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Research article

Lyapunov-type inequalities for systems of second order quasilinear differential equations

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Abstract: We establish Lyapunov-type inequalities for systems of second order quasilinear differential equations with Dirichlet boundary conditions. By incorporating the positive parts of the coefficient functions, we correct and extend several known inequalities for quasilinear systems. Our results cover constant and variable coefficient cases and lead to explicit lower bounds for generalized eigenvalues. These inequalities generalize classical scalar results and improve the existing estimates within the literature. As an application, we obtain explicit and computable lower bounds for generalized eigenvalues of coupled quasilinear operators.

Keywords: systems of nonlinear differential equations; Lyapunov-type inequalities; eigenvalue estimation

1. Introduction

Lyapunov-type inequalities play a fundamental role in the qualitative theory of differential equations. They provide necessary conditions for the existence of nontrivial solutions of boundary value problems (BVPs) and have been widely used in oscillation theory, disconjugacy, and the study of eigenvalue problems. In particular, such inequalities yield explicit lower bounds for eigenvalues of differential operators and are closely related to stability and resonance phenomena in nonlinear models. For the second order linear differential equation

$$u'' + f(x)u = 0, \tag{1.1}$$

where $f \in L^1(a, b)$, the classical Lyapunov inequality asserts that if Eq (1.1) admits a nontrivial solution that satisfies the Dirichlet boundary conditions (BCs) $u(a) = u(b) = 0$ and $u(x) \neq 0$ for $x \in (a, b)$, then

$$\int_a^b |f(x)|dx > \frac{4}{b-a}. \tag{1.2}$$

We refer the interested reader to [1, 2] for further details.

This result has been refined and extended in numerous directions. In particular, it was observed by Wintner [3] and later by several other authors that inequality (1.2) can be sharpened by replacing $|f(x)|$ with $f^+(x) := \max\{0, f(x)\}$, which is the nonnegative part of $f(x)$, to become the following:

$$\int_a^b f^+(x)dx > \frac{4}{b-a}. \tag{1.3}$$

Lyapunov-type inequalities have been extensively studied for various scalar equations. For more on the topic, we refer the interested reader to [4–11] and the references cited therein.

Subsequent generalizations of Lyapunov-type inequalities include extensions to variable coefficient equations of the form

$$(r(x)u')' + f(x)u = 0, \tag{1.4}$$

where $r \in C([a, b], \mathbb{R})$ such that $r(x) > 0$ for $x \in [a, b]$. Specifically, in [12, Chap. XI], Hartman established that if Eq (1.4) admits a nontrivial solution that satisfies the BCs $u(a) = u(b) = 0$ and $u(x) \neq 0$ for $x \in (a, b)$, then

$$\int_a^b f^+(x)dx > \frac{4}{\int_a^b r^{-1}(x)dx}. \tag{1.5}$$

More recently, Lyapunov-type inequalities have been developed for second order quasilinear equations that involve the p-Laplacian. For instance, Yang considered the following equation:

$$(r(x)\phi_p(u'))' + f(x)\phi_p(u) = 0, \tag{1.6}$$

where $\phi_p(x) = |x|^{p-1}x$ for $p > 0$. In [13], he proved that if Eq (1.6) admits a nontrivial solution that satisfies the BCs $u(a) = u(b) = 0$ and $u(x) \neq 0$ for $x \in (a, b)$, then

$$\int_a^b f^+(x)dx > \frac{2^{p+1}}{\left(\int_a^b r^{-\frac{1}{p}}(x)dx\right)^p}. \tag{1.7}$$

It immediately follows that for $p = 1$, (1.7) reduces to (1.5). At the time of writing, Yang's result was the most general result for a second order quasilinear equation. Dhar and Kelly [8] further generalized the results by establishing results for higher order, variable coefficient quasilinear equations.

Lyapunov-type results have also been extended to systems of differential equations, but several questions remain open. Most existing results for systems either require sign restrictions on the coefficient functions or rely on assumptions that prevent direct applications to problems with sign-changing nonlinearities. Moreover, some previously published inequalities for quasilinear systems implicitly assume positivity conditions that are not explicitly stated, which may lead to incorrect conclusions in general settings. Among the few results that are available for systems of linear or quasilinear differential equations, Napoli and Pinasco [14] first considered the following one-dimensional quasilinear elliptic system of resonant type that involve the (p_1, p_2)-Laplacian operators as follows:

$$\begin{aligned} -\left(|u_1'|^{p_1-2}u_1'\right)' &= f_1(x)|u_1|^{\alpha_1-2}u_1|u_2|^{\alpha_2} \\ -\left(|u_2'|^{p_2-2}u_2'\right)' &= f_2(x)|u_1|^{\alpha_1}|u_2|^{\alpha_2-2}u_2, \end{aligned} \tag{1.8}$$

where $f_1, f_2 \in L^1(a,b)$, $p_1, p_2 > 1$ and $\alpha_1, \alpha_2 > 0$ are such that

$$\frac{\alpha_1}{p_1} + \frac{\alpha_2}{p_2} = 1, \tag{1.9}$$

together with BCs

$$u_1(a) = u_1(b) = u_2(a) = u_2(b) = 0. \tag{1.10}$$

In [14], they established that if System (1.8) has a nontrivial solution (u_1, u_2) that satisfies (1.10), then

$$\left(\int_a^b f_1(x)dx\right)^{\frac{\alpha_1}{p_1}} \left(\int_a^b f_2(x)dx\right)^{\frac{\alpha_2}{p_2}} \geq \frac{2^{\alpha_1+\alpha_2}}{(b-a)^{\alpha_1+\alpha_2-1}}. \tag{1.11}$$

It is worth noting that in (1.11), the integrals do not involve the absolute value or the positive parts of the functions $f_1,\ f_2$. Thus, (1.11) may not be defined for certain functions, namely those where $\int_a^b f_i\, dx < 0$ for $i = 1, 2$ and the corresponding exponent is non-integer. Therefore, (1.11) does not apply to sign-changing coefficients in general. In this paper, we have restructured the proof and corrected (1.11) by replacing the functions $f_1,\ f_2$ by $f_1^+,\ f_2^+$ (i.e., the positive parts of $f_1,\ f_2$), which remain meaningful for sign-changing data and coincide with the earlier results in the special case $f_i \geq 0$. Our results both correct the formulation of the previous inequalities and enlarge the class of admissible coefficients. Later, we provide a generalized result for n quasilinear equations that involve the positive parts of these functions.

Furthermore, Çakmak and Tiryaki extended results to a system of $(p_1, p_2, \cdots, p_n)$-quasilinear equations in [15]. In particular, they considered the following system of n-quasilinear equations:

$$\begin{aligned}
-\left(|u_1'|^{p_1-2}u_1'\right)' &= f_1(x)|u_1|^{\alpha_1-2}u_1|u_2|^{\alpha_2}\cdots|u_n|^{\alpha_n} \\
-\left(|u_2'|^{p_2-2}u_2'\right)' &= f_2(x)|u_1|^{\alpha_1}|u_2|^{\alpha_2-2}u_2\cdots|u_n|^{\alpha_n} \\
&\vdots \\
-\left(|u_n'|^{p_n-2}u_n'\right)' &= f_n(x)|u_1|^{\alpha_1}|u_2|^{\alpha_2}\cdots|u_n|^{\alpha_n-2}u_n,
\end{aligned} \tag{1.12}$$

where $f_i \in L^1(a,b)$, $p_i > 1$ and α_i are such that

$$\sum_{i=1}^{n} \frac{\alpha_i}{p_i} = 1, \quad 1 \leq i \leq n, \tag{1.13}$$

together with BCs

$$u_i(a) = u_i(b) = 0, \quad 1 \leq i \leq n. \tag{1.14}$$

Unlike scalar equations, Lyapunov-type inequalities for systems exhibit genuine coupling effects, where the interaction between components cannot be reduced to independent scalar estimates. In particular, the balance condition (1.13) plays a critical role in canceling nonlinear interactions and reflects a natural scaling invariance of the system. This structural feature is absent in scalar problems and highlights the necessity of developing system-specific techniques.

By extending Lyapunov-type inequality results for systems, Çakmak and Tiryaki established the following inequality for non-zero solutions of System (1.12) that satisfies (1.13) and BCs (1.14):

$$\prod_{i=1}^{n}\left(\int_a^b f_i^+(x)\,dx\right)^{\frac{\alpha_i}{p_i}} \geq \prod_{i=1}^{n}\left[(c_i-a)^{1-p_i}+(b-c_i)^{1-p_i}\right]^{\frac{\alpha_i}{p_i}}. \tag{1.15}$$

Tang and He [16] considered the variable coefficient quasilinear elliptic system

$$\begin{aligned} -\left(r_1(x)|u_1'|^{p_1-2}u_1'\right)' &= f_1(x)|u_1|^{\alpha_1-2}u_1|u_2|^{\alpha_2}\cdots|u_n|^{\alpha_n}\\ -\left(r_2(x)|u_2'|^{p_2-2}u_2'\right)' &= f_2(x)|u_1|^{\alpha_1}|u_2|^{\alpha_2-2}u_2\cdots|u_n|^{\alpha_n}\\ &\vdots\\ -\left(r_n(x)|u_n'|^{p_n-2}u_n'\right)' &= f_n(x)|u_1|^{\alpha_1}|u_2|^{\alpha_2}\cdots|u_n|^{\alpha_n-2}u_n\,, \end{aligned} \tag{1.16}$$

where $1 < p_i < \infty$, $\alpha_i > 0$ satisfies (1.13), $r_i(x) > 0$ for all $t \in (a, b)$, and $f_i(x)$ is continuous on $\mathbb{R}$ for $i \in \{1, 2, \ldots, n\}$. For nontrivial solutions to System (1.16) that satisfy BCs (1.14),

$$\int_a^b \frac{(\int_a^x [r(\tau)]^{1/(1-p)}d\tau)^{p-1}(\int_x^b [r(\tau)]^{1/(1-p)}d\tau)^{p-1})}{(\int_a^x [r(\tau)]^{1/(1-p)}d\tau)^{p-1}+(\int_x^b [r(\tau)]^{1/(1-p)}d\tau)^{p-1})} f^+(x)\,dx > 1\,. \tag{1.17}$$

While the obtained result (1.17) is stronger than (1.15), the improvement comes with a corresponding increase in technical complexity. For additional results related to Lyapunov-type inequalities for systems of equations, see [17–20].

The main purpose of this paper is to establish new Lyapunov-type inequalities for systems of quasilinear differential equations subject to Dirichlet BCs. In Section 2, we establish Lyapunov-type inequalities for various quasilinear systems. First, we begin with a quasilinear system of two equations with constant coefficients; then, we generalize this result for equations which have variable coefficients and systems with n equations. A discussion of several special cases that illustrate the structure of the results is included. Our approach systematically incorporates the positive parts of the coefficient functions, which allows us to correct and extend several known inequalities in the literature, such as [14]. Additionally, our methods and results are simpler and easier to follow than that of [16]. For completeness, the results are obtained for both constant and variable coefficient cases and are valid for general n-dimensional systems of second order quasilinear differential equations. The constants obtained in our inequalities are consistent with known sharp constants in the scalar cases. Lastly, Section 3 is devoted to applications to eigenvalue problems, where explicit lower bounds are obtained.

2. Main results

Before stating the main results, we specify the functional setting used throughout the paper. For each $i \in \{1, \ldots, n\}$, we consider solutions $u_i \in W_0^{1,p_i}(a, b)$. When the expressions

$$(r_i(x)|u_i'(x)|^{p_i-2}u_i'(x))'$$

are distributionally interpreted, Systems (1.8), (1.12), and (1.16) are understood in the weak sense. In particular, for every test function $\varphi_i \in W_0^{1,p_i}(a,b)$, the ith equation is satisfied in the following variational form:

$$\int_a^b r_i(x)|u_i'|^{p_i-2}u_i'\varphi_i'\,dx = \int_a^b f_i(x)|u_i|^{\alpha_i-2}u_i\prod_{j\neq i}|u_j|^{\alpha_j}\varphi_i\,dx,$$

with $r_i \equiv 1$ in the constant-coefficient case. Whenever additional regularity is available, the weak solutions are classical, and all identities below may be read in the classical sense. Since our arguments only rely on the weak formulation, the boundary condition, and Sobolev-type estimates, the proofs remain valid in this framework. In particular, all mathematical tools used in the proofs are justified within this weak formulation.

Next, we establish Lyapunov-type inequalities for quasilinear systems with constant coefficients (i.e., $r_i \equiv 1$). We begin with the two-equation system and derive a sharp inequality using Sobolev-type estimates. Then, we extend the argument to general n-dimensional systems and finally present a unified framework that prepares for variable coefficient systems treated later. These results form the foundation for the eigenvalue estimates in Section 3. For all $u_i \in W_0^{1,p_i}(a,b)$, we define the following:

$$\|u_i\|_\infty = \sup_{x\in[a,b]}|u_i(x)|, \quad 1 < p_i < \infty.$$

It is clear that for all $x \in (a,b)$ and $\gamma > 0$, we have the following:

$$|u_i(x)|^\gamma \le \|u_i\|_\infty^\gamma\,. \tag{2.1}$$

In [8], the authors established a Sobolev–type inequality for a higher derivative of the function u. In particular, they established the following result in [8, Lemma 3.1].

Lemma 2.1. *Let $1 < p < \infty$ and $m \in \mathbb{N}$. Assume $u \in C^{(m)}(a,b)$ and u satisfies $u(a) = u(b) = 0$. Then,*

$$\|u\|_\infty^p < \frac{(b-a)^{mp-1}}{2^{mp}\,(\Gamma(m))^p}\left(\frac{p-1}{mp-1}\right)^{p-1}\int_a^b \left|u^{(m)}(x)\right|^p\,dx\,. \tag{2.2}$$

For the results presented in this paper, it is sufficient to restrict their results for the $u \in C^1(a,b)$. To that end, we only use the case $m = 1$ throughout this paper. Additionally, for $i \in \{1,\dots,n\}$, we define $p \equiv p_i$ and $u \equiv u_i$. Then, (2.2) gets reduced to the following:

$$\|u_i\|_\infty^p < \frac{(b-a)^{p-1}}{2^p}\int_a^b |u_i'|^p\,dx. \tag{2.3}$$

Now, we present the first Lyapunov-type inequalities for System (1.8). Our proofs rely on (2.3), and the novelty lies in the systematic use of the integrals of the positive parts of the functions f_1 and f_2, which corrects and strengthens the existing results. In particular, this approach allows us to establish Lyapunov-type inequalities for systems where the original coefficients change sign, a case not covered by earlier works.

Theorem 2.1. *Assume that* (1.9) *holds, and System* (1.8) *has a nontrivial solution $u(x) = (u_1(x), u_2(x))$ that satisfies BCs* (1.10)*. Then,*

$$\left(\int_a^b f_1^+\,dx\right)^{\frac{\alpha_1}{p_1}}\left(\int_a^b f_2^+\,dx\right)^{\frac{\alpha_2}{p_2}} > \frac{2^{\alpha_1+\alpha_2}}{(b-a)^{\alpha_1+\alpha_2-1}}. \tag{2.4}$$

Proof. By multiplying the first equation of System (1.8) by u_1 and using integration by parts together with $u_1(a) = u_1(b) = 0$, we have the following:

$$\int_a^b |u_1'|^{p_1}\,dx = \int_a^b f_1(x)|u_1|^{\alpha_1}|u_2|^{\alpha_2}\,dx. \tag{2.5}$$

Recall that $f_1^+(x) = \max\{0, f_1(x)\}$ and $\alpha_1, \alpha_2 > 0$. Then it follows that

$$\int_a^b |u_1'|^{p_1}\,dx \le \|u_1\|_\infty^{\alpha_1}\|u_2\|_\infty^{\alpha_2}\int_a^b f_1^+\,dx. \tag{2.6}$$

From (2.3), we have the following:

$$\|u_1\|_\infty^{p_1} < \frac{(b-a)^{p_1-1}}{2^{p_1}}\int_a^b |u_1'|^{p_1}\,dx. \tag{2.7}$$

Using the estimates from (2.7) in (2.6), we see that

$$\frac{2^{p_1}}{(b-a)^{p_1-1}} < \|u_1\|_\infty^{\alpha_1-p_1}\|u_2\|_\infty^{\alpha_2}\int_a^b f_1^+\,dx. \tag{2.8}$$

A similar process for u_2 yields the following:

$$\frac{2^{p_2}}{(b-a)^{p_2-1}} < \|u_1\|_\infty^{\alpha_1}\|u_2\|_\infty^{\alpha_2-p_2}\int_a^b f_2^+\,dx. \tag{2.9}$$

Now, we raise both sides of (2.8) and and (2.9) to the power $e_1 \neq 0$ and $e_2 \neq 0$, respectively. After multiplying the resulting equations, we obtain the following:

$$\begin{aligned}\frac{2^{p_1e_1+p_2e_2}}{(b-a)^{(p_1-1)e_1+(p_2-1)e_2}} &< \|u_1\|_\infty^{(\alpha_1-p_1)e_1+\alpha_1e_2}\|u_2\|_\infty^{\alpha_2e_1+(\alpha_2-p_2)e_2}\\ &\quad\times\left(\int_a^b f_1^+\,dx\right)^{e_1}\left(\int_a^b f_2^+\,dx\right)^{e_2}.\end{aligned} \tag{2.10}$$

We choose e_1 and e_2 such that the exponents of $\|u_1\|_\infty$ and $\|u_2\|_\infty$ become zero. In other words, we solve the following system of homogeneous linear equations:

$$\begin{aligned}(\alpha_1-p_1)e_1 + \alpha_1e_2 &= 0\\ \alpha_2e_1 + (\alpha_2-p_2)e_2 &= 0.\end{aligned} \tag{2.11}$$

This system admits a nontrivial solution if and only if the corresponding coefficient matrix has a determinant of 0. This is equivalent to

$$\frac{\alpha_1}{p_1} + \frac{\alpha_2}{p_2} = 1,$$

which is precisely condition (1.9) for the system. Hence, both the equations in (2.11) are equivalent to the following:

$$e_1\frac{\alpha_2}{p_2} = e_2\frac{\alpha_1}{p_1}.$$

Thus, we may choose $e_i = \frac{\alpha_i}{p_i}$ for $i \in \{1, 2\}$. Using this in (2.10), we see that (2.4) holds.

Remark 2.1. The proof of Theorem 2.1 mirrors that of [14, Theorem 1.5], thereby correcting the result by considering f_1^+ and f_2^+. Moreover, the constant appearing in Theorem 2.1 is consistent with the classical Lyapunov constant in the scalar case and coincides with the bound obtained in [14] when $f_i \geq 0$. This suggests that the inequality is sharp with respect to the interval length $b - a$.

Example 2.1. Consider the two-equation system (1.8) on $(0, \pi)$ with

$$p_1 = p_2 = 2, \qquad \alpha_1 = \alpha_2 = 1,$$

so that (1.9) holds. Let

$$f_1(x) = \sin x - \frac{3}{4}, \qquad f_2(x) = 2 + \cos x.$$

Then,

$$\int_0^\pi f_1(x)\, dx = 2 - \frac{3\pi}{4} < 0, \qquad \int_0^\pi f_2(x)\, dx = 2\pi.$$

Hence, the expression that appears in (1.11) is not meaningful over the reals, since it involves the following:

$$\left(\int_0^\pi f_1(x)\, dx\right)^{\frac{1}{2}} = \left(2 - \frac{3\pi}{4}\right)^{\frac{1}{2}}.$$

In contrast, Theorem 2.1 applies to the positive parts f_1^+ and f_2^+, and yields the following:

$$\left(\int_0^\pi f_1^+(x)\, dx\right)^{\frac{1}{2}} \left(\int_0^\pi f_2^+(x)\, dx\right)^{\frac{1}{2}} > \frac{2}{\pi}.$$

Thus the formulation of Theorem 2.1 remains well-defined for sign-changing coefficients and covers cases excluded by (1.11). This shows that the earlier inequality does not provide any information in this case, whereas Theorem 2.1 yields a valid and nontrivial bound.

Now, we extend the results from Theorem 2.1 to a general system with n quasilinear equations (i.e., System (1.12)).

Theorem 2.2. *Assume that* (1.13) *holds, and System* (1.12) *has a nontrivial solution* $u(x)$ *that satisfies BCs* (1.14)*. Then*

$$\prod_{i=1}^{n} \left(\int_a^b f_i^+\, dx\right)^{\frac{\alpha_i}{p_i}} > \frac{2^B}{(b-a)^{B-1}}, \tag{2.12}$$

where $B = \sum_{i=1}^{n} \alpha_i$.

Proof. First, we rewrite System (1.12) using the associated equation as follows:

$$-\left(|u_i'|^{p_i-2}u_i'\right)' = f_i(x)|u_i|^{\alpha_i-2}u_i \prod_{j=1, j\neq i}^{n} |u_j|^{\alpha_j}, \quad i \in \{1, \ldots, n\}. \tag{2.13}$$

By multiplying (2.13) by u_i and using integration by parts together with BCs (1.14), we have the following:

$$\int_a^b |u_i'|^{p_i}\, dx = \int_a^b f_i(x) \prod_{j=1}^{n} |u_j|^{\alpha_j}\, dx. \tag{2.14}$$

Rewriting (2.14), using (2.1) and $f_i^+(x) = \max\{0, f_i(x)\}$ yields the following:

$$\int_a^b |u_i'|^{p_i}\, dx \le \left(\prod_{j=1}^{n} \|u_j\|_\infty^{\alpha_j}\right) \int_a^b f_i^+\, dx. \tag{2.15}$$

Using (2.3) it follows that

$$\|u_i\|_\infty^{p_i} < \frac{(b-a)^{p_i-1}}{2^{p_i}} \left(\prod_{j=1}^{n} \|u_j\|_\infty^{\alpha_j}\right) \int_a^b f_i^+\, dx.$$

This, after rearrangement, yields the following:

$$\begin{aligned} \frac{2^{p_i}}{(b-a)^{p_i-1}} &< \frac{\prod_{j=1}^{n} \|u_j\|_\infty^{\alpha_j}}{\|u_i\|_\infty^{p_i}} \int_a^b f_i^+\, dx \\ &= \left(\prod_{j=1, j\neq i}^{n} \|u_j\|_\infty^{\alpha_j}\right) \|u_i\|_\infty^{\alpha_i - p_i} \int_a^b f_i^+\, dx\,. \end{aligned} \tag{2.16}$$

We raise both sides of (2.16) to a power, $e_i \neq 0$:

$$\frac{2^{p_i e_i}}{(b-a)^{(p_i-1)e_i}} < \left(\prod_{j=1, j\neq i}^{n} \|u_j\|_\infty^{\alpha_j}\right)^{e_i} \|u_i\|_\infty^{(\alpha_i - p_i)e_i} \left(\int_a^b f_i^+\, dx\right)^{e_i}. \tag{2.17}$$

The procedure outlined above is performed for each equation in System (1.12), that is, for each $i \in \{1, \ldots, n\}$. The result is n inequalities in the form of Eq (2.17). Multiplying these n equations yields the following:

$$\begin{aligned} \frac{2^{\sum_{i=1}^{n} p_i e_i}}{(b-a)^{\sum_{i=1}^{n} (p_i-1)e_i}} &< \prod_{i=1}^{n} \left(\|u_i\|_\infty^{(\alpha_i - p_i)e_i} \prod_{j=1, j\neq i}^{n} \|u_j\|_\infty^{\alpha_j e_i}\right) \prod_{i=1}^{n} \left(\int_a^b f_i^+\, dx\right)^{e_i} \\ &= \prod_{i=1}^{n} \left(\|u_i\|_\infty^{\left((\alpha_i - p_i)e_i + \sum_{j=1, j\neq i}^{n} \alpha_j e_j\right)} \left(\int_a^b f_i^+\, dx\right)^{e_i}\right). \end{aligned} \tag{2.18}$$

We choose $\{e_i\}_{i=1}^{n}$ for $i \in \{1, \ldots, n\}$ such that

$$(\alpha_i - p_i)e_i + \sum_{j=1, j\neq i}^{n} \alpha_j e_j = 0.$$

In other words, we aim to solve the following system of homogeneous linear equations:

$$\begin{aligned} (\alpha_1 - p_1)e_1 + \alpha_1 e_2 + \cdots + \alpha_1 e_n &= 0 \\ \alpha_2 e_1 + (\alpha_2 - p_2)e_2 + \cdots + \alpha_2 e_n &= 0 \\ &\vdots \\ \alpha_n e_n + \alpha_n e_2 + \cdots + (\alpha_n - p_n)e_n &= 0. \end{aligned} \tag{2.19}$$

This system admits a nontrivial solution if and only if the corresponding coefficient matrix has a determinant of 0. This is equivalent to

$$\sum_{i=1}^{n} \frac{\alpha_i}{p_i} = 1,$$

which is precisely condition (1.13) for the system. In fact, (1.13) is essential to balance the nonlinear coupling terms and ensures the cancellation of the supremum norms, which is crucial for obtaining an inequality independent of the solution in the proof. From a variational perspective, this condition reflects a natural scaling invariance of the system. Now, each of the equations in (2.19) may be rewritten as

$$e_i \sum_{j=1, j\neq i} \frac{\alpha_j}{p_j} = \frac{\alpha_i}{p_i} \sum_{j=1, j\neq i} e_j$$

for $i \in \{1, \ldots, n\}$. Thus, we may choose $e_i = \frac{\alpha_i}{p_i}$ for $i \in \{1, \ldots, n\}$. Then,

$$\sum_{i=1}^{n} p_i e_i = \sum_{i=1}^{n} \alpha_i = B$$

and

$$\sum_{i=1}^{n} (p_i - 1) e_i = \sum_{i=1}^{n} (p_i - 1) \frac{\alpha_i}{p_i} = \sum_{i=1}^{n} \alpha_i - \sum_{i=1}^{n} \frac{\alpha_i}{p_i} = B - 1.$$

It follows that (2.18) reduces to the desired Lyapunov inequality (2.12).

Remark 2.2. To illustrate the structure of Theorem 2.2, we consider the symmetric case. Consider the System (1.12) under the symmetric assumptions such as

$$\alpha_i = \alpha > 0, \quad p_i = p > 1, \quad f_i = f \in L(a, b), \quad \text{for all} \quad i = 1, 2, \cdots n.$$

Hence, (1.13) implies

$$\sum_{i=1}^{n} \frac{\alpha}{p} = 1 \iff \alpha = \frac{p}{n}$$

and

$$B = \sum_{i=1}^{n} \alpha = n\alpha = p.$$

Then, System (1.12) reduces to the following:

$$-\left(|u_i'|^{p-2} u_i'\right)' = f(x) |u_i|^{\frac{p}{n}-2} u_i \prod_{j=1, j\neq i}^{n} |u_j|^{\frac{p}{n}}, \quad i \in \{1, \ldots, n\}.$$

By applying Theorem 2.2 in this symmetric setting, we obtain the following:

$$\prod_{i=1}^{n} \left(\int_a^b f^+ \, dx \right)^{\frac{\alpha}{p}} > \frac{2^p}{(b-a)^{p-1}}.$$

Since each factor in the product is identical, this reduces to the following:

$$\left(\int_a^b f^+ \, dx\right)^{\frac{n\alpha}{p}} = \int_a^b f^+ \, dx > \frac{2^p}{(b-a)^{p-1}}.$$

Although the system consists of n coupled equations, the Lyapunov inequality only depends on the exponent p and the length of the interval. The number of equations in the system does not explicitly appear. The above inequality is the classical Lyapunov inequality for quasilinear equations, which was first obtained by Yang [13]. It is clear that for $p = 2$, we recover the result for the linear case (i.e., (1.3)), which was originally established by Wintner in [3].

Finally, we present a result for a general quasilinear system involving the coefficient functions $r_i(x)$. The result below is the most general and covers all the previous results in the literature.

Theorem 2.3. *Assume that* (1.13) *holds, System* (1.16) *has a nontrivial solution $u(x)$ that satisfies BCs* (1.14)*, and let t be such that* $1 < t \le \min\{p_1, \dots p_n\}$. *Then,*

$$\prod_{i=1}^{n}\left(\int_a^b f_i^+ \, dx\right)^{\frac{\alpha_i}{p_i}} > \frac{2^{B-t+1}}{(b-a)^{B-t}\prod\limits_{i=1}^{n}\left(\int_a^b r_i^{-\frac{1}{t-1}}(x)\, dx\right)^{(t-1)(\alpha_i/p_i)}}, \tag{2.20}$$

where $B = \sum\limits_{i=1}^{n} \alpha_i$.

Proof. As before, we consider the associated equation

$$-\left(r_i(x)|u_i'|^{p_i-2}u_i'\right)' = f_i(x)|u_i|^{\alpha_i-2}u_i \prod_{j=1, j\neq i}^{n} |u_j|^{\alpha_j}, \quad i \in \{1, \dots, n\}. \tag{2.21}$$

By multiplying (2.21) by u_i and applying integration by parts together with BCs (1.14), we have the following:

$$\int_a^b r_i(x)|u_i'|^{p_i} \, dx = \int_a^b f_i(x) \prod_{j=1}^{n} |u_j|^{\alpha_j} \, dx. \tag{2.22}$$

Utilizing (2.1) and substituting $f_i^+(x) = \max\{0, f_i(x)\}$ yields the following:

$$\int_a^b r_i(x)|u_i'|^{p_i} \, dx \le \left(\prod_{j=1}^{n} \|u_j\|_\infty^{\alpha_j}\right) \int_a^b f_i^+ \, dx. \tag{2.23}$$

Choose $1 < t \le \min\{p_1, \dots, p_n\}$. The restriction $t > 1$ ensures that the exponent $\frac{-1}{t-1}$ is well-defined, while $t \le p_i$ allows the weighted Sobolev estimate from [8, Lemma 4.1] to be applied to each component u_i. Hence, for each i, it follows that

$$\|u_i\|_\infty^{p_i} < \frac{(b-a)^{p_i-t}}{2^{p_i-t+1}} \left(\int_a^b r_i^{-\frac{1}{t-1}}(x)\right)^{t-1} \int_a^b r_i(x)|u_i'|^{p_i} \, dx. \tag{2.24}$$

Combining with (2.23) gives the following:

$$\|u_i\|_\infty^{p_i} < \frac{(b-a)^{p_i-t}}{2^{p_i-t+1}} \left(\int_a^b r_i^{-\frac{1}{t-1}}(x) \right)^{t-1} \left(\prod_{j=1}^{n} \|u_j\|_\infty^{\alpha_j} \right) \int_a^b f_i^+ \, dx; \tag{2.25}$$

rearrangement leads to the following:

$$\frac{2^{p_i-t+1}}{(b-a)^{p_i-t}} < \left(\prod_{j=1, j\neq i}^{n} \|u_j\|_\infty^{\alpha_j} \right) \|u_i\|_\infty^{\alpha_i - p_i} \left(\int_a^b r_i^{-\frac{1}{t-1}}(x) \right)^{t-1} \int_a^b f_i^+ \, dx. \tag{2.26}$$

We raise both sides of (2.26) to a non-zero power, e_i, and repeat this procedure for each equation in System (1.12). Multiplying the resulting n equations yields

$$\frac{2^{\sum_{i=1}^{n}(p_i-t+1)e_i}}{(b-a)^{\sum_{i=1}^{n}(p_i-t)e_i}} < \prod_{i=1}^{n} \|u_i\|_\infty^{(\alpha_i-p_i)e_i + \sum_{j=1, j\neq i}^{n} \alpha_j e_j} \times \prod_{i=1}^{n} \left(\int_a^b r_i^{-\frac{1}{t-1}} \, dx \right)^{(t-1)e_i} \prod_{i=1}^{n} \left(\int_a^b f_i^+ \, dx \right)^{e_i}. \tag{2.27}$$

As in Theorem 2.1, we choose $e_i = \frac{\alpha_i}{p_i}$. Using (1.13), we see that for each fixed i

$$(\alpha_i - p_i)e_i + \sum_{j=1, j\neq i}^{n} \alpha_j e_j = 0 \text{ for all } i = 1, \ldots, n.$$

Furthermore, a simple computation shows that

$$\sum_{i=1}^{n}(p_i - t + 1)e_i = \sum_{i=1}^{n}(p_i - t + 1)\frac{\alpha_i}{p_i} = \sum_{i=1}^{n} \alpha_i - (t-1)\sum_{i=1}^{n} \frac{\alpha_i}{p_i} = B - t + 1,$$

and similarly

$$\sum_{i=1}^{n}(p_i - t)e_i = \sum_{i=1}^{n} \alpha_i - t\sum_{i=1}^{n} \frac{\alpha_i}{p_i} = B - t,$$

where $B = \sum_{i=1}^{n} \alpha_i$. Therefore, (2.27) reduces to the desired Lyapunov inequality (2.20).

Remark 2.3. Theorem 2.3 extends the Lyapunov-type inequalities obtained in Theorem 2.2 to systems with variable coefficients $r_i(x)$. In particular, the balance condition (1.13) guarantees the correct scaling behavior of the associated eigenvalue bounds. When $r_i(x) \equiv 1$, the inequality reduces to the constant-coefficient case, thereby showing that Theorem 2.3 is a genuine generalization rather than a separate result. The presence of the weight functions $r_i(x)$ allows applications to a wider class of problems, including eigenvalue problems with non-homogeneous or weighted operators, as illustrated in Section 3. The parameter t enters the estimate through both the factor $2^{B-t+1}(b-a)^{-(B-t)}$ and the weighted integrals $\int_a^b r_i(x)^{-1/(t-1)} \, dx$. Accordingly, different admissible choices of t may lead to different lower bounds. In applications, one may optimize the right-hand side of (2.20) over $1 < t \leq \min\{p_1, \ldots, p_n\}$ whenever the coefficient functions r_i are specified.

3. Lower bounds for eigenvalues

In this section, we apply the Lyapunov inequality result derived in Theorem 2.3 to obtain a lower bound for the generalized eigenvalues for solutions of System (1.16) that satisfy BCs (1.14). We replace $f_i(x)$ in System (1.16) with $\lambda_i \alpha_i s_i(x)$. Call $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\{u_1, u_2, \ldots, u_n\}$ the generalized n-tuple eigenvalues and the generalized eigenfunctions, respectively, if $\{u_1, u_2, \ldots, u_n\}$ are the corresponding non-trivial solutions of System (1.16) satisfying BCs (1.14). Then, by utilizing techniques similar to those of Napoli and Pinasco [14], we obtain the following result.

Theorem 3.1. *Assume that* (1.13) *holds. There exists a function* $h(\lambda_1, \lambda_2, \ldots, \lambda_{n-1})$ *such that*

$$\lambda_n > h(\lambda_1, \lambda_2, \ldots, \lambda_{n-1})$$

for every generalized eigenvalue $(n-1)$*-tuple* $(\lambda_1, \lambda_2, \ldots, \lambda_{n-1})$*, where*

$$h(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) = \frac{1}{\alpha_n} \left(\frac{C}{\prod_{i=1}^{n} \int_a^b s_i^+(x)\,dx} \right)^{p_n/\alpha_n}, \tag{3.1}$$

where

$$C = \frac{2^{B-t+1}}{(b-a)^{B-t} \prod_{i=1}^{n} \left(\int_a^b r_i(x)^{-1/(t-1)}\,dx \right)^{(t-1)(\alpha_i/p_i)} \prod_{i=1}^{n-1} (\lambda_i \alpha_i)^{\alpha_i/p_i}}$$

with $B = \sum_{i=1}^{n} \alpha_i$*.*

Proof. Let $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a generalized n-tuple of eigenvalues and $\{u_1, u_2, \ldots, u_n\}$ be the corresponding non-trivial solutions of System (1.16) that satisfy BCs (1.14).

Define $M = \frac{2^{B-t+1}}{(b-a)^{B-t} \prod_{i=1}^{n} \left(\int_a^b r_i(x)^{-1/(t-1)}\,dx \right)^{(t-1)(\alpha_i/p_i)}}$ where $B = \sum_{i=1}^{n} \alpha_i$. For each $i \in \{1, 2, \ldots, n\}$, replace $f_i(x)$ in System (1.16) with $\lambda_i \alpha_i s_i(x)$. Then, Theorem 2.3 can be rewritten as follows:

$$M < \prod_{i=1}^{n} \left(\int_a^b f_i^+(x)\,dx \right)^{\alpha_i/p_i} \tag{3.2}$$

$$= \prod_{i=1}^{n} \left(\int_a^b \lambda_i \alpha_i s_i^+(x)\,dx \right)^{\alpha_i/p_i} \tag{3.3}$$

$$= \prod_{i=1}^{n} (\lambda_i \alpha_i)^{\alpha_i/p_i} \cdot \prod_{i=1}^{n} \left(\int_a^b s_i^+(x)\,dx \right)^{\alpha_i/p_i}; \tag{3.4}$$

rearranging gives the following:

$$\left(\frac{M}{\prod_{i=1}^{n-1} (\lambda_i \alpha_i)^{\alpha_i/p_i} \cdot \prod_{i=1}^{n} \int_a^b s_i^+(x)\,dx} \right)^{p_i/\alpha_i} < \lambda_n \alpha_n. \tag{3.5}$$

Hence, we have the following:

$$\lambda_n \geq \frac{1}{\alpha_n}\left(\frac{M}{\prod_{i=1}^{n-1}(\lambda_i\alpha_i)^{\alpha_n/p_n}\cdot\prod_{i=1}^{n}\int_a^b s_i^+(x)\,dx}\right)^{p_i/\alpha_i} \tag{3.6}$$

$$= \frac{1}{\alpha_n}\left(\frac{C}{\prod_{i=1}^{n}\int_a^b s_i^+(x)\,dx}\right)^{p_n/\alpha_n}, \tag{3.7}$$

where

$$C = \frac{M}{\prod_{i=1}^{n-1}(\lambda_i\alpha_i)^{\alpha_i/p_i}}. \tag{3.8}$$

In particular, for fixed $(\lambda_1,\ldots,\lambda_{n-1})$, Theorem 3.1 yields an explicit forbidden region

$$0 < \lambda_n \leq h(\lambda_1,\ldots,\lambda_{n-1}),$$

so the estimate provides a computable spectral exclusion principle for the coupled system, independent of the particular choice of eigenfunctions. This interpretation highlights how the coupling exponents α_i, the weights r_i, and the coefficient functions s_i jointly constrain the admissible generalized eigenvalue tuples.

When $n = 2$, $r_i(x) \equiv 1$, and $f_i = \lambda_i\alpha_i s_i(x)$ for $i \in \{1,2\}$, System (1.16) that satisfies BCs (1.14) reduces to System (1.8) that satisfies BCs (1.10) and Theorem 2.1 is applicable. The following corollary is immediate.

Corollary 3.1. *There exists a function $h(\lambda_1,\lambda_2)$ such that $\lambda_2 > h(\lambda_1)$ for every generalized eigenvalue pair (λ_1,λ_2), where*

$$h(\lambda_1) = \frac{1}{\alpha_2}\left(\frac{2^{\alpha_1+\alpha_2}}{(\lambda_1\alpha_1)^{\alpha_1/p_1}(b-a)^{\alpha_1+\alpha_2-1}\int_a^b s_1^+(x)\,dx}\right)^{p_2/\alpha_2}. \tag{3.9}$$

Remark 3.1. The result of Corollary 3.1 is consistent with the results of [14, Theorem 1.4].

Similarly, for $n > 1$ and $r_i(x) \equiv 1$, Theorem 2.2 applies and the result below immediately follows.

Corollary 3.2. *There exists a function $h(\lambda_1,\lambda_2,\ldots,\lambda_{n-1})$ such that $\lambda_n > h(\lambda_1,\lambda_2,\ldots,\lambda_{n-1})$ for every generalized eigenvalue $(n-1)$-tuple $(\lambda_1,\lambda_2,\ldots,\lambda_{n-1})$, where*

$$h(\lambda_1,\lambda_2,\ldots,\lambda_{n-1}) = \frac{1}{\alpha_n}\left(\frac{C}{\prod_{i=1}^{n}\int_a^b s_i^+(x)\,dx}\right)^{p_n/\alpha_n}, \tag{3.10}$$

where

$$C = \frac{2^{B-t+1}}{(b-a)^{B-t}\prod_{i=1}^{n-1}(\lambda_i\alpha_i)^{\alpha_i/p_i}}$$

with $B = \sum_{i=1}^{n}\alpha_i$.

These estimates show how the coupling parameters and coefficient functions influence the admissible eigenvalue region. These bounds can also be interpreted as exclusion regions in the generalized eigenvalue space, thus providing a priori spectral constraints for coupled quasilinear operators.

Use of AI tools declaration

No AI tools were used in the development of the scientific content of this paper; however, minor language and stylistic revisions were carried out using AI tools.

Conflict of interest

Sougata Dhar is a guest editor for *Electronic Research Archive* and was not involved in the editorial review or the decision to publish this article. The authors declare there is no conflict of interest.

References

1. A. M. Liapunov, Probleme general de la stabilite du mouvement, *Ann. Math. Stud.*, **17** (1947), 203–474.
2. G. Borg, On a Liapunoff criterion of stability, *Am. J. Math.*, **71** (1949), 67–70. https://doi.org/10.2307/2372093
3. A. Wintner, On the non-existence of conjugate points, *Am. J. Math.*, **73** (1951), 368–380. https://doi.org/10.2307/2372182
4. S. Dhar, Q. Kong, Liapunov-type inequalities for third-order half-linear equations and applications to boundary value problems, *Nonlin. Anal. TMA*, **110** (2014), 170–181. https://doi.org/10.1016/j.na.2014.07.020
5. S. Dhar, Q. Kong, Lyapunov-type inequalities for higher order half-linear differential equations, *Appl. Math. Comput.*, **273** (2016), 114–124. https://doi.org/10.1016/j.amc.2015.09.090
6. S. Dhar, Q. Kong, Lyapunov-type inequalities for third-order linear differential equations, *Math. Inequal. Appl.*, **19** (2016), 297–312. https://doi.org/10.7153/mia-19-22
7. S. Dhar, Q. Kong, Lyapunov-type inequalities for odd order linear differential equations, *Electr. J. Differ. Equation*, **2016** (2016), 1–10.
8. S. Dhar, J. S. Kelly, Lower bounds for eigenvalues of even ordered quasilinear differential equations, *Proc. Am. Math. Soc.*, **151** (2023), 647–661. https://doi.org/10.1090/proc/16122
9. J. P. Pinasco, Lower bounds for eigenvalues of the one-dimensional p-Laplacian, *Abstr. Appl. Anal.*, **2** (2004), 147–153. https://doi.org/10.1155/S108533750431002X
10. J. P. Pinasco, *Lyapunov-type Inequalities, with Applications to Eigenvalue Problems*, Springer Briefs in Mathematics, Springer, New York, 2013.
11. K. Watanabe, Lyapunov type inequality for the equation including 1-dim p-Laplacian, *Math. Inequal. Appl.*, **15** (2012), 657–662. https://doi.org/10.7153/mia-15-58
12. P. Hartman, *Ordinary Differential Equations*, Wiley, New York, 1964, and Birkhauser, Boston, 1982.
13. X. Yang, On inequalities of Lyapunov type, *Appl. Math. Comput.*, **134** (2003), 293–300. https://doi.org/10.1016/S0096-3003(01)00283-1

14. P. L. Napoli, J. P. Pinasco, Estimates for eigenvalues of quasilinear elliptic systems, *J. Differ. Equation*, **227** (2006), 102–115. https://doi.org/10.1016/j.jde.2006.01.004

15. D. Çakmak, A. Tiryaki, Lyapunov-type inequality for a class of Dirichlet quasilinear systems involving the $(p_1, p_2, \cdots, p_n)$–Laplacian, *J. Math. Anal. Appl.*, **369** (2010), 76–81. https://doi.org/10.1016/j.jmaa.2010.02.043

16. X. H. Tang, X. He, Lower bounds for generalized values of the quasilinear systems, *J. Math. Anal. Appl.*, **385** (2012), 72–85. https://doi.org/10.1016/j.jmaa.2011.06.026

17. M. F. Aktas, D. Çakmak, A. Tiryaki, A note on Tang and He's paper, *Appl. Math. Comput.*, **218** (2012), 4867–4871. https://doi.org/10.1016/j.amc.2011.10.050

18. D. Çakmak, A. Tiryaki, On Lyapunov-type inequality for quasilinear systems, *Appl. Math. Comput.*, **216** (2010), 3584–3591. https://doi.org/10.1016/j.amc.2010.05.004

19. G. Guseinov, B. Kaymakcalan, Lyapunov inequalities for discrete linear Hamiltonian system, *Comput. Math. Appl.*, **45** (2003), 1399–1416. https://doi.org/10.1016/S0898-1221(03)00095-6

20. I. Sim, Y. Lee, Lyapunov inequalities for one-dimensional p-Laplacian problems with singular weight function, *J. Inequal. Appl.*, **2010** (2010), 1–9. https://doi.org/10.1155/2010/865096