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Research article

The existence and averaging principle for second order stochastic differential systems with pure delay

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Abstract: Stochastic delay differential systems are widely applied in various fields, featuring stochasticity, time delay, and nonlinearity, which makes their analysis highly challenging. This paper investigates the existence, uniqueness, and averaging principle of solutions for a class of stochastic delay differential systems. First, by using delay matrix functions and rigorous theoretical derivation, we establish a theorem on the existence and uniqueness of solutions, which lays a foundation for further analysis. Second, under classical assumptions combined with inequality techniques and Itǒ's formula, an averaging principle is derived, showing that the solution of the original system can be well approximated by that of the averaged system. Finally, numerical simulations are conducted to verify the correctness and practicality of the theoretical results.

Keywords: second-order stochastic delay systems; delayed matrix function; existence and uniqueness; averaging principle

1. Introduction

As is widely recognized, stochastic processes are generally regarded as the dynamic components of probability theory. Driven by the demands of practical applications and the endeavors of mathematicians, this field has achieved vigorous development both theoretically and practically. It is worth noting that stochastic differential equations have long occupied a pivotal position in diverse fields of science and engineering. They are extensively applied in disciplines ranging from mechanics, physics, biology, and chemistry to economics and finance, with fundamental theories and key findings are well documented in [1–5]. Nevertheless, in numerous scenarios, stochastic processes are dependent not only on current states but also on historical states, and they incorporate time-delayed effects, which can be fully described by stochastic differential systems (SDSs) and stochastic delay differential systems (SDDSs). The reference therein provides results on relative controllability results in [6, 7], approximate controllability results in [8, 9], mean-square convergence results in [10, 11], stability results in [12, 13], averaging principle result in [14], and finite-time stability result in [15].

The averaging principle for stochastic delay differential equations serves as a key tool for simplifying complex multi-scale systems and uncovering macroscopic dynamics, holding significant theoretical and applied value.The interplay between time delay and noise induces the separation of fast and slow time scales, rendering direct analysis highly challenging. Through the elimination of high-frequency stochastic fluctuations, this principle yields a low-dimensional averaged system, thereby substantially simplifying the analysis. Research in this area focuses on mean-square convergence, asymptotic stability, and error estimation; rigorous convergence criteria are established through Itô's formula, martingale inequalities, and Lyapunov methods. This principle finds extensive application in population biology, financial modeling, intelligent control, and other domains, providing efficient methodologies for long-term behavior prediction, parameter identification, and control design. Consequently, it advances the development of both stochastic delay system theory and its engineering applications. The averaging principle was first put forward by Khasminskii in [16], where he demonstrated that solutions to the averaged equation are capable of converging to those of the complex system under appropriate conditions.

Gaussian noise is regarded as an ideal noise generator, as it can characterize normal diffusion and reproduce small range-fluctuations around the mean, but is incapable of capturing large-scale variations. In recent years, the qualitative theory of stochastic delay differential equations with Gaussian noise has been extensively investigated. For example, in [17], the authors established the averaging principle of stochastic differential equations with non-Gaussian Lévy noise by virtue of the Lipschitz condition, the linear growth condition, and three mean inequalities. In [18], the authors formulated an averaging principle that is applicable to the stochastic Korteweg-de Vries equation under a general averaging condition. In [19], The authors investigated the averaging principle for stochastic differential equations with slow and fast time scales, by virtue of the local Lipschitz condition and stopping time techniques. In [20], the authors established mixed functional Itô's formulas and the corresponding martingale representation, then developed an averaging principle using weak convergence methods. In [21], the research team formulated an averaging principle applicable to stochastic differential equations governed by time-changed Lévy noise and incorporating variable time delays. In [22], the authors studied the averaging principle applicable to the following neutral stochastic partial functional differential equations incorporating time delayed impulses.

Inspired by the previous discussion, we will devote this paper to deriving the existence and averaging principle for the following second-order SDDSs:

$$
\begin{cases} Z''(t) + \mathbf{A}Z(t-\tau) = h(t, Z(t)) + g(t, Z(t))\dfrac{dW(t)}{dt}, \ t \in J := [0, T], \tau > 0, \\ Z(t) = \Psi(t), \ Z'(t) = \Psi'(t), -\tau \le t \le 0, \end{cases} \tag{1.1}
$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$ is a constant matrix, the state vector $Z \in \mathbb{R}^n$ is a stochastic process, τ is a fixed delay time, $h : J \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : J \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are measurable continuous functions, and $W(t)$ is an m-dimensional Brownian motion on a complete probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$. The function $\Psi \in C^2([-\tau, 0], \mathbb{R}^n)$ satisfies $\mathbb{E}(|\Psi(t)|^2) < \infty$ and $T = k^*\tau$ for a fixed $k^* \in \mathbb{N}^+ := \{1, 2, \cdots\}$.

The key challenges of this paper lie in the following three aspects: First, the form of the solution for SDDSs with the delayed matrix function is complicated; second, handling the stochastic term and the delay term presents a notable challenge, and third, deriving norm estimates associated with delayed matrix functions proves tricky.

This paper proposes the following solutions to address the three major challenges associated with stochastic differential systems featuring delayed matrix functions:

(i). To address the challenges in norm estimation, tight upper bounds for delayed matrix functions are derived using delay-dependent integral inequalities and operator semigroup theory.

(ii). For the coupling of stochastic and delay terms, the stochastic averaging principle is adopted to separate fast and slow scales, and decoupling analysis is realized with Burkholder-Davis-Gundy martingale inequalities.

The subsequent sections of this paper are structured as follows. Section 2 outlines the preliminary concepts and lemmas essential to the subsequent analysis. In Section 3, we rigorously prove the existence and uniqueness of solutions for second-order stochastic delay differential equations (SDDEs) driven by Brownian motion. Section 4 is devoted to establishing an approximation theorem, which serves as an averaging principle for the solutions of the considered second-order SDDEs. Finally, Section 5 presents two numerical examples to demonstrate the effectiveness of the main theoretical results.

2. Preliminaries

For each $t \in [-\tau, T]$, let $\Upsilon = \mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ denote the space consisting of all $\mathcal{F}(t)$-measurable, mean square integrable functions $Z : \Omega \to \mathbb{R}^n$; we define the mean-square norm of $Z(t)$ as $\|Z(t)\|_{ms} := \sqrt{\sum_{i=1}^{n} \mathbb{E}(|Z_i(t)|^2)} = \sqrt{\mathbb{E}(\|Z(t)\|^2)}$, and $\|y\| = \sum_{i=1}^{n} |y_i|$ and $\|\mathbf{A}\| = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|$ are the vector norm and matrix norm, respectively. A process $Z : [-\tau, T] \to \mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ is said to be $\mathcal{F}(t)$-adapted if $Z(t) \in \Upsilon$.

Definition 2.1. *(see [23]) The delayed matrix function $\mathcal{H}_\tau(\mathbf{A}\cdot) : \mathbb{R} \to \mathbb{R}^{n\times n}$ is given by*

$$\mathcal{H}_\tau(\mathbf{A}t) = \begin{cases} \Theta, & -\infty < t < -\tau, \\ E, & -\tau \le t < 0, \\ E - \mathbf{A}\dfrac{t^2}{2!}, & 0 \le t < \tau, \\ E - \mathbf{A}\dfrac{t^2}{2!} + \cdots + (-1)^k \mathbf{A}^k \dfrac{(t-(k-1)\tau)^{2k}}{(2k)!}, & (k-1)\tau \le t < k\tau,\ k \in \mathbb{N}^+, \end{cases} \tag{2.1}$$

and $\mathcal{N}_\tau(\mathbf{A}\cdot) : \mathbb{R} \to \mathbb{R}^{n\times n}$ is given by

$$\mathcal{N}_\tau(\mathbf{A}t) = \begin{cases} \Theta, & -\infty < t < -\tau, \\ E(t+\tau), & -\tau \le t < 0, \\ E(t+\tau) - \mathbf{A}\dfrac{t^3}{3!}, & 0 \le t < \tau, \\ E(t+\tau) - \mathbf{A}\dfrac{t^3}{3!} + \cdots + (-1)^k \mathbf{A}^k \dfrac{(t-(k-1)\tau)^{2k+1}}{(2k+1)!}, & (k-1)\tau \le t < k\tau,\ k \in \mathbb{N}^+, \end{cases} \tag{2.2}$$

where Θ is a zero matrix and E is an identity matrix.

Definition 2.2. *(see [23]) An $\mathbb{R}^n$-value stochastic process $\{Z(t) : -\tau \le t \le T\}$ is called a solution of* (1.1) *if $Z(t)$ satisfies the following:*

$$Z(t)=\begin{cases}\mathcal{H}_\tau(\mathbf{A}t)\Psi(-\tau)+\mathcal{N}_\tau(\mathbf{A}t)\Psi'(-\tau)+\int_{-\tau}^{0}\mathcal{N}_\tau(\mathbf{A}(t-\tau-s))\Psi''(s)ds\\ \quad+\int_0^t\mathcal{N}_\tau(\mathbf{A}(t-\tau-s))h(s,Z(s))ds\\ \quad+\int_0^t\mathcal{N}_\tau(\mathbf{A}(t-\tau-s))g(s,Z(s))dW(s),\ t\in(0,T],\\ \Psi(t),\ -\tau\le t\le 0.\end{cases}\tag{2.3}$$

where $Z(t)$ is $\mathcal{F}(t)$-adapted and $\mathbb{E}\Big(\int_{-\tau}^{T}|Z(s)|^2ds\Big)<\infty$.

Lemma 2.3. *For arbitrary $t\in((k-1)\tau,k\tau]$ and $k\in\mathbb{N}^+$, we have*

$$\|\mathcal{H}_\tau(\mathbf{A}t)\|\le\cosh(\sqrt{\|\mathbf{A}\|}t),\quad \|\mathcal{N}_\tau(\mathbf{A}t)\|\le\Phi_1(t),$$

where

$$\Phi_1(t)=\begin{cases}\tau+\sum_{i=0}^{k}\|\mathbf{A}\|^i\frac{t^{2i+1}}{(2i+1)!},\|\mathbf{A}\|>1,\\ \tau+\frac{1}{2}(e^t-e^{-t})=\tau+\sinh(t),0<\|\mathbf{A}\|\le 1.\end{cases}$$

Proof. By Eqs (2.1) and (2.2), we have

$$\begin{aligned}\|\mathcal{H}_\tau(\mathbf{A}t)\| &\le 1+\|\mathbf{A}\|\frac{t^2}{2!}+\|\mathbf{A}\|^2\frac{(t-\tau)^4}{4!}+\cdots+\|\mathbf{A}\|^k\frac{(t-(k-1)\tau)^{2k}}{(2k)!}\\ &\le 1+\|\mathbf{A}\|\frac{t^2}{2!}+\|\mathbf{A}\|^2\frac{t^4}{4!}+\cdots+\|\mathbf{A}\|^k\frac{t^{2k}}{(2k)!}\\ &\le \frac{1}{2}\Big(e^{\sqrt{\|\mathbf{A}\|}t}+e^{-\sqrt{\|\mathbf{A}\|}t}\Big)\\ &= \cosh(\sqrt{\|\mathbf{A}\|}t),\end{aligned}$$

$$\begin{aligned}\|\mathcal{N}_\tau(\mathbf{A}t)\| &\le (t+\tau)+\|\mathbf{A}\|\frac{t^3}{3!}+\|\mathbf{A}\|^2\frac{(t-\tau)^5}{5!}+\cdots+\|\mathbf{A}\|^k\frac{(t-(k-1)\tau)^{2k+1}}{(2k+1)!}\\ &\le (t+\tau)+\|\mathbf{A}\|\frac{t^3}{3!}+\|\mathbf{A}\|^2\frac{t^5}{5!}+\cdots+\|\mathbf{A}\|^k\frac{t^{2k+1}}{(2k+1)!}\\ &\le \begin{cases}\tau+\sum_{i=0}^{k}\|\mathbf{A}\|^i\frac{t^{2i+1}}{(2i+1)!},\|\mathbf{A}\|>1,\\ \tau+\frac{1}{2}(e^t-e^{-t})=\tau+\sinh(t),0<\|\mathbf{A}\|\le 1.\end{cases}\end{aligned}$$

□

To ensure the smooth progression of the research, the following assumptions are proposed.

(H_1) For arbitrary $t \in J$ and $x, y \in \mathbb{R}^n$, there exists a positive constant K such that

$$\|h(t,x) - h(t,y)\|^2 \vee \|g(t,x) - g(t,y)\|^2 \leq K^2\|x - y\|^2,$$

where $\|\cdot\|$ is the norm of $\mathbb{R}^n$, and $x \vee y = \max\{x, y\}$.

(H_2) Let $g(\cdot, 0)$ be essentially bounded, i.e.,

$$\|g(\cdot,0)\|_\infty := \operatorname{ess}\sup_{t\in[0,\infty)} \|g(t,0)\| < +\infty,$$

and $h(\cdot, 0)$ be $\mathbb{L}^2$ integrable, i.e.,

$$\|h\|_{\mathbb{L}^2} = \int_0^{+\infty} \|h(t,0)\|^2 dt < +\infty.$$

3. Existence and uniqueness results

In this section, we establish the existence and uniqueness of solutions to the equation given by (1.1). Let $\mathbb{H}^2([0,T])$ denote the space consisting of all measurable, $\mathcal{F}(t)$-adapted processes Z that satisfy the condition $\|Z\|_{\mathbb{H}^2} := \sup_{0\leq t\leq T} \|Z(t)\|_{ms} < \infty$. It follows from [24] that $(\mathbb{H}^2([0,T]), \|\cdot\|_{\mathbb{H}^2})$ is a Banach space.

For every $t \in [-\tau, T]$ and $\Psi \in C([-\tau, 0], \mathbb{R}^n)$, we define an operator $\mathcal{F} : \mathbb{H}^2([0,T]) \to \mathbb{H}^2([0,T])$ by

$$\begin{aligned}(\mathcal{F}Z)(t) &= \mathcal{H}_\tau(\mathbf{A}t)\Psi(-\tau) + \mathcal{N}_\tau(\mathbf{A}t)\Psi'(-\tau) + \int_{-\tau}^0 \mathcal{N}_\tau(\mathbf{A}(t-\tau-s))\Psi''(s)ds \\ &\quad + \int_0^t \mathcal{N}_\tau(\mathbf{A}(t-\tau-s))h(s,Z(s))ds + \int_0^t \mathcal{N}_\tau(\mathbf{A}(t-\tau-s))g(s,Z(s))dW(s). \quad (3.1)\end{aligned}$$

To prove existence and uniqueness of solution for second-order SDDSs, we aim to demonstrate that the operator $\mathcal{F}$ introduced in (3.1) satisfies the contractivity property, which is achieved by employing an appropriate time-weighted norm (see [24, Theorem 1]), the same approach adopted to establish the existence and uniqueness of solutions for Caputo-type fractional stochastic differential equations.

Lemma 3.1. *Let $t \in [-\tau, T]$ and $\Psi \in C([-\tau, 0], \mathbb{R}^n)$. If (H_1) and (H_2) hold, then the operator $\mathcal{F}$ is well-defined.*

Proof. Let any $Z \in \mathbb{H}^2([0,T])$; by using (3.1) and the inequality $\|x_1 + x_2 + x_3 + x_4\|^2 \leq 4(\|x_1\|^2 + \|x_2\|^2 + \|x_3\|^2 + \|x_4\|^2)$ for any $x_i \in \mathbb{R}^n$, $i = 1, 2, 3, 4$, we have

$$\begin{aligned}\|(\mathcal{F}Z)(t)\|_{ms}^2 &\leq 4\mathbb{E}(\|\mathcal{H}_\tau(\mathbf{A}t)\Psi(-\tau) + \mathcal{N}_\tau(\mathbf{A}t)\Psi'(-\tau)\|^2) \\ &\quad +4\mathbb{E}\Big(\Big\|\int_{-\tau}^0 \mathcal{N}_\tau(\mathbf{A}(t-\tau-s))\Psi''(s)ds\Big\|^2\Big) \\ &\quad +4\mathbb{E}\Big(\Big\|\int_0^t \mathcal{N}_\tau(\mathbf{A}(t-\tau-s))h(s,Z(s))ds\Big\|^2\Big) \\ &\quad +4\mathbb{E}\Big(\Big\|\int_0^t \mathcal{N}_\tau(\mathbf{A}(t-\tau-s))g(s,Z(s))dW(s)\Big\|^2\Big)\end{aligned}$$

$$=: \quad J_1 + J_2 + J_3 + J_4. \tag{3.2}$$

For J_1, by Lemma 2.3, one can get

$$\begin{aligned} J_1 &= 4\mathbb{E}(\|\mathcal{H}_\tau(\mathbf{A}t)\Psi(-\tau) + \mathcal{N}_\tau(\mathbf{A}t)\Psi'(-\tau)\|^2) \\ &\leq 8\mathbb{E}(\|\mathcal{H}_\tau(\mathbf{A}t)\Psi(-\tau)\|^2 + \|\mathcal{N}_\tau(\mathbf{A}t)\Psi'(-\tau)\|^2) \\ &\leq 8(\|\Psi(-\tau)\|^2 \cosh(\sqrt{\|\mathbf{A}\|}T)^2 + \|\Psi'(-\tau)\|^2\Phi_1^2(T)). \end{aligned} \tag{3.3}$$

For J_2, by Lemma 2.3 and the Cauchy-Schwarz inequality, one can get

$$\begin{aligned} J_2 &= 4\mathbb{E}\Big(\Big\|\int_{-\tau}^0 \mathcal{N}_\tau(\mathbf{A}(t-\tau-s))\Psi''(s)ds\Big\|^2\Big) \\ &\leq 4\int_{-\tau}^0 \|\mathcal{N}_\tau(\mathbf{A}(t-\tau-s))\|^2 ds \cdot \mathbb{E}\Big(\int_{-\tau}^0 \|\Psi''(s)\|^2 ds\Big) \leq 4\Xi\tau\Phi_1^2(T), \end{aligned} \tag{3.4}$$

where $\Xi = \int_{-\tau}^0 \|\Psi''(s)\|^2 ds < \infty$. For J_3, according to (H_1), (H_2), and the Cauchy-Schwarz inequality, one can get

$$\begin{aligned} J_3 &= 4\mathbb{E}\Big(\Big\|\int_0^t \mathcal{N}_\tau(\mathbf{A}(t-\tau-s))h(s,Z(s))ds\Big\|^2\Big) \\ &\leq 4\int_0^t \|\mathcal{N}_\tau(\mathbf{A}(t-\tau-s))\|^2 ds \cdot \mathbb{E}\Big(\int_0^t \|h(s,Z(s)) - h(s,0) + h(s,0)\|^2 ds\Big) \\ &\leq 4T\Phi_1^2(T)\mathbb{E}\Big(\int_0^t K^2\|Z(s)\|^2 ds + \int_0^t \|h(s,0)\|^2 ds\Big) \\ &\leq 4T\Phi_1^2(T)\Big(TK^2 \sup_{0\leq t\leq T} \mathbb{E}(\|Z(t)\|^2) + \|h\|_{\mathbb{L}^2}\Big) \\ &\leq 4T\Phi_1^2(T)\Big(TK^2\|Z\|_{\mathbb{H}^2}^2 + \|h\|_{\mathbb{L}^2}\Big). \end{aligned} \tag{3.5}$$

For J_4, by using (H_1), (H_2), and Itô's isometry, we obtain

$$\begin{aligned} J_4 &= 4\mathbb{E}\Big(\Big\|\int_0^t \mathcal{N}_\tau(\mathbf{A}(t-\tau-s))g(s,Z(s))dW(s)\Big\|^2\Big) \\ &\leq 4\mathbb{E}\Big(\int_0^t \|\mathcal{N}_\tau(\mathbf{A}(t-\tau-s))\|^2\|g(s,Z(s))\|^2 ds\Big) \\ &\leq 4\int_0^t \|\mathcal{N}_\tau(\mathbf{A}(t-\tau-s))\|^2 ds \cdot \mathbb{E}\Big(\int_0^t \|g(s,Z(s)) - g(s,0) + g(s,0)\|^2 ds\Big) \\ &\leq 4\Phi_1^2(T)\mathbb{E}\Big(\int_0^t (K^2\|Z(s)\|^2 + \|g(t,0)\|^2)ds\Big) \\ &\leq 4T\Phi_1^2(T)(K^2\|Z\|_{\mathbb{H}^2}^2 + \|g(\cdot,0)\|_\infty^2). \end{aligned} \tag{3.6}$$

Substituting (3.3)–(3.6) into (3.2),

$$\|(\mathcal{F}Z)(t)\|_{ms}^2 \quad \leq \quad J_1 + J_2 + J_3 + J_4$$

$$\leq 8(\|\Psi(-\tau)\|^2 \cosh(\sqrt{\|\mathbf{A}\|}T)^2 + \|\Psi'(-\tau)\|^2\Phi_1^2(T))$$
$$+4\Xi\tau\Phi_1^2(T) + 4T\Phi_1^2(T)\Big(TK^2\|Z\|_{\mathbb{H}^2}^2 + \|h\|_{\mathbb{L}^2}\Big)$$
$$+4T\Phi_1^2(T)(K^2\|Z\|_{\mathbb{H}^2}^2 + \|g(\cdot,0)\|_\infty^2),$$

which implies that $\|\mathcal{F}Z\|_{\mathbb{H}^2} < \infty$. □

Theorem 3.2. *Suppose that* (H_1) *and* (H_2) *hold. Then,* (1.1) *has a unique solution* $Z \in \mathbb{H}^2([0,T])$.

Proof. For arbitrary $T > 0$, choose and fix a constant $\gamma > 0$ such that

$$\gamma > 2K^2\Phi_1^2(T)(T+1). \tag{3.7}$$

On the space $\mathbb{H}^2([0,T])$, we define a weighted norm $\|\cdot\|_\gamma$ as below

$$\|Z\|_\gamma = \sup_{t\in[0,T]} \sqrt{\frac{\mathbb{E}(\|Z(t)\|^2)}{\exp(\gamma t)}}, \text{ for any } Z \in \mathbb{H}^2([0,T]).$$

It can be deduced from [24, Theorem 1] that the norms $\|\cdot\|_{\mathbb{H}^2}$ and $\|\cdot\|_\gamma$ are equivalent, which implies that $(\mathbb{H}^2([0,T]), \|\cdot\|_\gamma)$ is a Banach space. It is straightforward to verify that $\mathcal{F} : \mathbb{H}^2([0,T]) \to \mathbb{H}^2([0,T])$ defined in (3.1) is a uniformly bounded operator by Lemma 3.1. Consequently, the only remaining step is to confirm that $\mathcal{F}$ is a contraction operator.

For every $Z, \widehat{Z} \in \mathbb{H}^2([0,T])$, from (3.1) and $\|x+y\|^2 \leq 2(\|x\|^2 + \|y\|^2)$, for all $x, y \in \mathbb{R}^n$, we have

$$\begin{aligned}
\mathbb{E}(\|\mathcal{F}Z(t) - \mathcal{F}\widehat{Z}(t)\|^2) &\leq 2\mathbb{E}\Big(\Big\|\int_0^t \mathcal{N}_\tau(\mathbf{A}(t-\tau-s))(h(s,Z(s)) - h(s,\widehat{Z}(s)))ds\Big\|^2\Big) \\
&\quad +2\mathbb{E}\Big(\Big\|\int_0^t \mathcal{N}_\tau(\mathbf{A}(t-\tau-s))(g(s,Z(s)) - g(s,\widehat{Z}(s)))dW(s)\Big\|^2\Big) \\
&\leq 2\int_0^t \|\mathcal{N}_\tau(\mathbf{A}(t-\tau-s))\|^2 ds \cdot \mathbb{E}\Big(\int_0^t \|h(s,Z(s)) - h(s,\widehat{Z}(s)\|^2 ds\Big) \\
&\quad +2\int_0^t \|\mathcal{N}_\tau(\mathbf{A}(t-\tau-s))\|^2 ds \cdot \mathbb{E}\Big(\int_0^t \|g(s,Z(s)) - g(s,\widehat{Z}(s)\|^2 ds\Big) \\
&\leq 2T\Phi_1^2(T)K^2 \int_0^t \mathbb{E}\Big(\|Z(s) - \widehat{Z}(s)\|^2\Big)ds \\
&\quad +2\Phi_1^2(T)K^2 \int_0^t \mathbb{E}\Big(\|Z(s) - \widehat{Z}(s)\|^2\Big)ds.
\end{aligned}$$

Consequently, for every given $t \in [0,T]$, one can get

$$\begin{aligned}
\frac{\mathbb{E}(\|\mathcal{F}Z(t) - \mathcal{F}\widehat{Z}(t)\|^2)}{\exp(\gamma t)} &\leq \frac{2K^2\Phi_1^2(T)(T+1)}{\exp(\gamma t)} \int_0^t \exp(\gamma s)ds\|Z - \widehat{Z}\|_\gamma^2 \\
&\leq \frac{2K^2\Phi_1^2(T)(T+1)}{\gamma}\|Z - \widehat{Z}\|_\gamma^2,
\end{aligned}$$

which implies that $\|\mathcal{F}Z - \mathcal{F}\widehat{Z}\|_\gamma^2 \leq \rho\|Z - \widehat{Z}\|_\gamma^2$, where $\rho = \sqrt{\frac{2K^2\Phi_1^2(T)(T+1)}{\gamma}}$.

According to (3.7), we have $\rho < 1$ and the operator $\mathcal{F}$ is contractive. By applying the Banach fixed point theorem, Eq (1.1) admits a unique solution. This completes the proof of the theorem. □

4. Averaging principle

In this section, we shall prove that the averaged solutions of second-order SDDSs converge to the standard solutions. For all $t \in J$, we now consider the standard form of Eq (1.1):

$$\begin{aligned} Z_\epsilon(t) &= \mathcal{H}_\tau(\mathbf{A}t)\Psi(-\tau) + \mathcal{N}_\tau(\mathbf{A}t)\Psi'(-\tau) \\ &\quad + \int_{-\tau}^{0} \mathcal{N}_\tau(\mathbf{A}(t-\tau-s))\Psi''(s)ds \\ &\quad + \epsilon \int_0^t \mathcal{N}_\tau(\mathbf{A}(t-\tau-s))h(s, Z_\epsilon(s))ds \\ &\quad + \sqrt{\epsilon}\int_0^t \mathcal{N}_\tau(\mathbf{A}(t-\tau-s))g(s, Z_\epsilon(s))dW(s), \end{aligned} \tag{4.1}$$

where $\epsilon \in (0, \epsilon_0]$ can be sufficiently small, and ϵ_0 is a fixed number.

Now, we consider the standard solution $Z_\epsilon(\cdot)$ in $\mathbb{R}^n$. Let $Z^*_\epsilon(\cdot)$ be the solution of the following averaged system:

$$\begin{aligned} Z^*_\epsilon(t) &= \mathcal{H}_\tau(\mathbf{A}t)\Psi(-\tau) + \mathcal{N}_\tau(\mathbf{A}t)\Psi'(-\tau) \\ &\quad + \int_{-\tau}^{0} \mathcal{N}_\tau(\mathbf{A}(t-\tau-s))\Psi''(s)ds \\ &\quad + \epsilon \int_0^t \mathcal{N}_\tau(\mathbf{A}(t-\tau-s))\widehat{h}(Z^*_\epsilon(s))ds \\ &\quad + \sqrt{\epsilon}\int_0^t \mathcal{N}_\tau(\mathbf{A}(t-\tau-s))\widehat{g}(Z^*_\epsilon(s))dW(s), \end{aligned} \tag{4.2}$$

where $\widehat{h} : \mathbb{R}^n \to \mathbb{R}^n$ and $\widehat{g} : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ satisfiy (H_1) and the following conditions:

(H_3) For all $t \in J$ and $z \in \mathbb{R}^n$, there exists positive bounded functions $\varphi_i(t)$, $i = 1, 2$ such that

$$\begin{aligned} \frac{1}{t}\int_0^t \|h(s,z) - \widehat{h}(z)\|^2 ds &\le \varphi_1(t)(1 + \|z\|^2), \\ \frac{1}{t}\int_0^t \|(g(s,z) - \widehat{g}(z))\|^2 ds &\le \varphi_2(t)(1 + \|z\|^2), \end{aligned}$$

where $\lim_{t\to\infty} \varphi_i(t) = 0$, $i = 1, 2$.

Next, we show that $Z_\epsilon \to Z^*_\epsilon$ in mean square as $\epsilon \to 0$.

Theorem 4.1. *Assume that (H_1), (H_2), and (H_3) hold, then for a given arbitrarily small $\varsigma > 0$, there exist constants $L > 0$, $\epsilon_1 \in (0, \epsilon_0]$ and $\xi \in (0, 1)$ such that for all $\epsilon \in (0, \epsilon_1]$,*

$$\mathbb{E}\Big(\sup_{t\in[0,L\epsilon^{-\xi}]} \|Z_\epsilon(t) - Z^*_\epsilon(t)\|^2\Big) \le \varsigma.$$

Proof. For any $t \in [0, u] \subset [0, T]$, by (4.1) and (4.2), we have

$$Z_\epsilon(t) - Z^*_\epsilon(t) = \Big(\mathcal{H}_\tau(\mathbf{A}t)\Psi(-\tau) + \mathcal{N}_\tau(\mathbf{A}t)\Psi'(-\tau)$$

$$
\begin{aligned}
& +\int_{-\tau}^{0} \mathcal{N}_\tau(\mathbf{A}(t-\tau-s))\Psi''(s)ds \\
& +\epsilon\int_0^t \mathcal{N}_\tau(\mathbf{A}(t-\tau-s))h(s,Z_\epsilon(s))ds \\
& +\sqrt{\epsilon}\int_0^t \mathcal{N}_\tau(\mathbf{A}(t-\tau-s))g(s,Z_\epsilon(s))dW(s)\Big) \\
& -\Big(\mathcal{H}_\tau(\mathbf{A}t)\Psi(-\tau)+\mathcal{N}_\tau(\mathbf{A}t)\Psi'(-\tau) \\
& +\int_{-\tau}^{0} \mathcal{N}_\tau(\mathbf{A}(t-\tau-s))\Psi''(s)ds \\
& +\epsilon\int_0^t \mathcal{N}_\tau(\mathbf{A}(t-\tau-s))\widehat{h}(Z_\epsilon^*(s))ds \\
& +\sqrt{\epsilon}\int_0^t \mathcal{N}_\tau(\mathbf{A}(t-\tau-s))\widehat{g}(Z_\epsilon^*(s))dW(s)\Big) \\
= \; & \epsilon\int_0^t \mathcal{N}_\tau(\mathbf{A}(t-\tau-s))(h(s,Z_\epsilon(s))-\widehat{h}(Z_\epsilon^*(s)))ds \\
& +\sqrt{\epsilon}\int_0^t \mathcal{N}_\tau(\mathbf{A}(t-\tau-s))(g(s,Z_\epsilon(s))-\widehat{g}(Z_\epsilon^*(s)))dW(s).
\end{aligned}
$$

Using the Jensen's inequality, we obtain

$$
\begin{aligned}
& \mathbb{E}\Big(\sup_{0\le t\le u}\|Z_\epsilon(t)-Z_\epsilon^*(t)\|^2\Big) \\
= \; & \mathbb{E}\Big(\sup_{0\le t\le u}\|\epsilon\int_0^t \mathcal{N}_\tau(\mathbf{A}(t-\tau-s))(h(s,Z_\epsilon(s))-\widehat{h}(Z_\epsilon^*(s)))ds \\
& +\sqrt{\epsilon}\int_0^t \mathcal{N}_\tau(\mathbf{A}(t-\tau-s))(g(s,Z_\epsilon(s))-\widehat{g}(Z_\epsilon^*(s)))dW(s)\|^2\Big) \\
\le \; & 2\epsilon^2\cdot\mathbb{E}\Big(\sup_{0\le t\le u}\Big\|\int_0^t \mathcal{N}_\tau(\mathbf{A}(t-\tau-s))(h(s,Z_\epsilon(s))-\widehat{h}(Z_\epsilon^*(s)))ds\Big\|^2\Big) \\
& +2\epsilon\cdot\mathbb{E}\Big(\sup_{0\le t\le u}\Big\|\int_0^t \mathcal{N}_\tau(\mathbf{A}(t-\tau-s))(g(s,Z_\epsilon(s))-\widehat{g}(Z_\epsilon^*(s)))dW(s)\Big\|^2\Big) \\
=: \; & I_1+I_2.
\end{aligned}
$$

For I_1, by using the Jensen's inequality again, one can get

$$
\begin{aligned}
I_1 \; \le \; & 4\epsilon^2\cdot\mathbb{E}\Big(\sup_{0\le t\le u}\Big\|\int_0^t \mathcal{N}_\tau(\mathbf{A}(t-\tau-s))(h(s,Z_\epsilon(s))-h(s,Z_\epsilon^*(s)))ds\Big\|^2\Big) \\
& +4\epsilon^2\cdot\mathbb{E}\Big(\sup_{0\le t\le u}\Big\|\int_0^t \mathcal{N}_\tau(\mathbf{A}(t-\tau-s))(h(s,Z_\epsilon^*(s))-\widehat{h}(Z_\epsilon^*(s)))ds\Big\|^2\Big) \\
=: \; & I_{11}+I_{12}.
\end{aligned}
$$

By using the Cauchy-Schwarz inequality, Lemma 2.3, and assumptions (H_1) and (H_3), we can obtain

$$
I_{11} \; \le \; 4\epsilon^2\cdot\mathbb{E}\Big(\sup_{0\le t\le u}\Big(\int_0^t \|\mathcal{N}_\tau(\mathbf{A}(t-\tau-s))\|\|(h(s,Z_\epsilon(s))-h(s,Z_\epsilon^*(s)))\|ds\Big)^2\Big)
$$

$$
\begin{aligned}
&\leq 4\epsilon^2 \cdot \mathbb{E}\Big(\sup_{0\leq t\leq u}\Big(\int_0^t \|\mathcal{N}_\tau(\mathbf{A}(t-\tau-s))\|^2 dt \cdot \int_0^t \|(h(s,Z_\epsilon(s)) - h(s,Z_\epsilon^*(s)))\|^2 ds\Big)\Big)\\
&\leq 4\epsilon^2\Phi_1^2(T-\tau)u \cdot \mathbb{E}\Big(\sup_{0\leq t\leq u}\Big(\int_0^t \|(h(s,Z_\epsilon(s)) - h(s,Z_\epsilon^*(s)))\|^2 ds\Big)\Big)\\
&\leq 4\epsilon^2\Phi_1^2(T-\tau)u \cdot \mathbb{E}\Big(\sup_{0\leq t\leq u}\int_0^t K^2\|Z_\epsilon(s) - Z_\epsilon^*(s)\|^2 ds\Big)\\
&\leq 4\epsilon^2\Phi_1^2(T-\tau)K^2 u\int_0^u \mathbb{E}\Big(\sup_{0\leq s_1\leq s}\|Z_\epsilon(s_1) - Z_\epsilon^*(s_1)\|^2\Big)ds, \qquad (4.3)
\end{aligned}
$$

and

$$
\begin{aligned}
I_{12} &\leq 4\epsilon^2 \cdot \mathbb{E}\Big(\sup_{0\leq t\leq u}\Big(\int_0^t \|\mathcal{N}_\tau(\mathbf{A}(t-\tau-s))\|\|(h(s,Z_\epsilon^*(s)) - \widehat{h}(Z_\epsilon^*(s)))\|ds\Big)^2\Big)\\
&\leq 4\epsilon^2 \cdot \mathbb{E}\Big(\sup_{0\leq t\leq u}\Big(\int_0^t \|\mathcal{N}_\tau(\mathbf{A}(t-\tau-s))\|^2 ds \cdot \int_0^t \|(h(s,Z_\epsilon^*(s)) - \widehat{h}(Z_\epsilon^*(s)))\|^2 ds\Big)\Big)\\
&\leq 4\epsilon^2\Phi_1^2(T-\tau)u^2\varphi_1(u)\Big(1 + \mathbb{E}\Big(\sup_{0\leq s\leq u}\|Z_\epsilon^*(s)\|^2\Big)\Big)\\
&\leq \Lambda_{12}u^2\epsilon^2, \qquad (4.4)
\end{aligned}
$$

where $\Lambda_{12} = 4\Phi_1^2(T-\tau)\sup_{0\leq u\leq T}(\varphi_1(u)(1 + \mathbb{E}(\sup_{0\leq s\leq u}\|Z_\epsilon^*(s)\|^2)))$.

For I_2, using an argument similar to that in I_1 yields

$$
\begin{aligned}
I_2 &\leq 4\epsilon \cdot \mathbb{E}\Big(\sup_{0\leq t\leq u}\Big\|\int_0^t \mathcal{N}_\tau(\mathbf{A}(t-\tau-s))(g(s,Z_\epsilon(s)) - g(s,Z_\epsilon^*(s)))dW(s)\Big\|^2\Big)\\
&\quad +4\epsilon \cdot \mathbb{E}\Big(\sup_{0\leq t\leq u}\Big\|\int_0^t \mathcal{N}_\tau(\mathbf{A}(t-\tau-s))(g(s,Z_\epsilon^*(s)) - \widehat{g}(Z_\epsilon^*(s)))dW(s)\Big\|^2\Big)\\
&=: I_{21} + I_{22}.
\end{aligned}
$$

By virtue of Doob's martingale inequality, Itô's formula, and the given condition (H_1) and (H_3), we can obtain

$$
\begin{aligned}
I_{21} &\leq 4\epsilon \cdot \mathbb{E}\Big(\sup_{0\leq t\leq u}\int_0^t \Big\|\mathcal{N}_\tau(\mathbf{A}(t-\tau-s))(g(s,Z_\epsilon(s)) - g(s,Z_\epsilon^*(s)))\Big\|^2 ds\Big)\\
&\leq 4\epsilon\Phi_1^2(T-\tau) \cdot \mathbb{E}\Big(\sup_{0\leq t\leq u}\int_0^u \|g(s,Z_\epsilon(s)) - g(s,Z_\epsilon^*(s))\|^2 ds\Big)\\
&\leq 4\epsilon\Phi_1^2(T-\tau)K^2\int_0^u \mathbb{E}\Big(\sup_{0\leq s_1\leq s}\|Z_\epsilon(s_1) - Z_\epsilon^*(s_1)\|^2\Big)ds. \qquad (4.5)
\end{aligned}
$$

and

$$
\begin{aligned}
I_{22} &\leq 4\epsilon \cdot \mathbb{E}\Big(\sup_{0\leq t\leq u}\int_0^t \Big\|\mathcal{N}_\tau(\mathbf{A}(t-\tau-s))(g(s,Z_\epsilon^*(s)) - \widehat{g}(Z_\epsilon^*(s)))\Big\|^2 ds\Big)\\
&\leq 4\epsilon\Phi_1^2(T-\tau) \cdot \mathbb{E}\Big(\int_0^u \|g(s,Z_\epsilon^*(s)) - \widehat{g}(Z_\epsilon^*(s))\|^2 ds\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\leq 4\epsilon\Phi_1^2(T-\tau)u\varphi_2(u)\cdot\Big(1+\mathbb{E}\Big(\sup_{0\leq s\leq u}\|Z_\epsilon^*(s)\|^2\Big)\Big)\\
&\leq \Lambda_{22}u\epsilon, \qquad (4.6)
\end{aligned}
$$

where $\Lambda_{22}=4\Phi_1^2(T-\tau)\sup_{0\leq u\leq T}(\varphi_2(u)(1+\mathbb{E}(\sup_{0\leq s\leq u}\|Z_\epsilon^*(s)\|^2)))$.

For any $u\in[0,T]$, it follows from (4.3)–(4.6) that

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{0\leq t\leq u}\|Z_\epsilon(t)-Z_\epsilon^*(t)\|^2\Big)\\
\leq\ &\Lambda_{12}u^2\epsilon^2+\Lambda_{22}u\epsilon+4\epsilon^2\Phi_1^2(T-\tau)K^2u\int_0^u\mathbb{E}\Big(\sup_{0\leq s_1\leq s}\|Z_\epsilon(s_1)-Z_\epsilon^*(s_1)\|^2\Big)ds\\
&+4\epsilon\Phi_1^2(T-\tau)K^2\int_0^u\mathbb{E}\Big(\sup_{0\leq s_1\leq s}\|Z_\epsilon(s_1)-Z_\epsilon^*(s_1)\|^2\Big)ds. \qquad (4.7)
\end{aligned}
$$

According to the Gronwall-Bellman inequality [25, Theorem 1], we can obtain

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{0\leq t\leq u}\|Z_\epsilon(t)-Z_\epsilon^*(t)\|^2\Big)\\
\leq\ &(\Lambda_{12}u^2\epsilon^2+\Lambda_{22}u\epsilon)\sum_{k=0}^{\infty}\frac{(4\Phi_1^2(T-\tau)K^2(u\epsilon^2+\epsilon)u)^k}{k!}\\
\leq\ &(\Lambda_{12}u^2\epsilon^2+\Lambda_{22}u\epsilon)\exp\big(4\Phi_1^2(T-\tau)K^2(u\epsilon^2+\epsilon)u\big).
\end{aligned}
$$

It thereby follows that we can pick a parameter $\xi\in(0,1)$ and $L>0$, for which the subsequent condition is satisfied by any $t\in[0,L\epsilon^{-\xi}]\subseteq J$:

$$
\mathbb{E}\Big(\sup_{0\leq t\leq u}\|Z_\epsilon(t)-Z_\epsilon^*(t)\|^2\Big)\leq C\epsilon^{1-\xi},
$$

where

$$
C=(\Lambda_{12}L^2\epsilon^{1-\xi}+\Lambda_{22}L)\exp(4\Phi_1^2(T-\tau)K^2(L^2\epsilon^{2-2\xi}+L\epsilon^{1-\xi})) \qquad (4.8)
$$

is a constant. Hence, for any given $\varsigma>0$, there exists $\epsilon_1\in(0,\epsilon_0]$ such that for each $\epsilon\in(0,\epsilon_1]$ and $t\in[0,L\epsilon^{-\xi}]\subseteq J$,

$$
\mathbb{E}\Big(\sup_{t\in[0,L\epsilon^{-\xi}]}\|Z_\epsilon(t)-Z_\epsilon^*(t)\|^2\Big)\leq\varsigma.
$$

□

Corollary 4.2. *Assume that (H_1), (H_2), and (H_3) hold, then for an any number $\widetilde{\varsigma}>0$ such that for $L>0$, $\epsilon_1\in(0,\epsilon_0]$ and $\xi\in(0,1)$, satisfying the following for all $\epsilon\in(0,\epsilon_1]$:*

$$
\lim_{\epsilon\to0}\mathbb{P}\Big(\sup_{t\in[0,L\epsilon^{-\xi}]}\|Z_\epsilon(t)-Z_\epsilon^*(t)\|^2>\widetilde{\varsigma}\Big)=0.
$$

Proof. By Theorem 4.1 and the Chebyshev-Markov inequality, for an arbitrary number $\widetilde{\varsigma} > 0$, we can obtain

$$\begin{aligned}\mathbb{P}\Big(\sup_{t\in[0,L\epsilon^{-\xi}]} \|Z_\epsilon(t) - Z^*_\epsilon(t)\|^2 > \widetilde{\varsigma}\Big) &\leq \frac{1}{\widetilde{\varsigma}^2}\mathbb{E}\Big(\sup_{t\in[0,L\epsilon^{-\xi}]} \|Z_\epsilon(t) - Z^*_\epsilon(t)\|^2\Big) \\ &\leq \frac{C\epsilon^{1-\xi}}{\widetilde{\varsigma}^2} \to 0 \ \text{ as } \epsilon \to 0,\end{aligned}$$

where C is defined in (4.8). □

5. Examples

Example 5.1. *Set* $\tau = 0.5$, $k^* = 10$, *and* $J := [0, 5]$. *Consider*

$$\begin{cases} Z''(t) + \mathbf{A}Z(t-\tau) = h(t, Z(t)) + g(t, Z(t))\dfrac{dW(t)}{dt}, \ t \in J := [0, 5], \\ Z(t) = \Psi(t), \ Z'(t) = \Psi'(t), -0.5 \leq t \leq 0, \end{cases} \tag{5.1}$$

where

$$\begin{aligned} Z(t) &= \begin{pmatrix} Z_1(t) \\ Z_2(t) \end{pmatrix}, \ \mathbf{A} = \begin{pmatrix} 1 & 1.5 \\ 1.5 & 2 \end{pmatrix}, \ \Psi(t) = \begin{pmatrix} t^2 + 1 \\ \frac{t^2}{2} + 1 \end{pmatrix}, \\ h(t, Z(t)) &= \begin{pmatrix} -\frac{1}{10}Z_1(t)\sin^2 t \\ -\frac{1}{10}Z_2(t)\sin^2 t \end{pmatrix}, g(t, Z(t)) = \begin{pmatrix} \frac{1}{2}Z_1(t)\cos^2 t + 1 \\ \frac{1}{2}Z_2(t)\cos^2 t + 2 \end{pmatrix}. \end{aligned} \tag{5.2}$$

For any $Z(t), \widehat{Z}(t) \in \Upsilon$ and $t \in [0, 5]$, we can get

$$\begin{aligned} \|h(t, Z(t)) - h(t, \widehat{Z}(t))\| &\leq \frac{1}{10}(|Z_1(t) - \widehat{Z}_1(t)|\sin^2 t + |Z_2(t) - \widehat{Z}_2(t)|\sin^2 t) \\ &\leq \frac{1}{10}\|Z - \widehat{Z}\|, \\ \|g(t, Z(t)) - g(t, \widehat{Z}(t))\| &\leq \frac{1}{2}(|Z_1(t) - \widehat{Z}_1(t)|\cos^2 t + |Z_2(t) - \widehat{Z}_2(t)|\cos^2 t) \\ &\leq \frac{1}{2}\|Z - \widehat{Z}\|. \end{aligned}$$

The functions h and g satisfy the assumptions (H_1) and (H_2).

By calculation, one has $\|\mathbf{A}\| = 3.5$, $\|h\|_{\mathbb{L}^2} = \int_0^{+\infty} \|h(t, 0)\|^2 dt = 0$, $K = \frac{1}{2}$, $\|g(\cdot, 0)\|_\infty := \operatorname{ess\,sup}_{t\in[0,\infty)} \|g(t, 0)\| = 3$, $\Xi = \int_{-0.5}^0 \|\Psi''(s)\|^2 ds = \int_{-0.5}^0 9dt = 4.5$, and $\Phi_1(5) = 74.5032$.

Therefore, we can choose a suitable value γ such that

$$12K^2\Phi_1^2(5) < \gamma.$$

By Theorem 3.2, for $t \in [0, 5]$, the unique solution of (5.1) has the following form:

$$Z(t) = \mathcal{H}_{0.5}(\mathbf{A}t)\Psi(-0.5) + \mathcal{N}_{0.5}(\mathbf{A}t)\Psi'(-0.5)$$

$$
\begin{aligned}
&+\int_{-0.5}^{0}\mathcal{N}_{0.5}(\mathbf{A}(t-0.5-s))\Psi''(s)ds\\
&+\int_{0}^{t}\mathcal{N}_{0.5}(\mathbf{A}(t-0.5-s))h(s,Z(s))ds\\
&+\int_{0}^{t}\mathcal{N}_{0.5}(\mathbf{A}(t-0.5-s))g(s,Z(s))dW(s).
\end{aligned}
$$

Example 5.2. *Set $\tau = 0.5$, $k^* = 10$, and $J := [0,5]$. Consider*

$$
\begin{cases}
Z''(t) + \mathbf{A}Z(t-\tau) = \epsilon h(t,Z(t)) + \sqrt{\epsilon}g(t,Z(t))\dfrac{dW(t)}{dt}, \ t \in J := [0,5],\\
Z(t) = \Psi(t), \ Z'(t) = \Psi'(t), -0.5 \le t \le 0,
\end{cases}
\tag{5.3}
$$

where $\mathbf{A}$ and $\Psi(\cdot)$ are defined in (5.2)*, and*

$$
Z_\epsilon(t) = \begin{pmatrix} Z_{1,\epsilon}(t)\\ Z_{2,\epsilon}(t)\end{pmatrix},\ h(t,Z_\epsilon(t)) = \begin{pmatrix} -\frac{1}{10}Z_{1,\epsilon}(t)\sin^2 t\\ -\frac{1}{10}Z_{2,\epsilon}(t)\cos^2 t\end{pmatrix},\ g(t,Z_\epsilon(t)) = \begin{pmatrix} \frac{1}{2}Z_{1,\epsilon}(t)\cos^2 t+1\\ \frac{1}{2}Z_{2,\epsilon}(t)\cos^2 t+2\end{pmatrix}.
$$

Under (H_1) and (H_2), by Theorem 3.2, the system (5.3) has a unique solution Z_ϵ given by

$$
\begin{aligned}
Z_\epsilon(t) \;=\; & \mathcal{H}_{0.5}(\mathbf{A}t)\Psi(-0.5) + \mathcal{N}_{0.5}(\mathbf{A}t)\Psi'(-0.5)\\
&+\int_{-0.5}^{0}\mathcal{N}_{0.5}(\mathbf{A}(t-0.5-s))\Psi''(s)ds\\
&+\epsilon\int_{0}^{t}\mathcal{N}_{0.5}(\mathbf{A}(t-0.5-s))h(s,Z_\epsilon(s))ds\\
&+\sqrt{\epsilon}\int_{0}^{t}\mathcal{N}_{0.5}(\mathbf{A}(t-0.5-s))g(s,Z_\epsilon(s))dW(s).
\end{aligned}
$$

Define

$$
\begin{aligned}
\widehat{h}(Z_\epsilon(s)) &= \frac{1}{\pi}\int_0^\pi h(s,Z_\epsilon(s))ds\\
&= \begin{pmatrix} -\frac{1}{10\pi}\int_0^\pi Z_{1,\epsilon}(s)\sin^2 s ds\\ -\frac{1}{10\pi}\int_0^\pi Z_{2,\epsilon}(s)\cos^2 s ds\end{pmatrix},\\
&= \begin{pmatrix} -\frac{1}{20}Z_{1,\epsilon}(s)\\ -\frac{1}{20}Z_{2,\epsilon}(s)\end{pmatrix},\\
\widehat{g}(Z_\epsilon(s)) &= \frac{1}{\pi}\int_0^\pi g(s,Z_\epsilon(s))ds\\
&= \begin{pmatrix} \frac{1}{\pi}\int_0^\pi(\frac{1}{2}Z_{1,\epsilon}(s)\cos^2 s+1)ds\\ \frac{1}{\pi}\int_0^\pi(\frac{1}{2}Z_{2,\epsilon}(s)\cos^2 s+2)ds\end{pmatrix},\\
&= \begin{pmatrix} \frac{1}{4}Z_{1,\epsilon}(s)+1\\ \frac{1}{4}Z_{2,\epsilon}(s)+2\end{pmatrix}.
\end{aligned}
$$

According to the above discussions, the assumption (H_3) holds. Then, the simplified second-order SDDSs can be defined as

$$
Z^*_\epsilon(t) \;=\; \mathcal{H}_{0.5}(\mathbf{A}t)\Psi(-0.5) + \mathcal{N}_{0.5}(\mathbf{A}t)\Psi'(-0.5)
$$

$$
\begin{aligned}
&+ \int_{-0.5}^{0} \mathcal{N}_{0.5}(\mathbf{A}(t-0.5-s))\Psi''(s)ds \\
&+\epsilon \int_{0}^{t} \mathcal{N}_{0.5}(\mathbf{A}(t-0.5-s))\widehat{h}(Z_{\epsilon}^{*}(s))ds \\
&+ \sqrt{\epsilon} \int_{0}^{t} \mathcal{N}_{0.5}(\mathbf{A}(t-0.5-s))\widehat{g}(Z_{\epsilon}^{*}(s))dW(s).
\end{aligned}
$$

Upon verification, all conditions required by Theorem 4.1 and Corollary 4.2 are satisfied. As a consequence, the primitive solutions $Z_{\epsilon}(\cdot)$ converge to $Z_{\epsilon}^{*}(\cdot)$ in mean square and in probability as $\epsilon \to 0$. In addition, we introduce the error term $E_r = (\|Z_{\epsilon}(t) - Z_{\epsilon}^{*}(t)\|^2)^{\frac{1}{2}}$. The results of Theorem 4.1 are illustrated and validated by Figure 1 with $\epsilon = 0.001$ and Figure 2 with $\epsilon = 0.0001$.

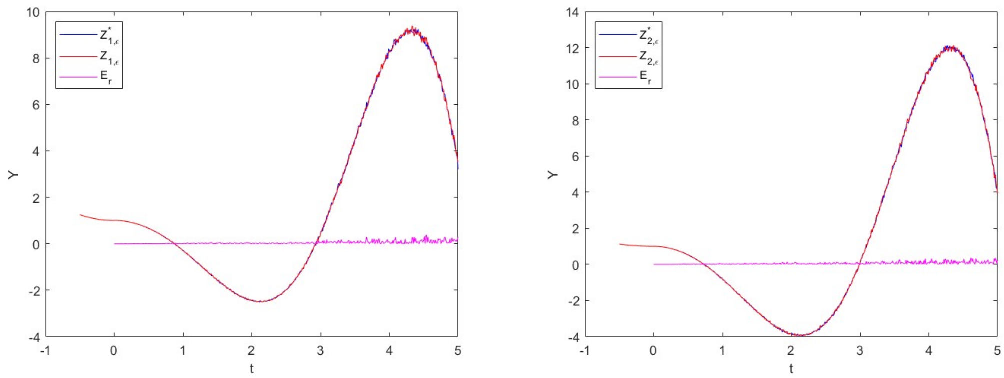

Figure 1. Comparison of $Z_{\epsilon}(\cdot)$ and $Z_{\epsilon}^{*}(\cdot)$ with $\epsilon = 0.001$.

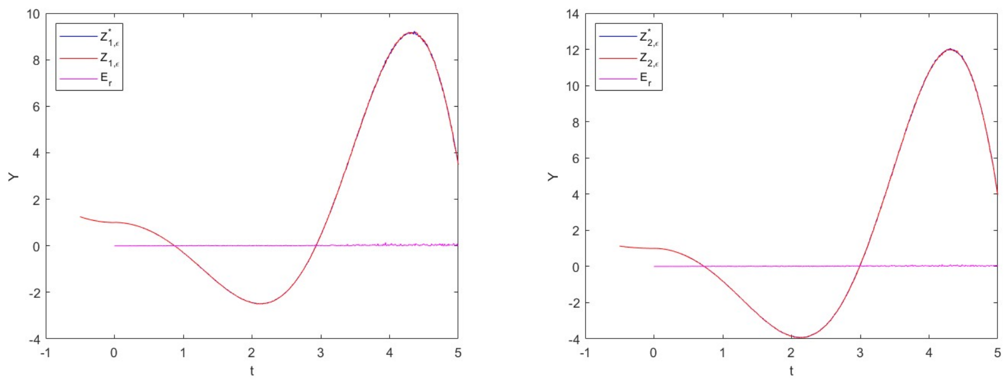

Figure 2. Comparison of $Z_{\epsilon}(\cdot)$ and $Z_{\epsilon}^{*}(\cdot)$ with $\epsilon = 0.0001$.

Remark 5.3. *As stated in Theorem 4.1, the constant parameter ϵ is required to approach zero. For values of ϵ that are not sufficiently small, the original solutions $Z_{\epsilon}(\cdot)$ might not exhibit sufficient proximity to the simplified solution $Z_{\epsilon}^{*}(\cdot)$. This point is illustrated by Figure 3 (with $\epsilon = 1$) and Figure 4 (with $\epsilon = 0.1$).*

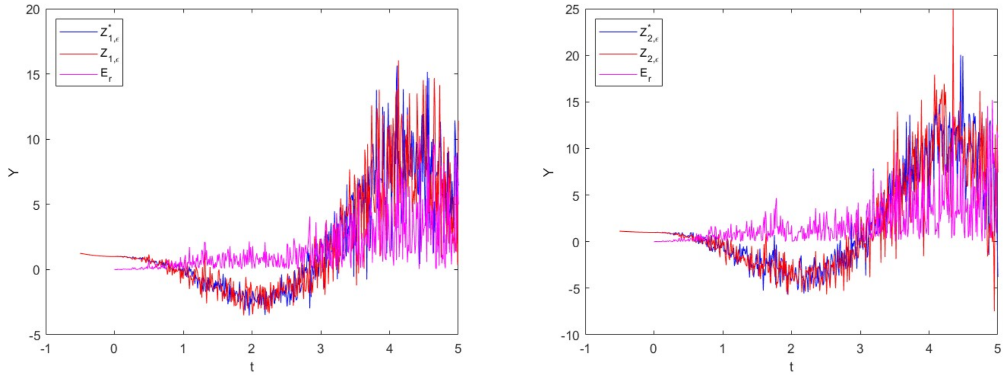

Figure 3. Comparison of $Z_\epsilon(\cdot)$ and $Z^*_\epsilon(\cdot)$ with $\epsilon = 1$.

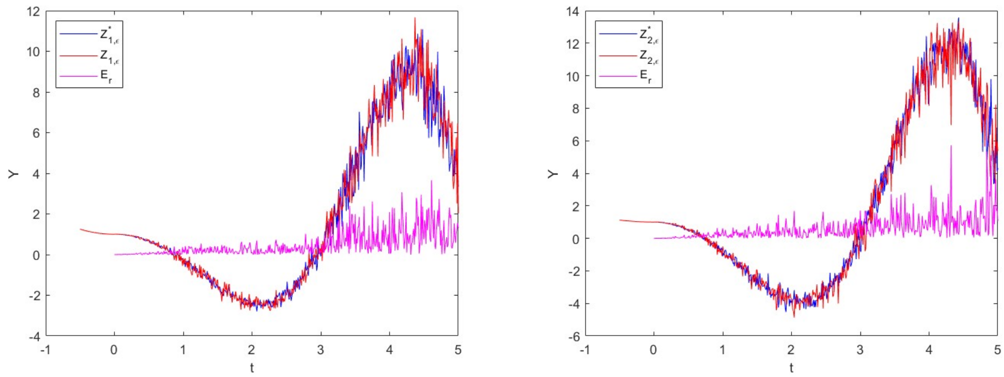

Figure 4. Comparison of $Z_\epsilon(\cdot)$ and $Z^*_\epsilon(\cdot)$ with $\epsilon = 0.1$.

6. Conclusions

This paper focuses on second-order stochastic differential systems with pure delay. By leveraging delayed matrix functions, inequality techniques, and operator theory, it overcomes the challenges posed by stochastic terms, delay terms, and norm estimation of delayed matrix functions. Under the local Lipschitz condition and linear growth condition, the existence and uniqueness of solutions are proven. A system containing a small parameter and a corresponding averaged system are constructed, and the averaging principle is established. The validity of the theoretical results is verified through numerical simulations. This research enriches the theoretical system of the relevant field but has limitations: It does not consider complex factors such as non-Gaussian noise and impulsive effects, and only targets the pure delay scenario, resulting in a limited scope of application. Future research directions can be expanded as follows: incorporating non-Gaussian noise and impulsive terms into the model; and extending to scenarios with multiple delays or time-varying delays; introducing new game subjects such as consumers to build a multi-agent interaction model; combining fuzzy mathematics theory to optimize the handling of parameter uncertainty, thereby enhancing the practicality and universality of the model.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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