Electronic Research Archive

https://www.aimspress.com/journal/era

ERA, 34(6): 3611–3625.
DOI: 10.3934/era.2026162
Received: 20 February 2026
Revised: 13 April 2026
Accepted: 16 April 2026
Published: 28 April 2026

Research article

Left focal points for Caputo fractional differential equations

Paul Eloe[1,*], **Yulong Li**[2] **and Jeffrey Neugebauer**[3]

[1] Department of Mathematics, University of Dayton, Dayton, OH 45469, USA
[2] Department of Mathematics, University of Dayton, Dayton, OH 45469, USA
[3] Department of Mathematics and Statistics, Eastern Kentucky University, Richmond, Kentucky 40475, USA

* **Correspondence:** Email: peloe1@udayton.edu.

Abstract: Let $\beta > 0$ and assume $0 < b \leq \beta$. Let $n \geq 2$ denote an integer and let $n-1 < \alpha \leq n$. The theory of u_0-positive operators with respect to a cone in a Banach space was applied to study eigenvalue problems for left focal boundary value problems for Caputo fractional linear differential equations. Under suitable conditions, it was first established that there exists $0 < \delta < \beta$ such that if $0 < b < \delta$, the left focal boundary value problem had a unique solution, $u \equiv 0$. Then, the left focal point of the left focal boundary value problem was defined and criteria was established to characterize the left focal point with respect to the spectral radius of an associated compact linear fractional integral operator. In order to establish the criteria, properties of related families of Green's functions were observed. The article closed with an application to a nonlinear boundary value problem.

Keywords: Caputo fractional differential equation; two−point boundary value problem; left focal point; u_0−positive operator; spectral radius

1. Introduction

Let $\beta > 0$. Let $n \geq 2$ be an integer and assume $n-1 < \alpha \leq n$. For each $0 < b \leq \beta$, consider the family of boundary value problems (BVPs) of the form

$$ {}^{C}D^{\alpha}_{0^+}u(t) + p(t)u(t) = 0, \quad 0 \leq t \leq b, \tag{1.1}$$

$$ u^{(i)}(0) = 0, \quad i = 1, \dots, n-1, \qquad u(b) = 0. \tag{1.2$_b$}$$

The operator, ${}^{C}D^{\alpha}_{0^+}$, defined precisely in Section 2, denotes a Caputo fractional derivative, and p is a continuous nonnegative function on $[0,\beta]$ which does not vanish identically on any open subinterval of $[0,\beta]$ of positive length.

For ordinary differential equations, the two−point boundary conditions (1.2_b) are contained in the family of disfocal or left disfocal boundary conditions [1,2]. For convenience, we shall refer to (1.2_b) as left focal boundary conditions. We shall study properties of a particular $b_0 \in (0,\beta]$, if it exists, such that the BVP, (1.1), (1.2_{b_0}), has a nontrivial solution, and for $0 < b < b_0$, the BVP, (1.1), (1.2_b), is uniquely solvable, with unique solution $u \equiv 0$. If such a b_0 exists, we shall define b_0 to be the left focal point of (1.1) corresponding to the left focal boundary conditions, (1.2_b).

For ordinary differential equations, it is well known that if that if p is continuous and nonnegative on $(0,\infty)$ and b_0 is the conjugate point of the BVP, $y''(t) + p(t)y(t) = 0$, $y(0) = y(b) = 0$, then there exists a solution y of the BVP, $y''(t) + p(t)y(t) = 0$, $y(0) = y(b_0) = 0$ such that $y(t) \neq 0$ for $0 < t < b_0$. The result is obtained by elementary methods, and we refer the reader to the authoritative monograph [3] for more context. Employing the theory of cones, Schmitt and Smith [4] extended this principle to second order, m−dimensional systems of two-point conjugate BVPs. Since then, their methods [4] have extended to BVPs for various functional equations such as higher order scalar ordinary differential equations [5–8] or BVPs for equations on various time scales [9,10]. Our purpose is to extend these methods to the BVP (1.1), (1.2_{b_0}).

More recently, positive operator theory has been successfully applied to study the eigenvalue problem for two−point BVPs for Riemann−Liouville fractional differential equations. Beginning in 2014, sufficient conditions were established to first imply the existence of a principal eigenvalue for a two−point conjugate BVP for a Riemann−Liouville fractional differential equation [11], and second, characterize the conjugate point for a family of two−point BVPs with respect to the spectral radius of an associated integral operator [12]. Further applications in the realm of fractional calculus can be found in [13,14].

The work here is motivated by [11,12]. We shall first establish the existence of the principal eigenvalue for an eigenvalue problem,

$$^{C}D^{\alpha}_{0^+}u(t) + \lambda p(t)u(t) = 0, \quad 0 \leq t \leq b,$$

coupled with (1.2_b), using the now standard methods of u_0- positive operators. With the observation that the principal eigenvalue produces the spectral radius of an associated integral operator, we then characterize the left focal point of the BVP (1.1), (1.2_b) with the corresponding spectral radius. We do point out that in the case $n = 2$, Henderson and Kosmatov [15] studied the principal eigenvalue of the BVP (1.1), (1.2_b).

In Section 2, we provide the definition of the left focal point, and for the sake of self-containment, we provide the necessary definitions and theorems that are employed. A family of Green's functions is constructed and analyzed in Section 3. In Section 4, the theory of u_0- positive operators is applied to obtain the existence of principal eigenvalues which are the spectral radii of the integral operators. A universal Banach space is also defined so that the spectral radii can be compared. In the primary result, Theorem 4.3, the left focal point of (1.1), (1.2_b) is characterized by the spectral radius. We close Section 4 with a remark, which we believe to be of interest, in which the kernel of an integral operator, recently studied in [16], which depends on $1 \leq \gamma \leq \alpha$, reduces to the Green's function employed here for the Caputo fractional differential equation at $\gamma = 1$, and reduces to the Green's function employed in [11] for the Riemann−Liouville fractional differential equation at $\gamma = \alpha$. An application to a nonlinear BVP is presented in Section 5.

2. Preliminaries

Definition 2.1. Let n denote a positive integer and assume $n-1<\alpha\leq n$. Let $t_0\in\mathbb{R}$ and assume $t_0<T\in\mathbb{R}$. For $u\in L_1[t_0,T]$, the Riemann−Liouville $\alpha-$th order fractional integral of u, originating at $t_0\in\mathbb{R}$, denoted as $I^{\alpha}_{t_0^+}u$, is defined as

$$I^{\alpha}_{t_0^+}u(t)=\int_{t_0}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)\,ds,\quad t_0\leq t\leq T.$$

The Riemann−Liouville $\alpha-$th order fractional derivative of u, originating at $t_0\in\mathbb{R}$, denoted as $D^{\alpha}_{0^+}u$, is defined as

$$D^{\alpha}_{t_0^+}u(t)=D^nI^{n-\alpha}_{t_0^+}u(t),\quad t_0\leq t\leq T,$$

provided the righthand side exists, where D^n denotes the usual nth order derivative. The Caputo $\alpha-$th order fractional derivative of u, originating at $t_0\in\mathbb{R}$, denoted as ${}^CD^{\alpha}_{t_0^+}u$, is defined as

$${}^CD^{\alpha}_{t_0^+}u(t)=D^{\alpha}_{t_0^+}\left(u(t)-\sum_{k=0}^{n-1}\frac{u^{(k)}(t_0)(t-t_0)^k}{k!}\right),\quad t_0\leq t\leq T,$$

provided the righthand side exists.

For our purposes, $2\leq n$, $t_0=0$, and $0<b=T$. Moreover [17],

$${}^CD^{\alpha}_{0^+}I^{\alpha}_{0^+}u(t)=D^{\alpha}_{0^+}I^{\alpha}_{0^+}u(t)=u(t), 0<t<b,\ \text{if } u\in L_1[0,b], \tag{2.1}$$

$$I^{\alpha}_{0+}D^{\alpha}_{0+}u(t)=u(t)+\sum_{k=1}^{n}c_kt^{\alpha-(n-(k-1))}, 0<t<b,\ \text{if } D^{\alpha}_{0+}u\in L_1[0,b], \tag{2.2}$$

$$\begin{aligned}I^{\alpha}_{0^+}({}^CD^{\alpha}_{0^+})u(t)&=u(t)+\sum_{k=1}^{n}c_kt^{n-(n-(k-1))}, 0<t<b,\\ &=u(t)+\sum_{k=1}^{n}c_kt^{k-1}, 0<t<b,\ \text{if } D^{\alpha}_{0+}u\in L_1[0,b].\end{aligned} \tag{2.3}$$

In (2.2), $c_k=\frac{-\lim_{x\to0^+}D^{k-1}I^{n-\alpha}_{0^+}u(x)}{\Gamma(\alpha-(n-k))}$, $k=1,\ldots,n$, and in (2.3), $c_k=\frac{-D^{k-1}u(0)}{\Gamma(k)}$, $k=1,\ldots,n$, and D^{k-1} denotes the usual integer order derivative.

Definition 2.2. We shall say that $0\leq b_0$ is the left focal point of the BVP (1.1), (1.2_b) if

$$b_0=\inf\{b>0:(1.1),(1.2_b)\text{ has a nontrivial solution}\}.$$

Definition 2.3. Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$. A closed nonempty subset $\mathcal{P}$ of $\mathcal{B}$ is said to be a cone provided

(i) $c_1u+c_2v\in\mathcal{P}$, for all $u,v\in\mathcal{P}$ and all $c_1,c_2\geq0$, and

(ii) $u\in\mathcal{P}$ and $-u\in\mathcal{P}$ implies $u=0$.

Definition 2.4. Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$ and assume $\mathcal{P} \subset \mathcal{B}$ is a cone. The cone $\mathcal{P}$ is a solid cone if the interior, Int $\mathcal{P}$, of $\mathcal{P}$, is nonempty. The cone $\mathcal{P}$ is reproducing if for each $w \in \mathcal{B}$, there exist $u, v \in \mathcal{P}$ such that $w = u - v$.

Remark 2.1. According to Krasnosel'skiĭ [18], every solid cone is a reproducing cone.

Definition 2.5. Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$ and assume $\mathcal{P} \subset \mathcal{B}$ is a cone. We say $u \leq v$ with respect to $\mathcal{P}$, if $u, v \in \mathcal{B}$, and $v - u \in \mathcal{P}$. If $M, N : \mathcal{B} \to \mathcal{B}$ are bounded linear operators, we say $M \leq N$ with respect to $\mathcal{P}$ if $Mu \leq Nu$ for all $u \in \mathcal{P}$.

Definition 2.6. [18] A bounded linear operator $M : \mathcal{B} \to \mathcal{B}$ is u_0-positive with respect to $\mathcal{P}$ if there exists $u_0 \in \mathcal{P}\backslash\{0\}$ such that for each $u \in \mathcal{P}\backslash\{0\}$, there exist constants $k_1(u) > 0$ and $k_2(u) > 0$ such that $k_1 u_0 \leq Mu \leq k_2 u_0$ with respect to $\mathcal{P}$.

Let $r(N)$ denote the spectral radius of a bounded linear operator N.

The application of u_0-positive operator theory is summarized in the following results. Theorem 2.3 is found in [19]. The last three results and proofs are found in [18, 20–22].

Theorem 2.1. *Let $\mathcal{B}$ denote a Banach space over the reals and let $\mathcal{P} \subset \mathcal{B}$ be a solid cone. If $N : \mathcal{B} \to \mathcal{B}$ is a linear operator such that $N : \mathcal{P} \setminus \{0\} \to$ Int $\mathcal{P}$, then N is u_0−positive with respect to $\mathcal{P}$.*

Theorem 2.2 (Krein Rutman Theorem [20])**.** *Let $\mathcal{B}$ denote a Banach space over the reals and assume $\mathcal{P} \subset \mathcal{B}$ is a reproducing cone. Assume $N : \mathcal{B} \to \mathcal{B}$ is a compact linear operator that is u_0−positive with respect to $\mathcal{P}$. Then, N has an eigenvalue that is simple, positive, and larger in modulus than any other eigenvalue of N. Moreover, a corresponding eigenvector is an element of $\mathcal{P}$ and the dimension of the eigenspace is 1. In particular, this eigenvalue is $r(N)$, the spectral radius of $N : \mathcal{B} \to \mathcal{B}$.*

Theorem 2.3. *[19] Let N_b, $\eta \leq b \leq \beta$ be a family of compact, linear operators on a Banach space such that the mapping $b \mapsto N_b$ is continuous in the uniform operator topology. Then, the mapping $b \mapsto r(N_b)$ is continuous.*

Theorem 2.4. *Assume $\mathcal{B}$ is a real Banach space and assume $\mathcal{P} \subset \mathcal{B}$ is a reproducing cone. Assume $N : \mathcal{B} \to \mathcal{B}$ is compact, linear, and positive with respect to $\mathcal{P}$, and assume $r(N) > 0$. Then, $r(N)$ is an eigenvalue of N, and there is a corresponding eigenvector in $\mathcal{P}$.*

Theorem 2.5. *Assume $\mathcal{B}$ is a real Banach space and assume $\mathcal{P} \subset \mathcal{B}$ is a reproducing cone. Assume $N_1, N_2 : \mathcal{B} \to \mathcal{B}$ are compact, linear, and positive with respect to $\mathcal{P}$, and assume $N_1 \leq N_2$ with respect to $\mathcal{P}$. Then, $r(N_1) \leq r(N_2)$.*

Theorem 2.6. *[21] Assume $\mathcal{B}$ is a real Banach space, assume $\mathcal{P} \subset \mathcal{B}$ is a reproducing cone and assume $N : \mathcal{B} \to \mathcal{B}$ is compact, linear, and positive with respect to $\mathcal{P}$. Suppose there exists $\gamma > 0$, $u \in \mathcal{B}$, $-u \notin \mathcal{P}$ such that $\gamma u \leq Nu$ with respect to $\mathcal{P}$. Then, N has an eigenvector in $\mathcal{P}$ which corresponds to an eigenvalue λ with $\lambda \geq \gamma$.*

3. Green's functions and associated properties

The Green's function for a BVP ${}^C D_{0^+}^{\alpha} u(t) + f(t) = 0$, (1.2_b) is given by

$$G(b; t, s) = \begin{cases} \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq b, \\ \\ \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq b. \end{cases} \tag{3.1}$$

To see this, employ (2.3) and apply the operator $I_{0^+}^{\alpha}$ to ${}^CD_{0^+}^{\alpha}u(t) + f(t) = 0$ to obtain

$$u(t) = -I_{0^+}^{\alpha}f(t) - \sum_{k=1}^{n} c_k t^{k-1}.$$

The boundary conditions at $t_0 = 0$ in (1.2_b) imply $c_k = 0, k = 2, \ldots, n$. So,

$$u(t) = -I_{0^+}^{\alpha}f(t) - c_1.$$

Then, the boundary condition $u(b) = 0$ implies $c_1 = -I_{0^+}^{\alpha}f(b)$ and

$$u(t) = -I_{0^+}^{\alpha}f(t) + I_0^{\alpha}f(b).$$

Extract the kernel of this integral equation to obtain (3.1).

Note also that for each $i \in \{1, \ldots, n-1\}$,

$$\frac{\partial^i}{\partial t^i}G(b; t, s) = \begin{cases} -\frac{(t-s)^{\alpha-1-i}}{\Gamma(\alpha-i)}, & 0 \le s \le t \le b, \\ 0, & 0 \le t \le s \le b. \end{cases}$$

We point out some obvious properties of G, which we state in a theorem without proof.

Theorem 3.1. *Note the following:*

(1) $G(b; t, s) > 0$ *on* $[0, b) \times [0, b)$.

(2) $\frac{\partial^i}{\partial t^i}G(b; t, s) \le 0$ *on* $[0, b) \times [0, b)$, *for* $i = 1, \ldots, n-1$.

(3) $G(b; b, s) = 0$ *on* $[0, b)$.

(4) $\frac{\partial}{\partial t}G(b; b, s) < 0$ *on* $[0, b)$.

(5) If $0 < b_1 < b_2$, *then* $G(b_2; t, s) > G(b_1; t, s) \ge 0$ *on* $[0, b_1] \times [0, b_1]$.

(6) If $0 < b$, *and* $f \in C[0, b]$, *then* $\int_0^b G(b; \cdot, s)f(s)\,ds \in C^{n-1}[0, b]$.

We shall employ Theorem 3.1 (1), (3)–(5) in the applications of Theorems 2.1, 2.2, 2.4–2.6 in Section 4. Theorem 3.1 (6) is employed to prove the following result that is fundamental for the purposes of applying fixed point theorems to obtain sufficient conditions for the existence of solutions of nonlinear BVPs, ${}^CD_{0^+}^{\alpha}u(t) + f(t, u(t), u'(t), \ldots, u^{(n-1)}(t)) = 0$, (1.2_b).

Theorem 3.2. *Assume* $f : [0, b] \times \mathbb{R}^n \to \mathbb{R}$ *is continuous. Then,* $u \in C^{n-1}[0, b]$ *is a solution of the BVP*

$${}^CD_{0^+}^{\alpha}u(t) + f(t, u(t), u'(t), \ldots, u^{(n-1)}(t)) = 0, \quad 0 < t < b,$$

$$u^{(i)}(0) = 0, i = 1, \ldots, n-1, \; u(b) = 0,$$

if, and only if, $u \in C^{n-1}[0, b]$ *is a solution of the integral equation*

$$u(t) = \int_0^b G(b; t, s)f(s, u(s), u'(s), \ldots, u^{(n-1)}(s))\,ds.$$

4. Criteria for left focal points

For each $0 < b \le \beta$, define a Banach space

$$\mathcal{B}(b) = \{u : [0,b] \to \mathbb{R} : u \in C^{n-1}[0,b], u^{(i)}(0) = 0, i = 1,\dots,n-1, u(b) = 0\},$$

with the norm

$$\|u\|_b = \max_{i=0,\dots,n-1}\{\sup_{t\in[0,b]} |u^{(i)}(t)|\}.$$

We shall also have need for the Banach space $\mathcal{B} = C[0,\beta]$ with $\|u\| = \sup_{t\in[0,\beta]} |u(t)|$.

For each $0 < b \le \beta$, define a cone $\mathcal{P}(b) \subset \mathcal{B}(b)$ by

$$\mathcal{P}(b) = \{u \in \mathcal{B}(b) : u(t) \ge 0 \text{ for } t \in [0,b]\}.$$

It is straightforward to show that Int $\mathcal{P}(b)$ is nonempty and, in fact, it is straightforward to show that

$$\operatorname{Int}\mathcal{P}(b) = \{u \in \mathcal{P}(b) : u(t) > 0, 0 \le t < b, u'(b) < 0\}. \tag{4.1}$$

Thus, $\mathcal{P}(b)$ is a solid cone and, hence, reproducing by Remark 2.1. Likewise, define the cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \{u \in \mathcal{B} : u(t) \ge 0, \text{ for } 0 \le t \le \beta\}.$$

Note that if $u \in \mathcal{B}$, then $u = u_1 - u_2$ where

$$u_1(t) = \max\{0, u(t)\} \in \mathcal{B}, \text{ and } u_2(t) = \max\{0, -u(t)\} \in \mathcal{B}.$$

Thus, $\mathcal{P}$ is also a reproducing cone.

For the remainder of the article, assume p is a continuous nonnegative function on $[0,\beta]$ which does not vanish identically on any open subinterval of $[0,\beta]$ of positive length, and set $P = \max_{0\le t\le\beta} |p(t)|$. Define $\bar{N}_0 u(t) := 0, 0 \le t \le \beta$, and for each $0 < b \le \beta$, define $\bar{N}_b : \mathcal{B} \to \mathcal{B}$ by

$$\bar{N}_b u(t) := \begin{cases} \int_0^b G(b;t,s)p(s)u(s)\,ds, & 0 \le t \le b, \\ 0, & b < t \le \beta. \end{cases} \tag{4.2}$$

For each $0 < b \le \beta$, we shall also consider $N_b : \mathcal{B}(b) \to \mathcal{B}(b)$ defined by

$$N_b u(t) := \int_0^b G(b;t,s)p(s)u(s)\,ds, \ 0 \le t \le b.$$

In the following theorem, the contraction mapping principle is employed to argue that b_0, the left focal point of (1.1), (1.2_b), is positive, if it exists.

Theorem 4.1. *There exists $0 < \delta \le \beta$ such that if $0 < b < \delta$, there exists a unique solution of the BVP, (1.1), (1.2_b); in particular, if $0 < b < \delta$, then $u \equiv 0$ is the only solution of (1.1), (1.2_b).*

Proof. Let $u_1, u_2 \in \mathcal{B}(b)$. For $t \in [0, b]$,

$$(N_b u_1 - N_b u_2)(t) = \int_0^b \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)}(pu_1 - pu_2)(s)\,ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(pu_1 - pu_2)(s)\,ds,$$

$$(N_b u_1 - N_b u_2)^{(i)}(t) = -\int_0^t \frac{(t-s)^{\alpha-1-i}}{\Gamma(\alpha-i)}(pu_1 - pu_2)(s)\,ds, \quad i = 1, \ldots, n-1.$$

Then,

$$\|N_b u_1 - N_b u_2\|_b \le \max\left\{\frac{2b^\alpha}{\Gamma(\alpha+1)}, \max_{i=1,\ldots,n-1}\{\frac{b^{\alpha-i}}{\Gamma(\alpha-i+1)}\}\right\} P\|u_1 - u_2\|_b.$$

Note that

$$\lim_{b\to 0+} \frac{2b^\alpha}{\Gamma(\alpha+1)} = 0, \quad \lim_{b\to 0+} \frac{b^{\alpha-i}}{\Gamma(\alpha-i+1)} = 0, \quad i = 1, \ldots, n-1;$$

thus, choose $\delta > 0$ such that if $0 < b < \delta$,

$$\max\left\{\frac{2b^\alpha}{\Gamma(\alpha+1)}, \max_{i=1,\ldots,n-1} \frac{b^{\alpha-i}}{\Gamma(\alpha-i+1)}\right\} < \frac{1}{2P}.$$

Then, $N_b : \mathcal{B}(b) \to \mathcal{B}(b)$ is a contraction map.

Lemma 4.1. *If* $0 < b \le \beta$, *then, the linear operator* $N_b : \mathcal{B}(b) \to \mathcal{B}(b)$ *is compact.*

Proof. Linearity is clear.

Let $u \in \mathcal{B}(b)$. Then, $pu \in C[0, b]$ and

$$N_b u = I_{0^+}^\alpha pu(b) - I_{0^+}^\alpha pu \in C^{n-1}[0, b],$$

by Theorem 3.1 (6). Moreover, by construction, $(N_b u)^{(i)}(0) = 0, i = 1, \ldots n-1$, and $N_b u(b) = 0$. Thus,

$$N_b : \mathcal{B}(b) \to \mathcal{B}(b).$$

To prove that $N_b : \mathcal{B}(b) \to \mathcal{B}(b)$ is compact, let $\{u_n\}$ denote an arbitrary sequence in the unit ball of $\mathcal{B}(b)$. We show that for each $i = 0, \ldots n-1$, the sequence $\{(N_b u_n)^{(i)}\}$ is uniformly bounded and equicontinuous and the proof of Lemma 4.1 will be complete.

For uniform boundedness, note that since $0 \le t \le b$,

$$|N_b u_n(t)| \le P\int_0^b \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds + P\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds \le \frac{2Pb^\alpha}{\Gamma(\alpha+1)}, \tag{4.3}$$

and for $i = 1, \ldots, n-1$,

$$|(N_b u_n)^{(i)}(t)| \le P\int_0^b \frac{(t-s)^{\alpha-1-i}}{\Gamma(\alpha-i)}\,ds \le \frac{Pb^{\alpha-i}}{\Gamma(\alpha-i+1)}.$$

So each sequence $\{(N_b u_n)^{(i)}\}$, $i = 0, \ldots, n-1$ is uniformly bounded.

For equicontinuity, first note that for each $i = 0, \ldots, n-2$, the uniform equicontinuity of $\{(N_b u_n)^{(i)}\}$ follows immediately from the mean value theorem and the uniform boundedness of $\{(N_b u_n)^{(i+1)}\}$ since for $t_1, t_2 \in [0, b]$, there exists $c_{n,i}$ between t_1 and t_2 such that

$$|(N_b u_n)^{(i)}(t_2) - (N_b u_n)^{(i)}(t_1)| = |(N_b u_n)^{(i+1)}(c_{n,i})||t_2 - t_1| \le \frac{Pb^{\alpha-i-1}}{\Gamma(\alpha-i)}|t_2 - t_1|.$$

To address the equicontinuity of $\{(N_b u_n)^{(n-1)}\}$, assume for convenience that $0 \le t_1 < t_2 \le b$. Then,

$$\begin{aligned}|(N_b u_n)^{(n-1)}(t_2) - (N_b u_n)^{(n-1)}(t_t)| &= \left|\int_0^{t_2} \frac{(t_2 - s)^{\alpha-n}}{\Gamma(\alpha - n + 1)} p(s)u_n(s)\, ds - \int_0^{t_1} \frac{(t_1 - s)^{\alpha-n}}{\Gamma(\alpha - n + 1)} p(s)u_n(s)\, ds\right| \\ &= \left|\int_0^{t_1} (\frac{(t_2 - s)^{\alpha-n}}{\Gamma(\alpha - n + 1)} - \frac{(t_1 - s)^{\alpha-n}}{\Gamma(\alpha - n + 1)}) p(s)u_n(s)\, ds + \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-n}}{\Gamma(\alpha - n + 1)} p(s)u_n(s)\, ds\right| \\ &\le P \int_0^{t_1} \left|\frac{(t_2 - s)^{\alpha-n}}{\Gamma(\alpha - n + 1)} - \frac{(t_1 - s)^{\alpha-n}}{\Gamma(\alpha - n + 1)}\right| ds + P\int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-n}}{\Gamma(\alpha - n + 1)}\, ds \\ &= P \int_0^{t_1} \left(\frac{(t_1 - s)^{\alpha-n}}{\Gamma(\alpha - n + 1)} - \frac{(t_2 - s)^{\alpha-n}}{\Gamma(\alpha - n + 1)}\right) ds + P\int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-n}}{\Gamma(\alpha - n + 1)}\, ds \\ &= P\left(\frac{t_1^{\alpha-n+1}}{\Gamma(\alpha - n + 2)} - \frac{t_2^{\alpha-n+1}}{\Gamma(\alpha - n + 2)} + 2\frac{(t_2 - t_1)^{\alpha-n+1}}{\Gamma(\alpha - n + 2)}\right).\end{aligned}$$

The function $f(x) = \frac{x^{\alpha-n+1}}{\Gamma(\alpha-n+2)}$, $0 \le x \le b$ is uniformly continuous on $[0, b]$. Thus, for $\epsilon > 0$, there exists $\delta > 0$ such that if $|t_2 - t_1| < \delta, t_1, t_2 \in [0, b]$, then

$$\left|\frac{t_1^{\alpha-n+1}}{\Gamma(\alpha - n + 2)} - \frac{t_2^{\alpha-n+1}}{\Gamma(\alpha - n + 2)}\right| < \frac{\epsilon}{2P} \text{ and } \frac{|(t_2 - t_1)|^{\alpha-n+1}}{\Gamma(\alpha - n + 2)} = \frac{|(t_2 - t_1) - 0|^{\alpha-n+1}}{\Gamma(\alpha - n + 2)} < \frac{\epsilon}{4P},$$

and the equicontinuity of $\{(N_b u_n)^{(n-1)}\}$ is established. This completes the proof that the map $N_b : \mathcal{B}(b) \to \mathcal{B}(b)$ is a compact map.

Lemma 4.2. *If* $0 < b \le \beta$, *then,* $N_b : \mathcal{P}_b \setminus \{0\} \to$ *Int* $\mathcal{P}_b$.

Proof. This is a straightforward consequence of Theorem 3.1 (1) and (4), the sign properties of p, and the characterization of Int $\mathcal{P}_b$ in (4.1). Let $u \in \mathcal{P}_b$. First, if $0 \le t < b$, then Theorem 3.1 (1) implies

$$N_b u(t) = \int_0^b G(b; t, s)p(s)u(s)\, ds > 0.$$

Second, if $t = b$, then Theorem 3.1 (4) implies

$$(N_b u)'(b) = \int_0^b \frac{\partial}{\partial t} G(b; b, s)p(s)u(s)\, ds < 0.$$

Corollary 4.1. *If* $0 < b \le \beta$, *then,* $r(N_b) > 0$. *Moreover,* $r(N_b)$ *is an eigenvalue of* N_b *with corresponding eigenvector,* $u \in$ *Int* $\mathcal{P}(b)$.

Proof. This is an immediate application of Theorems 2.1 and 2.2.

Corollary 4.2. *Let* $0 < b \le \beta$. *Then,* λ *is an eigenvalue of* N_b *if, and only if,* λ *is an eigenvalue of* $\bar{N}$. *Moreover,* $r(\bar{N}_b) = r(N_b) > 0$, *with corresponding eigenvector* $\bar{u}_b \in \mathcal{P} \setminus \{0\}$.

Proof. The equivalence expressed in this corollary is an easy consequence of the observation that if u is an eigenfunction for $\bar{N}_b$, then $u(t) = 0$, for $b < t \le \beta$, which implies

$$N_b u(t) + \lambda p u(t) = 0, 0 \le t \le b \text{ if, and only if, } \bar{N}_b u(t) + \lambda p u(t) = 0, 0 \le t \le \beta.$$

Lemma 4.3. *The family of operators* $\bar{N}_b : \mathcal{B} \to \mathcal{B}$, $0 \le b \le \beta$ *is a family of compact, linear operators on* $\mathcal{B}$ *and the mapping* $b \mapsto \bar{N}_b$ *is continuous in the uniform topology.*

Proof. Linearity is clear. Since $\mathcal{B} = C[0,\beta]$, the Arzela-Ascoli theorem only needs to be applied to $\{\bar{N}_b u_n\}$, where $\{u_n\}$ is contained in the unit ball of $\mathcal{B}$. A calculation, very similar to that in (4.3), implies

$$|\bar{N}_b u_n(t)| \le \frac{2Pb^\alpha}{\Gamma(\alpha+1)}, \quad 0 \le t \le \beta.$$

For equicontinuity, one considers each of three cases, $0 \le t_1 < t_2 \le b$, $0 \le t_1 < b < t_2 \le \beta$, and $b \le t_1 < t_2 \le \beta$. Straightforward calculations imply

$$|\bar{N}_b u(t_2) - \bar{N}_b u(t_1)| \le \frac{Pb^{\alpha-2}}{\Gamma(\alpha-1)}|t_2 - t_1|.$$

So, $\bar{N}_b : \mathcal{B} \to \mathcal{B}$ is a linear compact map.

To see that the mapping $b \mapsto \bar{N}_b$ is continuous in the uniform operator topology, assume $u \in \mathcal{B}$ with $\|u\| = 1$. Assume for simplicity that $0 \le b_1 < b_2 \le \beta$.

First, assume the case $b_1 = 0$. Then,

$$\begin{aligned} |(\bar{N}_{b_2} - \bar{N}_0)u(t)| &\le \left|\int_0^{b_2} \frac{(b_2-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)\, ds\right| + \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)\, u(s)\, ds\right| \\ &\le \frac{2Pb_2^\alpha}{\Gamma(\alpha+1)} = 2P\left(\frac{b_2^\alpha}{\Gamma(\alpha+1)} - \frac{b_1^\alpha}{\Gamma(\alpha+1)}\right). \end{aligned}$$

Now assume the case $0 < b_1$. For this case, first assume $0 < t \le b_1$. Then

$$\begin{aligned} |(\bar{N}_{b_2} - \bar{N}_{b_1})u(t)| &= \left|\int_0^{b_2} \frac{(b_2-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)\, ds - \int_0^{b_1} \frac{(b_1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)\, ds\right| \\ &= \left|\int_0^{b_1} (\frac{(b_2-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(b_1-s)^{\alpha-1}}{\Gamma(\alpha)}) p(s)u(s)\, ds + \int_{b_1}^{b_2} \frac{(b_2-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)\, ds\right| \\ &\le P\left(\int_0^{b_1} \left(\frac{(b_2-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(b_1-s)^{\alpha-1}}{\Gamma(\alpha)}\right) ds + \int_{b_1}^{b_2} \frac{(b_2-s)^{\alpha-1}}{\Gamma(\alpha)}\, ds\right) \\ &= P\left(\frac{b_2^\alpha}{\Gamma(\alpha+1)} - \frac{b_1^\alpha}{\Gamma(\alpha+1)}\right). \end{aligned}$$

Second, assume $b_1 < t \le b_2$. Then,

$$\begin{aligned} |(\bar{N}_{b_2} - \bar{N}_{b_1})u(t)| &= \left|\int_0^{b_2} \frac{(b_2-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)\, ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)\, ds\right| \\ &\le P\left(\frac{b_2^\alpha}{\Gamma(\alpha+1)} - \frac{t^\alpha}{\Gamma(\alpha+1)}\right) \le P\left(\frac{b_2^\alpha}{\Gamma(\alpha+1)} - \frac{b_1^\alpha}{\Gamma(\alpha+1)}\right). \end{aligned}$$

Finally, for $b_2 < t \le \beta$,

$$|(\bar{N}_{b_2} - \bar{N}_{b_1})u(t)| = 0 \le P\left(\frac{b_2^\alpha}{\Gamma(\alpha+1)} - \frac{b_1^\alpha}{\Gamma(\alpha+1)}\right).$$

In particular, if $0 \le b_1 < b_2 \le \beta$, apply the mean value theorem and

$$\|(\bar{N}_{b_2} - \bar{N}_{b_1})u\| \le \frac{2P\beta^{\alpha-1}}{\Gamma(\alpha)}|b_2 - b_1|.$$

Thus, the mapping $b \to N_b$ is continuous in the uniform topology.

Corollary 4.3. *The mapping $b \mapsto r(\bar{N}_b)$ is continuous.*

Proof. Apply Theorem 2.3 and the preceding lemma.

Theorem 4.2. *The mapping $b \mapsto r(\bar{N}_b)$ is strictly increasing as a function of b.*

Proof. First note that $r(\bar{N}_0) = 0$. Let $b \in (0,\beta]$. Then, $r(\bar{N}_b) = r(N_b) > 0$ is an eigenvalue of $N_b : \mathcal{B}(b) \to \mathcal{B}(b)$ with corresponding eigenfunction $u_b \in \operatorname{Int} \mathcal{P}(b)$. Extend $u_b \in C^{n-1}[0,b]$ to $\bar{u}_b \in C[0,\beta]$ by

$$\bar{u}_b(t) = \begin{cases} u_b(t), & 0 \le t \le b, \\ \\ 0, & b < t \le \beta, \end{cases} \tag{4.4}$$

and note that $\bar{u}_b \in \mathcal{P} \setminus \{0\}$. Perform this extension for each $b \in (0,\beta]$.

If $0 < b_1 < b_2 \le \beta$, then, $r(N_{b_1}) > 0$ exists. Moreover, there exists $u_{b_1} \in \mathcal{P}_{b_1} \setminus \{0\}$ such that

$$N_{b_1} u_{b_1}(t) = r(N_{b_1}) u_{b_1}(t), \quad 0 \le t \le b_1.$$

Let $\bar{u}_{b_1}$ denote the extension of u_{b_1} to $[0,\beta]$ according to (4.4). Set $y_1 = \bar{N}_{b_1}\bar{u}_{b_1}(t)$, $0 \le t \le \beta$, and note $y_1 \in \mathcal{P} \setminus \{0\}$; set $y_2 = \bar{N}_{b_2}\bar{u}_{b_1} \in \mathcal{P} \setminus \{0\}$. For $0 \le t \le b_1$,

$$\begin{aligned} (y_2 - y_1)(t) &= \int_0^{b_2} G(b_2; t, s) p\bar{u}_{b_1}(s)\, ds - \int_0^{b_1} G(b_1; t, s) p\bar{u}_{b_1}(s)\, ds \\ &= \int_0^{b_1} (G(b_2; t, s) - G(b_1; t, s) p u_{b_1}(s)\, ds. \end{aligned}$$

Note that $u_{b_1} \in \operatorname{Int} \mathcal{P}(b_1)$ implies $(y_2 - y_1)(t) > 0$ for $0 \le t \le b_1$ by Theorem 3.1 (5). Moreover, $(y_2 - y_1) \in C[0,\beta]$, and so set $m = \min_{0 \le t \le b_1}(y_2 - y_1)(t) > 0$. Likewise, $u_{b_1} \in C[0, b_1]$, and so set $M = \max_{0 \le t \le b_1} u_{b_1}(t) > 0$. Set $\delta = \frac{m}{M} > 0$ and $(y_2 - y_1)(t) \ge \delta u_{b_1}(t)$ for $0 \le t \le b_1$. For $b_1 < b \le \beta$, $y_2(t) \ge 0$ and $y_1(t) = 0$. Then, $(y_2 - y_1)(t) \ge \delta \bar{u}_{b_1}(t)$, for all $0 \le t \le \beta$; in particular, $(y_2 - y_1) \ge \delta \bar{u}_{b_1}$ with respect to the cone $\mathcal{P}$. Thus, with respect to $\mathcal{P}$,

$$y_2 \ge y_1 + \delta \bar{u}_{b_1} = (r(\bar{N}_{b_1}) + \delta)\bar{u}_{b_1}.$$

Apply Theorem 2.6 and $r(\bar{N}_{b_2}) \ge r(\bar{N}_{b_1}) + \delta > r(\bar{N}_{b_1})$.

The next theorem characterizes the left focal point of the BVP (1.1), (1.2_b) with respect to a corresponding positive eigenfunction and with respect to a normalized spectral radius.

Theorem 4.3. *The following are equivalent:*

(1) b_0 is the left focal point of the BVP (1.1), (1.2_b)*;*

(2) there exists a nontrivial solution u of the BVP (1.1), (1.2_{b_0}) *such that $u \in \mathcal{P}(b_0)$;*

(3) $r(N_{b_0}) = r(\bar{N}_{b_0}) = 1$.

Proof. (3) $\implies$ (2) is true due to Theorem 2.4.

To see that (2) $\implies$ (1), let $u \in \mathcal{P}_{b_0} \setminus \{0\}$ satisfy (1.1), (1.2_{b_0}). Let $\bar{u}$ denote the extension of u according to (4.4). Then, $\bar{u}(t) = \bar{N}_{b_0}\bar{u}(t)$, $0 \le t \le \beta$, and $r(\bar{N}_{b_0}) \ge 1$.

If $r(\bar{N}_{b_0}) = 1$, apply Theorem 4.2. Let $0 < b < b_0$. Then, $r(N_b) = r(\bar{N}_b) < 1$ and the BVP (1.1), (1.2_b) has only the trivial solution. By definition, b_0 is the left focal point of the BVP (1.1), (1.2_b) and (2) $\implies$ (1) is proved.

If $r(\bar{N}_{b_0}) > 1$, Let $\bar{v} \in \mathcal{P} \setminus \{0\}$ be an eigenfunction corresponding to $r(\bar{N}_{b_0})$ for the operator $\bar{N}_{b_0}$. Let $v(t)$ denote the restriction of $v(t) = \bar{v}$ to $[0, b_0]$. Then, for $0 \le t \le b_0$, $N_{b_0}v(t) = r(\bar{N}_{b_0})v(t)$. Moreover, $v \in \operatorname{Int}\mathcal{P}(b_0)$. Thus, there exists $\delta > 0$ such that $\delta v \le u$ with respect to the cone $\mathcal{P}_{b_0}$. Thus, $\delta\bar{v} \le \bar{u}$, with respect to the cone $\mathcal{P}$. Let $\bar{\delta} = \sup\{\delta > 0$ such that $\delta\bar{v} \le \bar{u}\}$. Then,

$$\bar{u} = \bar{N}_{b_0}\bar{u} \ge N_{b_0}\bar{\delta}\bar{v} = \bar{\delta}\bar{N}_{b_0}\bar{v} = \bar{\delta}r(\bar{N}_{b_0})\hat{v},$$

a contradiction if $r(\bar{N}_{b_0}) > 1$. Thus, $r(\bar{N}_{b_0}) = 1$.

To prove (1) $\implies$ (3), observe that the continuity of the mapping $b \mapsto r(\bar{N}_b)$ implies $\lim_{b \to 0^+} r(\bar{N}_b) = 0$. Note that (1) implies $r(\bar{N}_{b_0}) \ge 1$. If $r(\bar{N}_{b_0}) > 1$, then the continuity of r implies there exists $0 < b < b_0$ such that $r(\bar{N}_b) = 1$, contradicting (1).

Remark 4.1. Assume $1 \le \gamma \le \alpha$. Recently in [16], an eigenvalue problem

$$u(t) = \lambda I^{\alpha}_{0^+}u(1)t^{\gamma-1} - \lambda I_{0^+}u(t) = \lambda \int_0^1 G(\alpha, \gamma; t, s)u(s)\, ds, \quad 0 \le t \le 1,$$

has been studied using u_0−positive operator theory for the purpose of studying the real roots of two−parameter Mittag−Leffler functions. So, on an interval $[0, b]$, consider an eigenvalue problem

$$u(t) = \lambda I^{\alpha}_{0^+}u(b)t^{\gamma-1} - \lambda I_{0^+}u(t) = \lambda \int_0^b G(\alpha, \gamma, b; t, s)u(s)\, ds, \quad 0 \le t \le b,$$

where, for $1 \le \gamma \le \alpha$,

$$G(\alpha, \gamma, b; t, s) = \begin{cases} \frac{t^{\gamma-1}(b-s)^{\alpha-1}}{b^{\gamma-1}\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le b, \\ \frac{t^{\gamma-1}(b-s)^{\alpha-1}}{b^{\gamma-1}\Gamma(\alpha)}, & 0 \le t \le s \le b. \end{cases}$$

Set $\gamma = 1$, $G(\alpha, 1, b; t, s) = G(b; t, s)$ as in (3.1), the Green's functions for the two−point left focal BVP for Caputo fractional differential equations. Set $\gamma = \alpha$, and $G(\alpha, \alpha, b; t, s)$ is precisely the family of Green's functions employed in [11] and [12] in the study of a two−point conjugate BVPs for a Riemann−Liouville fractional differential equation. The kernel $G(\alpha, \gamma, b; t, s)$ provides an interesting algebraic connection between focal BVPs for Caputo fractional differential equations and conjugate BVPs for Riemann-Liouville fractional differential equations.

5. Application to a nonlinear problem

Motivated by [4], the above results are applied to a nonlinear problem. This application employs a fixed point theorem, proved in [23] or [4].

Consider a BVP for a nonlinear fractional differential equation of the form,

$$ {}^{C}D^{\alpha}_{0^+}u(t) + f(t,u) = 0, \quad 0 < t < b, \tag{5.1} $$

with boundary conditions (1.2_b), where $f(t,u) : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous, and $f(t,0) \equiv 0$. In addition, assume f is differentiable in u at $u = 0$. Assume $p = (\frac{\partial f}{\partial u})(\cdot, 0)$ is continuous on $[0,\infty)$ and does not vanish identically on each open subinterval of $[0,\infty)$ of positive length. Then, the variational equation along the zero solution of (5.1) is

$$ {}^{C}D^{\alpha}_{0^+}u(t) + p(t)u(t) = 0, \quad 0 < t < b. \tag{5.2} $$

Thus, assume in addition that $p(t) \geq 0$, $t \in (0,\infty)$ so that if $u \in \mathcal{P}(b)$, then $pu(t) \geq 0$, for $0 \leq t \leq b$.

We shall apply the following fixed point theorem.

Theorem 5.1. *[23] Let $\mathcal{B}$ be a real Banach space and assume $\mathcal{P} \subset \mathcal{B}$ is a reproducing cone. Assume $M : \mathcal{B} \to \mathcal{B}$ is a completely continuous operator such that $M : \mathcal{P} \to \mathcal{P}$ and $M(0) = 0$. Assume M is Fréchet differentiable at $u = 0$ whose Fréchet derivative $N = M'(0)$ has the property:*

(A) There exist $w \in \mathcal{P}$ and $\mu > 1$ such that $Nw = \mu w$, and $Nu = u$ implies $u \notin \mathcal{P}$. Further, there exists $\rho > 0$ such that, if $u = (\frac{1}{\lambda})Mu, u \in \mathcal{P}$ and $\|u\| = \rho$, then $\lambda \leq 1$.
Then, the equation $u = Mu$ has a solution $u \in \mathcal{P} \setminus \{0\}$.

We shall apply Theorem 5.1 and the results of Section 4 to prove the following theorem. As above, let $\beta > 0$, $\mathcal{B} = C[0,\beta]$, and $\mathcal{P} = \{u \in \mathcal{B} : u(t) \geq 0, \text{ for } 0 \leq t \leq \beta.\}$.

Theorem 5.2. *Assume b_0 is the left focal point of (5.2), (1.2_b). For each $b_0 < b \leq \beta$ assume the property:*

($\hat{A}$) There exists $\rho(b) > 0$ such that if $u(t)$ is a nontrivial solution of the BVP,

$$ {}^{C}D^{\alpha}_{0^+}u + \left(\frac{1}{\lambda}\right) f(t,u) = 0, \quad 0 < t < b, $$

with boundary conditions (1.2_b) and if $u \in \mathcal{P}_b$, with $\|u\|_b = \rho(b)$, then $\lambda \leq 1$.
Then, for each $b_0 < b$, the BVP (5.1), (1.2_b) has a nontrivial solution $u \in \mathcal{P}_b$.

Proof. For each $b_0 < b$, let $\bar{N}_b : \mathcal{B} \to \mathcal{B}$ be defined by (4.2) where $p(t) \equiv (\frac{\partial f}{\partial u})(t,0)$, $0 \leq t \leq b$, and define the nonlinear operator, $M_b : \mathcal{B} \to \mathcal{B}$, by

$$ M_b u(t) = \begin{cases} \int_0^b G(b;t,s) f(s,u(s))\, ds, & 0 \leq t \leq b, \\ 0, & b \leq t < \infty. \end{cases} $$

The assumed conditions on f imply that M_b is Fréchet differentiable at $u = 0$, and $M_b'(0) = N_b$. To see this, employ the definition of Fréchet differentiable and the mean value theorem to obtain

$$\begin{aligned}\left|\int_0^b G(b;t,s)[f(s,u(s)) - f(s,0) - p(s)u(s)]\,ds\right| &= \left|\int_0^b G(b;t,s)[f(s,u(s)) - p(s)u(s)]\,ds\right| \\ &= \left|\int_0^b G(b;t,s)[f_u(s,c(s)) - p(s)]u(s)\,ds\right| \\ &\le \frac{b^{\alpha-1}}{\Gamma(\alpha)}\int_0^b |f_u(s,c(s)) - p(s)|\,ds\|u\|,\end{aligned}$$

where $0 \le c(s) \le u(s)$ for $0 \le s \le b$. Moreover, $M_b'(0) = N_b$.

By Theorem 4.3, $r(\bar{N}_{b_0}) = 1$, and by Theorem 4.2, $r(\bar{N}_b) > 1$ if $b_0 < b$. Moreover, since b_0 is the left focal point of (5.2), (1.2_b), Theorem 4.3 implies that if $b_0 < b$, and if $\bar{N}_b u = u$, $u \neq 0$, then $u \notin \mathcal{P}$. Thus, Condition $(\hat{A})$ implies that Theorem 5.1 applies, and there exists $u \in \mathcal{P} \setminus \{0\}$ such that $u = M_b u$.

Remark 5.1. One may question if Condition $(\hat{A})$ can be verified. Schmitt and Smith [4] addressed that question for the case $\alpha = 2$ and for the case of conjugate boundary conditions. Their calculations are readily modified to show that if for each $0 < b \le \beta$, there exist $K(b) > 0$, and $\gamma \in (0,1)$ such that

$$|f(t,u)| \le |u|^\gamma, \text{ for } |u| \ge K(b), \quad 0 \le t \le b,$$

then Condition $(\hat{A})$ is satisfied.

6. Results and discussion

In this paper, we considered a family high order left focal two−point BVPs for a Caputo fractional differential equation. Employing the theory of u_0−positive operators, the existence of a principal eigenvalue with corresponding eigenfunction living in the interior of a cone is obtained. Since the principal eigenvalue corresponds to the spectral radius of an associated integral operator, the left focal point is defined and studied in relation to the spectral radius of the associated integral operator. A fundamental part of the analysis is to invert the BVPs and observe properties of a family of Green's functions. The article closes with an application of a fixed point theorem to a nonlinear BVP.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

Jeffrey Neugebauer is a guest editor for *Electronic Research Archive* and was not involved in the editorial review or the decision to publish this article. The authors declare there is no conflicts of interest.

References

1. Z. Nehari, Nonlinear techniques for linear oscillation problems, *Trans. Am. Math. Soc.*, **210** (1975), 387–406. http://doi.org/10.1090/S0002-9947-1975-0372327-3
2. Z. Nehari, Green's functions and disconjugacy, *Arch. Ration. Mech. Anal.*, **62** (1976), 53–76. http://doi.org/10.1007/BF00251856
3. W. Coppel, *Disconjugacy*, in *Lecture Notes in Mathematics*, Springer-Verlag, New York/Berlin, 1971. https://doi.org/10.1007/BFb0058618
4. K. Schmitt, H. L. Smith, Positive solutions and conjugate points for systems of differential equations, *Nonlinear Anal.*, **2** (1978), 93–105. http://doi.org/10.1016/0362-546X(78)90045-7
5. P. W. Eloe, J. Henderson, Comparison of eigenvalues for a class of two-point boundary value problems, *Appl. Anal.*, **34** (1989), 25–34. https://doi.org/10.1080/00036818908839881
6. R. D. Gentry, C. C. Travis, Comparison of eigenvalues associated with linear differential equations of arbitrary order, *Trans. Am. Math. Soc.*, **223** (1976), 167–179. https://doi.org/10.1090/S0002-9947-1976-0425241-X
7. C. C. Travis, Comparison of eigenvalues for linear differential equations of order $2n$, *Trans. Am. Math. Soc.*, **177** (1973), 363–374. http://doi.org/10.1090/S0002-9947-1973-0316808-5
8. J. R. L. Webb, Uniqueness of the principal eigenvalue in nonlocal boundary value problems, *Discrete Contin. Dyn. Syst. - Ser. S* , **1** (2008), 177–186. http://doi.org/10.3934/dcdss.2008.1.177
9. C. J. Chyan, J. M. Davis, J. Henderson, W. K. C. Yin, Eigenvalue comparisons for differential equations on a measure chain, *Electron. J. Differ. Equations*, **1998** (1998), 1–7.
10. D. Hankerson, A. Peterson, Comparison of eigenvalues for focal point problems for nth order difference equations, *Differ. Integr. Equations*, **3** (1990), 363–380. https://doi.org/10.57262/die/1371586150
11. P. W. Eloe, J. T. Neugebauer, Existence and comparison of smallest eigenvalues for a fractional boundary value problem, *Electron. J. Differ. Equations*, **2014** (2014), 1–10.
12. P. W. Eloe, J. T. Neugebauer, Conjugate points for fractional differential equations, *Fract. Calc. Appl. Anal.*, **17** (2014), 11–18. https://doi.org/10.2478/s13540-014-0201-5
13. J. T. Neugebauer, Classifying first extremal points for a fractional boundary value problem with a fractional boundary condition, *Mediterr. J. Math.*, **14** (2017), 11. https://doi.org/10.1007/s00009-017-0974-y
14. J. Henderson, J. T. Neugebauer, First extremal point comparison for a fractional boundary value problem with a fractional boundary condition, *Proc. Am. Math. Soc.*, **147** (2019), 5323–5327. https://doi.org/10.1090/proc/14648
15. J. Henderson, N. Kosmatov, Eigenvalue comparison for fractional boundary value problems with the Caputo derivative, *Fract. Calc. Appl. Anal.*, **17** (2017), 872–880. https://doi.org/10.2478/s13540-014-0202-4
16. P. W. Eloe, Y. Li, On the first root of two-parametric Mittag–Leffler functions: a functional perspective, *Integr. Transforms Special Funct.*, **36** (2025), 776–806. https://doi.org/10.1080/10652469.2025.2455502

17. K. Diethelm, *The Analysis of Fractional Differential Equations. An Application-oriented Exposition Using Differential Operators of Caputo Type (Lecture Notes in Mathematics, 2004)*, Springer-Verlag, Berlin, 2010. https://doi.org/10.1007/978-3-642-14574-2
18. M. Krasnosel'skiĭ, *Positive Solutions of Operator Equations*, Fizmatgiz, Moscow, 1962; English Translation P. Noordhoff Ltd. Gronigen, The Netherlands, 1964.
19. R. D. Nussbaum, Periodic solutions of some nonlinear integral equations, in *Dynamical Systems: Proceedings of a University of Florida International Symposium*, Gainesville, FL, 1976. https://doi.org/10.1016/B978-0-12-083750-2.50021-7
20. M. G. Krein, M. A. Rutman, Linear operators leaving a cone invariant in a Banach space, *Am. Math. Soc. Transl.*, **1950** (1950), 128.
21. M. Keener, C. C. Travis, Positive cones and focal points for a class of nth order differential equations, *Trans. Am. Math. Soc.*, **237** (1978), 331–351. https://doi.org/10.1090/S0002-9947-1978-0479377-X
22. H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM Rev.*, **18** (1976), 620–709. https://doi.org/10.1137/1018114
23. K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, New York, 1985. https://doi.org/10.1007/978-3-662-00547-7