(n+u) + 2(n-1)(n+u-1)) = n^2 + 10un + 9u^2 - 4u - 1 \geq 0.$$

At $n = 1$, there is only one off-diagonal entry, and

$$a_1(u) = 9u^2 + 8u + 2 > \sqrt{2}\,(u+1) = \sqrt{c_1(u)}.$$

Thus, $T_N(u)$ is strictly diagonally dominant and has positive diagonal. Since $T_N(u)$ is real and symmetric, all its eigenvalues are real. By Gershgorin's theorem, every eigenvalue lies in an interval $[a_i - r_i, a_i + r_i]$, where a_i is a diagonal entry and r_i is the sum of the absolute values of the off-diagonal entries in the same row [18, Section 6.1]. Strict diagonal dominance gives $a_i - r_i > 0$ for every row, so all these intervals are contained in $(0, \infty)$. Hence, all eigenvalues are positive, and $T_N(u)$ is positive definite. By Sylvester's criterion, its leading principal minors are positive [18, Section 7.2]. These leading principal minors satisfy the same determinant recurrence as $q_N(u)$, and hence $q_N(u) = \det T_N(u) > 0$.

To derive the increment formula, we use the standard determinant identity for J-fractions. It follows by applying the recurrence to $p_n q_{n-1} - p_{n-1} q_n$. This gives

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1} \prod_{j=0}^{n-1} b_j(u)}{q_n q_{n-1}} = \frac{(-1)^{n-1} \prod_{j=1}^{n-1} b_j(u)}{q_n q_{n-1}}$$

since $b_0(u) = 1$. For $j \geq 1$,

$$b_j(u) = -2j(2j-1)(j+u)^2.$$

Hence,

$$\prod_{j=1}^{n-1} b_j(u) = (-1)^{n-1} \Big(\prod_{j=1}^{n-1} 2j(2j-1) \Big) \Big(\prod_{j=1}^{n-1} (j+u)^2 \Big) = (-1)^{n-1} (2n-2)! (u+1)_{n-1}^2.$$

The signs cancel, so

$$X_n(u) - X_{n-1}(u) = \frac{(2n-2)!(u+1)_{n-1}^2}{q_n q_{n-1}}.$$

Substituting $q_n = (2n)! \widetilde{P}_u(n)$ gives

$$X_n(u) - X_{n-1}(u) = \frac{(u+1)_{n-1}^2}{(2n)! \widetilde{P}_u(n) \widetilde{P}_u(n-1)}.$$

Finally,

$$(u+1)_{n-1} = (n-1)! \binom{n+u-1}{u}, \qquad \frac{(n-1)!^2}{(2n)!} = \frac{1}{n^2 \binom{2n}{n}},$$

which yields (2.3).

To obtain the recurrence, substitute $q_n = (2n)! \widetilde{P}_u(n)$ into (2.1), divide by $(2n)!$, and use

$$b_{n-1}(u) = -2(n-1)(2n-3)(n+u-1)^2.$$

Then

$$\frac{b_{n-1}(u)(2n-4)!}{(2n)!} = -\frac{(n+u-1)^2}{(2n)(2n-1)},$$

which gives (2.4). The initial values follow from $q_0 = 1$ and $q_1 = a_1(u)$.

3. A parameter-raising operator for the parametric family

The operator $\mathcal{L}_u$ is a gauge-equivalent form of the normalized denominator recurrence. This form is useful because it makes it possible to compare the recurrence at level u with the recurrence at level $u+1$. The parameter-raising operator Ψ_u connects these two recurrence operators by a precise algebraic identity. Together with Ψ_u, the operator $\mathcal{L}_u$ converts the fixed-parameter recurrence into an induction on the parameter u.

For a function $F : \mathbb{Z} \to \mathbb{Q}$, define

$$(\mathcal{L}_u F)(n) = 2(2n-1)(n+u)F(n) - a_n(u)F(n-1) + (n-1)(n+u-1)F(n-2). \tag{3.1}$$

Also define

$$\mathscr{A}_0(n,u) = \frac{(n+u+1)(-n^2-6nu-6n+3u^2-2)}{3}, \tag{3.2}$$

$$\mathscr{A}_1(n,u) = \frac{2(2n+1)(n^2+10nu+8n+21u^2+34u+13)}{3}, \tag{3.3}$$

and let

$$(\Psi_u F)(n) := \mathscr{A}_0(n,u)F(n) + \mathscr{A}_1(n,u)F(n+1).$$

Theorem 3.1. *For every function $F : \mathbb{Z} \to \mathbb{Q}$,*

$$(\mathcal{L}_{u+1}(\Psi_u F))(n) = \mathscr{B}(n,u)(\mathcal{L}_u F)(n) + \mathscr{C}(n,u)(\mathcal{L}_u F)(n+1), \tag{3.4}$$

where

$$\mathscr{B}(n,u) = \frac{(n+u)(-n^2-6nu-2n+3u^2+12u+6)}{3}, \tag{3.5}$$

$$\mathscr{C}(n,u) = \frac{2(2n-1)(n^2+10nu+8n+21u^2+34u+13)}{3}. \tag{3.6}$$

In particular, if Q_u satisfies $\mathcal{L}_u(Q_u) = 0$, then $Q_{u+1} := \Psi_u(Q_u)$ satisfies $\mathcal{L}_{u+1}(Q_{u+1}) = 0$.

The factors $1/3$ in (3.2), (3.3), (3.5), and (3.6) are only part of our normalization convention.

Proof. The identity (3.4) is an operator identity. We prove it by applying both sides to an arbitrary function F and comparing the coefficients of the shifted values of F.

Write

$$F_{+1} := F(n+1), \qquad F_0 := F(n), \qquad F_{-1} := F(n-1), \qquad F_{-2} := F(n-2).$$

Since $\Psi_u F$ uses only $F(n)$ and $F(n+1)$, the expression $\mathcal{L}_{u+1}(\Psi_u F)$ can involve only

$$F_{+1}, \quad F_0, \quad F_{-1}, \quad F_{-2}.$$

The right-hand side of (3.4) involves the same four shifted values. Therefore, the difference between the two sides has the form

$$E_{+1}F_{+1} + E_0F_0 + E_{-1}F_{-1} + E_{-2}F_{-2}.$$

Here, each E_j is a polynomial in n and u after clearing the common factor $1/3$.

For example, the coefficient of F_{+1} comes from the $F(n+1)$-part of $\mathcal{L}_{u+1}(\Psi_u F)$ and from the term $\mathscr{C}(n,u)(\mathcal{L}_u F)(n+1)$. Thus, this coefficient is

$$E_{+1} = 2(2n-1)(n+u+1)\mathscr{A}_1(n,u) - 2(2n+1)(n+u+1)\mathscr{C}(n,u).$$

Using (3.3) and (3.6), this is identically zero.

The remaining three shifted coefficients are computed analogously. After substituting $\mathscr{A}_0, \mathscr{A}_1, \mathscr{B}, \mathscr{C}$, one obtains

$$E_{+1} = E_0 = E_{-1} = E_{-2} = 0.$$

The explicit expressions before cancellation are recorded in Appendix A. Hence, (3.4) holds for every function F.

Finally, if $\mathcal{L}_u(F) = 0$, then both terms on the right-hand side of (3.4) vanish. Therefore,

$$\mathcal{L}_{u+1}(\Psi_u F) = 0.$$

Thus, the parameter-raising operator sends a solution of the recurrence at level u to a solution of the recurrence at level $u+1$.

We start from $Q_0(n) \equiv 1$. Repeated use of Ψ_u gives a chain of solutions. The normalization used next makes $P_u(0) = 1$ and matches the normalized denominator sequence in Proposition 2.2.

We now prove Theorem 1.1. Starting from the constant solution $Q_0(n) \equiv 1$, we define

$$Q_{u+1} := \Psi_u(Q_u), \qquad u \geq 0.$$

By Theorem 3.1, each Q_u satisfies

$$\mathcal{L}_u(Q_u) = 0.$$

We now identify this operator-generated family with the normalized denominator polynomials from Proposition 2.2. Evaluate $\mathcal{L}_u(Q_u) = 0$ at $n = 1$:

$$0 = 2(u+1)Q_u(1) - a_1(u)Q_u(0), \qquad Q_u(1) = \frac{a_1(u)}{2(u+1)}Q_u(0).$$

Now evaluate the raising recursion $Q_{u+1} = \Psi_u(Q_u)$ at $n = 0$:

$$Q_{u+1}(0) = \mathscr{A}_0(0,u)Q_u(0) + \mathscr{A}_1(0,u)Q_u(1).$$

Substituting the formula for $Q_u(1)$ and simplifying yields

$$Q_{u+1}(0) = 8(2u+1)^3 Q_u(0), \qquad Q_0(0) = 1.$$

Therefore,

$$Q_u(0) = \prod_{k=0}^{u-1} 8(2k+1)^3 = \left(\frac{(2u)!}{u!}\right)^3 \tag{3.7}$$

because

$$\prod_{k=0}^{u-1}(2k+1) = \frac{(2u)!}{2^u u!}.$$

Define

$$D_u := u!\, Q_u(0) = \frac{(2u)!^3}{(u!)^2}, \qquad P_u(n) := \frac{(n+1)_u}{D_u} Q_u(n).$$

Then

$$P_u(0) = \frac{(1)_u\, Q_u(0)}{D_u} = 1.$$

To identify P_u with the normalized denominator family, rewrite

$$Q_u(n) = \frac{D_u P_u(n)}{(n+1)_u}$$

and substitute into $\mathcal{L}_u(Q_u) = 0$. Multiplying by $(n+1)_u$ gives

$$2(2n-1)(n+u)P_u(n) - a_n(u)\frac{(n+1)_u}{(n)_u}P_u(n-1) + (n-1)(n+u-1)\frac{(n+1)_u}{(n-1)_u}P_u(n-2) = 0.$$

Using

$$\frac{(n+1)_u}{(n)_u} = \frac{n+u}{n}, \qquad \frac{(n+1)_u}{(n-1)_u} = \frac{(n+u)(n+u-1)}{n(n-1)},$$

we obtain

$$2(2n-1)(n+u)P_u(n) - a_n(u)\frac{n+u}{n}P_u(n-1) + \frac{(n+u)(n+u-1)^2}{n}P_u(n-2) = 0.$$

Multiplying by $n/(n+u)$ yields

$$(2n)(2n-1)P_u(n) = a_n(u)P_u(n-1) - (n+u-1)^2 P_u(n-2),$$

which is exactly (2.4). The initial value at $n = 1$ follows from the earlier relation

$$Q_u(1) = \frac{a_1(u)}{2(u+1)} Q_u(0).$$

Indeed,

$$P_u(1) = \frac{(2)_u}{D_u} Q_u(1) = \frac{(u+1)!}{D_u} \cdot \frac{a_1(u)}{2(u+1)} Q_u(0) = \frac{a_1(u)}{2}.$$

Therefore, P_u and $\widetilde{P}_u$ satisfy the same second-order recurrence for $n \geq 2$ with the same initial values, so

$$P_u(n) = \widetilde{P}_u(n) \qquad \text{for all } n \geq 0.$$

This proves Theorem 1.1.

The degree and the leading coefficient of P_u are also transparent from the raising recursion. Write

$$Q_u(n) = \kappa_u n^{3u} + O(n^{3u-1}).$$

From (3.2) and (3.3),

$$\mathscr{A}_0(n,u) = -\frac{1}{3}n^3 + O(n^2), \qquad \mathscr{A}_1(n,u) = \frac{4}{3}n^3 + O(n^2),$$

and

$$Q_u(n+1) = \kappa_u n^{3u} + O(n^{3u-1}).$$

Hence,

$$Q_{u+1}(n) = \mathscr{A}_0(n,u)Q_u(n) + \mathscr{A}_1(n,u)Q_u(n+1) = \kappa_u n^{3u+3} + O(n^{3u+2}),$$

so $\kappa_{u+1} = \kappa_u$ and $\deg Q_{u+1} = 3u+3$. Since $Q_0 \equiv 1$, we have $\kappa_u = 1$ and $\deg Q_u = 3u$ for all u. Therefore,

$$P_u(n) = \frac{(n+1)_u}{D_u} Q_u(n)$$

has degree $4u$, and its leading coefficient is $1/D_u$.

Finally, the continued fraction converges. By Theorem 1.1,

$$X_N(u) = \sum_{n=1}^{N} \frac{\binom{n+u-1}{u}^2}{n^2\binom{2n}{n}} \cdot \frac{1}{P_u(n)P_u(n-1)}.$$

Using Stirling's estimate,

$$\binom{2n}{n} \asymp \frac{4^n}{\sqrt{\pi n}}, \qquad \frac{1}{n^2\binom{2n}{n}} \ll 4^{-n} n^{-3/2}.$$

Also

$$\binom{n+u-1}{u}^2 \ll n^{2u}, \qquad P_u(n)P_u(n-1) \asymp n^{8u},$$

so the summand is

$$O(4^{-n} n^{-3/2-6u}).$$

The series therefore converges absolutely, and the limit

$$X(u) := \lim_{N\to\infty} X_N(u)$$

exists.

4. Telescoping and evaluation of the full family

We now pass from the parameter-raising construction to the value of the continued fraction. Lemma 4.1 compares the summands at levels u and $u+1$. Theorem 4.2 then sums this relation and gives the evaluation for all $u \geq 0$.

Set

$$F_u(n) := \binom{n+u-1}{u}^2, \qquad S_u(n) := \frac{F_u(n)}{n^2\binom{2n}{n}P_u(n)P_u(n-1)}, \qquad C_u := \binom{2u}{u}^3.$$

By Theorem 1.1,

$$X(u) = \sum_{n\geq 1} S_u(n).$$

Also put

$$\lambda_u := \frac{C_{u+1}}{C_u} = \frac{8(2u+1)^3}{(u+1)^3}, \qquad \Delta_u := 8(u+1)(2u+1)^3.$$

We now prepare the induction step from u to $u+1$. Since

$$\lambda_u = \frac{C_{u+1}}{C_u},$$

the term $\lambda_u S_u(n)$ has the correct next coefficient C_{u+1} in front of the u-independent central-binomial factor. Thus, it remains to prove that the difference

$$S_{u+1}(n) - \lambda_u S_u(n)$$

is telescoping.

We choose the denominator of $\Theta_u(n)$ to contain

$$P_u(n-1)P_{u+1}(n-1).$$

After replacing n by $n+1$, this becomes

$$P_u(n)P_{u+1}(n),$$

which is the denominator needed in the comparison of $S_{u+1}(n)$ and $S_u(n)$. The shifted raising formula at $m = n-1$ gives

$$P_{u+1}(n-1) = \frac{(n+u)\mathscr{A}_0(n-1,u)}{\Delta_u} P_u(n-1) + \frac{n\mathscr{A}_1(n-1,u)}{\Delta_u} P_u(n).$$

Thus, the coefficient $\mathscr{A}_1(n-1,u)$ is the one that enters the cancellation. The remaining normalization is fixed by

$$\frac{\Delta_u}{\lambda_u} = \frac{8(u+1)(2u+1)^3}{8(2u+1)^3/(u+1)^3} = (u+1)^4.$$

This leads to the following definition of $\Theta_u(n)$.

Since

$$\mathscr{A}_1(n-1,u) = \frac{2(2n-1)}{3}(n^2 + (10u+6)n + 21u^2 + 24u + 6),$$

we put

$$\ell_u(n) := n^2 + (10u+6)n + 21u^2 + 24u + 6$$

and define

$$\Theta_u(n) := -\frac{F_u(n)\mathscr{A}_1(n-1,u)}{(u+1)^4\, n\binom{2n}{n} P_u(n-1)P_{u+1}(n-1)}. \tag{4.1}$$

Equivalently,

$$\Theta_u(n) = -\frac{2F_u(n)(2n-1)\ell_u(n)}{3(u+1)^4\, n\binom{2n}{n} P_u(n-1)P_{u+1}(n-1)}.$$

Lemma 4.1. *For every $u \in \mathbb{N}_0$ and every $n \geq 1$,*

$$S_{u+1}(n) - \lambda_u S_u(n) = \Theta_u(n) - \Theta_u(n+1). \tag{4.2}$$

Proof. The proof has three steps. We first express the two values of P_{u+1} through two values of P_u. Then we divide out the common hypergeometric factor. Finally, the identity reduces to a quadratic polynomial in $A = P_u(n-1)$ and $B = P_u(n)$.

Let

$$A := P_u(n-1), \qquad B := P_u(n), \qquad U := P_{u+1}(n-1), \qquad V := P_{u+1}(n).$$

Recall that

$$\Delta_u = 8(u+1)(2u+1)^3.$$

Step 1: shifted raising formula. We first derive the shifted raising formula for the normalized polynomials. By definition,

$$P_{u+1}(m) = \frac{(m+1)_{u+1}}{D_{u+1}} Q_{u+1}(m), \qquad Q_{u+1}(m) = \mathscr{A}_0(m,u)Q_u(m) + \mathscr{A}_1(m,u)Q_u(m+1).$$

Using

$$Q_u(k) = \frac{D_u P_u(k)}{(k+1)_u}$$

gives

$$P_{u+1}(m) = \frac{D_u}{D_{u+1}} (m+1)_{u+1} \left(\frac{\mathscr{A}_0(m,u)P_u(m)}{(m+1)_u} + \frac{\mathscr{A}_1(m,u)P_u(m+1)}{(m+2)_u} \right).$$

Now

$$\frac{(m+1)_{u+1}}{(m+1)_u} = m+u+1, \qquad \frac{(m+1)_{u+1}}{(m+2)_u} = m+1,$$

and

$$\frac{D_{u+1}}{D_u} = 8(u+1)(2u+1)^3 = \Delta_u.$$

Therefore,

$$P_{u+1}(m) = \frac{(m+u+1)\mathscr{A}_0(m,u)P_u(m) + (m+1)\mathscr{A}_1(m,u)P_u(m+1)}{\Delta_u}. \tag{4.3}$$

Using (4.3) at $m = n-1$ gives

$$U = \alpha A + \beta B,$$

where

$$\alpha = \frac{(n+u)\mathscr{A}_0(n-1,u)}{\Delta_u}, \qquad \beta = \frac{n\mathscr{A}_1(n-1,u)}{\Delta_u}.$$

Using (4.3) at $m = n$ and eliminating $P_u(n+1)$ by

$$P_u(n+1) = \frac{a_{n+1}(u)P_u(n) - (n+u)^2 P_u(n-1)}{2(n+1)(2n+1)},$$

we obtain

$$V = \gamma A + \delta B,$$

where

$$\gamma = -\frac{(n+u)^2 \mathscr{A}_1(n,u)}{2(2n+1)\Delta_u}$$

and

$$\delta = \frac{(n+u+1)\mathscr{A}_0(n,u)}{\Delta_u} + \frac{\mathscr{A}_1(n,u)a_{n+1}(u)}{2(2n+1)\Delta_u}.$$

Step 2: reduction to an algebraic identity. We now verify (4.2). Divide both sides by the common factor

$$\frac{F_u(n)}{\binom{2n}{n}},$$

and use

$$\frac{F_{u+1}(n)}{F_u(n)} = \frac{(n+u)^2}{(u+1)^2}, \qquad \frac{F_u(n+1)}{F_u(n)} = \frac{(n+u)^2}{n^2},$$

together with

$$\frac{\binom{2n}{n}}{(n+1)\binom{2n+2}{n+1}} = \frac{1}{2(2n+1)}.$$

After this division, the two boundary terms become

$$\frac{\Theta_u(n)}{F_u(n)/\binom{2n}{n}} = -\frac{\mathscr{A}_1(n-1,u)}{(u+1)^4 nAU}$$

and

$$\frac{\Theta_u(n+1)}{F_u(n)/\binom{2n}{n}} = -\frac{(n+u)^2\mathscr{A}_1(n,u)}{2(2n+1)(u+1)^4 n^2 BV}.$$

Hence, the desired identity becomes equivalent to

$$\frac{(n+u)^2}{(u+1)^2 n^2 UV} - \frac{\lambda_u}{n^2 AB} = -\frac{\mathscr{A}_1(n-1,u)}{(u+1)^4 nAU} + \frac{(n+u)^2\mathscr{A}_1(n,u)}{2(2n+1)(u+1)^4 n^2 BV}.$$

Multiplying by $(u+1)^4 n^2 ABUV$, this is equivalent to

$$\begin{aligned}&(u+1)^2(n+u)^2 AB - \Delta_u UV \\ &\quad = -nBV\mathscr{A}_1(n-1,u) + \frac{(n+u)^2 AU\mathscr{A}_1(n,u)}{2(2n+1)}.\end{aligned} \tag{4.4}$$

Step 3: coefficient comparison. Substitute $U = \alpha A + \beta B$ and $V = \gamma A + \delta B$ into (4.4). The coefficient of B^2 vanishes because

$$\Delta_u\beta = n\mathscr{A}_1(n-1,u).$$

The coefficient of A^2 vanishes because

$$\Delta_u\gamma = -\frac{(n+u)^2\mathscr{A}_1(n,u)}{2(2n+1)}.$$

It remains to verify the coefficient of AB. We need

$$\Delta_u(\alpha\delta - \beta\gamma) = (u+1)^2(n+u)^2. \tag{4.5}$$

The remaining AB-coefficient vanishes precisely by (4.5). After substituting $\mathscr{A}_0$, $\mathscr{A}_1$, and a_{n+1} and clearing denominators, the difference between the two sides is the zero polynomial in n and u; see Appendix B. Thus, all coefficients vanish, and (4.2) follows.

Theorem 4.2. *For every integer $u \geq 0$,*

$$X(u) = \binom{2u}{u}^3 \frac{\pi^2}{18} + \rho_u, \qquad \rho_u \in \mathbb{Q}.$$

Moreover, the rational correction is determined by

$$\rho_0 = 0$$

and

$$\rho_{u+1} = \frac{8(2u+1)^3}{(u+1)^3}\rho_u - \frac{21u^2 + 34u + 13}{3(u+1)^4}, \qquad u \geq 0. \tag{4.6}$$

Proof. The induction is carried out at the level of summands. We show that $S_u(n)$ is the u-independent central-binomial factor multiplied by C_u, plus a telescoping difference. We prove by induction on u that there exists a sequence $G_u(n)$, built recursively from $G_0 = 0$ and the terms $\Theta_0, \ldots, \Theta_{u-1}$, such that

$$S_u(n) = C_u \frac{1}{n^2\binom{2n}{n}} + G_u(n) - G_u(n+1), \qquad G_u(n) \to 0. \tag{4.7}$$

For $u = 0$, we have $P_0(n) = 1$, $F_0(n) = 1$, and $C_0 = 1$. Hence,

$$S_0(n) = \frac{1}{n^2\binom{2n}{n}},$$

so (4.7) holds with $G_0(n) = 0$.

Assume (4.7) holds for a fixed u. By Lemma 4.1,

$$S_{u+1}(n) = \lambda_u S_u(n) + \Theta_u(n) - \Theta_u(n+1).$$

Substituting the induction hypothesis gives

$$S_{u+1}(n) = \lambda_u C_u \frac{1}{n^2\binom{2n}{n}} + (\lambda_u G_u(n) + \Theta_u(n)) - (\lambda_u G_u(n+1) + \Theta_u(n+1)).$$

Since $\lambda_u C_u = C_{u+1}$, the induction continues with

$$G_{u+1}(n) := \lambda_u G_u(n) + \Theta_u(n).$$

It remains to check the tail condition. For $u = 0$, $G_0(n) = 0$. Suppose $G_u(n) \to 0$. Since

$$P_u(n) = \frac{1}{D_u} n^{4u} + O(n^{4u-1}), \qquad P_{u+1}(n) = \frac{1}{D_{u+1}} n^{4u+4} + O(n^{4u+3}),$$

the denominator $P_u(n-1)P_{u+1}(n-1)$ has polynomial growth. On the other hand,

$$\binom{2n}{n}^{-1} = O(4^{-n} n^{1/2}).$$

The remaining factors in (4.1) are polynomial in n for fixed u. Hence, $\Theta_u(n) \to 0$, and therefore

$$G_{u+1}(n) = \lambda_u G_u(n) + \Theta_u(n) \to 0.$$

Thus, the tail condition holds for every $u \geq 0$ by induction.

Summing (4.7) from $n = 1$ to N gives

$$\sum_{n=1}^{N} S_u(n) = C_u \sum_{n=1}^{N} \frac{1}{n^2\binom{2n}{n}} + G_u(1) - G_u(N+1).$$

Letting $N \to \infty$ and using

$$\sum_{n\geq 1} \frac{1}{n^2\binom{2n}{n}} = \frac{\pi^2}{18},$$

we obtain

$$X(u) = C_u \frac{\pi^2}{18} + G_u(1).$$

Finally, $G_u(1) \in \mathbb{Q}$ because all quantities used in the recursion defining G_u are rational at $n = 1$. Thus the theorem follows with $\rho_u = G_u(1)$.

The recurrence for ρ_u follows by evaluating the recursion

$$G_{u+1}(n) = \lambda_u G_u(n) + \Theta_u(n)$$

at $n = 1$. Since

$$F_u(1) = 1, \qquad P_u(0) = P_{u+1}(0) = 1, \qquad \binom{2}{1} = 2,$$

and

$$\ell_u(1) = 21u^2 + 34u + 13,$$

we have

$$\Theta_u(1) = -\frac{21u^2 + 34u + 13}{3(u+1)^4}.$$

Therefore,

$$\rho_{u+1} = \lambda_u \rho_u + \Theta_u(1) = \frac{8(2u+1)^3}{(u+1)^3}\rho_u - \frac{21u^2 + 34u + 13}{3(u+1)^4},$$

as claimed.

Examples. The first cases illustrate the evaluation formula.

Example 4.3. For $u = 0$, one has $P_0(n) = 1$ and $F_0(n) = 1$. Hence,

$$S_0(n) = \frac{1}{n^2\binom{2n}{n}}.$$

Since $C_0 = 1$ and $\rho_0 = 0$, Theorem 4.2 gives

$$X(0) = \frac{\pi^2}{18}.$$

Example 4.4. For $u = 1$, the normalized polynomial is

$$P_1(n) = \frac{(n+1)(n^3 + 9n^2 + 20n + 8)}{8}, \qquad C_1 = 8.$$

The correction term comes from (4.6):

$$\rho_1 = 8\rho_0 - \frac{13}{3} = -\frac{13}{3}.$$

Therefore,

$$X(1) = C_1 \frac{\pi^2}{18} + \rho_1 = \frac{4\pi^2}{9} - \frac{13}{3}.$$

Example 4.5. For $u = 2$, the normalized polynomial is

$$\begin{aligned} P_2(n) = \frac{(n+1)(n+2)}{3456}(n^6 + 33n^5 + 397n^4 + 2175n^3 \\ + 5506n^2 + 5712n + 1728), \end{aligned}$$

and $C_2 = 216$. The correction term is again obtained from (4.6):

$$\rho_2 = 27\rho_1 - \frac{17}{12} = -\frac{1421}{12}.$$

Hence,

$$X(2) = C_2 \frac{\pi^2}{18} + \rho_2 = 12\pi^2 - \frac{1421}{12}.$$

5. Conclusions

We have proved the evaluation of the family (1.1) for every integer $u \geq 0$:

$$X(u) = \binom{2u}{u}^3 \frac{\pi^2}{18} + \rho_u, \qquad \rho_u \in \mathbb{Q}.$$

The parameter-raising operator constructs the normalized denominator polynomials P_u, and the parameter-shift telescoping identity turns this construction into an induction on u. Thus, the polynomial identities are not only auxiliary computations but form the algebraic structure behind the induction.

It remains natural to ask whether the same idea can be used for other Apéry-like constants, for example $\zeta(3)$. That case seems more rigid because the usual kernels involve higher-order recurrences and more delicate cancellations.

Use of AI tools declaration

The authors declare that generative AI tools were used only for language editing during the preparation of this manuscript.

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Conflict of interest

The authors declare there are no conflicts of interest.

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Appendix

A. Coefficient check for the parameter-raising identity

This appendix records the coefficient comparison used in Theorem 3.1. After applying both sides of (3.4) to an arbitrary F, the four coefficients of $F(n+1)$, $F(n)$, $F(n-1)$, and $F(n-2)$ are as follows. With

$$P(n,u) := n^2 + 10nu + 8n + 21u^2 + 34u + 13,$$

the common coefficient of $F(n+1)$ on the two sides of (3.4) is

$$\frac{4}{3}(2n-1)(2n+1)(n+u+1)P(n,u).$$

For the coefficient of $F(n)$, both sides reduce to

$$-\frac{2}{3}(2n-1)Q_0(n,u),$$

where

$$\begin{aligned} Q_0(n,u) = {}& 6n^4 + 72n^3u + 48n^3 + 264n^2u^2 + 342n^2u + 109n^2 \\ & + 384nu^3 + 732nu^2 + 458nu + 94n \\ & + 186u^4 + 462u^3 + 425u^2 + 172u + 26. \end{aligned}$$

For the coefficient of $F(n-1)$, both sides reduce to

$$\frac{n+u}{3}Q_{-1}(n,u),$$

where

$$\begin{aligned} Q_{-1}(n,u) = {}& 9n^4 + 84n^3u + 36n^3 + 162n^2u^2 + 54n^2u - n^2 \\ & + 12nu^3 - 216nu^2 - 110nu \\ & - 27u^4 - 90u^3 + 15u^2 + 24u - 6. \end{aligned}$$

Finally, the common coefficient of $F(n-2)$ is

$$-\frac{1}{3}(n-1)(n+u)(n+u-1)(n^2 + 6nu + 2n - 3u^2 - 12u - 6).$$

Thus, all four shift coefficients agree. This proves identity (3.4) without imposing any additional condition on the function F.

B. Algebraic check in the parameter-shift identity

This appendix records the algebraic check used in Lemma 4.1. We need to verify

$$\Delta_u(\alpha\delta - \beta\gamma) = (u+1)^2(n+u)^2.$$

Recall that

$$\Delta_u = 8(u+1)(2u+1)^3.$$

Write

$$\mathscr{A}_0(n,u) = \frac{(n+u+1)B_0(n,u)}{3}, \qquad \mathscr{A}_1(n,u) = \frac{2(2n+1)B_1(n,u)}{3},$$

where

$$B_0(n,u) = -n^2 - 6nu - 6n + 3u^2 - 2$$

and

$$B_1(n,u) = n^2 + 10nu + 8n + 21u^2 + 34u + 13.$$

We also use the shifted quantities

$$B_0^-(n,u) := B_0(n-1,u) = -n^2 - 6nu - 4n + 3u^2 + 6u + 3,$$

$$B_1^-(n,u) := B_1(n-1,u) = n^2 + (10u+6)n + 21u^2 + 24u + 6,$$

and

$$a_+(n,u) := a_{n+1}(u) = 5(n+1)^2 + (14u-4)(n+1) + (3u-1)^2.$$

With this notation, the four coefficients become

$$\alpha = \frac{(n+u)^2 B_0^-(n,u)}{3\Delta_u},$$

$$\beta = \frac{2n(2n-1)B_1^-(n,u)}{3\Delta_u},$$

$$\gamma = -\frac{(n+u)^2 B_1(n,u)}{3\Delta_u},$$

and

$$\delta = \frac{(n+u+1)^2 B_0(n,u) + B_1(n,u)a_+(n,u)}{3\Delta_u}.$$

Therefore, the desired determinant identity is equivalent, after cancelling the common factor $(n+u)^2/(9\Delta_u)$, to

$$\begin{aligned} B_0^-(n,u)\Big((n+u+1)^2 B_0(n,u) + B_1(n,u)a_+(n,u)\Big) \\ + 2n(2n-1)B_1^-(n,u)B_1(n,u) \\ = 72(u+1)^3(2u+1)^3. \end{aligned}$$

To make the cancellation visible, define

$$\begin{aligned} \mathcal{E}(n,u) := B_0^-(n,u)\Big((n+u+1)^2 B_0(n,u) + B_1(n,u)a_+(n,u)\Big) \\ + 2n(2n-1)B_1^-(n,u)B_1(n,u). \end{aligned}$$

A direct expansion gives

$$\mathcal{E}(n,u) = 576u^6 + 2592u^5 + 4752u^4 + 4536u^3 + 2376u^2 + 648u + 72.$$

Equivalently,

$$\mathcal{E}(n,u) = 72(u+1)^3(2u+1)^3.$$

Thus, all coefficients of n^k, $k \geq 1$ cancel. This proves the required determinant identity

$$\Delta_u(\alpha\delta - \beta\gamma) = (u+1)^2(n+u)^2.$$