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Research article

Spatial decay estimate for the MGT-Fourier model on a semi-infinite cylinder in $\mathbb{R}^2$

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Abstract: This study investigates the spatial decay estimate for a coupled Moore-Gibson-Thompson (MGT)-Fourier system on a semi-infinite cylinder in $\mathbb{R}^2$. The system is comprised of a third-order MGT displacement equation and a parabolic heat equation with symmetric coupling. Using a weighted energy functional and an integral-differential inequality, we establish the exponential decay along the axial direction. This result offers a generalized Saint-Venant principle for coupled dissipative thermoviscoelastic models, thus extending the classical analysis to higher-order systems and advancing our understanding of MGT equations in an unbounded domain.

Keywords: Saint-Venant principle; MGT-Fourier model; integral-differential inequality; semi-infinite cylinder

1. Introduction

Over the past five decades, the Saint-Venant principle has remained a fundamental concept in both applied mathematics and continuum mechanics, thereby providing essential insights into the spatial decay behavior of solutions in elastic structures.The classical formulation of this principle has undergone substantial theoretical extensions through extensive investigative efforts. For systematic examinations of advancements in this domain, readers are directed to foundational works by Horgan [1, 2] and collaborative studies with Knowles [3]. The established Saint-Venant-type theorems demonstrate that energy experiences exponential decay with an increasing axial distance from the proximal end toward infinity along a semi-infinite strip or cylindrical geometries (cf. [4–7]). A critical prerequisite for deriving these decay properties involves imposing the a priori condition that solutions exhibit algebraic attenuation to zero at asymptotic distances. Recently, some authors have begun to study the Phragmén-Lindelöf alternative principle, thereby investigating the spatial properties of solutions to higher-order partial differential equations without imposing decay assumptions at infinity. In such cases, they obtained results consistent with the classical Phragmén-Lindelöf theorem, thereby demonstrating that as the spatial coordinate tends to infinity, the energy either grows or decays exponentially, with only one of these behaviors being admissible. In parallel with studies on spatial decay, the Phragmén-Lindelöf alternative principle was extensively developed for higher-order partial differential equations without imposing decay assumptions at infinity. Li and Chen [8] established Phragmén-Lindelöf alternative results for a time-dependent double-diffusive Darcy plane flow, thereby characterizing the admissible growth or decay behavior of solutions in unbounded domains. Lin and Payne [9] investigated a class of quasilinear second-order parabolic problems and derived alternative results demonstrating that solutions either grow or decay exponentially, with only one behavior being possible. Leseduarte and Quintanilla [10] considered the Laplace equation with dynamic boundary conditions and obtained Phragmén–Lindelöf type alternatives, thereby extending classical results to problems with time-dependent boundary effects. Horgan and Payne [11] analyzed harmonic functions with nonlinear boundary conditions, thereby deriving Phragmén-Lindelöf type results that reveal the influence of boundary nonlinearities on the spatial behavior of solutions.

In recent years, the Moore-Gibson-Thompson (MGT) equation has received much attention. Early investigations into nonlinear acoustics, particularly those that examined second sound phenomena in viscous thermally relaxing fluids (as initially explored in seminal works [12–14]), adopted an approximate framework based on the fully compressible Navier-Stokes-Cattaneo system under irrotational flow conditions. This line of inquiry led to the derivation of the MGT equation, which is a third-order partial differential equation that captures both damping and thermal relaxation effects in high-intensity ultrasound propagation. Kaltenbacher et al. [15] established the well-posedness and exponential stability of the nonlinear Jordan–Moore–Gibson–Thompson equation, thus providing a rigorous foundation for its analysis in high-intensity ultrasound propagation. Subsequently, Kaltenbacher and Nikolic [16] extended these results by incorporating quadratic gradient nonlinearities and examining the singular limit as the relaxation time vanishes, further clarifying the relationship between the MGT equation and classical wave models. From a dynamical systems perspective, Conejero et al. [17] demonstrated the existence of chaotic behavior in MGT-type equations, thereby revealing complex long-term dynamics. Marchand et al. [18] developed an abstract semigroup framework for the MGT equation, thereby reformulating it as a first-order evolution system and establishing analyticity and exponential stability. The work [19] provides a rigorous mathematical understanding of the long-time behavior of the MGT equation in the dissipative setting, thus contributing to the broader theory of higher-order hyperbolic-parabolic coupled systems. The paper [20] investigates the MGT heat equation, a third-order in time partial differential equation that models heat propagation by accounting for thermal relaxation effects. More recently, Chen and Gong [21] analyzed the viscous MGT equation within the L^q framework, thereby deriving refined asymptotic profiles and decay estimates.

In parallel with these developments, increasing attention has been directed toward coupled systems that involve the MGT equation and thermal effects. The coupling between the MGT displacement equation and the Fourier heat equation introduces additional mathematical complexity, particularly when the spatial domain is unbounded. Such coupled models naturally arise in thermoviscoelasticity, where mechanical and thermal dissipation mechanisms interact, and require careful analyses of both temporal stability and spatial decay characteristics [22–25]. Moreover, the incorporation of higher-order operators, such as the biharmonic term, further complicates the construction of appropriate energy functionals and the derivation of decay estimates [7]. Despite these challenges, understanding the spatial behavior of solutions in unbounded domains is essential for applications that involve wave propagation in elongated structures and to validate numerical approximations in truncated computational domains.

In the context of thermoviscoelasticity, the coupling between mechanical vibrations and heat conduction plays a fundamental role. Alves et al. [26] considered a linear standard solid model of viscoelasticity coupled with a Fourier heat equation, thereby deriving exponential stability in bounded domains via multiplier techniques. Their work highlights how thermal effects contribute to the dissipation of mechanical energy, a feature also present in the MGT-Fourier system studied here.

In [27], the authors addressed the energy transfer in the differential system made by a Moore-Gibson-Thompson equation in the supercritical regime, hence antidissipative, coupled with the classical heat equation. The system obtained by coupling the traditional MGT equation with the classical Fourier heat equation is as follows:

$$\begin{aligned} u_{,ttt} + au_{,tt} - b\Delta u_{,t} - c\Delta u &= -d\Delta\theta \\ \theta_{,t} - k\Delta\theta &= d\Delta u_{,tt} + \alpha d\Delta u_{,t} \end{aligned} \tag{1.1}$$

The MGT-Fourier system (1.1) is physically meaningful: as discussed in [27], the MGT equation models vibrations in a standard linear viscoelastic solid, while the coupling terms describe thermal stress effects in a deformable heat conductor.In this work, we investigate the spatial decay properties of a coupled MGT-Fourier system defined on a two-dimensional semi-infinite cylinder. The system comprises a third-order MGT displacement equation and a parabolic heat equation with symmetric coupling terms. Unlike the bounded-domain analysis in [26, 27], the present work focuses on a semi-infinite cylindrical domain and establishes spatial decay estimates, thereby extending the physical understanding of thermoviscoelastic coupling to unbounded geometries. From a mathematical perspective, the main contribution of this paper is the derivation of a generalized Saint-Venant principle for a higher-order coupled MGT-Fourier system. By constructing suitable weighted energy functionals and employing an integral-differential inequality method, we obtain explicit exponential decay rates along the axial direction. This approach overcomes the lack of compactness typical of unbounded domains and provides a rigorous framework to analyze the spatial behavior in dissipative coupled systems. The results not only generalize classical spatial decay principles but also offer a foundation to study structural stability and more complex models involving biharmonic operators in unbounded domains. The study of spatial behavior in unbounded domains has also been pursued through the Phragmén–Lindelöf alternative principle and related decay estimates. Leseduarte and Quintanilla [25] analyzed the Laplace equation with dynamic boundary conditions, thereby establishing alternative results that extend classical spatial decay principles to problems with time-dependent boundary data. Horgan and Payne [28] investigated harmonic functions subject to nonlinear boundary conditions, thereby deriving Phragmén–Lindelöf type results that highlight the role of boundary nonlinearities in determining the spatial behavior. Furthermore, the integral-differential inequality method proposed in this paper may also be applicable to the study of the Saint-Venant principle for other classes of partial differential equations.

To formulate the problem under consideration, we define the following semi-infinite strip:

$$\Omega_0 := \{(x_1, x_2) \mid x_1 > 0, 0 \leq x_2 \leq h\},$$

where h is a fixed constant. Define the cross-section at axial coordinate z by the following:

$$L_z = \{(x_1, x_2) \mid x_1 = z \geq 0, 0 \leq x_2 \leq h\}.$$

Here, the variable z serves as the longitudinal coordinate aligned with the x_1-axis.

The initial boundary conditions are as follows:

$$\theta(x_1, 0, t) = u(x_1, 0, t) = u_{,t}(x_1, 0, t) = u_{,tt}(x_1, 0, t) = 0, \quad x_1 > 0, t > 0, \tag{1.2}$$

$$\theta(x_1, h, t) = u(x_1, h, t) = u_{,t}(x_1, h, t) = u_{,tt}(x_1, h, t) = 0, \quad x_1 > 0, t > 0, \tag{1.3}$$

$$\theta(0, x_2, t) = g_1(x_2, t), \quad 0 \leq x_2 \leq h, t > 0, \tag{1.4}$$

$$u(0, x_2, t) = g_2(x_2, t), \quad 0 \leq x_2 \leq h, t > 0, \tag{1.5}$$

$$u_{,t}(0, x_2, t) = g_3(x_2, t), \quad 0 \leq x_2 \leq h, t > 0, \tag{1.6}$$

$$u_{,tt}(0, x_2, t) = g_4(x_2, t), \quad 0 \leq x_2 \leq h, t > 0, \tag{1.7}$$

and

$$\theta(x_1, x_2, 0) = u(x_1, x_2, 0) = u_{,t}(x_1, x_2, 0) = u_{,tt}(x_1, x_2, 0) = 0, \quad 0 \leq x_2 \leq h, x_1 > 0. \tag{1.8}$$

We further impose the following asymptotic decay assumptions on the solution for each fixed $t > 0$ as $x_1 \to \infty$ uniformly in x_2:

$$\theta(x_1, x_2, t), u(x_1, x_2, t), u_{,tt}(x_1, x_2, t), u_{,t}(x_1, x_2, t) \to 0 \quad as \quad x_1 \to \infty. \tag{1.9}$$

In this work, a comma-based subscript notation is adopted to signify partial differentiation. Specifically, differentiation with respect to the spatial coordinate x_k is denoted as k; thus, $v_{,i}$ denotes $\frac{\partial v}{\partial x_i}$ and $\dot{\theta}$ denotes $\frac{\partial \theta}{\partial t}$. The usual summation convention is employed with repeated Greek subscripts α summed from 1 to 2. Hence, $u_{,\alpha}u_{,\alpha} = u_{,1}^2 + u_{,2}^2$.

This paper is organized as follows: in Section 2, we define the energy functionals and derive their explicit representations, establishing the analytical framework for our subsequent analysis; Section 3 presents the construction of the key integral-differential inequality that form the foundation of our decay estimate; in Section 4, we obtain the main spatial decay estimate through a systematic analysis of the energy function; and finally, Section 5 concludes the paper with a discussion of future research directions, including the investigation of structural stability properties and the extension to models that incorporate biharmonic operators in unbounded spatial domains.

2. The definition of the energy

In this section, we define the energy functional that will be employed to derive our main result. In the following discussions, η denotes the time variable.

Proposition 2.1: Given a smooth solution (u, θ) of (1.1) subject to the initial-boundary conditions (1.2)–(1.8), we define the functional as follows:

$$\begin{aligned} F_1(z,t) = {} & -b\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta\eta}u_{,1\eta}dx_2d\eta - c\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta\eta}u_{,1}dx_2d\eta \\ & + d\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta\eta}\theta_{,1}dx_2d\eta. \end{aligned} \tag{2.1}$$

Additionally, we can also obtain the following:

$$\begin{aligned} F_1(z,t) = {} & \left(\frac{\omega}{2}+a\right)\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(u_{,\eta\eta})^2dAd\eta + \frac{1}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)(u_{,tt})^2dA \\ & + \left(\frac{\omega b}{2}-c\right)\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}dAd\eta + \frac{b}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha t}u_{,\alpha t}dA \\ & + \frac{c\omega^2}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha}u_{,\alpha}dAd\eta + \frac{c\omega}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)u_{,\alpha}u_{,\alpha}dA \\ & - d\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha\eta\eta}\theta_{,\alpha}dAd\eta + c\int_z^\infty\int_{L_\xi}\exp(-\omega t)u_{,\alpha t}u_{,\alpha}dA, \end{aligned} \tag{2.2}$$

where ω is an arbitrary positive constant.

Proof: Multiplying $(1.1)_1$ by $\exp(-\omega\eta)u_{,\eta\eta}$ and integrating leads to the followings:

$$0 = \int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\eta\eta}(u_{,\eta\eta\eta} + au_{,\eta\eta} - bu_{,\alpha\alpha\eta} - cu_{,\alpha\alpha} + d\theta_{,\alpha\alpha})dAd\eta. \tag{2.3}$$

The first term on the right hand side of (2.3) can be rewritten as follows:

$$\begin{aligned} \int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\eta\eta}u_{,\eta\eta\eta}dAd\eta = {} & \frac{\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(u_{,\eta\eta})^2dAd\eta \\ & + \frac{1}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)(u_{,tt})^2dA. \end{aligned} \tag{2.4}$$

The third term on the right hand side of (2.3) can be rewritten as follows:

$$\begin{aligned} & -b\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\eta\eta}u_{,\alpha\alpha\eta}dAd\eta \\ & = b\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha\eta\eta}u_{,\alpha\eta}dAd\eta + b\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta\eta}u_{,1\eta}dx_2d\eta \\ & = \frac{\omega b}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}dAd\eta + \frac{b}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)u_{,\alpha t}u_{,\alpha t}dA \\ & \quad + b\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta\eta}u_{,1\eta}dx_2d\eta. \end{aligned} \tag{2.5}$$

The fourth term on the right hand side of (2.3) can be rewritten as follows:

$$\begin{aligned}
&-c\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\eta\eta}u_{,\alpha\alpha}dAd\eta\\
&=c\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha\eta\eta}u_{,\alpha}dAd\eta+c\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta\eta}u_{,1}dx_2d\eta\\
&=-c\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}dAd\eta+c\omega\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha}dAd\eta\\
&\quad+c\int_z^\infty\int_{L_\xi}\exp(-\omega t)u_{,\alpha t}u_{,\alpha}dA+c\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta\eta}u_{,1}dx_2d\eta\\
&=-c\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}dAd\eta+\frac{c\omega^2}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha}u_{,\alpha}dAd\eta\\
&\quad+\frac{c\omega}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)u_{,\alpha}u_{,\alpha}dA+c\int_z^\infty\int_{L_\xi}\exp(-\omega t)u_{,\alpha t}u_{,\alpha}dA\\
&\quad+c\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta\eta}u_{,1}dx_2d\eta.
\end{aligned}\tag{2.6}$$

The last term on the right hand side of (2.3) can be rewritten as follows:

$$\begin{aligned}
&d\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\eta\eta}\theta_{,\alpha\alpha}dAd\eta\\
&=-d\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha\eta\eta}\theta_{,\alpha}dAd\eta-d\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta\eta}\theta_{,1}dx_2d\eta.
\end{aligned}\tag{2.7}$$

By combining (2.3)–(2.7), we have the following:

$$\begin{aligned}
&-b\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta\eta}u_{,1\eta}dx_2d\eta-c\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta\eta}u_{,1}dx_2d\eta\\
&\quad+d\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta\eta}\theta_{,1}dx_2d\eta\\
&=\left(\frac{\omega}{2}+a\right)\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(u_{,\eta\eta})^2dAd\eta+\frac{1}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)(u_{,tt})^2dA\\
&\quad+\left(\frac{\omega b}{2}-c\right)\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}dAd\eta+\frac{b}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)u_{,\alpha t}u_{,\alpha t}dA\\
&\quad+\frac{c\omega^2}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha}u_{,\alpha}dAd\eta+\frac{c\omega}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)u_{,\alpha}u_{,\alpha}dA\\
&\quad-d\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha\eta\eta}\theta_{,\alpha}dAd\eta+c\int_z^\infty\int_{L_\xi}\exp(-\omega t)u_{,\alpha t}u_{,\alpha}dA.
\end{aligned}\tag{2.8}$$

If we define $F_1(z,t)$ as (2.1), then we can easily get (2.2) from (2.8).

Proposition 2.2: Given a smooth solution (u, θ) of (1.1) subject to the initial-boundary conditions (1.2)–(1.8), we define the functional as follows:

$$\begin{aligned} F_2(z,t) = &-k\int_0^t\int_{L_z}\exp(-\omega\eta)\theta\theta_{,1}dx_2d\eta - d\int_0^t\int_{L_z}\exp(-\omega\eta)\theta u_{,1\eta\eta}dx_2d\eta \\ &- ad\int_0^t\int_{L_z}\exp(-\omega\eta)\theta u_{,1\eta}dx_2d\eta. \end{aligned} \tag{2.9}$$

Additionally, we can obtain the following:

$$\begin{aligned} F_2(z,t) = &\frac{\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta^2dAd\eta + \frac{1}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)\theta^2dA \\ &+ k\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta_{,\alpha}\theta_{,\alpha}dAd\eta + d\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta_{,\alpha}u_{,\alpha\eta\eta}dAd\eta \\ &+ ad\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta_{,\alpha}u_{,\alpha\eta}dAd\eta. \end{aligned} \tag{2.10}$$

Proof: Multiplying $(1.1)_2$ by $\exp(-\omega\eta)\theta$ and integrating leads to the following:

$$0 = \int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta(\theta_{,\eta} - k\theta_{,\alpha\alpha} - du_{,\alpha\alpha\eta\eta} - adu_{,\alpha\alpha\eta})dAd\eta. \tag{2.11}$$

The first term on the right hand side of (2.11) can be rewritten as follows:

$$\begin{aligned} \int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta\theta_{,\eta}dAd\eta = &\frac{\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta^2dAd\eta \\ &+ \frac{1}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)\theta^2dA. \end{aligned} \tag{2.12}$$

The second term on the right hand side of (2.11) can be rewritten as follows:

$$\begin{aligned} -k\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta\theta_{,\alpha\alpha}dAd\eta = &k\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta_{,\alpha}\theta_{,\alpha}dAd\eta \\ &+ k\int_0^t\int_{L_z}\exp(-\omega\eta)\theta\theta_{,1}dx_2d\eta. \end{aligned} \tag{2.13}$$

The third term on the right hand side of (2.11) can be rewritten as follows:

$$\begin{aligned} -d\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta u_{,\alpha\alpha\eta\eta}dAd\eta = &d\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta_{,\alpha}u_{,\alpha\eta\eta}dAd\eta \\ &+ d\int_0^t\int_{L_z}\exp(-\omega\eta)\theta u_{,1\eta\eta}dx_2d\eta. \end{aligned} \tag{2.14}$$

The last term on the right hand side of (2.11) can be rewritten as follows:

$$\begin{aligned} -ad\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta u_{,\alpha\alpha\eta}dAd\eta = &ad\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta_{,\alpha}u_{,\alpha\eta}dAd\eta \\ &+ ad\int_0^t\int_{L_z}\exp(-\omega\eta)\theta u_{,1\eta}dx_2d\eta. \end{aligned} \tag{2.15}$$

By combining (2.11)–(2.15), we have the following:

$$\begin{aligned}
&-k\int_0^t\int_{L_z}\exp(-\omega\eta)\theta\theta_{,1}dx_2d\eta-d\int_0^t\int_{L_z}\exp(-\omega\eta)\theta u_{,1\eta\eta}dx_2d\eta\\
&-ad\int_0^t\int_{L_z}\exp(-\omega\eta)\theta u_{,1\eta}dx_2d\eta\\
&=\frac{\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta^2dAd\eta+\frac{1}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)\theta^2dA\\
&\quad+k\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta_{,\alpha}\theta_{,\alpha}dAd\eta+d\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta_{,\alpha}u_{,\alpha\eta\eta}dAd\eta\\
&\quad+ad\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta_{,\alpha}u_{,\alpha\eta}dAd\eta.
\end{aligned}\tag{2.16}$$

If we define $F_2(z,t)$ as (2.9), then we can easily get (2.10) from (2.16).

By combining the results of Propositions 2.1 and 2.2, we obtain the following results.

Proposition 2.3: If we define $F(z,t)=F_1(z,t)+F_2(z,t)$, then we can easily get either

$$\begin{aligned}
F(z,t)=&-b\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta\eta}u_{,1\eta}dx_2d\eta-c\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta\eta}u_{,1}dx_2d\eta\\
&+d\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta\eta}\theta_{,1}dx_2d\eta-k\int_0^t\int_{L_z}\exp(-\omega\eta)\theta\theta_{,1}dx_2d\eta\\
&-ad\int_0^t\int_{L_z}\exp(-\omega\eta)\theta u_{,1\eta}dx_2d\eta\\
&-d\frac{\partial}{\partial z}\left[\int_0^t\int_{L_z}\exp(-\omega\eta)\theta u_{,\eta\eta}dx_2d\eta\right]+d\int_0^t\int_{L_z}\exp(-\omega\eta)\theta_{,1}u_{,\eta\eta}dx_2d\eta,
\end{aligned}\tag{2.17}$$

or

$$\begin{aligned}
F(z,t)=&\left(\frac{\omega}{2}+a\right)\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(u_{,\eta\eta})^2dAd\eta\\
&+\frac{1}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)(u_{,tt})^2dA+\left(\frac{\omega b}{2}-c\right)\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}dAd\eta\\
&+\frac{b}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)u_{,\alpha t}u_{,\alpha t}dA+\frac{c\omega^2}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha}u_{,\alpha}dAd\eta\\
&+\frac{c\omega}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)u_{,\alpha}u_{,\alpha}dA+\frac{\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta^2dAd\eta\\
&+\frac{1}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)\theta^2dA+k\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta_{,\alpha}\theta_{,\alpha}dAd\eta\\
&+c\int_z^\infty\int_{L_\xi}\exp(-\omega t)u_{,\alpha t}u_{,\alpha}dA+ad\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta_{,\alpha}u_{,\alpha\eta}dAd\eta.
\end{aligned}\tag{2.18}$$

3. Some basic inequalities

In this paper, we will employ the integral-differential inequality method to obtain our result. To derive our result, it is necessary to construct some important inequalities.

Proposition 3.1: For the energy function $F(z,t)$ defined in (2.18), we have the following estimates:

$$\begin{aligned} G(z,t) = &\left(\frac{\omega}{2}+a\right)\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)(u_{,\eta\eta})^2dAd\eta+\frac{1}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)(u_{,tt})^2dA\\ &+\left(\frac{\omega b}{2}-c-\frac{a^2d^2}{2k}\right)\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}dAd\eta+\frac{b}{4}\int_z^\infty\int_{L_\xi}\exp(-\omega t)u_{,\alpha t}u_{,\alpha t}dA\\ &+\frac{c\omega^2}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha}u_{,\alpha}dAd\eta+\left(\frac{c\omega}{2}-\frac{c^2}{b}\right)\int_z^\infty\int_{L_\xi}\exp(-\omega t)u_{,\alpha}u_{,\alpha}dA\\ &+\frac{\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta^2dAd\eta+\frac{1}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)\theta^2dA\\ &+\frac{k}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta_{,\alpha}\theta_{,\alpha}dAd\eta\\ &\le F(z,t), \end{aligned}\tag{3.1}$$

and

$$\begin{aligned} -\frac{\partial F(z,t)}{\partial z}\ge &\left(\frac{\omega}{2}+a\right)\int_0^t\int_{L_z}\exp(-\omega\eta)(u_{,\eta\eta})^2dx_2d\eta+\frac{1}{2}\int_{L_z}\exp(-\omega t)(u_{,tt})^2dx_2\\ &+\left(\frac{\omega b}{2}-c-\frac{a^2d^2}{2k}\right)\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}dx_2d\eta+\frac{b}{4}\int_{L_z}\exp(-\omega t)u_{,\alpha t}u_{,\alpha t}dx_2\\ &+\frac{c\omega^2}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha}u_{,\alpha}dx_2d\eta+\left(\frac{c\omega}{2}-\frac{c^2}{b}\right)\int_{L_z}\exp(-\omega t)u_{,\alpha}u_{,\alpha}dx_2\\ &+\frac{\omega}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)\theta^2dx_2d\eta+\frac{1}{2}\int_{L_z}\exp(-\omega t)\theta^2dx_2\\ &+\frac{k}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)\theta_{,\alpha}\theta_{,\alpha}dx_2d\eta. \end{aligned}\tag{3.2}$$

If we choose $\omega\ge\frac{4c}{b}+\frac{2a^2d^2}{k}$, then we can obtain $G(z,t)$ and $-\frac{\partial F(z,t)}{\partial z}$ are all positive.

Proof: Using the Schwarz inequality for terms in (2.18), we obtain the following:

$$\begin{aligned} \left|c\int_z^\infty\int_{L_\xi}\exp(-\omega t)u_{,\alpha t}u_{,\alpha}dA\right|\le &\frac{b}{4}\int_z^\infty\int_{L_\xi}\exp(-\omega t)u_{,\alpha t}u_{,\alpha t}dA\\ &+\frac{c^2}{b}\int_z^\infty\int_{L_\xi}\exp(-\omega t)u_{,\alpha}u_{,\alpha}dA, \end{aligned}\tag{3.3}$$

and

$$\begin{aligned} \left|ad\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta_{,\alpha}u_{,\alpha\eta}dAd\eta\right|\le &\frac{k}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta_{,\alpha}\theta_{,\alpha}dAd\eta\\ &+\frac{(ad)^2}{2k}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}dAd\eta. \end{aligned}\tag{3.4}$$

By inserting (3.3) and (3.4) into (2.18), we obtained the desired result (3.1).

Differentiating (2.18) with respect to z leads to the following:

$$\begin{aligned}-\frac{\partial F(z,t)}{\partial z} &= \left(\frac{\omega}{2}+a\right)\int_0^t\int_{L_z}\exp(-\omega\eta)(u_{,\eta\eta})^2dx_2d\eta+\frac{1}{2}\int_{L_z}\exp(-\omega t)(u_{,tt})^2dx_2\\ &+\left(\frac{\omega b}{2}-c\right)\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}dx_2d\eta+\frac{b}{2}\int_{L_z}\exp(-\omega t)u_{,\alpha t}u_{,\alpha t}dx_2\\ &+\frac{c\omega^2}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha}u_{,\alpha}dx_2d\eta+\frac{c\omega}{2}\int_{L_z}\exp(-\omega t)u_{,\alpha}u_{,\alpha}dx_2\\ &+\frac{\omega}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)\theta^2dx_2d\eta+\frac{1}{2}\int_{L_z}\exp(-\omega t)\theta^2dx_2\\ &+k\int_0^t\int_{L_z}\exp(-\omega\eta)\theta_{,\alpha}\theta_{,\alpha}dx_2d\eta+c\int_{L_z}\exp(-\omega t)u_{,\alpha t}u_{,\alpha}dx_2\\ &+ad\int_0^t\int_{L_z}\exp(-\omega\eta)\theta_{,\alpha}u_{,\alpha\eta}dx_2d\eta.\end{aligned} \tag{3.5}$$

Following the same procedures as deriving (3.1), we have the following:

$$\begin{aligned}\left|c\int_{L_z}\exp(-\omega t)u_{,\alpha t}u_{,\alpha}dx_2\right| &\le \frac{b}{4}\int_{L_z}\exp(-\omega t)u_{,\alpha t}u_{,\alpha t}dx_2\\ &+\frac{c^2}{b}\int_{L_z}\exp(-\omega t)u_{,\alpha}u_{,\alpha}dx_2\end{aligned} \tag{3.6}$$

$$\begin{aligned}\left|ad\int_0^t\int_{L_z}\exp(-\omega\eta)\theta_{,\alpha}u_{,\alpha\eta}dx_2d\eta\right| &\le \frac{k}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)\theta_{,\alpha}\theta_{,\alpha}dx_2d\eta\\ &+\frac{(ad)^2}{2k}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}dx_2d\eta\end{aligned} \tag{3.7}$$

Inserting (3.6) and (3.7) into (3.5), we obtain the desired result (3.2).

Proposition 3.2: For the energy function $F(z,t)$ defined in (2.17) and (2.18), we have the following estimate:

$$\frac{\sqrt{(\omega+2a)\omega}}{2d}\int_z^\infty F(\xi,t)d\xi+\frac{\partial F(z,t)}{\partial z}\le k_1F(z,t), \tag{3.8}$$

with

$$\begin{aligned}k_1=\frac{\sqrt{(\omega+2a)\omega}}{2d}\Big(&2\sqrt{c}\frac{1}{\sqrt{\omega+2a}}\cdot\frac{1}{\omega}+2b\frac{1}{\sqrt{\omega+2a}\sqrt{\omega b-2c}}+2\sqrt{2}d\frac{1}{\sqrt{\omega+2a}\sqrt{\omega b-2c}}\\ &+2\sqrt{2}d\frac{1}{\sqrt{\omega+2a}}\cdot\frac{1}{\sqrt{k}}+2ad\frac{1}{\sqrt{\omega}\sqrt{\omega b-2c}}+\sqrt{2}k\frac{1}{\sqrt{\omega k}}\Big).\end{aligned}$$

Proof: Integrating (2.17) with respect to z from z to ∞, we have the following:

$$\begin{aligned}\int_z^\infty F(\xi,t)d\xi = &-b\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\eta\eta}u_{,1\eta}dAd\eta - c\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\eta\eta}u_{,1}dAd\eta\\ &+2d\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\eta\eta}\theta_{,1}dAd\eta - k\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta\theta_{,1}dAd\eta\\ &-ad\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta u_{,1\eta}dAd\eta - d\int_0^t\int_{L_z}\exp(-\omega\eta)\theta u_{,\eta\eta}dx_2d\eta.\end{aligned} \tag{3.9}$$

Using the Schwarz inequality, we have the following:

$$\begin{aligned}-d\int_0^t\int_{L_z}\exp(-\omega\eta)\theta u_{,\eta\eta}dx_2d\eta \le & d\left(\int_0^t\int_{L_z}\exp(-\omega\eta)\theta^2dx_2d\eta\right)^{\frac{1}{2}}\\ &\left(\int_0^t\int_{L_z}\exp(-\omega\eta)(u_{,\eta\eta})^2dx_2d\eta\right)^{\frac{1}{2}}.\end{aligned} \tag{3.10}$$

Using (3.2), we have the following:

$$-d\int_0^t\int_{L_z}\exp(-\omega\eta)\theta u_{,\eta\eta}dx_2d\eta \le \frac{2d}{\sqrt{\omega(\omega+2a)}}\left(-\frac{\partial F(z,t)}{\partial z}\right). \tag{3.11}$$

Following the same procedures and using (3.1), we can obtain the following:

$$-c\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\eta\eta}u_{,1}dAd\eta \le \frac{2\sqrt{c}}{\sqrt{\omega(\omega+2a)}}\cdot\frac{1}{\omega}F(z,t), \tag{3.12}$$

$$-b\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\eta\eta}u_{,1\eta}dAd\eta \le \frac{2b}{\sqrt{(\omega+2a)(\omega b-2c)}}F(z,t), \tag{3.13}$$

$$2d\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\eta\eta}\theta_{,1}dAd\eta \le \frac{2\sqrt{2}d}{\sqrt{\omega+2a}}\cdot\frac{1}{\sqrt{k}}F(z,t), \tag{3.14}$$

$$-ad\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta u_{,1\eta}dAd\eta \le \frac{2ad}{\sqrt{\omega(\omega b-2c)}}F(z,t), \tag{3.15}$$

and

$$-k\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)\theta\theta_{,1}dAd\eta \le \sqrt{2}k\frac{1}{\sqrt{\omega k}}F(z,t). \tag{3.16}$$

A combination of (3.11)–(3.16) gives the desired result (3.8).

4. Spatial decay estimate

It is very difficult to directly solve (3.8), so we must introduce two auxiliary functions. Now, we define the following two functions:

$$\psi(z,t) = e^{-k_1z}F(z,t), \tag{4.1}$$

and

$$H(z,t) = \psi(z,t) + \tau \int_z^\infty e^{k_1(\xi - z)} \psi(\xi,t) d\xi. \tag{4.2}$$

In order to obtain our result, we will use the following Propositions.

Proposition 4.1: Recall that a, b, c, d, and k are given positive system parameters, ω is a chosen positive constant that statisfies the condition in Proposition 3.1, and k_1 is defined in Proposition 3.2. For the function $H(z,t)$ defined in (4.2), we have the following inequality:

$$\frac{\partial H(z,t)}{\partial z} + \tau_1 H(z,t) \leq 0, \tag{4.3}$$

with

$$\tau_1 = \frac{k_1 + \sqrt{k_1^2 + \frac{2\sqrt{(\omega+2a)\omega}}{d}}}{2}.$$

Proof: Differentiating (4.2) with respect to z leads to the following:

$$\begin{aligned}\frac{\partial H(z,t)}{\partial z} &= \frac{\partial \psi(z,t)}{\partial z} - k_1\tau \int_z^\infty e^{k_1(\xi-z)} \psi(\xi,t) d\xi - \tau\psi(z,t)\\ &= -k_1 e^{-k_1 z} F(z,t) + e^{-k_1 z} \frac{\partial F(z,t)}{\partial z}\\ &\quad - k_1\tau \int_z^\infty e^{-k_1 z} F(\xi,t) d\xi - \tau e^{-k_1 z} F(z,t).\end{aligned} \tag{4.4}$$

Thus, we obtain the following:

$$\begin{aligned}\frac{\partial H(z,t)}{\partial z} + \tau H(z,t) &= -k_1 e^{-k_1 z} F(z,t) + e^{-k_1 z} \frac{\partial F(z,t)}{\partial z}\\ &\quad - k_1\tau \int_z^\infty e^{-k_1 z} F(\xi,t) d\xi - \tau e^{-k_1 z} F(z,t)\\ &\quad + \tau e^{-k_1 z} F(z,t) + \tau^2 \int_z^\infty e^{-k_1 z} F(\xi,t) d\xi.\end{aligned} \tag{4.5}$$

From the result of Proposition 3.2, we have the following:

$$-k_1 e^{-k_1 z} F(z,t) + e^{-k_1 z} \frac{\partial F(z,t)}{\partial z} \leq -\frac{\sqrt{(\omega+2a)\omega}}{2d} e^{-k_1 z} \int_z^\infty F(\xi,t) d\xi. \tag{4.6}$$

Inserting (4.6) into (4.5), we obtain the following:

$$\frac{\partial H(z,t)}{\partial z} + \tau H(z,t) \leq \left(\tau^2 - k_1\tau - \frac{\sqrt{(\omega+2a)\omega}}{2d}\right) \int_z^\infty e^{-k_1 z} F(\xi,t) d\xi. \tag{4.7}$$

If we choose

$$\tau_1 = \frac{k_1 + \sqrt{k_1^2 + \frac{2\sqrt{(\omega+2a)\omega}}{d}}}{2},$$

then we have

$$\tau_1^2 - k_1\tau_1 - \frac{\sqrt{(\omega+2a)\omega}}{2d} = 0.$$

Thus, we obtain the desired result (4.3).

Proposition 4.2: For the functions $F(z,t)$ defined in (2.18) and $H(z,t)$ defined in (4.2), we have the following estimate:

$$F(z,t) \le H(0,t)e^{-(\tau_1-k_1)z}. \tag{4.8}$$

Proof: Integrating (4.3), we have the following:

$$H(z,t) \le H(0,t)e^{-\tau_1 z}. \tag{4.9}$$

From the definition of $H(z,t)$ in (4.2), we obtain the following:

$$e^{-k_1 z}F(z,t) \le H(0,t)e^{-\tau_1 z}. \tag{4.10}$$

Thus, we obtain the desired result (4.8).

Proposition 4.3: For $H(z,t)$ defined in (4.2), we have the following estimate:

$$H(0,t) \le \frac{2\tau_1 - k_1}{\tau_1 - k_1}F(0,t). \tag{4.11}$$

Proof: From the definition of $H(z,t)$ defined in (4.2), we have the following:

$$\begin{aligned} H(0,t) &= \psi(0,t) + \tau_1\int_0^\infty e^{k_1\xi}\psi(\xi,t)d\xi \\ &= F(0,t) + \tau_1\int_0^\infty F(\xi,t)d\xi. \end{aligned} \tag{4.12}$$

We must give a bound for $\int_0^\infty F(\xi,t)d\xi$.

Using the result of (4.9), we have the following:

$$F(z,t) + \tau_1\int_z^\infty F(\xi,t)d\xi \le e^{-(\tau_1-k_1)z}H(0,t). \tag{4.13}$$

Multiplying $e^{-\tau_1 z}$ on both sides of (4.13) leads to the following:

$$e^{-\tau_1 z}F(z,t) + \tau_1 e^{-\tau_1 z}\int_z^\infty F(\xi,t)d\xi \le e^{-\tau_1 z}e^{-(\tau_1-k_1)z}H(0,t). \tag{4.14}$$

(4.14) can be rewritten as follows:

$$-\frac{\partial}{\partial z}\left[e^{-\tau_1 z}\int_z^\infty F(\xi,t)d\xi\right] \le H(0,t)e^{-(2\tau_1-k_1)z}. \tag{4.15}$$

Integrating (4.15) from 0 to ∞, we have the following:

$$\int_0^\infty F(\xi,t)d\xi \le \frac{H(0,t)}{2\tau_1 - k_1}. \tag{4.16}$$

Inserting $H(0,t) = F(0,t) + \tau_1 \int_0^\infty F(\xi,t)d\xi$ into (4.16), we have the following:

$$\int_0^\infty F(\xi,t)d\xi \leq \frac{F(0,t) + \tau_1 \int_0^\infty F(\xi,t)d\xi}{2\tau_1 - k_1}. \tag{4.17}$$

Thus, we obtain the following:

$$\int_0^\infty F(\xi,t)d\xi \leq \frac{F(0,t)}{\tau_1 - k_1}. \tag{4.18}$$

Inserting (4.18) into (4.12), we can obtain the desired result (4.11).

By combining results of the above Propositions 4.2 and 4.3, we can obtain the following Theorem:

Theorem 4.1: For the energy function $G(z,t)$ defined in (3.1), we have the following spatial decay estimate:

$$G(z,t) \leq F(z,t) \leq \frac{2\tau_1 - k_1}{\tau_1 - k_1} F(0,t) e^{-(\tau_1 - k_1)z}. \tag{4.19}$$

Inequality (4.19) demonstrates the exponential decay with respect to the variable z.

Remark 4.1: We note that the decay rate of (4.19) is $\tau_1 - k_1 = \frac{-k_1 + \sqrt{k_1^2 + \frac{2\sqrt{(\omega+2a)\omega}}{d}}}{2}$. Since ω is an arbitrary positive constant, we can obtain that the solution can decay at an arbitrarily fast rate when w is taken sufficiently large. This is a striking and physically significant feature of the MGT-Fourier coupling, as it reveals the strong dissipative effect of the model.

Remark 4.2: The exponential decay rate in Theorem 4.1 may not be sharp. As noted in [27, Appendix], standard energy estimates cannot yield the true decay rate for such systems. Therefore, our result is limited by the method itself, but it provides a first and robust estimate for the coupled system on an unbounded domain. Sharper estimates are deferred to future research.

5. Conclusions

In this work, we specifically focused on investigating the spatial decay properties of the solution, thereby deriving an explicit estimate that characterizes its behavior over distance. While this analysis provides valuable insight into the solution's qualitative features, a natural and significant extension lies in examining its structural stability. This will form a core direction for our future research. Investigating the structural stability is particularly challenging due to the unbounded nature of the spatial domain under consideration. Results that concern the structural stability in unbounded domains are notably scarce in the existing literature, largely because the lack of compactness and the difficulties in formulating appropriate function spaces considerably complicate the analysis.

Furthermore, we aim to explore more complex formulations of the problem in subsequent studies. A particularly promising avenue involves incorporating a biharmonic operator into the governing equation. The introduction of such a higher-order term fundamentally alters the nature of the model, thus leading to a new class of problems with potentially different physical interpretations. However, this modification also introduces substantial mathematical challenges. The construction of suitable energy functionals and Lyapunov-type functions, which are essential tools to prove both the spatial decay and the stability estimates, become significantly more intricate with the presence of a biharmonic operator. The increased order of the derivatives demands more delicate estimates and often requires the development of new analytical techniques.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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