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Research article

Dedekind σ-complete partially ordered rings

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Abstract: In this paper, I present the properties and structures of Dedekind σ-complete directed partially ordered rings and algebras over $\mathbb{R}$. Under certain conditions, such as property P_1, chain conditions on convex ideals, and strong Archimedean, structures are isomorphic to the real number field $\mathbb{R}$ with the usual total order or finite product or subdirect product of $\mathbb{R}$.

Keywords: Archimedean; Dedekind σ-complete; idempotent; infinite prime; maximal partial order; strong Archimedean; total order

1. Introduction

All rings in the paper are associative, with $1 \neq 0$, and of characteristic 0. A *partially ordered ring* $(R, \geq)$ is a ring R equipped with a partial order $\geq$ that satisfies

$$\forall a, b, c \in R, \text{ if } a \geq b, \text{ then } a + c \geq b + c \text{ and if } a \geq 0, b \geq 0, \text{ then } ab \geq 0.$$

The *positive cone* of the partially ordered ring $(R, \geq)$ is defined as $R^+ = \{a \in R \mid a \geq 0\}$. R^+ is closed under the addition and multiplication of R, and $R^+ \cap -R^+ = \{0\}$. The elements in R^+ are called *positive* (including 0). On the other hand, let R be a ring and P be a subset of R that satisfies $P + P \subseteq P$, $PP \subseteq P$, and $P \cap -P = \{0\}$. Then, the partial order defined by for all $x, y \in R$, $y \geq x$ if $y - x \in P$ makes R into a partially ordered ring $(R, \geq)$ with the positive cone P. Because of this connection, we will denote a partially ordered ring either by $(R, \geq)$, where $\geq$ is a partial order, or by (R, P), where P is the positive cone of a partial order, and we will say that $\geq$ or P is a partial order on the ring R. A partial order P on a ring R is called *directed* if for any $a \in R$, and there exist $b, c \in P$ such that $a = b - c$. If the partial order P on R is a *lattice order*, that is, any two elements have the least upper bound and greatest lower bound, then (R, P) is called a *lattice-ordered ring* (ℓ*-ring*). An f-ring is an ℓ-ring in which for any $a \geq 0$, $x \wedge y = 0$ implies $ax \wedge y = xa \wedge y = 0$. In what follows, $\mathbb{R}^+$ denotes the usual total order on the field $\mathbb{R}$ of real numbers and $\mathbb{Q}^+ = \mathbb{Q} \cap \mathbb{R}^+$, where $\mathbb{Q}$ is the field of rational numbers. For more information on partially ordered rings and ℓ-rings, the reader is referred to [1–4].

A nonempty subset S of a ring R is called a *preprime* if $S + S \subseteq S$, $SS \subseteq S$, and $-1 \notin S$. A maximal preprime is called a *prime*. By Zorn's Lemma, each preprime is contained in a prime. A prime S is called *infinite* if $1 \in S$, otherwise S is called *finite*. An infinite prime S of R is called *full* if $R = S - S = \{a - b \mid a, b \in S\}$. An infinite prime S (or a partial order P with $1 \in P$) is called *strong Archimedean* if for any $a \in S$ (resp. $a \in P$), and there exists a positive integer n such that $n - a \in S$ (resp. $n - a \in P$). A partially ordered ring $(R, \geq)$ is called *Archimedean* if for any $a, b \in R$, $a \geq nb$ for all integers n implies that $b = 0$. An infinite prime S is called *conic* if $S \cap -S = \{0\}$. Moreover, a conic infinite prime is a maximal partial order. For more information on primes for rings, the reader is referred to [5].

A partially ordered ring $(R, \geq)$ is called *Dedekind σ-complete* if $\{x_n\}$ is a sequence in R such that $x_1 \geq x_2 \geq \cdots \geq 0$, so that $\inf\{x_n\}$ exists. DeMarr and Dai initiated the study of Dedekind σ-complete directed partially ordered algebras over $\mathbb{R}$. They call a directed partially ordered algebra over $\mathbb{R}$ as a *pola* (partially ordered linear algebra). For more information on a Dedekind σ-complete pola, the reader is referred to [6–9].

The topics in each section are as followes. In Section 2, it is shown that if R is a Dedekind σ-complete directed partially ordered domain with $1 > 0$ that satisfies property P_1, then R is isomorphic to $\mathbb{R}$ with the usual total order. In Section 3, we consider the structures of Dedekind σ-complete real directed partially ordered algebras A with $1 > 0$ that have property P_1. It is proved, for instance, that if A is strong Archimedean, and the intersection of maximal ideals is zero, then A is isomorphic to a subdirect product of $\mathbb{R}$. In Section 4, we introduce the subalgebra bounded by integers that is strong Archimedean. In Section 5, we give a characterization of Dedekind σ-complete finite-dimensional ℓ-algebras. In Section 6, we consider the bounded inversion property for commutative ring with $1 \neq 0$. In particular, it is shown that, under certain conditions, the directed maximal partial order is an intersection of full infinite primes. In Section 7, we show that if the exponential and logarithmic functions are positive with respect to a lattice order with $1 > 0$ on $\mathbb{R}$, then the lattice order must be the usual total order.

2. Isomorphic to $(\mathbb{R}, \mathbb{R}^+)$

Recall that a partially ordered ring $(R, \geq)$ is called *integrally closed* if, for all $a, b \in R$, $na \leq b$ for all $n \geq 1$ implies that $a \leq 0$. An integral closed partially ordered ring must be Archimedean. For ℓ-rings, two concepts are equivalent [4, Theorem 2.3.1].

The following result was proved by DeMarr for a Dedekind σ-complete pola.

Lemma 1. *Let $(R, \geq)$ be a Dedekind σ-complete directed partially ordered ring with $1 > 0$. If for any positive integer n $(= \sum_{i=1}^{n} 1)$, the inverse $\frac{1}{n}$ exists and $\frac{1}{n} \geq 0$, then $(R, \geq)$ is integrally closed.*

Proof. Suppose that $a, b \in R$ such that $na \leq b$ for all positive integers n. We show $a \leq 0$. Since $\frac{1}{n} \geq 0$, $na \leq b$, $\forall n \geq 1$ implies $a \leq \frac{1}{n}b$, $\forall n \geq 1$. Since $(R, \geq)$ is directed, $b \leq c$ for some $0 \leq c \in R$ and hence $a \leq \frac{1}{n}c$, $\forall n \geq 1$. Now

$$c \geq \frac{1}{2}c \geq \cdots \geq \frac{1}{n}c \geq \cdots \geq 0$$

implies that $\inf\{\frac{1}{n}c\}$ exists. Let $w = \inf\{\frac{1}{n}c\}$. Then $w \geq 0$. It follows from $w \leq \frac{1}{n}c$, $\forall n \geq 1$ that $nw \leq c$, $\forall n \geq 1$. Thus $w \leq 2w \leq \cdots \leq nw \leq \cdots \leq c$ implies that $\sup\{nw\}$ exists. Let $z = \sup\{nw\}$. Then

$$w \leq 2w \leq \cdots \leq nw \leq \cdots \leq z \Rightarrow 0 \leq w \leq \cdots \leq (n-1)w \leq \cdots \leq z - w.$$

Hence, $z \leq z - w$, so $w \leq 0$. Therefore, $w = 0$ and $a \leq \frac{1}{n}c$, $\forall n \geq 1$ implies that $a \leq w = 0$.

We notice that the condition "for any positive integer n, $\frac{1}{n}$ exists and is positive" in Lemma 1 is equivalent to that $\mathbb{Q} \subseteq R$ and for any $0 \neq q \in \mathbb{Q}^+$, $q \geq 0$.

DeMarr [7, Theorem] proved that every Dedekind σ-complete directed partially ordered division ring with $1 > 0$ that satisfies if $x > 0$, then $x^{-1} > 0$ is isomorphic to $\mathbb{R}$ with the usual total order. We notice that the condition "if $x > 0$, then x^{-1} exists and $x^{-1} > 0$" implies property P_1: if $x \geq 1$ then x^{-1} exists and $x^{-1} > 0$.

Theorem 1. *Let $(R, \geq)$ be a Dedekind σ-complete directed partially ordered ring with $1 > 0$ that has property P_1. Then R is isomorphic to $\mathbb{R}$ with the usual total order if and only if R contains only trivial idempotent elements 1 and 0.*

Proof. We need to only show that if R has only trivial idempotent elements, then R is isomorphic to $(\mathbb{R}, \mathbb{R}^+)$. By Lemma 1 and [6, Theorem 3.2], $(R, \geq)$ is an Archimedean commutative f-ring, and for any $x \in R$, $x \vee 0 = a_x x$, where a_x is an idempotent element [6, Theorem 3.1]. Thus, $a_x = 1$ or 0. If $a_x = 1$, then $x \geq 0$; and if $a_x = 0$, then $x \leq 0$. Therefore, $(R, \geq)$ is totally ordered.

Let $0 < a \in R$. There exists a positive integer n such that $a \leq n1$ since $\geq$ is an Archimedean total order, so $\frac{1}{n} > 0$ implies that $\frac{1}{n}a \leq 1$. Additionally, there exists a positive integer m such that $ma \geq 1$. Hence, $(nm1)(\frac{1}{n}a) = ma \geq 1$. By [8, Proposition 3], $\frac{1}{n}a$ has an inverse and hence a^{-1} exists. Thus, $(R, \geq)$ is a totally ordered field, so $(R, \geq)$ is isomorphic to $(\mathbb{R}, \mathbb{R}^+)$ by [7, Theorem].

The following result is a direct consequence of Theorem 1 that generalizes DeMarr's result from division rings to rings without nonzero zero divisors.

Corollary 1. *Let $(R, \geq)$ be a Dedekind σ-complete directed partially ordered ring with $1 > 0$ that does not contain nonzero zero divisors and has property P_1. Then R is isomorphic to $\mathbb{R}$ with the usual total order.*

The conditions in Corollary 1 cannot be dropped. Let $\mathbb{Z}$ be the ring of integers with the usual total order. Then, $\mathbb{Z}$ is Dedekind σ-complete, but $\mathbb{Z}$ does not have property P_1. Additionally, $(\mathbb{Q}, \mathbb{Q}^+)$ has property P_1, but it is not Dedekind σ-complete.

3. Idempotents in a Dedekind σ-complete pola that has P_1

Recall that two idempotents α, β in a ring are called *orthogonal* if $\alpha\beta = \beta\alpha = 0$. An idempotent $e \neq 0$ is called *primitive* if e has no decomposition into $e = \alpha + \beta$, where α, β are nonzero orthogonal idempotents. We need the following results in the proof later.

Lemma 2. *([10, Lemma 5]) Let R be an ℓ-ring in which the square of each element is positive. If $e \in R$ is an idempotent, then eR is a right ℓ-ideal of R.*

Proof. Suppose $|a| \leq |b|$ and $b \in eR$. Then $b = er$ for some $r \in R$, so $|a| \leq |b| \leq e|r|$ implies that $(1-e)|a| = 0$, so $|a| = e|a| \in eR$. It follows that $a^+, a^- \in eR$ since $a^+, a^- \leq |a|$ and hence $a \in eR$.

Lemma 3. *(cf. [11, Proposition 3.4] and [12, Theorem 4]) Let A be an Archimedean totally ordered algebra with $1 \neq 0$ over $\mathbb{R}$ with the usual total order. Then A is isomorphic to $\mathbb{R}$ as totally ordered algebras.*

Proof. Since A is totally ordered, $1 > 0$. From the proof of [11, Proposition 3.4], for any $a \in A$, there exists a unique $\lambda \in \mathbb{R}$ such that $a = \lambda 1$. Then the mapping $\varphi : A \to \mathbb{R}$ defined by $\varphi(a) = \lambda$ is an isomorphic between totally ordered algebras A and $\mathbb{R}$ with the usual total order.

Theorem 2. *Let $(A, \geq)$ be a Dedekind σ-complete real directed partially ordered algebra with $1 > 0$ that has property P_1 and let $e \in A$ be an idempotent.*

(1). The ℓ-ideal eA is an Archimedean Dedekind σ-complete real f-algebra and has property P_1.

(2). The ℓ-ideal eA is isomorphic to $\mathbb{R}$ with the usual total order if and only if e is primitive.

Proof. (1) Since A is an Archimedean commutative f-algebra, the square of each element is positive, so by Lemma 2, eA is an ℓ-ideal. Suppose that $\{x_n\}$ is a sequence in eA such that $x_1 \geq x_2 \geq \cdots \geq 0$. We show that $\inf\{x_n\}$ exists in eA. Let $x = \inf\{x_n\}$ in A. By [6, Lemma 1.1], $ex = \inf\{ex_n\} = \inf\{x_n\} = x$, so $x \in eA$. Thus, x is the greatest lower bound of $\{x_n\}$ in eA as well, and hence eA is Dedekind σ-complete. Therefore, eA is a Dedekind σ-complete f-subalgebra over $\mathbb{R}$. Let $x \in eA$ and $x \geq e$. Then, $x + (1-e) \geq 1$, so there is $y \in A$, $y > 0$ such that $y(x + 1 - e) = 1$ and hence $(ey)x = e$. Thus, eA has property P_1.

(2) Suppose that eA is isomorphic to $\mathbb{R}$ with the usual total order. Moreover, the identity element e is primitive. Now let $x \in eA$ be an idempotent element. Then, $e = x + (e - x)$ and $x(e - x) = 0$ and hence e is primitive implies that $x = 0$ or $e - x = 0$, so eA contains only trivial idempotent elements 0 and e. By (1), eA is Dedekind σ-complete that has P_1, so eA is an Archimedean totally ordered algebra over $\mathbb{R}$. Thus, eA is isomorphic to $\mathbb{R}$ as totally ordered algebras over $\mathbb{R}$ with the usual total order by Lemma 3.

Corollary 2. *Let $(A, \geq)$ be a Dedekind σ-complete real directed partially ordered algebra with $1 > 0$ that has property P_1. Then, $(A, \geq)$ is isomorphic to a finite direct product of copies of $\mathbb{R}$ with the usual total order if and only if 1 can be written as a sum of orthogonal primitive idempotents.*

Proof. "$\Rightarrow$" It is clear. "$\Leftarrow$" Let $1 = e_1 + \cdots + e_n$, where each e_i is a primitive idempotent, $n \geq 1$, and $e_ie_j = e_je_i = 0$. Then $A = e_1A + \cdots + e_nA$ as a direct product of ℓ-ideals e_iA of A. By Theorem 2, each e_iA is isomorphic to $\mathbb{R}$ as algebras over $\mathbb{R}$ with the usual total order.

In Corollary 2, property P_1 is essential. The $n \times n$ matrix algebra over $\mathbb{R}$ with the entrywise order is Dedekind σ-complete and does not have property P_1. Additionally, the identity matrix is a sum of n orthogonal primitive idempotents.

A partially ordered ring is called *left* (*right*) artinian if there is no infinite, strictly decreasing chain of convex left (right) ideals. A partially ordered ring is called *left* (*right*) noetherian if there is no infinite, strictly increasing chain of convex left (right) ideals.

Theorem 3. *Let $(A, \geq)$ be a Dedekind σ-complete real directed partially ordered algebra with $1 \neq 0$ that has property P_1. If $(A, \geq)$ is left (or right) artinian or noetherian, then $(A, \geq)$ is isomorphic to a finite direct product of copies of $\mathbb{R}$ with the usual total order.*

Proof. We show that 1 is a sum of orthogonal primitive idempotents. Then, the result follows from Corollary 2. Suppose that 1 is not a sum of orthogonal primitive idempotents, then 1 is not primitive, so $1 = e_1 + f_1$, where e_1, f_1 are nonzero idempotents and $e_1 f_1 = f_1 e_1 = 0$. Notice that $A \supsetneq e_1 A$ and $A \supsetneq f_1 A$. Since 1 is not a sum of orthogonal primitive idempotents, without loss of generality, we may assume that e_1 is not a sum of orthogonal primitive idempotents. Then, $e_1 = e_2 + f_2$, where e_2, f_2 are nonzero idempotents and $e_2 f_2 = f_2 e_2 = 0$. We have $A \supsetneq e_1 A \supsetneq e_2 A$, $A \supsetneq e_1 A \supsetneq f_2 A$ and $f_1 A \subsetneq f_1 A + e_2 A$, $f_1 A \subsetneq f_1 A + f_2 A$. Since e_1 is not a sum of orthogonal primitive idempotents, we may assume that $f_2 = e_3 + f_3$, where e_3, f_3 are nonzero idempotents and $e_3 f_3 = f_3 e_3 = 0$. Then we have

$$A \supsetneq e_1 A \supsetneq f_2 A \supsetneq e_3 A; \; A \supsetneq e_1 A \supsetneq f_2 A \supsetneq f_3 A$$

and

$$f_1 A \subsetneq f_1 A + e_2 A \subsetneq f_1 A + e_2 A + e_3 A; \; f_1 A \subsetneq f_1 A + e_2 A \subsetneq f_1 A + e_2 A + f_3 A.$$

Repeating the above procedure leads to a strictly decreasing chain of ℓ-ideals and a strictly increasing chain of ℓ-ideals, a contradiction. Thus, 1 must be a sum of orthogonal primitive idempotents.

Let $(R, \geq)$ be a partially ordered ring with $1 \geq 0$. An interesting property studied in [9] is P_2: if $x \in R$, then there exist $y, z \in R$ such that $y \geq 0, z \geq 0$, $yz = 0$, and $x = y - z$. DeMarr showed that for a Dedekind σ-complete real directed partially ordered algebra, the properties P_1 and P_2 are equivalent. Using the weaker condition Archimedean instead of Dedekind σ-complete, we have the following characterization on property P_2.

Theorem 4. *Let $(R, \geq)$ be an Archimedean partially ordered ring with $1 > 0$. Then, R has property P_2 if and only if $(R, \geq)$ is an f-ring.*

Proof. "$\Leftarrow$" If R is an f-ring, then for any $x \in R$, $x = x^+ - x^-$ and $x^+ x^- = x^- x^+ = 0$, where $x^+ = x \vee 0$ and $x^- = (-x) \vee 0$. Thus property P_2 is true.

"$\Rightarrow$" We first show that if $0 \leq a \in R$, $a^2 = 0$, then $a = 0$. Let $1 - a = b - c$ with $b \geq 0$, $c \geq 0$ and $bc = 0$. Then, $a = ab - ac \geq 0$ implies $ac \leq ab$, so $ac^2 = 0$ since $bc = 0$. It follows that $ac = abc - ac^2 = 0$ and hence $1 - a = b - c$ implies that $0 \leq c = -c^2 \leq 0$, so $c = 0$. Therefore, $a \leq 1$. For any positive integer n, $a^2 = 0$ implies that $(na)^2 = 0$ and hence $na \leq 1$ by the above argument. Since $(R, \geq)$ is Archimedean, we have $a = 0$.

For $x \in R$, by property P_2, $x = y - z$ for some $y \geq 0$, $z \geq 0$ and $yz = 0$. Since $(zy)^2 = 0$, we have $zy = 0$ by the previous argument and hence $x^2 = y^2 + z^2 \geq 0$. That is, for any $x \in R$, $x^2 \geq 0$. Let z be a nilpotent element of R. Then, for any positive integer n, $(1 \pm nz)^2 \geq 0$ implies $\pm 2nz \leq 1$, so $z = 0$ since R is Archimedean. Thus, R does not have nonzero nilpotent elements, and hence $(R, \geq)$ is an f-ring [13, Theorem].

4. The subalgebra of a pola that is bounded by integers

Let $(A, \geq)$ be a real directed partially ordered algebra with $1 > 0$. Define

$$B = \{x \in A \mid -n \leq x \leq n, \text{ for some positive integer } n\}. \quad (*)$$

Lemma 4. *Let $(A, \geq)$ be a real directed partially ordered algebra with $1 > 0$. Then, B is a convex directed partially ordered subalgebra of A with $1 > 0$.*

Proof. Let $a, b \in B$. Then $-n \leq a \leq n$ and $-m \leq b \leq m$ for some positive integers n, m. Thus, $a - b \in B$. Since

$$(a+n)(b+m) \geq 0 \;\Rightarrow\; ab + ma + nb + nm \geq 0,$$
$$(a-n)(b-m) \geq 0 \;\Rightarrow\; ab - ma - nb + nm \geq 0,$$

we have $2(ab + nm) \geq 0$, so $ab + nm = \frac{1}{2}(2(ab + nm)) \geq 0$. Similarly,

$$(a+n)(b-m) \leq 0 \;\Rightarrow\; ab - ma + nb - nm \leq 0,$$
$$(a-n)(b+m) \leq 0 \;\Rightarrow\; ab + ma - nb - nm \leq 0,$$

implies $ab - nm \leq 0$. Thus, $ab \in B$. Let $a \in B$. Suppose that $-n \leq a \leq n$ for some positive integer n. Take $0 \neq \beta \in \mathbb{R}$. If $\beta \in \mathbb{R}^+$, $-n\beta \leq \beta a \leq n\beta$. Since $\mathbb{R}^+$ is Archimedean, there exists a positive integer m such that $n\beta < m$, so $-m \leq \beta a \leq m$. Thus, $\beta a \in B$. If $\beta \in -\mathbb{R}^+$, then $(-\beta)a \in B$ by the above argument, and hence $\beta a \in B$ for any $\beta \in \mathbb{R}$. Thus, B is a subalgebra of A over $\mathbb{R}$.

Let P be the positive cone of $\geq$. As such, $(B, P \cap B)$ is a partially ordered algebra over $\mathbb{R}$. Let $a \in B$. Then, $-n \leq a \leq n$ for some positive integer n. Let $b = n - a$. Then $0 \leq b \leq n$ implies that $b \in B$, so $a = n - b$ with $n, b \in P \cap B$. Thus, $(B, P \cap B)$ is directed with $1 > 0$. As such, $0 \leq x \leq y \in B$ for some $x, y \in R$ implies that $x \in B$, that is, B is convex.

Theorem 5. *Let $(A, \geq)$ be a real Dedekind σ-complete directed partially ordered algebra with $1 > 0$ and P be the positive cone of $\geq$. Then, $(B, P \cap B)$ is a real Dedekind σ-complete directed partially ordered algebra that has property P_1.*

Proof. We first confirm that $(B, P \cap B)$ is Dedekind σ-complete. Let $\{x_n\} \subseteq B$ such that $x_1 \geq_{P\cap B} x_2 \geq_{P\cap B} \cdots \geq_{P\cap B} 0$. Then $x_1 \geq x_2 \geq \cdots \geq 0$ in $(A, \geq)$. So $\inf\{x_n\} = x \in A$ exists with respect to $\geq$. Since $0 \leq x \leq x_n \in B$ and B is convex with respect to $\geq$, $x \in B$, so $x \leq_{P\cap B} x_n$, $\forall n \geq 1$. Let $y \in B$ be a lower bound of $\{x_n\}$ with respect to $P \cap B$. Then, y is a lower bound of $\{x_n\}$ in A with respect to $\geq$, so $y \leq x$, but $x, y \in B$ implies that $y \leq_{P\cap B} x$. Thus, $x = \inf\{x_n\}$ in $(B, P \cap B)$ and hence $(B, P \cap B)$ is Dedekind σ-complete.

Let $a \in B$ and $a \geq_{P\cap B} 1$. Then, there exists a positive integer n such that $-n \leq a \leq n$. It follows that $\frac{1}{n}a \leq_{P\cap B} 1$ and $n(\frac{1}{n}a) = a \geq_{P\cap B} 1$, so $\frac{1}{n}a$ has a positive inverse [8, Proposition 3], and hence a has a positive inverse a^{-1}. Thus $(B, P \cap B)$ has property P_1.

Theorem 6. *Let $(A, \geq)$ be a Dedekind σ-complete real directed partially ordered algebra with $1 > 0$ that is strong Archimedean.*

(1). $(A, \geq)$ has property P_1.

(2). For a maximal ideal M, A/M is isomorphic to $\mathbb{R}$ with the usual total order.

Proof. (1) If A is strong Archimedean, then $A = B$ that is defined in $(*)$, so A has property P_1 by Theorem 5.

(2) By (1), $(A, \geq)$ is an Archimedean commutative f-algebra with $1 > 0$. Since $(A, \geq)$ has property P_1, each maximal ideal is an ℓ-ideal by Lemma 5 or [14, Lemma 1.1]. Let M be a maximal ideal of A.

Since M is convex, M is closed under scalar multiplication. In fact, for any $r \in \mathbb{R}$, $r \geq 0$, there exists a positive integer m such that $0 \leq r \leq m$ in $\mathbb{R}$. Thus, $0 \leq rx \leq nx$ in $(A, \geq)$, for any positive element $x \in A$, so $rx \in M$ if $x \in M$. Hence, A/M is an algebra over $\mathbb{R}$ and a totally ordered field. We show that A/M is also strong Archimedean. Let $a + M \geq 0$ in A/M. We may assume $a \geq 0$ in $(A, \geq)$. Then, there exists a positive integer n such that $n - a \geq 0$ in $(A, \geq)$, so $n(1 + M) - (a + M) = (n - a) + M \geq 0$ in A/M. Thus, A/M is strong Archimedean. Finally, we show that A/M is Archimedean. Let $x, y \in A/M$ with $x \geq 0$, $y \geq 0$, and $nx \leq y$ in A/M, for all $n \geq 1$. If $x \neq 0$, then $n \leq x^{-1}y < m$ for all $n \geq 1$ and some positive integer m, a contradiction. Hence, A/M is Archimedean. Since A/M is an Archimedean totally ordered algebra over $\mathbb{R}$, A/M is isomorphic to $\mathbb{R}$ with the usual total order by Lemma 3.

The condition that $(A, \geq)$ is strong Archimedean in Theorem 6 cannot be omitted. The $n \times n$ matrix algebra over $\mathbb{R}$ with the entrywise order is not strong Archimedean and does not have property P_1.

Let $(A, \geq)$ be a Dedekind σ-complete real directed partially ordered algebra with $1 > 0$ and B be defined in $(*)$. We notice that B is a Dedekind σ-complete real directed partially ordered algebra with $1 > 0$ that is strong Archimedean. If the intersection of all maximal ideals of B is zero, by Theorem 6 (2), $(A, \geq)$ contains a subdirect product of the copies of $\mathbb{R}$ with the usual total order. A special case is that if $(A, \geq)$ is a finite-dimensional Dedekind σ-complete real directed partially ordered algebra with $1 > 0$, then $(A, \geq)$ contains a direct product of a finite copies of $\mathbb{R}$ with the usual total order. More details will be given in the following section.

5. Dedekind σ-complete finite dimensional real ℓ-algebras

We first provide a characterization for Dedekind σ-complete finite-dimensional ℓ-algebras over $\mathbb{R}$ with $1 > 0$.

Theorem 7. *Let A be an ℓ-algebra over $\mathbb{R}$ with $1 > 0$. Then, A is Dedekind σ-complete and n-dimensional over $\mathbb{R}$ if and only if as a vector lattice over $\mathbb{R}$, A is isomorphic to a direct product of n copies of $\mathbb{R}$ with the usual total order.*

Proof. Suppose that A is Dedekind σ-comlete and n-dimensional over $\mathbb{R}$. By Lemma 1, A is Archimedean that implies that A is Archimedean over $\mathbb{R}$ [1, Definition, page 50], so as a vector space over $\mathbb{R}$, A is a finite direct product of maximal convex totally ordered subspaces of A over $\mathbb{R}$ [3, Corollary 1.3]. Since A is Archimedean, each maximal convex totally ordered subspace of A is also Archimedean, so by [11, Proposition 3.4], each maximal convex totally ordered subspace of A is isomorphic to $\mathbb{R}$ with the usual total order as vector lattices and hence as a vector lattice A is isomorphic to a direct product of n copies of $\mathbb{R}$ with the usual total order.

Conversely, suppose that as a vector lattice over $\mathbb{R}$, $A = \mathbb{R} \oplus \cdots \oplus \mathbb{R}$ is a direct product of n copies of $\mathbb{R}$ with the usual total order. Let $x_1 \geq x_2 \geq \cdots \geq 0$ be a sequence in A. Denote by $(x_k)_i$ the i^{th} component of x_k in the direct product, $i = 1, \cdots, n$. Then for any i, $(x_1)_i \geq (x_2)_i \geq \cdots \geq 0$ in $\mathbb{R}$, so $\inf\{(x_k)_i\}_{k=1}^{\infty}$ exists. Let $z_i = \inf\{(x_k)_i\}_{k=1}^{\infty}$, $i = 1, \cdots, n$. Then $(z_1, \cdots, z_n) = \inf\{x_k\}_{k=1}^{\infty}$. Therefore, A is Dedekind σ-complete and n-dimensinal over $\mathbb{R}$.

Corollary 3. *Let A be an ℓ-algebra over $\mathbb{R}$ with $1 > 0$ that is Dedekind σ-complete and n-dimensional. Then, the following are equivalent.*

(1). A has property P_1.

(2). A is an f-algebra.

(3). A is isomorphic to a direct product of n copies of $\mathbb{R}$ as ℓ-algebras over $\mathbb{R}$ with the usual total order.

Proof. (1) $\Rightarrow$ (2) By [6, Theorem 3.2]. (2) $\Rightarrow$ (3) By Theorem 7, $A = \mathbb{R} \oplus \cdots \oplus \mathbb{R}$ with n copies of $\mathbb{R}$ as vector lattice over $\mathbb{R}$ with the usual total order. Let $1 = e_1 + \cdots + e_n$. Since $1 > 0$, each $e_i > 0$ and $e_i \wedge e_j = 0$ for any $i \neq j$. Then, that A is an f-algebra implies $e_i e_j = 0$ for any $i \neq j$, so each e_i is an idempotent. Thus $A = \mathbb{R} \oplus \cdots \oplus \mathbb{R}$ as an algebra over $\mathbb{R}$. (3) $\Rightarrow$ (1) is clear.

Let us look at some examples.

Example 1. *(1) Let $M_n(\mathbb{R})$ be the $n \times n$ matrix algebras over $\mathbb{R}$, $n \geq 2$. The only lattice order on $M_n(\mathbb{R})$ to make it an ℓ-algebra with the positive identity matrix is the entrywise order $M_n(\mathbb{R}^+)$ [15, Theorem 12]. By Theorem 7, $(M_n(\mathbb{R}), M_n(\mathbb{R}^+))$ is a Dedekind σ-complete real ℓ-algebra. Thus, the B defined in $(*)$ consists of all diagonal matrices over $\mathbb{R}$.*

(2) Let $T_2(\mathbb{R})$ be the 2×2 upper triangular matrix algebra over $\mathbb{R}$. There are four lattice orders on $T_2(\mathbb{R})$ such that the identity matrix is positive [16, Theorem 2.2]. One of the four lattice orders is not Archimedean, so it is not Dedekind σ-complete. Another lattice order is the entrywise lattice order and B also consists of all diagonal matrices. For the other two cases [16, Theorem 2.2 (1)(b) and (2)], $B = \mathbb{R}$. We look at more details for one of them. Let

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then $\{a, b, c\}$ is a basis of $T_2(\mathbb{R})$ over $\mathbb{R}$. Define an element $\alpha a + \beta b + \gamma c$ positive if $\alpha, \beta, \gamma \in \mathbb{R}^+$. Then $T_2(\mathbb{R})$ becomes a Dedekind σ-complete ℓ-algebra with $1 > 0$. It is straightforward to verify that for any element $x \in T_2(\mathbb{R})$, if $x \geq 1$, then x^{-1} exists. However, property P_1 is not satisfied, for instance, $(a + c)(a - c) = a$ implies that $(a + c)^{-1} = a - c \not\geq 0$.

6. The bounded inversion property

Let $(R, \geq)$ be a partially ordered commutative ring with $1 > 0$. The R is said to have *bounded inversion property* if whenever $x \geq 1$ for $x \in R$, then x has an inverse. This concept was first introduced in [14] for f-rings. For a commutative f-ring, it has the bounded inversion property if and only if each maximal ideal is an ℓ-ideal [14, Lemma 1.1]. We notice that for an f-ring, the bounded inversion property is equivalent to property P_1. In this section, we study the connection between directed maximal partial orders that have bounded inversion property and full infinite primes.

Lemma 5. *Let $(R, \geq)$ be a directed partially ordered commutative ring with $1 > 0$. Then, R has bounded inversion property if and only if each maximal ideal is convex.*

Proof. "$\Rightarrow$" Let M be a maximal ideal of R and $0 \leq a \leq b \in M$. If $a \notin M$, then $R = aR + M$, so $1 = ar + m$ for some $r \in R$ and $m \in M$. Since $(R, \geq)$ is directed, $-s \leq r \leq s$ for some $s > 0$ and hence

$$1 = ar + m \leq as + m \leq bs + m \in M$$

implies that $bs + m$ is invertible. Thus, $1 \in M$, a contradiction.

"$\Leftarrow$" Suppose that $x \geq 1$. Then, the ideal xR is not contained in any maximal ideal since each maximal ideal is convex. Thus, $xR = R$, so x is invertible.

Lemma 6. *Let R be a commutative ring and P be the positive cone of a directed maximal partial order on R. Suppose that* (R, P) *has bounded inversion property.*

(1). For each maximal ideal M, there exists a full infinite prime S_M *such that* $M = S_M \cap -S_M$.

(2). If the intersection of all maximal ideals is zero, then $P = \bigcap_{M_i} S_{M_i}$, *where* M_i *runs through all maximal ideals of R.*

Proof. (1) By Lemma 5, M is convex. Consider $P + M$. As such, it is closed under the addition and multiplication. If $-1 \in P + M$, then $-1 = a + m$, for some $a \in P$ and $m \in M$, implies that $1 + a = -m \in M$, so $1 \in M$ since $0 \leq_P 1 \leq_P 1 + a$, a contradiction. Thus, $-1 \notin P + M$ and hence $P + M$ is a preprime of R. Then, $P + M \subseteq S_M$ for some infinite prime S_M. Since P is directed, S_M is full, and hence $S_M \cap -S_M$ is a prime ideal of R [5, Proposition 2.5]. It follows from $M \subseteq S_M \cap -S_M$ that $M = S_M \cap -S_M$.

(2) Define $S = \bigcap_{M_i} S_{M_i}$, where M_i runs through all maximal ideals of R. It is straightforward to check that $S + S \subseteq S$, $SS \subseteq S$ and $S \cap -S = \bigcap M_i = \{0\}$. Hence, S is a partial order. It follows that $P = S$ since $P \subseteq S$ and P is a maximal partial order.

The condition that (R, P) has bounded inversion property in Lemma 6 cannot be dropped. For example, $\mathbb{Z}^+$ is the only full infinite prime in $\mathbb{Z}$. Let us look at an application of Lemma 6.

Theorem 8. *Let R be a commutative ring with* $1 \neq 0$ *and P be the positive cone of a directed maximal partial order on R. Then,* (R, P) *is isomorphic to a finite direct product of subfields of* $\mathbb{R}$ *with the usual total order if and only if the following conditions are satisfied.*

(1). (R, P) *is strong Archimedean.*

(2). R has bounded inversion property.

(3). R has a finite number of maximal ideals with the zero intersection.

Proof. "$\Rightarrow$" is clear. "$\Leftarrow$" Suppose that $M_1, \cdots, M_n$ are the all maximal ideals of R. By Lemma 6(1), there exist full infinite primes $S_1, \cdots, S_n$ such that $S_i \cap -S_i = M_i$ for $i = 1, \cdots n$. For each M_i, R/M_i is a field and $\overline{S_i} = \{a + M_i \mid a \in S_i\}$ is a conic full infinite prime for R/M_i.

By [17, Corollary 2.27], $\theta : R \to R/M_1 \times \cdots \times R/M_n$ defined by $\theta(r) = (r + M_1, \cdots, r + M_n)$ is a ring isomorphism, $\forall r \in R$. By Lemma 6, $P = \bigcap S_i$. Thus, if $r \in P$, then $r \in S_i$ implies each $r + M_i \in \overline{S_i}$. Suppose that $(r_1 + M_1, \cdots, r_n + M_n)$ with $r_i \in S_i$, $i = 1, \cdots, n$. Then there exists $r \in R$ such that $r + M_i = r_i + M_i$, $i = 1, \cdots, n$, so $r = r_i + m_i, m_i \in M_i$, $i = 1, \cdots, n$ implies that $r \in S_i$ for $i = 1, \cdots, n$, and hence $r \in \bigcap_{i=1}^n S_i = P$. Thus, θ is an isomorphism between two partially ordered rings. Since (R, P) is strong Archimedean, each $(R/M_i, \overline{S_i})$ is strong Archimedean, $i = 1, \cdots, n$. It follows from [5, Proposition 1.7] that there exists an embedding $\varphi_i : R/M_i \to \mathbb{R}$ such that $\overline{S_i} = \varphi^{-1}(\mathbb{R}^+)$. Thus, R/M_i is isomorphic to a subfield of $\mathbb{R}$ with the usual total order.

7. Positive exponential and logarithmic functions on $\mathbb{R}$

Let $f(x) = e^x$ be the exponential function and $g(x) = \ell n(x)$ be the logarithmic function on $\mathbb{R}$. It is well known that for any $x, y \in \mathbb{R}$, $f(x+y) = f(x)f(y)$, and $x \geq_{\mathbb{R}^+} y$ implies that $f(x) \geq_{\mathbb{R}^+} f(y)$; for any $a, b >_{\mathbb{R}^+} 0$, $g(ab) = g(a) + g(b)$, $a \geq_{\mathbb{R}^+} b$ implies that $g(a) \geq_{\mathbb{R}^+} g(b)$, and $f(g(a)) = a$. We may call $f(x)$ and $g(x)$ as positive functions to $\mathbb{R}^+$.

In 1976, Wilson proved that $\mathbb{R}$ has infinitely many lattice orders [18]. All lattice orders constructed in [18] are contained in $\mathbb{R}^+$ with positive identity element. In this section, we show that the usual total order is the only lattice order on $\mathbb{R}$ with $1 > 0$ such that $f(x) = e^x$ and $g(x) = \ell n(x)$ are positive functions.

Recall that a positive element a in an ℓ-ring is called a *d-element* if $x \wedge y = 0$ implies $ax \wedge ay = xa \wedge ya = 0$.

Theorem 9. *Let $(\mathbb{R}, \geq)$ be an ℓ-field with $1 > 0$ and P be the positive cone of $\geq$. If there are functions $f : \mathbb{R} \to \mathbb{R}$ and $g : P \setminus \{0\} \to \mathbb{R}$ such that*

(1). $\forall x, y \in \mathbb{R}$, $f(x+y) = f(x)f(y)$.

(2). $\forall x, y \in \mathbb{R}$, $x \geq y$ implies that $f(x) \geq f(y)$.

(3). $\forall x, y \in P \setminus \{0\}$, $g(xy) = g(x) + g(y)$.

(4). $\forall x, y \in P \setminus \{0\}$, $x \geq y$ implies that $g(x) \geq g(y)$.

(5). $\forall x \in P \setminus \{0\}$, $f(g(x)) = x$.

Then $\geq$ is the usual total order on $\mathbb{R}$.

Proof. We show that each positive element is a d-element. It follows from $g(1) = g(1) + g(1)$ that $g(1) = 0$, so $f(0) = f(g(1)) = 1$. We first show that for any $0 \leq x \in \mathbb{R}$, $f(x) \geq 1$ is an d-element. By the definition of f,

$$1 = f(0) = f(x - x) = f(x + (-x)) = f(x)f(-x).$$

Since $-x \leq 0$, $f(-x) \leq f(0) = 1$. Let $a = 1 - f(-x)$. Then, $a \geq 0$. If $a = 0$, then $f(-x) = 1$, so $f(x) = 1$ is a d-element. Suppose that $a > 0$. Since

$$f(-2x) = f(-x)f(-x) = (1 - a)^2 \text{ and } -x \geq -2x,$$

$1-a = f(-x) \geq f(-2x) = (1-a)^2$, so $a \geq a^2 > 0$. Thus, $g(a) \geq g(a^2) = g(a)+g(a)$, and hence $g(a) \leq 0$. It follows that $1 = f(0) \geq f(g(a)) = a$. Since $0 < a \leq 1$, $0 \leq f(-x) = 1 - a \leq 1$. By [3, Theorem 1.20], $f(x)$ is a d-element since $f(x)^{-1} = f(-x) \geq 0$. Thus, for any $0 \leq x \in \mathbb{R}$, $f(x)$ is a d-element.

Now take $x \in \mathbb{R}$ such that $x \geq 1$. Then $g(x) \geq g(1) = 0$ implies that $x = f(g(x))$ is a d-element by the above argument. Therefore, for any $0 \leq x \in \mathbb{R}$, $x \leq x + 1$ implies that x is a d-element. Hence, each positive element is a d-element, so $(\mathbb{R}, \geq)$ is a totally ordered field [1], and hence $\geq$ is the usual total order on $\mathbb{R}$.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares there are no conflicts of interest.

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