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Research article

Finite-time stability of mild solutions for Caputo-Katugampola fractional differential equations

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Abstract: This paper focuses on studying the existence and robust finite-time stability (FTS) for a more generalized class of nonlinear Caputo-Katugampola fractional differential equations with perturbation. The existence of mild solutions is proved by using the Sadovskii fixed-point theorem (SFPT). Then, sufficient conditions for robust FTS are obtained by applying the Jensen inequality and Hölder inequality. Finally, our theoretical results are supported by the research on the Lasota-Wazewska red blood cell (LWRBC) model.

Keywords: Caputo-Katugampola fractional differential equation; finite-time stability; Sadovskii fixed-point theorem; Jensen inequality; Grönwall inequality

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1. Introduction

In the past, it was acknowledged that integer-order differential equations (DEs), because of their local nature, are not effective in accurately modeling the dynamic processes of biological systems. Consequently, fractional differential equations (FDEs) were suggested for further investigation. In contrast to integer-order differential equations, FDEs exhibit hereditary characteristics. These equations also allow for the consideration of problems with memory effects and non-locality. As a result, FDEs are suitable for the description of dynamic processes. Present studies on FDEs mainly focus on applied models in economics, physics, etc. [1–7]. Several formulations of fractional derivatives (FDs) have been developed in fractional calculus. The most prevalent include the Riemann-Liouville FD [8], the Caputo FD [8], the Yang-Srivastava-Machado FD [9], and local FDs [10]. It is worth noting that the most commonly used two are the Riemann-Liouville and the Caputo FDs.

In the research fields of fractional calculus, fuzzy control, and related areas, the existence and stability of solutions are two fundamental properties that characterize the dynamic behavior of systems and ensure their reliable operation, which have long attracted extensive attention from the academic community [11–20]. For different types of complex systems, scholars have carried out a great deal of targeted research. Anusha et al. [12] investigated the existence and uniqueness of solutions for nonlinear impulsive neutral Hammerstein mixed integral dynamic equations on finite time-scales by means of the Banach contraction principle and Picard iterative operator, and presented a unified and generalized approach to study Ulam-type stability. Chalishajar et al. [14] studied the new solution representation and Ulam-Hyers-Rassias stability of Hilfer fractional stochastic impulsive differential equations with nonlocal conditions driven by time-changing fractional Brownian motion. With the help of the fixed point theorem, the well-posedness of the solution to the system in finite-dimensional space was proved. Li et al. [16] investigated the stability and stabilization of Takagi-Sugeno (T-S) fuzzy systems with time-varying delay. They proposed an improved augmented Lyapunov-Krasovskii functional based on the fuzzy line integral and derived a less conservative stability criterion by combining a free-matrix method dependent on fuzzy rules and the derivative of time delay.

In practical applications, the dynamic behavior of systems over a finite time interval is often of primary interest. Finite-time stability (FTS) describes the transient performance within a prescribed time interval and thus has important practical value. This concept was first proposed by Kamenkov in 1953. In 1965, Weiss and Infante systematically analyzed FTS for nonlinear systems and introduced finite-time contractive stability. Since then, FTS has attracted extensive attention in the academic community. Du and Lu [17] proposed a fractional-order Grönwall inequality with time delay, and based on this inequality, derived a new criterion for the FTS of fractional-order time-delay systems. Li et al. [20] proved the existence of almost periodic solutions for the fractional-order Lasota-Wazewska red blood cell (LWRBC) model using fixed points on a positive cone and demonstrated the FTS of the solutions through inequality techniques.

Recently, Katugampola [21, 22] introduced the Katugampola fractional integral and derivative and provided their relevant properties. Based on the research results of Katugampola, Almeida et al. [23] proposed the Caputo-Katugampola fractional differential equations (CKFDEs), which combine the advantages of the Caputo derivative and the Katugampola derivative. Since then, the CKFDE has attracted a widespread attention [24–29]. Dai et al. [26] established theorems on the existence and boundedness of solutions for Caputo-Katugampola (CK) fractional delay projection neural networks by applying Sadovskii's fixed point theorem, Banach fixed point theorem, and the generalized Grönwall inequality, and also studied the FTS of the system. Jmal et al. [28] employed inequality techniques to investigate the FTS of equilibrium points for CK fractional neural networks with time delay, and established their existence and uniqueness. Makhlouf and Nagy [29] investigated the linear CK fractional delay equation and obtained sufficient conditions for FTS by using two methods: Grönwall's inequality and Cauchy-Schwarz's inequality. It is worth noting that although some scholars have carried out research on the existence and FTS of solutions to CKFDEs, most of these studies are limited to specific models or equation types.

Inspired by the above literatures, this paper investigates the existence and FTS of mild solutions for a more generalized class of nonlinear CKFDEs with perturbation w:

$$\begin{cases} {}^{C}\mathcal{D}_{\tau}^{\varepsilon,\rho} u(\tau) = f(\tau, u(\tau), w(\tau)) + g(\tau, u(\tau - \kappa(\tau))), \ \tau \in [0, T], \\ \qquad u(\tau) = \phi(\tau), \ \tau \in [-\varrho, 0], \end{cases} \tag{1.1}$$

where $u \in C([-\varrho, T], \mathbb{R}) \cap C^1([0, T], \mathbb{R})$, the delay function $\kappa : [0, T] \to \mathbb{R}$ is continuous with $0 \leq \kappa(\tau) < \varrho$, and the perturbation $w : [0, T] \to \mathbb{R}$ is continuous on $[0, T]$, i.e., for $\tau \in [0, T]$, there exists a positive ζ such that $|w(\tau)| \leq \zeta$, $\phi \in C([-\varrho, 0]; \mathbb{R})$ is the given data. Additionally, $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ are both continuous. We use the Sadovskii fixed-point theorem (SFPT) to study the existence of its mild solutions and obtain sufficient conditions for the robust FTS of its mild solutions by using the Jensen inequality and Hölder inequality.

The layout of the manuscript is as follows. The definitions and preliminary findings in lemmas are presented in Section 2. The robust FTS of mild solutions for the CKFDEs is discussed in Section 3. Our theoretical results are applied to the LWRBC models in Section 4. Our results are summarized in Section 5.

2. Preliminaries

Let $\mathbb{R} = (-\infty, +\infty)$ and $L^1[0, T]$, $C([-\varrho, T], \mathbb{R})$, and $C^1([0, T], \mathbb{R})$ be Banach spaces containing all Lebesgue integrable functions on $[0, T]$, all continuous functions from $[-\varrho, T]$ into $\mathbb{R}$, and all continuously differentiable functions from $[0, T]$ into $\mathbb{R}$, respectively. In this paper, for the continuous function $f \in C([-\varrho, T], \mathbb{R})$, we define the norm $\|f\| = \sup_{\tau \in [-\varrho, T]} |f(\tau)|$.

Definition 2.1. *[21] Given a function $u \in L^1[0, T]$ and the classical Gamma function $\Gamma(\varepsilon)$, the Katugampola fractional integral is given by*

$$\mathcal{I}_{\tau}^{\varepsilon,\rho} u(\tau) = \frac{\rho^{1-\varepsilon}}{\Gamma(\varepsilon)} \int_0^{\tau} \frac{t^{\rho-1} u(t)}{(\tau^{\rho} - t^{\rho})^{1-\varepsilon}} dt,$$

where $\varepsilon > 0$ and $\rho > 0$.

Definition 2.2. *[23,27] Let $u \in C^1([0, T], \mathbb{R})$, $\varepsilon \in (0, 1]$, and $\rho > 0$. The CKFD of function $u : [0, T] \to \mathbb{R}$ with a parameter ρ is defined by*

$$\begin{aligned} {}^{C}\mathcal{D}_{\tau}^{\varepsilon,\rho} u(\tau) &= \frac{\rho^{\varepsilon}}{\Gamma(1-\varepsilon)} \int_0^{\tau} \frac{u'(t)}{(\tau^{\rho} - t^{\rho})^{\varepsilon}} dt \\ &= \mathcal{I}_{\tau}^{1-\varepsilon,\rho} \left(t^{1-\rho} \frac{d}{d\tau} u \right)(\tau). \end{aligned}$$

Lemma 2.3. *[23] Let $u \in C^1([0, T], \mathbb{R})$, then*
(1) $\mathcal{I}_{\tau}^{\varepsilon,\rho}\, {}^{C}\mathcal{D}_{\tau}^{\varepsilon,\rho} u(\tau) = u(\tau) - u(0)$;
(2) ${}^{C}\mathcal{D}_{\tau}^{\varepsilon,\rho} \mathcal{I}_{\tau}^{\varepsilon,\rho} u(\tau) = u(\tau)$.

Lemma 2.4. *[30] Let X be a Banach space, F_1 be a contraction operator, and F_2 be a completely continuous operator. If the sum operator $F = F_1 + F_2 : X \to X$, then F is a condensing operator.*

Lemma 2.5. (*SFPT [30]*) *Let $\mathcal{B}$ be a convex, closed, and bounded set of Banach space X. If a condensing operator $\mathcal{N}$ maps $\mathcal{B}$ into itself, i.e., $N(\mathcal{B}) \subseteq \mathcal{B}$, then $\mathcal{N}$ admits at least one fixed point in $\mathcal{B}$.*

Lemma 2.6. *(Jensen inequality [31]) If $u_1, u_2, ..., u_n$ are non-negative real numbers, and $n \in \mathbb{N}$, then*

$$\left(\sum_{j=1}^{n} u_j \right)^{p} \leq n^{p-1} \sum_{j=1}^{n} u_j^{p}, \quad \textit{for } p > 1.$$

Lemma 2.7. *[32, 33] Let function k be positive and integrable on $[0, T]$, and two functions u and h be continuous on $[0, T]$. If*

$$u(\tau) \leq h(\tau) + \int_0^\tau k(t)u(t)dt,$$

then

$$u(\tau) \leq h(\tau) + \int_0^\tau h(t)k(t)e^{\int_t^\tau k(s)ds}dt, \;\; \tau \in [0, T].$$

3. Finite-time stability

This section proves the robust FTS of mild solutions for CKFDEs (1.1).

Furthermore, we need to make the following sublinear-growth assumption (H):

$$\begin{aligned} &|f(\tau, u, w)| \leq L_1(\tau)\|u\| + L_2(\tau)\|w\|, \; (\tau, u, w) \in [0, T] \times \mathbb{R} \times \mathbb{R}, \\ &|g(\tau, u) - g(\tau, v)| \leq L_g\|u - v\|, \; (\tau, u), \; (\tau, v) \in [0, T] \times \mathbb{R}, \end{aligned}$$

where the non-negative functions $L_1(\tau)$, $L_2(\tau)$ are continuous on $[0, T]$ and L_g is a positive constant. Additionally, we assume that for all $\tau \in [0, T]$, the function satisfies $g(\tau, 0) = 0$.

To enable the following analysis, we shall establish the subsequent notations:

$$\xi = \sup_{\tau\in[0,T]} L_1(\tau), \;\; \eta = \sup_{\tau\in[0,T]} L_2(\tau).$$

According to Lemma 2.3 and Definition 1.1 in reference [34], we are able to define a mild solution of (1.1).

Definition 3.1. *Function $u \in C([-\varrho, T], \mathbb{R}) \cap C^1([0, T], \mathbb{R})$ is a mild solution to Eq (1.1) if it fulfills the specified fractional integral equations as follows:*

$$u(\tau) = \begin{cases} \phi(0) + \dfrac{\rho^{1-\varepsilon}}{\Gamma(\varepsilon)} \displaystyle\int_0^\tau \dfrac{t^{\rho-1}}{(\tau^\rho - t^\rho)^{1-\varepsilon}}(f(t, u(t), w(t)) + g(t, u(t - \kappa(t))))dt, \; \tau \in [0, T], \\ \phi(\tau), \; \tau \in [-\varrho, 0]. \end{cases} \tag{3.1}$$

Before determining the FTS of mild solutions for Eq (1.1), we first prove the existence by utilizing the SFPT.

Theorem 3.2. *Suppose that (H) is satisfied. Equation (1.1) possesses at least one mild solution $u \in C([-\varrho, T], \mathbb{R}) \cap C^1([0, T], \mathbb{R})$, given that*

$$(\xi + L_g)T^{\rho\varepsilon} < \Gamma(\varepsilon + 1)\rho^\varepsilon. \tag{3.2}$$

Proof. Based on Definition 3.1, we consider the operator

$$\Phi : C([-\varrho, T], \mathbb{R}) \cap C^1([0, T], \mathbb{R}) \to C([-\varrho, T], \mathbb{R}) \cap C^1([0, T], \mathbb{R}),$$

defined by

$$(\Phi u)(\tau) = \begin{cases} \phi(0) + \dfrac{\rho^{1-\varepsilon}}{\Gamma(\varepsilon)} \displaystyle\int_0^\tau \dfrac{t^{\rho-1}}{(\tau^\rho - t^\rho)^{1-\varepsilon}}(f(t, u(t), w(t)) + g(t, u(t - \kappa(t))))dt, \; \tau \in [0, T], \\ \phi(\tau), \; \tau \in [-\varrho, 0]. \end{cases}$$

It is clear that the operator Φ has a fixed point, implying that a mild solution to Eq (1.1) exists.

Let $B_\delta = \{u \in C([-\varrho, T], \mathbb{R}) \cap C^1([0, T], \mathbb{R}) : \|u\| \leq \delta\}$. Straightforwardly, B_δ is a bounded, closed, and convex subset of $C([-\varrho, T], \mathbb{R}) \cap C^1([0, T], \mathbb{R})$, where

$$\delta \geq \frac{\|\phi\|\Gamma(\varepsilon+1)\rho^\varepsilon + \eta\zeta T^{\rho\varepsilon}}{\Gamma(\varepsilon+1)\rho^\varepsilon - (\xi + L_g)T^{\rho\varepsilon}}.$$

Next, we show that there exists at least one fixed point for Φ on B_δ.

Step 1: Prove that Φ is a self-map on B_δ. For $u \in B_\delta$, we obtain

$$\begin{aligned}
|(\Phi u)(\tau)| &\leq |\phi| + \frac{\rho^{1-\varepsilon}}{\Gamma(\varepsilon)} \int_0^\tau \frac{t^{\rho-1}}{(\tau^\rho - t^\rho)^{1-\varepsilon}} (|f(t, u(t), w(t))| + |g(t, u(t-\kappa(t)))|) dt \\
&\leq \|\phi\| + \frac{\rho^{1-\varepsilon}}{\Gamma(\varepsilon)} \int_0^\tau \frac{t^{\rho-1}}{(\tau^\rho - t^\rho)^{1-\varepsilon}} \left(L_1(t)\|u\| + L_2(t)\|w\| + L_g\|u\|\right) dt \\
&\leq \|\phi\| + \frac{\rho^{1-\varepsilon}}{\Gamma(\varepsilon)} \int_0^\tau \frac{t^{\rho-1}}{(\tau^\rho - t^\rho)^{1-\varepsilon}} dt ((\xi + L_g)\delta + \eta\zeta) \\
&\leq \|\phi\| + \frac{(\xi + L_g)\delta + \eta\zeta}{\Gamma(\varepsilon+1)\rho^\varepsilon} T^{\rho\varepsilon} \\
&\leq \delta.
\end{aligned}$$

According to the definition of the norm, we have $\|(\Phi u)\| \leq \delta$. This implies that Φ is a self-map on B_δ.

In order to show that Φ is a condensing operator, we define two operators $\mathcal{F}$ and $\mathcal{G}$ on $C([-\varrho, T], \mathbb{R}) \cap C^1([0, T], \mathbb{R})$ by

$$(\mathcal{F}u)(\tau) = \begin{cases} \phi(0) + \dfrac{\rho^{1-\varepsilon}}{\Gamma(\varepsilon)} \displaystyle\int_0^\tau \frac{t^{\rho-1}}{(\tau^\rho - t^\rho)^{1-\varepsilon}} g(t, u(t-\kappa(t))) dt, \ \tau \in [0, T], \\ 0, \ \tau \in [-\varrho, 0], \end{cases}$$

and

$$(\mathcal{G}u)(\tau) = \begin{cases} \dfrac{\rho^{1-\varepsilon}}{\Gamma(\varepsilon)} \displaystyle\int_0^\tau \frac{t^{\rho-1}}{(\tau^\rho - t^\rho)^{1-\varepsilon}} f(t, u(t), w(t)) dt, \ \tau \in [0, T], \\ \phi(\tau), \ \tau \in [-\varrho, 0]. \end{cases}$$

Clearly, $\Phi = \mathcal{F} + \mathcal{G}$.

Subsequently, we demonstrate that $\mathcal{F}$ is a contraction operator and $\mathcal{G}$ is a completely continuous operator.

Step 2: Prove that $\mathcal{F}$ is a contraction operator on B_δ. For $u, v \in B_\delta$, one has

$$\begin{aligned}
|(\mathcal{F}u)(\tau) - (\mathcal{F}v)(\tau)| &\leq \frac{\rho^{1-\varepsilon}}{\Gamma(\varepsilon)} \int_0^\tau \frac{t^{\rho-1}}{(\tau^\rho - t^\rho)^{1-\varepsilon}} |g(t, u(t-\kappa(t))) - g(t, v(t-\kappa(t)))| dt \\
&\leq \frac{L_g\rho^{1-\varepsilon}}{\Gamma(\varepsilon)} \int_0^\tau \frac{t^{\rho-1}}{(\tau^\rho - t^\rho)^{1-\varepsilon}} dt \|u - v\| \\
&\leq k\|u - v\|,
\end{aligned}$$

where $k = \frac{L_g}{\Gamma(\varepsilon+1)\rho^\varepsilon} T^{\rho\varepsilon}$. Thus, $\|(\mathcal{F}u) - (\mathcal{F}v)\| \leq k\|u - v\|$ for $u, v \in B_\delta$. From (3.2), we deduce that $k < 1$, and then $\mathcal{F}$ is a contraction operator defined on B_δ.

Step 3: Prove that $\mathcal{G}$ is a completely continuous operator on B_δ.

(i) $\mathcal{G}(B_\delta)$ is uniformly bounded on B_δ. For $u \in B_\delta$, we have

$$\begin{aligned}
|(\mathcal{G}u)(\tau)| &\leq \frac{\rho^{1-\varepsilon}}{\Gamma(\varepsilon)} \int_0^\tau \frac{t^{\rho-1}}{(\tau^\rho - t^\rho)^{1-\varepsilon}} |f(t, u(t), w(t))| dt \\
&\leq \frac{\rho^{1-\varepsilon}}{\Gamma(\varepsilon)} \int_0^\tau \frac{t^{\rho-1}}{(\tau^\rho - t^\rho)^{1-\varepsilon}} (L_1(t)\|u\| + L_2(t)\|w\|) dt \\
&\leq \frac{\rho^{1-\varepsilon}}{\Gamma(\varepsilon)} \int_0^\tau \frac{t^{\rho-1}}{(\tau^\rho - t^\rho)^{1-\varepsilon}} (\xi\delta + \eta\zeta) dt \\
&\leq \frac{\xi\delta + \eta\zeta}{\Gamma(\varepsilon + 1)\rho^\varepsilon} T^{\rho\varepsilon}.
\end{aligned}$$

Thus, we can obtain $\|\mathcal{G}u\| \leq \frac{\xi\delta+\eta\zeta}{\Gamma(\varepsilon+1)\rho^\varepsilon} T^{\rho\varepsilon}$. Therefore, $\mathcal{G}(B_\delta)$ is uniformly bounded on B_δ.

(ii) $\mathcal{G}$ is equicontinous on B_δ. For $u \in B_\delta$, we have

$$\begin{aligned}
&|(\mathcal{G}u)(\mu) - (\mathcal{G}u)(\nu)| \\
=& \left| \frac{\rho^{1-\varepsilon}}{\Gamma(\varepsilon)} \int_0^\mu \frac{t^{\rho-1}}{(\mu^\rho - t^\rho)^{1-\varepsilon}} f(t, u(t), w(t)) dt - \frac{\rho^{1-\varepsilon}}{\Gamma(\varepsilon)} \int_0^\mu \frac{t^{\rho-1}}{(\nu^\rho - t^\rho)^{1-\varepsilon}} f(t, u(t), w(t))) dt \right. \\
&\left. - \frac{\rho^{1-\varepsilon}}{\Gamma(\varepsilon)} \int_\mu^\nu \frac{t^{\rho-1}}{(\nu^\rho - t^\rho)^{1-\varepsilon}} f(t, u(t), w(t)) dt \right| \\
\leq& (\xi\delta + \eta\zeta) \frac{\rho^{1-\varepsilon}}{\Gamma(\varepsilon)} \left(\left| \int_0^\mu \frac{t^{\rho-1}}{(\mu^\rho - t^\rho)^{1-\varepsilon}} - \frac{t^{\rho-1}}{(\nu^\rho - t^\rho)^{1-\varepsilon}} dt \right| + \left| \int_\mu^\nu \frac{t^{\rho-1}}{(\nu^\rho - t^\rho)^{1-\varepsilon}} dt \right| \right).
\end{aligned}$$

When $\mu \longrightarrow \nu$, the right side approaches zero, indicating that $\mathcal{G}u$ is equicontinuous on B_δ.

We conclude that $\mathcal{G}(B_\delta)$ is relatively compact via the Arzelà-Ascoli theorem. In addition, obviously, $\mathcal{G}$ is continuous. Hence, referring to Lemma 2.4, we conclude that Φ is a condensing operator.

Consequently, Φ satisfies all conditions of Lemma 2.5. As a result, Φ has at least one fixed point on B_δ.

Subsequently, to investigate the robust FTS of Eq (1.1), according to Definition 1.2 in reference [34], we present the corresponding definition for Eq (1.1).

Definition 3.3. *Let $u(t)$ be a mild solution of Eq (1.1) with the initial value function ϕ. Given positive numbers c_1, c_2, and T with $c_1 \leq c_2$, Eq (1.1) is said to be robust FTS with respect to c_1, c_2, and T if*

$$\|\phi\| \leq c_1 \Rightarrow \|u\| \leq c_2,$$

for all perturbations w satisfying $\|w\| \leq \zeta$.

Theorem 3.4. *Suppose that (H) is satisfied. Given positive numbers c_1, c_2, and T with $c_1 \leq c_2$, Eq (1.1) has robust FTS with respect to c_1, c_2, and T if*

$$3^{\frac{1}{\varepsilon}}(c_1^r + h(T)^r)(r + me^{(r+m)\frac{T^\rho}{\rho}}) \leq (r + m)c_2^r, \tag{3.3}$$

where $m = \dfrac{3^{\frac{1}{\varepsilon}}(\xi + L_g)^r \lambda^r}{\Gamma^r(\varepsilon)}$, $h(T) = \dfrac{\eta\zeta}{\Gamma(\varepsilon + 1)\rho^\varepsilon} T^{\varepsilon\rho}$, $l = 1 + \varepsilon$, $r = 1 + \dfrac{1}{\varepsilon}$, *and* $\lambda = (\dfrac{\Gamma(\varepsilon^2)}{l^{\varepsilon^2}})^{\frac{1}{l}}$.

Proof. According to Definition 3.1, a mild solution of Eq (1.1) is expressed as follows:

$$u(\tau) = \phi(0) + \frac{\rho^{1-\varepsilon}}{\Gamma(\varepsilon)} \int_0^\tau \frac{t^{\rho-1}}{(\tau^\rho - t^\rho)^{1-\varepsilon}} (f(t, u(t), w(t)) + g(t, u(t-\kappa(t))))dt,\ \tau \in [0, T].$$

By (H) and $\|w\| \le \zeta$ for a positive ζ, we have

$$\begin{aligned} |u(\tau)| &\le \|\phi\| + \frac{\rho^{1-\varepsilon}}{\Gamma(\varepsilon)} \int_0^\tau \frac{t^{\rho-1}}{(\tau^\rho - t^\rho)^{1-\varepsilon}} (L_1(t)\|u(t)\| + L_2(t)\|w\| + L_g\|u(t-\kappa(t))\|)dt \\ &\le \|\phi\| + \frac{\rho^{1-\varepsilon}}{\Gamma(\varepsilon)} \int_0^\tau \frac{t^{\rho-1}}{(\tau^\rho - t^\rho)^{1-\varepsilon}} \Big((\xi + L_g)x(t) + \eta\zeta\Big)\, dt, \end{aligned}$$

where $x(\tau) = max_{-\varrho\le\theta\le\tau}|u(\theta)|$ for all $\tau \ge 0$.

Since the right-hand side of the above inequality is increasing with respect to $\tau \in [0, T]$ and the identity $e^{\frac{t^\rho}{\rho}} e^{-\frac{t^\rho}{\rho}} = 1$, we have

$$\begin{aligned} x(\tau) &= \max\left\{ \sup_{\theta\in[-\varrho,0]} |u(\theta)|, \sup_{i\in[0,\tau]} |u(i)| \right\} \\ &\le \max\left\{ \|\phi\|, \sup_{i\in[0,\tau]} \left(\|\phi\| + \frac{\rho^{1-\varepsilon}}{\Gamma(\varepsilon)} \int_0^i \frac{t^{\rho-1}}{(i^\rho - t^\rho)^{1-\varepsilon}} ((\xi + L_g)x(t) + \eta\zeta)dt \right) \right\} \\ &= \|\phi\| + \frac{\rho^{1-\varepsilon}}{\Gamma(\varepsilon)} \int_0^\tau \frac{t^{\rho-1}}{(\tau^\rho - t^\rho)^{1-\varepsilon}} \Big((\xi + L_g)x(t) + \eta\zeta\Big)\, dt \\ &\le \|\phi\| + h(\tau) + \frac{\xi + L_g}{\Gamma(\varepsilon)} \int_0^\tau (\frac{\tau^\rho}{\rho} - \frac{t^\rho}{\rho})^{\varepsilon-1} e^{\frac{t^\rho}{\rho}} e^{-\frac{t^\rho}{\rho}} x(t) d\frac{t^\rho}{\rho}, \end{aligned}$$

where $h(\tau) = \frac{\eta\zeta}{\Gamma(\varepsilon+1)\rho^\varepsilon}\tau^{\varepsilon\rho}$.

Obviously, it is easy to know that

$$\int_0^\tau (\frac{\tau^\rho}{\rho} - \frac{t^\rho}{\rho})^{l(\varepsilon-1)} e^{l\frac{t^\rho}{\rho}} d\frac{t^\rho}{\rho} = \frac{\Gamma(\varepsilon^2)}{l^{\varepsilon^2}} e^{l\frac{\tau^\rho}{\rho}}.$$

We select conjugate exponents $l = 1 + \varepsilon$ and $r = 1 + \frac{1}{\varepsilon}$ for the Hölder inequality, ensuring the CK fractional integral is estimated. In view of the Hölder inequality [35], we derive the following inequality:

$$\begin{aligned} \sup_{\tau\in[0,T]} |x(\tau)| &\le \|\phi\| + h(\tau) + \frac{\xi + L_g}{\Gamma(\varepsilon)} (\int_0^\tau (\frac{\tau^\rho}{\rho} - \frac{t^\rho}{\rho})^{l(\varepsilon-1)} e^{l\frac{t^\rho}{\rho}} d\frac{t^\rho}{\rho})^{\frac{1}{l}} (\int_0^\tau e^{-r\frac{t^\rho}{\rho}} \sup_{t\in[0,T]} |x(t)| d\frac{t^\rho}{\rho})^{\frac{1}{r}} \\ &= \|\phi\| + h(\tau) + \frac{\xi + L_g}{\Gamma(\varepsilon)} (\frac{\Gamma(\varepsilon^2)}{l^{\varepsilon^2}} e^{l\frac{\tau^\rho}{\rho}})^{\frac{1}{l}} (\int_0^\tau e^{-r\frac{t^\rho}{\rho}} \sup_{t\in([0,T]} |x(t)| d\frac{t^\rho}{\rho})^{\frac{1}{r}}. \end{aligned}$$

Let $\lambda = (\frac{\Gamma(\varepsilon^2)}{l^{\varepsilon^2}})^{\frac{1}{l}}$, and one has

$$\sup_{\tau\in[0,T]} |x(\tau)| \le \|\phi\| + h(\tau) + \frac{(\xi + L_g)\lambda}{\Gamma(\varepsilon)} e^{\frac{\tau^\rho}{\rho}} (\int_0^\tau e^{-r\frac{t^\rho}{\rho}} \sup_{t\in[0,T]} |x(t)| d\frac{t^\rho}{\rho})^{\frac{1}{r}}.$$

By using the Jensen inequality (Lemma 2.6), we derive the following inequality:

$$\begin{aligned}
[\sup_{\tau\in[0,T]}|x(\tau)|]^r &\leq 3^{\frac{1}{\varepsilon}}(||\phi||^r + h(\tau)^r + \frac{(\xi+L_g)^r\lambda^r e^{r\frac{\tau^\rho}{\rho}}}{\Gamma^r(\varepsilon)}\int_0^\tau e^{-r\frac{t^\rho}{\rho}}[\sup_{t\in[0,T]}|x(t)|]^r d\frac{t^\rho}{\rho})\\
&= 3^{\frac{1}{\varepsilon}}||\phi||^r + 3^{\frac{1}{\varepsilon}}h(\tau)^r + \frac{3^{\frac{1}{\varepsilon}}(\xi+L_g)^r\lambda^r e^{r\frac{\tau^\rho}{\rho}}}{\Gamma^r(\varepsilon)}\int_0^\tau e^{-r\frac{t^\rho}{\rho}}t^{\rho-1}[\sup_{t\in[0,T]}|x(t)|]^r dt.
\end{aligned}$$

Let $m = \frac{3^{\frac{1}{\varepsilon}}(\xi+L_g)^r\lambda^r}{\Gamma^r(\varepsilon)}$, and multiply both sides by $e^{-r\frac{\tau^\rho}{\rho}}$ to convert the inequality into the standard form for the Grönwall inequality, then one gets

$$e^{-r\frac{\tau^\rho}{\rho}}[\sup_{\tau\in[0,T]}|x(\tau)|]^r \leq 3^{\frac{1}{\varepsilon}}||\phi||^r e^{-r\frac{\tau^\rho}{\rho}} + 3^{\frac{1}{\varepsilon}}h(\tau)^r e^{-r\frac{\tau^\rho}{\rho}} + m\int_0^\tau e^{-r\frac{t^\rho}{\rho}}t^{\rho-1}[\sup_{t\in[0,T]}|x(t)|]^r dt.$$

Using the Grönwall inequality (Lemma 2.7) yields the following inequality:

$$\begin{aligned}
e^{-r\frac{\tau^\rho}{\rho}}[\sup_{\tau\in[0,T]}|x(\tau)|]^r &\leq 3^{\frac{1}{\varepsilon}}(||\phi||^r + h(\tau)^r)e^{-r\frac{\tau^\rho}{\rho}} + \int_0^\tau mt^{\rho-1}3^{\frac{1}{\varepsilon}}(||\phi||^r + h(t)^r)e^{-r\frac{t^\rho}{\rho}}e^{m\int_t^\tau \tau^{\rho-1}d\tau}dt\\
&\leq 3^{\frac{1}{\varepsilon}}(||\phi||^r + h(T)^r)e^{-r\frac{\tau^\rho}{\rho}} + 3^{\frac{1}{\varepsilon}}(||\phi||^r + h(T)^r)m\int_0^\tau e^{-(r+m)\frac{t^\rho}{\rho}}e^{m\frac{\tau^\rho}{\rho}}d\frac{t^\rho}{\rho}\\
&= 3^{\frac{1}{\varepsilon}}(||\phi||^r + h(T)^r)e^{-r\frac{\tau^\rho}{\rho}} + 3^{\frac{1}{\varepsilon}}(||\phi||^r + h(T)^r)me^{m\frac{\tau^\rho}{\rho}}\frac{1-e^{-(r+m)\frac{\tau^\rho}{\rho}}}{r+m}\\
&= \frac{3^{\frac{1}{\varepsilon}}(||\phi||^r + h(T)^r)(re^{-r\frac{\tau^\rho}{\rho}} + me^{m\frac{\tau^\rho}{\rho}})}{r+m}.
\end{aligned}$$

If $||\phi|| \leq c_1$, $\tau \leq T$ and

$$3^{\frac{1}{\varepsilon}}(c_1^r + h(T)^r)(r + me^{(r+m)\frac{T^\rho}{\rho}}) \leq (r+m)c_2^r,$$

we have

$$||u|| \leq x(T) \leq \left(\frac{3^{\frac{1}{\varepsilon}}(||\phi||^r + h(T)^r)(r + me^{(r+m)\frac{T^\rho}{\rho}})}{r+m}\right)^{\frac{1}{r}} \leq c_2.$$

Hence, we conclude with robust FTS of Eq (1.1).

Remark 3.5. *Since $L_1(\tau)$ and $L_2(\tau)$ are bounded on $[0,T]$ and $\eta = \sup_{\tau\in[0,T]}L_2(\tau), \xi = \sup_{\tau\in[0,T]}L_1(\tau)$, and L_g are constants, thus*

$$m = \frac{3^{\frac{1}{\varepsilon}}(\xi+L_g)^r\lambda^r}{\Gamma^r(\varepsilon)}$$

and

$$h(T) = \frac{\eta\zeta}{\Gamma(\varepsilon+1)\rho^\varepsilon}T^{\varepsilon\rho}$$

are also specific values.

Substituting the above constants into

$$\left(\frac{3^{\frac{1}{\varepsilon}}(\|\phi\|^r + h(T)^r)(r + me^{(r+m)\frac{T^\rho}{\rho}})}{r+m}\right)^{\frac{1}{r}}, \tag{3.4}$$

the value of the expression (3.4) can be rigorously derived through explicit parameters, guaranteeing that there exists a positive constant c_2 *for which* $\|u\| \leq c_2$.

4. Case study

Let us examine the following CK fractional LWRBC models:

$$\begin{cases} {}^C\mathscr{D}_\tau^{\varepsilon,\rho} u(\tau) = -au(\tau) + b(\tau)e^{-\beta(\tau)u(\tau-\kappa(\tau))} + I(\tau), \ \tau \in [0,T], \\ \quad u(\tau) = \phi(\tau), \ \tau \in [-\varrho, 0], \end{cases}$$

where $u(\tau)$ represents the red blood cell (RBC) count at time τ, while the parameter a signifies the mortality rate of these cells. The parameters β and b are associated with the RBC production rate, and κ indicates the duration necessary for the generation of one RBC. Additionally, $I(\tau)$ serves as a disturbance term. This model analyzes the behavior of RBC quantities, with the variables u,β, b, κ, and I needing to be non-negative to maintain the biological relevance of the model. This model is a classical LWRBC model describing the dynamics of red blood cell counts, which has been widely studied in biological system modeling [20].

According to Definition 3.1, we get

$$u(\tau) = \begin{cases} \phi(0) + \frac{\rho^{1-\varepsilon}}{\Gamma(\varepsilon)} \int_0^\tau \frac{t^{\rho-1}}{(\tau^\rho - t^\rho)^{1-\varepsilon}} (-au(t) + b(t)e^{-\beta(t)u(t-\kappa(t))} + I(t))dt, \ \tau \in [0,T], \\ \phi(\tau), \ \tau \in [-\varrho, 0]. \end{cases} \tag{4.1}$$

The utilization of the mean value theorem allows us to express a time-based relationship regarding the count of blood cells:

$$\begin{aligned} u(\tau) &= \phi(0) + \frac{\rho^{1-\varepsilon}}{\Gamma(\varepsilon)} \int_0^\tau \frac{t^{\rho-1}}{(\tau^\rho - t^\rho)^{1-\varepsilon}} \left(-au(t) + b(t)e^{-\beta(t)u(t-\kappa(t))} + I(t)\right) dt \\ &= \phi(0) + \frac{\rho^{1-\varepsilon}}{\Gamma(\varepsilon)} \int_0^\tau \frac{t^{\rho-1}}{(\tau^\rho - t^\rho)^{1-\varepsilon}} \left(-au(t) + b(t)e^{-\theta(t)}\beta(t)u(t-\kappa(t)) + b(t) + I(t)\right) dt, \end{aligned}$$

with the condition that $\beta(t)u(t-\kappa(t)) \geq 0$, where θ varies between 0 and $\beta(t)u(t-\kappa(t))$.

Consequently, we rewrite Eq (4.1) into

$$u(\tau) = \begin{cases} \phi(0) + \frac{\rho^{1-\varepsilon}}{\Gamma(\varepsilon)} \int_0^\tau \frac{t^{\rho-1}}{(\tau^\rho - t^\rho)^{1-\varepsilon}} \left(-au(t) + b(t)e^{-\theta(t)}\beta(t)u(t-\kappa(t)) + b(t) + I(t)\right) dt, \ \tau \in [0,T], \\ \phi(\tau), \ \tau \in [-\varrho, 0], \end{cases}$$

where $f(\tau, u(\tau), w(\tau)) = -au(\tau) + w(\tau)$, $g(\tau, u(\tau-\kappa(\tau))) = b(\tau)e^{-\theta(\tau)}\beta(\tau)u(\tau-\kappa(\tau)) \leq b(\tau)\beta(\tau)u(\tau-\kappa(\tau))$, and $w(\tau) = b(\tau) + I(\tau)$.

To aid in the following analysis, we introduce the subsequent notations:

$$y^+ = \sup_{\tau\in[0,T]} |y(\tau)| \; and \; y^- = \inf_{\tau\in[0,T]} |y(\tau)|,$$

where the function $y : P \to \mathbb{R}$ is bounded. Since

$$\begin{aligned} |f(\tau, u, w)| &= |-au(\tau) + w(\tau)| \\ &\le a\|u\| + \|w\| \end{aligned}$$

and

$$\begin{aligned} |g(\tau, u) - g(\tau, v)| &= |b(\tau)e^{-\theta(\tau)}\beta(\tau)(u(\tau - \kappa(\tau)) - v(\tau - \kappa(\tau)))| \\ &\le b^+\beta^+\|u - v\|, \end{aligned}$$

f and g satisfy Assumption (H).

Let

$$\varepsilon = 0.7,\ \ a = 0.05,\ \ T = 1,\ \ \rho = 2,\ \ b(\tau) = 0.05\cos\tau + 0.3,\ c_2 = 1.4,\ \varrho = \frac{\pi}{2},$$
$$\beta(\tau) = \frac{1}{3}\sin\tau + \frac{2}{3},\ \ I(\tau) = 0.05\sin\tau + 0.3,\ \ \kappa(\tau) = \frac{1}{4}(\cos\tau + 1),\ \ \phi(\tau) = 0.4\cos\tau.$$

Obviously,

$$\begin{aligned} &\beta^+ = 1,\ I^+ = 0.35,\ b^+ = 0.35,\ l = 1.7,\ r = \frac{17}{7}, \zeta = b^+ + I^+ = 0.7, \\ &L_g = b^+\beta^+ = 0.35,\ \eta = 1,\ \xi = 0.05,\ c_1 = 0.4. \end{aligned}$$

Therefore,

$$\frac{(\xi + L_g)T^{\rho\varepsilon}}{\Gamma(\varepsilon + 1)\rho^\varepsilon} = 0.2710 < 1,$$

i.e., (3.2) is satisfied. According to Theorem 3.2, we see that the CK fractional LWRBC models have at least one mild solution.

Moreover,

$$\lambda = (\frac{\Gamma(\varepsilon^2)}{l^{\varepsilon^2}})^{\frac{1}{l}} \approx 1.2158,\ \ m = \frac{3^{\frac{1}{\varepsilon}}(\xi + L_g)^r\lambda^r}{\Gamma^r(\varepsilon)} \approx 0.4427,\ \ h(T) = \frac{\eta\zeta}{\Gamma(\varepsilon + 1)\rho^\varepsilon}T^{\varepsilon\rho} \approx 0.4742.$$

Substituting λ, m, and $h(T)$ into (3.3), one can obtain

$$\frac{3^{\frac{1}{\varepsilon}}(c_1^r + h(T)^r)(r + me^{(r+m)\frac{T^\rho}{\rho}})}{r + m} \approx 1.9473 \le c_2^r \approx 2.2640.$$

According to Theorem 3.4, the CK fractional LWRBC models have robust FTS with respect to parameters c_1, c_2, and T. Figure 1 presents the variation of the solution u of LWRBC and its theoretical upper bound U under given parameters. The results show that the numerical solution is consistent with the theoretical upper bound, verifying the robust FTS stability of the LWRBC models.

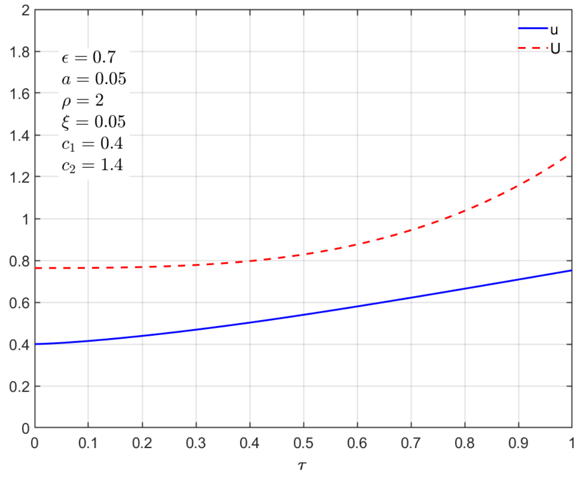

Figure 1. Variation of the solution u and its upper bound function U with τ ($\tau \in (0, 1]$).

5. Conclusions

We systematically investigate the existence of mild solutions for a more generalized class of CKFDEs through the SFPT, and focus on analyzing the robust FTS by using the Jensen inequality and Hölder inequality. Stability is crucial for describing the dynamic behavior of FDEs, and existing stability types include FTS, Hyers-Ulam stability, and Mittag-Leffler stability, among others. In the future, further exploration of other types of stability for mild solutions of CKFDEs is feasible and holds significant research value.

Use of AI tools declaration

This work was not written with AI.

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Conflict of interest

The authors declare there is no conflict of interest.

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