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Research article

On the existence of solutions for multi-term p-Laplacian fractional differential equations with anti-periodic boundary conditions

Wei Zhang*, Feiyan Xie and Jinbo Ni

School of Mathematics and Big Data, Anhui University of Science and Technology, Huainan 232001, China

* **Correspondence:** Email: zhangwei_azyw@163.com.

Abstract: This paper investigates the anti-periodic boundary value problem for multi-term fractional differential equations that involve the p-Laplacian operator. We establish existence results by separately applying the Leray-Schauder nonlinear alternative and the Krasnosel'skii's fixed point theorem. Finally, two illustrative examples are presented to demonstrate the effectiveness of the main results.

Keywords: multi-term fractional differential equation; anti-periodic boundary condition; fixed point theorem; existence

1. Introduction

Fractional differential equations (FDEs) have demonstrated strong application-driven momentum in recent years. Their distinctive modeling capability enables researchers to more accurately characterize the dynamical behavior of complex systems, particularly in the presence of long-memory effects, nonlocal interactions, and anomalous diffusion mechanisms. Numerous studies have shown that fractional models exhibit significant advantages in diverse fields such as robust control design in control engineering, electrochemical process modeling, financial market volatility analyses, and biomedical signal processing [1]. Compared with traditional integer-order models, fractional approaches provide a more precise fitting of experimental data, reveal the hidden nonlinear features of systems, and enhance the predictive performance. Therefore, conducting systematic qualitative analyses of FDEs is of great importance for theoretical research, while also laying a solid mathematical foundation for complex system modeling and applications.

A differential equation (DE) that involves more than one fractional-order derivative is referred to as a multi-term FDE. Compared with single-term fractional models, the multi-term formulation allows the incorporation of multiple fractional derivatives (FDs) of different orders, thereby providing a more comprehensive description of the multi-scale memory effects and nonlocal dynamical properties of complex systems. Typical examples include the Bagley-Torvik equation and the Basset equation, which have been widely applied in areas such as viscoelastic material dynamics and anomalous diffusion in fluid mechanics [2]. In recent years, the investigation into the existence and uniqueness (E&U) of solutions for multi-term fractional boundary value problems (FBVPs) has drawn significant research interest and has seen continuous advancements. Recent progress in the study of multi-term FBVPs can be categorized by the types of equations considered. Systems of nonlinear multi-term FDEs have been investigated in several works [3–5]. Further contributions addressed multi-term fractional pantograph equations [6]. In addition, FBVPs that involve generalized fractional derivatives, such as ψ-Caputo and ψ-Hilfer operators, have been analyzed [7, 8]. For example, Ahmad et al. [5] investigated a coupled system of nonlinear multi-term FDEs subject to anti-periodic type coupled nonlocal boundary conditions (BCs):

$$\begin{cases} \mu_1 {}^C\mathscr{D}^{\epsilon_1}_{0+}\mu(\zeta) + \mu_2 {}^C\mathscr{D}^{\varepsilon_1}_{0+}\mu(\zeta) = \mathfrak{f}(\zeta, \mu(\zeta), \nu(\zeta)), \quad \zeta \in (0, 1), \\ \mu_3 {}^C\mathscr{D}^{\epsilon_2}_{0+}\nu(\zeta) + \mu_4 {}^C\mathscr{D}^{\varepsilon_2}_{0+}\nu(\zeta) = \mathfrak{g}(\zeta, \mu(\zeta), \nu(\zeta)), \quad \zeta \in (0, 1), \\ \mu(0) + \mu(1) = \sum\limits_{i=1}^{m} a_i \nu(\eta_i),\ \mu'(0) + \mu'(1) = \sum\limits_{i=1}^{m} b_i \nu'(\eta_i), \\ \nu(0) + \nu(1) = \sum\limits_{i=1}^{m} c_i \mu(\eta_i),\ \nu'(0) + \nu'(1) = \sum\limits_{i=1}^{m} d_i \mu'(\eta_i), \end{cases} \tag{1.1}$$

where $1 < \varepsilon_i < \epsilon_i$, $1 < \epsilon_i \le 2$, ${}^C\mathscr{D}^{\epsilon_i}_{0+}$, and ${}^C\mathscr{D}^{\varepsilon_i}_{0+}$ denote the Caputo FD operators of orders ϵ_i and ε_i, respectively, and $\mathfrak{f}, \mathfrak{g} : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions. By applying the Banach contraction mapping principle (BCMP), Krasnosel'skii's fixed point theorem (KFPT), and the Leray-Schauder alternative (LSA), the authors established E&U results for the boundary value problems (BVPs) (1.1).

Xu et al. [7] investigated two-term FDEs with Dirichlet BCs:

$$\begin{cases} {}^C\mathscr{D}^{\epsilon;\psi}_{0+}\mu(\zeta) - \xi {}^C\mathscr{D}^{\varepsilon;\psi}_{0+}\mu(\zeta) + \mathfrak{f}(\zeta, \mu(\zeta)) = 0, \quad \zeta \in (0, 1), \\ \mu(0) = \mu(1) = 0, \end{cases} \tag{1.2}$$

where $0 < \varepsilon < \epsilon$, $1 < \epsilon < 2$, $\psi(0) = 0, \psi(1) = 1$, ${}^C\mathscr{D}^{\epsilon;\psi}_{0+}$, and ${}^C\mathscr{D}^{\varepsilon;\psi}_{0+}$ are the ψ-Caputo FDs, and $\mathfrak{f} : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous. The authors applied the LSA and the BCMP to establish E&U results for the BVP (1.2), and subsequently investigated the Hyers-Ulam stability of the corresponding solutions.

On the other hand, the study of p-Laplacian DEs originates from mathematical modeling of nonlinear diffusion phenomena, such phenomena as gas flow in porous media, and turbulence challenges. Since the 1980s, the p-Laplacian operator $\phi_p(u) = |u|^{p-2}u$ $(u \neq 0)$, $\phi_p(0) = 0$, $p > 1$, has provided a unified framework for the analysis of various nonlinear DEs [9]. When $p = 2$, it reduces to the linear case, while for a general p, it captures essential features of nonlinear diffusion, non-Newtonian fluid dynamics, and porous medium flows [10, 11]. In view of its significant applications, p-Laplacian FBVPs have long been regarded as one of the major research directions that has attracted sustained attention from scholars, with extensive studies devoted to the E&U and qualitative properties of solutions under different BCs, including anti-periodic BCs [12–14], integral BCs [15, 16], mixed two-point and Sturm-Liouville type BCs [17, 18], and multi-point BCs [19, 20]. Among these contributions, Chen and Liu [12] studied the following p-Laplacian FDE with anti-periodic BCs:

$$\begin{cases} {}^C\mathscr{D}^{\varepsilon}_{0+}\phi_p({}^C\mathscr{D}^{\epsilon}_{0+}\mathfrak{x}(\zeta)) = \mathfrak{f}(\zeta,\mathfrak{x}(\zeta)), \quad \zeta\in(0,1),\\ \mathfrak{x}(0) = -\mathfrak{x}(1), \quad {}^C\mathscr{D}^{\epsilon}_{0+}\mathfrak{x}(0) = -{}^C\mathscr{D}^{\epsilon}_{0+}\mathfrak{x}(1),\end{cases} \tag{1.3}$$

where $0<\epsilon,\varepsilon\leq 1$, $1<\epsilon+\varepsilon\leq 2$, ${}^C\mathscr{D}^{\epsilon}_{0+}$, and ${}^C\mathscr{D}^{\varepsilon}_{0+}$ represent the Caputo FDs, and $\mathfrak{f}:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous. The authors established the existence of solutions for the BVP (1.3) by applying Schaefer's fixed point theorem (SFPT).

Dien and Viet [13] studied a class of p-Laplacian fractional Langevin equations subject to anti-periodic type BCs given by the following:

$$\begin{cases} {}^C\mathscr{D}^{\varepsilon;\psi}_{a+}\phi_p[({}^C\mathscr{D}^{\epsilon;\psi}_{a+}+\mu)\mathfrak{x}(\zeta)] = \mathfrak{f}(\zeta,\psi,\mathfrak{x}(\zeta)), \quad \zeta\in(a,b),\\ \sigma_1\mathfrak{x}(a)+\sigma_2\mathfrak{x}(b)=0, \quad \sigma_1{}^C\mathscr{D}^{\epsilon;\psi}_{a+}\mathfrak{x}(a)+\sigma_2{}^C\mathscr{D}^{\epsilon;\psi}_{a+}\mathfrak{x}(b)=0,\end{cases} \tag{1.4}$$

where $0<\epsilon,\varepsilon\leq 1$, ${}^C\mathscr{D}^{\epsilon;\psi}_{0+}$, and ${}^C\mathscr{D}^{\varepsilon;\psi}_{0+}$ are defined as the ψ-Caputo FDs, and they established E&U results for the BVP (1.4) by applying SFPT, the Leray-Schauder nonlinear alternative (LSNA), and the BCMP.

We are not aware of extensive research on multi-term FBVPs that incorporate the p-Laplacian operator. Inspired by the studies mentioned above, in this paper, we further study a class of multi-term p-Laplacian FDEs with anti-periodic BCs. To be more precise, we study the subsequent multi-term p-Laplacian FBVP:

$$\begin{cases} \xi{}^C\mathscr{D}^{\epsilon}_{0+}\phi_p({}^C\mathscr{D}^{\gamma}_{0+}\mathfrak{x}(\zeta))+(1-\xi){}^C\mathscr{D}^{\varepsilon}_{0+}\phi_p({}^C\mathscr{D}^{\gamma}_{0+}\mathfrak{x}(\zeta))=\mathfrak{f}(\zeta,\mathfrak{x}(\zeta),{}^C\mathscr{D}^{\gamma}_{0+}\mathfrak{x}(\zeta)), \ \zeta\in(0,\iota),\\ \mathfrak{x}(0)=-\mathfrak{x}(\iota), \ {}^C\mathscr{D}^{\gamma}_{0+}\mathfrak{x}(0)=-{}^C\mathscr{D}^{\gamma}_{0+}\mathfrak{x}(\iota), \ (\phi_p({}^C\mathscr{D}^{\gamma}_{0+}\mathfrak{x}(0)))'=-(\phi_p({}^C\mathscr{D}^{\gamma}_{0+}\mathfrak{x}(\iota)))',\end{cases} \tag{1.5}$$

where $0<\varepsilon<1+\varepsilon<\epsilon<2$, $0<\gamma<1$, and ${}^C\mathscr{D}^{\rho}_{0+}$ denotes the Caputo FD operator of order $\rho(\rho=\epsilon,\varepsilon,\gamma)$, $\mathfrak{f}\in C([0,\iota]\times\mathbb{R}^2,\mathbb{R})$, $\phi_p(\varrho)=|\varrho|^{p-2}\varrho(\varrho\neq 0)$, $\phi_p(0)=0$ is the p-Laplacian operator, $\iota>0$, $0\neq\xi\in\mathbb{R}$ satisfying $\Delta=2a_1-\iota a_2\neq 0$, and

$$a_1 = 2+\frac{(1-\xi)\iota^{\epsilon-\varepsilon}}{\xi\Gamma(\epsilon-\varepsilon+1)}, \quad a_2=\frac{(1-\xi)\iota^{\epsilon-\varepsilon-1}}{\xi\Gamma(\epsilon-\varepsilon)}.$$

The main contributions of this paper, which build upon and extend the existing studies discussed above, are highlighted as follows.

- Extension of existing results: we extend the study of multi-term FBVPs discussed in the literature (e.g., [4,5,7]) to the case that involves the p-Laplacian operator, thus advancing the study of FDEs with more general nonlinear structures.

- Generalization from single-term to multi-term: we generalize previous works on BVPs for single-term FDEs with the p-Laplacian operator (e.g., [12]) to the multi-term case, thus offering a natural continuation of the existing results.

- Existence results via multiple fixed point techniques: for Problem (1.5), we establish distinct existence results of solutions by employing different fixed point approaches, namely the LSNA and KFPT.

- Overcoming difficulties due to the quasi-linear characteristics of the p-Laplacian operator: for Problem (1.5), the quasi-linear characteristics of the p-Laplacian operator prevent a direct reduction to an integral equation fixed point problem, unlike the case without the p-Laplacian. We resolve this difficulty and successfully utilize fixed point theorems to establish the existence of solutions.

The remainder of this paper is structured as follows: Section 2 recalls some preliminaries of fractional calculus and introduces the main analytical tools employed in this work, namely the LSNA and KFPT; in Section 3, under specific conditions imposed on the nonlinear term, we apply these two fixed point theorems to establish the existence results for Problem (1.5); to demonstrate the applicability of the main results, Section 4 presents two illustrative examples; and finally, the paper is concluded in Section 5.

2. Preliminaries

Definition 2.1. ([21]) The Riemann-Liouville fractional integral of order κ of a function $\varphi : (0, +\infty) \to \mathbb{R}$ is given by the following:

$$\mathscr{I}^{\kappa}_{0+}\varphi(\zeta) = \frac{1}{\Gamma(\kappa)} \int_0^{\zeta} (\zeta - \varrho)^{\kappa-1}\varphi(\varrho)d\varrho,$$

provided that the right-hand side is defined pointwise on $(0, +\infty)$.

Definition 2.2. ([21]) For $\varphi : (0, +\infty) \to \mathbb{R}$, the Caputo FD of order κ is defined as follows:

$${}^{C}\mathscr{D}^{\kappa}_{0+}\varphi(\zeta) = \mathscr{I}^{n-\kappa}_{0+}\frac{d^n\varphi(\zeta)}{d\zeta^n} = \frac{1}{\Gamma(n-\kappa)} \int_0^{\zeta} (\zeta - \varrho)^{n-\kappa-1}\varphi^{(n)}(\varrho)d\varrho,$$

where $n = [\kappa] + 1,$ provided the right-hand side is pointwise defined on $(0, +\infty)$.

Lemma 2.1. ([21]) Let $\kappa > 0$ and $\iota > 0$; if $\varphi \in AC^n[0, \iota]$, then

$$\mathscr{I}^{\kappa}_{0+}\,{}^{C}\mathscr{D}^{\kappa}_{0+}\varphi(\zeta) = \varphi(\zeta) + \sum_{i=0}^{n-1} c_i\zeta_i,$$

where $c_i = -\frac{\varphi^{(i)}(0)}{i!},\ i = 0, 1, 2, \cdots, n-1$, and $n = [\kappa] + 1$.

Lemma 2.2. ([21]) Let $\iota, \kappa > 0$, and let $\varphi(\zeta) \in C[0, \iota]$; then, the following identities hold:

$$\mathscr{I}^{\iota}_{0+}\mathscr{I}^{\kappa}_{0+}\varphi(\zeta) = \mathscr{I}^{\iota+\kappa}_{0+}\varphi(\zeta), \quad \mathscr{I}^{\iota}_{0+}\zeta^{\kappa-1} = \frac{\Gamma(\kappa)}{\Gamma(\iota+\kappa)}\zeta^{\iota+\kappa-1},$$
$${}^{C}\mathscr{D}^{\iota}_{0+}\mathscr{I}^{\iota}_{0+}\varphi(t) = \varphi(\zeta), \quad {}^{C}\mathscr{D}^{\iota}_{0+}t^{\kappa-1} = \frac{\Gamma(\kappa)}{\Gamma(\kappa-\iota)}\zeta^{\kappa-\iota-1}.$$

Theorem 2.1. (LSNA [22]) Consider a Banach space X, an open bounded subset $\Omega \in X$ with $\theta \in \Omega$, and a completely continuous operator $A : \overline{\Omega} \to X$. Then, either A possesses a fixed point in Ω, or one can find $\mu \in \partial\Omega$ and $\nu \in (0, 1)$ such that $\mu = \nu A\mu$.

Theorem 2.2. (KFPT [22]) Suppose M is a nonempty, closed, convex, and bounded subset of a Banach space X. Consider two operators A_1 and A_2 that satisfy the following:

(a) $A_1\mathfrak{x} + A_2\mathfrak{y} \in M$ for all $\mathfrak{x}, \mathfrak{y} \in M$;

(b) A_1 is a completely continuous operator;

(c) A_2 is a contraction mapping.

Then, there exists $\mathfrak{z} \in M$ so that $\mathfrak{z} = A_1\mathfrak{z} + A_2\mathfrak{z}$.

3. Main result

It should be emphasized that the p-Laplacian operator is a quasilinear operator, which makes it difficult to directly employ the properties of fractional calculus to equivalently transform the BVP (1.5) into a fixed point problem of an integral equation. Observe that the p-Laplacian operator $\phi_p(\cdot)$ is invertible, with the inverse denoted by $\phi_q(\cdot)$, where p and q are conjugate exponents that satisfy $\frac{1}{p}+\frac{1}{q}=1$. Accordingly, by setting $\phi_p({}^C D^{\gamma}_{0+}\mathfrak{x}(\zeta)) = \mathfrak{y}(\zeta)$, one immediately obtains ${}^C D^{\gamma}_{0+}\mathfrak{x}(\zeta) = \phi_q(\mathfrak{y}(\zeta))$. Therefore, we consider reformulating the BVP (1.5) equivalently as the following system:

$$\begin{cases} {}^C\mathscr{D}^{\gamma}_{0+}\mathfrak{x}(\zeta) = \phi_q(\mathfrak{y}(\zeta)), \quad \zeta \in (0,\iota), \\ \xi\,{}^C\mathscr{D}^{\epsilon}_{0+}\mathfrak{y}(\zeta) + (1-\xi)\,{}^C\mathscr{D}^{\varepsilon}_{0+}\mathfrak{y}(\zeta) = \mathfrak{f}\left(\zeta, \mathfrak{x}(\zeta), \phi_q(\mathfrak{y}(\zeta))\right), \quad \zeta \in (0,\iota), \\ \mathfrak{x}(0) = -\mathfrak{x}(\iota), \quad \mathfrak{y}(0) = -\mathfrak{y}(\iota), \quad \mathfrak{y}'(0) = -\mathfrak{y}'(\iota). \end{cases} \tag{3.1}$$

It is clear that if u=$(\mathfrak{x}, \mathfrak{y})$ is a solution of Problem (3.1), then $\mathfrak{x}$ must be a solution of Problem (1.5). Therefore, to prove that Problem (1.5) admits solutions, it is sufficient to demonstrate the existence of solutions to the system BVP (3.1).

Consider the Banach space $Z = C([0,\iota],\mathbb{R})$ with the norm

$$\|\mathfrak{z}\|_\infty = \max_{\zeta\in[0,\iota]} |\mathfrak{z}(\zeta)|.$$

Define the space

$$U = \{u = (\mathfrak{x}, \mathfrak{y}) : \mathfrak{x}, \mathfrak{y} \in Z\},$$

equipped with the norm

$$\|u\|_U = \|\mathfrak{x}\|_\infty + \|\mathfrak{y}\|_\infty .$$

From the perspective of a linear functional analysis, proving that U is a Banach space is straightforward.

Lemma 3.1. For $\mathfrak{g}(\zeta) \in C([0,\iota],\mathbb{R})$, the unique solution of the linear system

$$\begin{cases} {}^C\mathscr{D}^{\gamma}_{0+}\mathfrak{x}(\zeta) = \phi_q(\mathfrak{y}(\zeta)), \quad \zeta \in (0,\iota), \\ \xi\,{}^C\mathscr{D}^{\epsilon}_{0+}\mathfrak{y}(\zeta) + (1-\xi)\,{}^C\mathscr{D}^{\varepsilon}_{0+}\mathfrak{y}(\zeta) = \mathfrak{g}(\zeta), \quad \zeta \in (0,\iota), \\ \mathfrak{x}(0) = -\mathfrak{x}(\iota), \quad \mathfrak{y}(0) = -\mathfrak{y}(\iota), \quad \mathfrak{y}'(0) = -\mathfrak{y}'(\iota), \end{cases} \tag{3.2}$$

is formulated as a pair of integral equations

$$\begin{aligned} \mathfrak{x}(\zeta) &= \mathscr{I}^{\gamma}_{0+}\phi_q(\mathfrak{y}(\zeta)) - \frac{1}{2}\mathscr{I}^{\gamma}_{0+}\phi_q(\mathfrak{y}(\zeta))|_{\zeta=\iota} \\ &= \frac{1}{\Gamma(\gamma)}\int_0^{\zeta} (\zeta-\varrho)^{\gamma-1}\phi_q(\mathfrak{y}(\varrho))d\varrho - \frac{1}{2\Gamma(\gamma)}\int_0^{\iota} (\iota-\varrho)^{\gamma-1}\phi_q(\mathfrak{y}(\varrho))d\varrho, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
\mathfrak{y}(\zeta)=&\frac{1}{\Delta}\left[1+\frac{(1-\xi)\zeta^{\epsilon-\varepsilon}}{\xi\Gamma(\epsilon-\varepsilon+1)}\right]\left\{\frac{2}{\xi}[(1-\xi)\mathscr{I}^{\epsilon-\varepsilon}_{0+}\mathfrak{y}(\zeta)-\mathscr{I}^{\epsilon}_{0+}\mathfrak{g}(\zeta)]|_{\zeta=\iota}-\frac{\iota}{\xi}[(1-\xi)\mathscr{I}^{\epsilon-\varepsilon-1}_{0+}\mathfrak{y}(\zeta)\right.\\
&\left.-\mathscr{I}^{\epsilon-1}_{0+}\mathfrak{g}(\zeta)]|_{\zeta=\iota}\right\}+\frac{\zeta}{\Delta}\left\{\frac{a_1}{\xi}[(1-\xi)\mathscr{I}^{\epsilon-\varepsilon-1}_{0+}\mathfrak{y}(\zeta)-\mathscr{I}^{\epsilon-1}_{0+}\mathfrak{g}(\zeta)]|_{\zeta=\iota}\right.\\
&\left.-\frac{a_2}{\xi}[(1-\xi)\mathscr{I}^{\epsilon-\varepsilon}_{0+}\mathfrak{y}(\zeta)-\mathscr{I}^{\epsilon}_{0+}\mathfrak{g}(\zeta)]|_{\zeta=\iota}\right\}-\frac{1-\xi}{\xi}\mathscr{I}^{\epsilon-\varepsilon}_{0+}\mathfrak{y}(\zeta)+\frac{1}{\xi}\mathscr{I}^{\epsilon}_{0+}\mathfrak{g}(\zeta)\\
=&\frac{1}{\Delta}\left[1+\frac{(1-\xi)\zeta^{\epsilon-\varepsilon}}{\xi\Gamma(\epsilon-\varepsilon+1)}\right]\left\{\frac{2}{\xi}\left[\frac{1-\xi}{\Gamma(\epsilon-\varepsilon)}\int_0^\iota(\iota-\varrho)^{\epsilon-\varepsilon-1}\mathfrak{y}(\varrho)d\varrho\right.\right.\\
&\left.-\frac{1}{\Gamma(\epsilon)}\int_0^\iota(\iota-\varrho)^{\epsilon-1}\mathfrak{g}(\varrho)d\varrho\right]-\frac{\iota}{\xi}\left[\frac{1-\xi}{\Gamma(\epsilon-\varepsilon-1)}\int_0^\iota(\iota-\varrho)^{\epsilon-\varepsilon-2}\mathfrak{y}(\varrho)d\varrho\right.\\
&\left.\left.-\frac{1}{\Gamma(\epsilon-1)}\int_0^\iota(\iota-\varrho)^{\epsilon-2}\mathfrak{g}(\varrho)d\varrho\right]\right\}+\frac{\zeta}{\Delta}\left\{\frac{a_1}{\xi}\left[\frac{1-\xi}{\Gamma(\epsilon-\varepsilon-1)}\int_0^\iota(\iota-\varrho)^{\epsilon-\varepsilon-2}\mathfrak{y}(\varrho)d\varrho\right.\right.\\
&\left.-\frac{1}{\Gamma(\epsilon-1)}\int_0^\iota(\iota-\varrho)^{\epsilon-2}\mathfrak{g}(\varrho)d\varrho\right]-\frac{a_2}{\xi}\left[\frac{1-\xi}{\Gamma(\epsilon-\varepsilon)}\int_0^\iota(\iota-\varrho)^{\epsilon-\varepsilon-1}\mathfrak{y}(\varrho)d\varrho\right.\\
&\left.\left.-\frac{1}{\Gamma(\epsilon)}\int_0^\iota(\iota-\varrho)^{\epsilon-1}\mathfrak{g}(\varrho)d\varrho\right]\right\}-\frac{1-\xi}{\xi\Gamma(\epsilon-\varepsilon)}\int_0^\zeta(\zeta-\varrho)^{\epsilon-\varepsilon-1}\mathfrak{y}(\varrho)d\varrho\\
&+\frac{1}{\xi\Gamma(\epsilon)}\int_0^\zeta(\zeta-\varrho)^{\epsilon-1}\mathfrak{g}(\varrho)d\varrho.
\end{aligned}\tag{3.4}$$

where $\Delta=2a_1-\iota a_2\neq0$, and

$$a_1=2+\frac{(1-\xi)\iota^{\epsilon-\varepsilon}}{\xi\Gamma(\epsilon-\varepsilon+1)},\quad a_2=\frac{(1-\xi)\iota^{\epsilon-\varepsilon-1}}{\xi\Gamma(\epsilon-\varepsilon)}.$$

Proof. According to Lemma 2.1, applying the operator $\mathscr{I}^{\gamma}_{0+}$ on both sides of the first equation of (3.2) yields the following:

$$\mathfrak{x}(\zeta)=\mathfrak{x}(0)+\mathscr{I}^{\gamma}_{0+}\phi_q(\mathfrak{y}(\zeta)).\tag{3.5}$$

Using the BC $\mathfrak{x}(0)=-\mathfrak{x}(\iota)$, then

$$\mathfrak{x}(0)=-\frac{1}{2}\mathscr{I}^{\gamma}_{0+}\phi_q(\mathfrak{y}(\zeta))|_{\zeta=\iota}.\tag{3.6}$$

Substituting (3.6) in (3.5) yields (3.3). On the other hand, according to Lemmas 2.1 and 2.2, by applying the operator $\mathscr{I}^{\epsilon}_{0+}$ on both sides of the second equation of (3.2), we deduce the following:

$$\mathfrak{y}(\zeta)=\mathfrak{y}(0)+\mathfrak{y}'(0)\zeta-\frac{1-\xi}{\xi}\left[\mathscr{I}^{\epsilon-\varepsilon}_{0+}\mathfrak{y}(\zeta)-\frac{\mathfrak{y}(0)\zeta^{\epsilon-\varepsilon}}{\Gamma(\epsilon-\varepsilon+1)}\right]+\frac{1}{\xi}\mathscr{I}^{\epsilon}_{0+}\mathfrak{g}(\zeta).\tag{3.7}$$

Therefore,

$$\mathfrak{y}'(\zeta)=\mathfrak{y}'(0)-\frac{1-\xi}{\xi}\left[\mathscr{I}^{\epsilon-\varepsilon-1}_{0+}\mathfrak{y}(\zeta)-\frac{\mathfrak{y}(0)\zeta^{\epsilon-\varepsilon-1}}{\Gamma(\epsilon-\varepsilon)}\right]+\frac{1}{\xi}\mathscr{I}^{\epsilon-1}_{0+}\mathfrak{g}(\zeta).$$

Form the BCs $\mathfrak{y}(0)=-\mathfrak{y}(\iota)$, $\mathfrak{y}'(0)=-\mathfrak{y}'(\iota)$, it follows that

$$\begin{cases}a_1\mathfrak{y}(0)+\iota\mathfrak{y}'(0)=\dfrac{1-\xi}{\xi}\mathscr{I}^{\epsilon-\varepsilon}_{0+}\mathfrak{y}(\zeta)|_{\zeta=\iota}-\dfrac{1}{\xi}\mathscr{I}^{\epsilon}_{0+}\mathfrak{g}(\zeta)|_{\zeta=\iota},\\ a_2\mathfrak{y}(0)+2\mathfrak{y}'(0)=\dfrac{1-\xi}{\xi}\mathscr{I}^{\epsilon-\varepsilon-1}_{0+}\mathfrak{y}(\zeta)|_{\zeta=\iota}-\dfrac{1}{\xi}\mathscr{I}^{\epsilon-1}_{0+}\mathfrak{g}(\zeta)|_{\zeta=\iota},\end{cases}$$

that is,

$$\begin{aligned}\mathfrak{y}(0)&=\frac{1}{\Delta}\left\{\frac{2}{\xi}[(1-\xi)\mathscr{I}_{0+}^{\epsilon-\varepsilon}\mathfrak{y}(\zeta)-\mathscr{I}_{0+}^{\epsilon}\mathfrak{g}(\zeta)]|_{\zeta=\iota}-\frac{\iota}{\xi}[(1-\xi)\mathscr{I}_{0+}^{\epsilon-\varepsilon-1}\mathfrak{y}(\zeta)-\mathscr{I}_{0+}^{\epsilon-1}\mathfrak{g}(\zeta)]|_{\zeta=\iota}\right\},\\ \mathfrak{y}'(0)&=\frac{1}{\Delta}\left\{\frac{a_1}{\xi}[(1-\xi)\mathscr{I}_{0+}^{\epsilon-\varepsilon-1}\mathfrak{y}(\zeta)-\mathscr{I}_{0+}^{\epsilon-1}\mathfrak{g}(\zeta)]|_{\zeta=\iota}-\frac{a_2}{\xi}[(1-\xi)\mathscr{I}_{0+}^{\epsilon-\varepsilon}\mathfrak{y}(\zeta)-\mathscr{I}_{0+}^{\epsilon}\mathfrak{g}(\zeta)]|_{\zeta=\iota}\right\}.\end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.7) yields (3.4). Conversely, by Lemma 2.2, it can be established that $(\mathfrak{x}(\zeta), \mathfrak{y}(\zeta))$ satisfies (3.2). □

According to Lemma 3.1, we define the following operator $\mathfrak{I} : U \to U$,

$$\mathfrak{I}(\mathfrak{x}(\zeta), \mathfrak{y}(\zeta)) = (\mathfrak{I}_1(\mathfrak{x}, \mathfrak{y})(\zeta), \mathfrak{I}_2(\mathfrak{x}, \mathfrak{y})(\zeta)),$$

where

$$\mathfrak{I}_1(\mathfrak{x}, \mathfrak{y})(\zeta) = \frac{1}{\Gamma(\gamma)}\int_0^{\zeta} (\zeta-\varrho)^{\gamma-1}\phi_q(\mathfrak{y}(\varrho))d\varrho - \frac{1}{2\Gamma(\gamma)}\int_0^{\iota} (\iota-\varrho)^{\gamma-1}\phi_q(\mathfrak{y}(\varrho))d\varrho,$$

and

$$\begin{aligned}\mathfrak{I}_2(\mathfrak{x}, \mathfrak{y})(\zeta) = {}& \frac{1}{\Delta}\left[1+\frac{(1-\xi)\zeta^{\epsilon-\varepsilon}}{\xi\Gamma(\epsilon-\varepsilon+1)}\right]\left\{\frac{2}{\xi}\left[\frac{1-\xi}{\Gamma(\epsilon-\varepsilon)}\int_0^{\iota}(\iota-\varrho)^{\epsilon-\varepsilon-1}\mathfrak{y}(\varrho)d\varrho\right.\right.\\ &\left.-\frac{1}{\Gamma(\epsilon)}\int_0^{\iota}(\iota-\varrho)^{\epsilon-1}\mathfrak{f}(\varrho,\mathfrak{x}(\varrho),\phi_q(\mathfrak{y}(\varrho)))d\varrho\right]-\frac{\iota}{\xi}\left[\frac{1-\xi}{\Gamma(\epsilon-\varepsilon-1)}\int_0^{\iota}(\iota-\varrho)^{\epsilon-\varepsilon-2}\mathfrak{y}(\varrho)d\varrho\right.\\ &\left.\left.-\frac{1}{\Gamma(\epsilon-1)}\int_0^{\iota}(\iota-\varrho)^{\epsilon-2}\mathfrak{f}(\varrho,\mathfrak{x}(\varrho),\phi_q(\mathfrak{y}(\varrho)))d\varrho\right]\right\}+\frac{\zeta}{\Delta}\left\{\frac{a_1}{\xi}\left[\frac{1-\xi}{\Gamma(\epsilon-\varepsilon-1)}\int_0^{\iota}(\iota-\varrho)^{\epsilon-\varepsilon-2}\mathfrak{y}(\varrho)d\varrho\right.\right.\\ &\left.-\frac{1}{\Gamma(\epsilon-1)}\int_0^{\iota}(\iota-\varrho)^{\epsilon-2}\mathfrak{f}(\varrho,\mathfrak{x}(\varrho),\phi_q(\mathfrak{y}(\varrho)))d\varrho\right]-\frac{a_2}{\xi}\left[\frac{1-\xi}{\Gamma(\epsilon-\varepsilon)}\int_0^{\iota}(\iota-\varrho)^{\epsilon-\varepsilon-1}\mathfrak{y}(\varrho)d\varrho\right.\\ &\left.\left.-\frac{1}{\Gamma(\epsilon)}\int_0^{\iota}(\iota-\varrho)^{\epsilon-1}\mathfrak{f}(\varrho,\mathfrak{x}(\varrho),\phi_q(\mathfrak{y}(\varrho)))d\varrho\right]\right\}-\frac{1-\xi}{\xi\Gamma(\epsilon-\varepsilon)}\int_0^{\zeta}(\zeta-\varrho)^{\epsilon-\varepsilon-1}\mathfrak{y}(\varrho)d\varrho\\ &+\frac{1}{\xi\Gamma(\epsilon)}\int_0^{\zeta}(\zeta-\varrho)^{\epsilon-1}\mathfrak{f}(\varrho,\mathfrak{x}(\varrho),\phi_q(\mathfrak{y}(\varrho)))d\varrho.\end{aligned}$$

Then, the existence of a solution to Eq (3.1) is equivalent to $\mathfrak{I}$ that possesses a fixed point.

For convenience in what follows, we introduce the following notation:

$$\begin{aligned}b_1 &= \frac{|1-\xi|\iota^{\epsilon-\varepsilon}}{|\xi|\Gamma(\epsilon-\varepsilon+1)},\ b_2 = \frac{\iota^{\epsilon}}{|\xi|\Gamma(\epsilon+1)},\\ b_3 &= \frac{|1-\xi|\iota^{\epsilon-\varepsilon}}{|\xi|\Gamma(\epsilon-\varepsilon)},\ b_4 = \frac{\iota^{\epsilon}}{|\xi|\Gamma(\epsilon)},\ b_5 = \frac{\iota^{\gamma}}{\Gamma(\gamma+1)},\\ \Delta_1 &= \frac{1}{|\Delta|}[(1+b_1)(2b_1+b_3)+|a_1|b_3+\iota|a_2|b_1]+b_1,\\ \Delta_2 &= \frac{1}{|\Delta|}[(1+b_1)(2b_2+b_4)+|a_1|b_4+\iota|a_2|b_2]+b_2.\end{aligned}$$

Now, we state our first main result, which uses the LSNA to prove the existence of solutions to the BVP (3.1).

Theorem 3.1. Assume that $\mathfrak{f} : [0,\iota] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and satisfies the following conditions:

(C_1) There exist nonnegative continuous functions $a(\zeta), b(\zeta), c(\zeta) \in C[0,\iota]$ such that

$$|\mathfrak{f}(\zeta, \mathfrak{x}(\zeta), \phi_q(\mathfrak{y}(\zeta)))| \leq a(\zeta) + b(\zeta)|\mathfrak{x}(\zeta)|^{p-1} + c(\zeta)|\mathfrak{y}(\zeta)|;$$

(C_2) There exists a constant $M > 0$ such that

$$\frac{\Delta_2 a + \Delta_2 b M^{p-1} + 3 \times 2^{-1} b_5 M^{q-1}}{(1 - \Delta_1 - \Delta_2 c)M} < 1,$$

where

$$a = \max_{\zeta\in[0,\iota]} |a(\zeta)|,\ b = \max_{\zeta\in[0,\iota]} |b(\zeta)|,\ c = \max_{\zeta\in[0,\iota]} |c(\zeta)|.$$

Then, there exists at least one solution to the coupled system (3.1) on $[0,\iota]$, provided that

$$\Delta_2 c + \Delta_1 < 1.$$

Proof. For any $\rho > 0$, define a subset of U by the following:

$$B_\rho = \{(\mathfrak{x},\mathfrak{y}) \in U : \|(\mathfrak{x},\mathfrak{y})\|_U < \rho\}.$$

We split the proof into two steps.

Step 1. Prove that the operator $\Im : \bar{B}_\rho \to U$ is completely continuous. Since $\mathfrak{f}$ and $\phi_q(\cdot)$ are continuous, it follows that the operator $\Im$ is continuous. Hence, it remains to establish that $\Im$ is compact on $\bar{B}_\rho$. To this end, we first show that $\Im$ is uniformly bounded on $\bar{B}_\rho$. Indeed, by Condition (C_1), for any $\zeta \in [0,\iota]$ and $(\mathfrak{x},\mathfrak{y}) \in \bar{B}_\rho$, we have

$$|\Im_1(\mathfrak{x},\mathfrak{y})(\zeta)| \leq \frac{1}{\Gamma(\gamma)} \int_0^\zeta (\zeta - \varrho)^{\gamma-1} |\phi_q(\mathfrak{y}(\varrho))| d\varrho + \frac{1}{2\Gamma(\gamma)} \int_0^\iota (\iota - \varrho)^{\gamma-1} |\phi_q(\mathfrak{y}(\varrho))| d\varrho \leq \frac{3}{2} b_5 \rho^{q-1},$$

and

$$\begin{aligned}
|\Im_2(\mathfrak{x},\mathfrak{y})(\zeta)| \leq{}& \frac{1}{|\Delta|}\left[1 + \frac{|1-\xi|\iota^{\epsilon-\varepsilon}}{|\xi|\Gamma(\epsilon-\varepsilon+1)}\right]\left\{\frac{2}{|\xi|}\left[\frac{|1-\xi|}{\Gamma(\epsilon-\varepsilon)}\int_0^\iota (\iota-\varrho)^{\epsilon-\varepsilon-1}|\mathfrak{y}(\varrho)|d\varrho\right.\right.\\
&\left. + \frac{1}{\Gamma(\epsilon)}\int_0^\iota (\iota-\varrho)^{\epsilon-1}|\mathfrak{f}(\varrho,\mathfrak{x}(\varrho),\phi_q(\mathfrak{y}(\varrho)))|d\varrho\right] + \frac{\iota}{|\xi|}\left[\frac{|1-\xi|}{\Gamma(\epsilon-\varepsilon-1)}\int_0^\iota (\iota-\varrho)^{\epsilon-\varepsilon-2}|\mathfrak{y}(\varrho)|d\varrho\right.\\
&\left.\left. + \frac{1}{\Gamma(\epsilon-1)}\int_0^\iota (\iota-\varrho)^{\epsilon-2}|\mathfrak{f}(\varrho,\mathfrak{x}(\varrho),\phi_q(\mathfrak{y}(\varrho)))|d\varrho\right]\right\} + \frac{\iota}{|\Delta|}\left\{\frac{|a_1|}{|\xi|}\left[\frac{|1-\xi|}{\Gamma(\epsilon-\varepsilon-1)}\int_0^\iota (\iota-\varrho)^{\epsilon-\varepsilon-2}|\mathfrak{y}(\varrho)|d\varrho\right.\right.\\
&\left. + \frac{1}{\Gamma(\epsilon-1)}\int_0^\iota (\iota-\varrho)^{\epsilon-2}|\mathfrak{f}(\varrho,\mathfrak{x}(\varrho),\phi_q(\mathfrak{y}(\varrho)))|d\varrho\right] + \frac{|a_2|}{|\xi|}\left[\frac{|1-\xi|}{\Gamma(\epsilon-\varepsilon)}\int_0^\iota (\iota-\varrho)^{\epsilon-\varepsilon-1}|\mathfrak{y}(\varrho)|d\varrho\right.\\
&\left.\left. + \frac{1}{\Gamma(\epsilon)}\int_0^\iota (\iota-\varrho)^{\epsilon-1}|\mathfrak{f}(\varrho,\mathfrak{x}(\varrho),\phi_q(\mathfrak{y}(\varrho)))|d\varrho\right]\right\} + \frac{|1-\xi|}{|\xi|\Gamma(\epsilon-\varepsilon)}\int_0^\iota (\iota-\varrho)^{\epsilon-\varepsilon-1}|\mathfrak{y}(\varrho)|d\varrho\\
& + \frac{1}{|\xi|\Gamma(\epsilon)}\int_0^\iota (\iota-\varrho)^{\epsilon-1}|\mathfrak{f}(\varrho,\mathfrak{x}(\varrho),\phi_q(\mathfrak{y}(\varrho)))|d\varrho\\
\leq{}& \left\{\frac{1}{|\Delta|}[(1+b_1)(2b_1+b_3) + |a_1|b_3 + \iota|a_2|b_1] + b_1\right\}\|\mathfrak{y}\|_\infty\\
& + \left\{\frac{1}{|\Delta|}[(1+b_1)(2b_2+b_4) + |a_1|b_4 + \iota|a_2|b_2] + b_2\right\}[a + b\|\mathfrak{x}\|_\infty^{p-1} + c\|\mathfrak{y}\|_\infty]\\
\leq{}& (\Delta_1 + \Delta_2 c)\rho + \Delta_2 b\rho^{p-1} + \Delta_2 a.
\end{aligned}$$

Hence,

$$\|\mathfrak{J}_1(\mathfrak{x},\mathfrak{y})(\zeta)\|_\infty \leq \frac{3}{2}b_5\rho^{q-1}, \tag{3.9}$$

and

$$\|\mathfrak{J}_2(\mathfrak{x},\mathfrak{y})(\zeta)\|_\infty \leq (\Delta_1+\Delta_2 c)\rho + \Delta_2 b\rho^{p-1} + \Delta_2 a. \tag{3.10}$$

Using (3.9) together with (3.10), we deduce the following:

$$\begin{aligned} \|\mathfrak{J}(\mathfrak{x},\mathfrak{y})(\zeta)\|_U &= \|\mathfrak{J}_1(\mathfrak{x},\mathfrak{y})(\zeta)\|_\infty + \|\mathfrak{J}_2(\mathfrak{x},\mathfrak{y})(\zeta)\|_\infty \\ &\leq (\Delta_1+\Delta_2 c)\rho + \Delta_2(a+b\rho^{p-1}) + \frac{3}{2}b_5\rho^{q-1}. \end{aligned} \tag{3.11}$$

Therefore, the operator $\mathfrak{J}$ is uniformly bounded on $\bar{B}_\rho$. Now, we proceed to demonstrate that $\mathfrak{J}$ is equicontinuous on $\bar{B}_\rho$. For any $(\mathfrak{x},\mathfrak{y}) \in \bar{B}_\rho$ and $\zeta_1,\zeta_2 \in [0,\iota]$, assume without a loss of generality that $0 \leq \zeta_1 < \zeta_2 \leq \iota$. Since ζ^γ and ζ are uniformly continuous on $[\zeta_1,\zeta_2]$, then we have the following:

$$\begin{aligned} |\mathfrak{J}_1(\mathfrak{x},\mathfrak{y})(\zeta_1) - \mathfrak{J}_1(\mathfrak{x},\mathfrak{y})(\zeta_2)| &\leq \frac{1}{\Gamma(\gamma)}\left(\int_0^{\zeta_1} |(\zeta_2-\varrho)^{\gamma-1} - (\zeta_1-\varrho)^{\gamma-1}||\phi_q(\mathfrak{y}(\varrho))|d\varrho + \int_{\zeta_1}^{\zeta_2} |(\zeta_2-\varrho)^{\gamma-1}||\phi_q(\mathfrak{y}(\varrho))|d\varrho\right) \\ &\leq \frac{\rho^{q-1}}{\Gamma(\gamma+1)}\left[(\zeta_1^\gamma - \zeta_2^\gamma) + 2(\zeta_2-\zeta_1)^\gamma\right] \to 0, \text{ as } \zeta_1 \to \zeta_2, \end{aligned}$$

which shows that $\mathfrak{J}_1$ is equicontinuous on $\bar{B}_\rho$.
On the other hand,

$$\begin{aligned} |\mathfrak{J}_2(\mathfrak{x},\mathfrak{y})(\zeta_1) - \mathfrak{J}_2(\mathfrak{x},\mathfrak{y})(\zeta_2)| \leq{} & \frac{|1-\xi|(\zeta_2^{\epsilon-\varepsilon} - \zeta_1^{\epsilon-\varepsilon})}{|\xi\Delta|\Gamma(\epsilon-\varepsilon+1)}\left\{\frac{2}{|\xi|}\left[\frac{|1-\xi|}{\Gamma(\epsilon-\varepsilon)}\int_0^\iota (\iota-\varrho)^{\epsilon-\varepsilon-1}\mathfrak{y}(\varrho)d\varrho\right.\right. \\ &\left. + \frac{1}{\Gamma(\epsilon)}\int_0^\iota (\iota-\varrho)^{\epsilon-1}\mathfrak{f}(\varrho,\mathfrak{x}(\varrho),\phi_q(\mathfrak{y}(\varrho)))d\varrho\right] + \frac{\iota}{|\xi|}\left[\frac{|1-\xi|}{\Gamma(\epsilon-\varepsilon-1)}\int_0^\iota (\iota-\varrho)^{\epsilon-\varepsilon-2}\mathfrak{y}(\varrho)d\varrho\right. \\ &\left.\left. + \frac{1}{\Gamma(\epsilon-1)}\int_0^\iota (\iota-\varrho)^{\epsilon-2}\mathfrak{f}(\varrho,\mathfrak{x}(\varrho),\phi_q(\mathfrak{y}(\varrho)))d\varrho\right]\right\} + \frac{\zeta_2-\zeta_1}{|\Delta|}\left\{\frac{|a_1|}{|\xi|}\left[\frac{|1-\xi|}{\Gamma(\epsilon-\varepsilon-1)}\int_0^\iota (\iota-\varrho)^{\epsilon-\varepsilon-2}|\mathfrak{y}(\varrho)|d\varrho\right.\right. \\ &\left. + \frac{1}{\Gamma(\epsilon-1)}\int_0^\iota (\iota-\varrho)^{\epsilon-2}|\mathfrak{f}(\varrho,\mathfrak{x}(\varrho),\phi_q(\mathfrak{y}(\varrho)))|d\varrho\right] + \frac{|a_2|}{|\xi|}\left[\frac{|1-\xi|}{\Gamma(\epsilon-\varepsilon)}\int_0^\iota (\iota-\varrho)^{\epsilon-\varepsilon-1}|\mathfrak{y}(\varrho)|d\varrho\right. \\ &\left.\left. + \frac{1}{\Gamma(\epsilon)}\int_0^\iota (\iota-\varrho)^{\epsilon-1}|\mathfrak{f}(\varrho,\mathfrak{x}(\varrho),\phi_q(\mathfrak{y}(\varrho)))|d\varrho\right]\right\} \\ & + \frac{|1-\xi|}{|\xi|\Gamma(\epsilon-\varepsilon)}\left|\int_0^{\zeta_1} (\zeta_1-\varrho)^{\epsilon-\varepsilon-1}\mathfrak{y}(\varrho)d\varrho - \int_0^{\zeta_2} (\zeta_2-\varrho)^{\epsilon-\varepsilon-1}\mathfrak{y}(\varrho)d\varrho\right| \\ & + \frac{1}{|\xi|\Gamma(\epsilon)}\left|\int_0^{\zeta_1} (\zeta_1-\varrho)^{\epsilon-1}\mathfrak{f}(\varrho,\mathfrak{x}(\varrho),\phi_q(\mathfrak{y}(\varrho)))d\varrho - \int_0^{\zeta_2} (\zeta_2-\varrho)^{\epsilon-1}\mathfrak{f}(\varrho,\mathfrak{x}(\varrho),\phi_q(\mathfrak{y}(\varrho)))d\varrho\right|. \end{aligned}$$

Note that

$$\begin{aligned} &\left|\int_0^{\zeta_2} (\zeta_2-\varrho)^{\epsilon-\varepsilon-1}\mathfrak{y}(\varrho)d\varrho - \int_0^{\zeta_1} (\zeta_1-\varrho)^{\epsilon-\varepsilon-1}\mathfrak{y}(\varrho)d\varrho\right| \\ &\leq \int_0^{\zeta_1} [(\zeta_2-\varrho)^{\epsilon-\varepsilon-1} - (\zeta_1-\varrho)^{\epsilon-\varepsilon-1}]|\mathfrak{y}(\varrho)|d\varrho + \int_{\zeta_1}^{\zeta_2} (\zeta_2-\varrho)^{\epsilon-\varepsilon-1}|\mathfrak{y}(\varrho)|d\varrho \\ &\leq \frac{\|\mathfrak{y}\|_\infty}{\epsilon-\varepsilon}(\zeta_2^{\epsilon-\varepsilon} - \zeta_1^{\epsilon-\varepsilon}) \to 0, \text{ as } \zeta_1 \to \zeta_2, \end{aligned}$$

and similarly

$$\begin{aligned}&\left|\int_0^{\zeta_2}(\zeta_2-\varrho)^{\epsilon-1}\mathfrak{f}(\varrho,\mathfrak{x}(\varrho),\phi_q(\mathfrak{y}(\varrho)))d\varrho-\int_0^{\zeta_1}(\zeta_1-\varrho)^{\epsilon-1}\mathfrak{f}(\varrho,\mathfrak{x}(\varrho),\phi_q(\mathfrak{y}(\varrho)))d\varrho\right|\\ &\leq\int_0^{\zeta_1}[(\zeta_2-\varrho)^{\epsilon-1}-(\zeta_1-\varrho)^{\epsilon-1}]|\mathfrak{f}(\varrho,\mathfrak{x}(\varrho),\phi_q(\mathfrak{y}(\varrho)))|d\varrho+\int_{\zeta_1}^{\zeta_2}(\zeta_2-\varrho)^{\epsilon-1}|\mathfrak{f}(\varrho,\mathfrak{x}(\varrho),\phi_q(\mathfrak{y}(\varrho)))|d\varrho\\ &\leq\frac{1}{\epsilon}(a+b\|\mathfrak{x}\|_\infty^{p-1}+c\|\mathfrak{y}\|_\infty)(\zeta_2^\epsilon-\zeta_1^\epsilon)\to 0,\ \text{as } \zeta_1\to\zeta_2.\end{aligned}$$

Moreover, since $\zeta^{\epsilon-\varepsilon}$ and ζ are uniformly continuous on $[\zeta_1,\zeta_2]$, we obtain

$$|\Im_2(\mathfrak{x},\mathfrak{y})(\zeta_1)-\Im_2(\mathfrak{x},\mathfrak{y})(\zeta_2)|\to 0,\ \text{as } \zeta_1\to\zeta_2,$$

thereby establishing that $\Im_2$ is equicontinuous on $\bar{B}_\rho$. Consequently, the operator $\Im=(\Im_1,\Im_2)$ is equicontinuous on $\bar{B}_\rho$. By the Arzelà-Ascoli theorem, we deduce that $\Im$ is compact on B_σ. Together with the continuity of $\Im$, this establishes that $\Im$ is completely continuous. It follows that $\Im$ is completely continuous.

Step 2. We assert that the operator $\Im$ admits a fixed point. To this end, we define the following set:

$$S=\{(\mathfrak{x},\mathfrak{y})\in U,\ \|(\mathfrak{x},\mathfrak{y})\|_U<M\}.$$

First, we show that $\Im(\bar{S})\subset\bar{S}$. Indeed, for any $(\mathfrak{x},\mathfrak{y})\in\bar{S}$ and $\zeta\in[0,\iota]$, by an argument analogous to (3.11), we derive the following:

$$\begin{aligned}\|\Im(\mathfrak{x},\mathfrak{y})(\zeta)\|_U&=\|\Im_1(\mathfrak{x},\mathfrak{y})(\zeta)\|_\infty+\|\Im_2(\mathfrak{x},\mathfrak{y})(\zeta)\|_\infty\\ &\leq(\Delta_1+\Delta_2c)M+\Delta_2bM^{p-1}+\frac{3}{2}b_5M^{q-1}+\Delta_2a.\end{aligned}$$

By combining Condition (C_2), we obtain the following:

$$\|\Im(\mathfrak{x},\mathfrak{y})(\zeta)\|_U<M.$$

Hence, $\Im(\bar{S})\subset\bar{S}$. Now, we show that for any $(\mathfrak{x},\mathfrak{y})\in\partial S,\ \omega\in(0,1)$, the relation $(\mathfrak{x},\mathfrak{y})\neq\omega\Im(\mathfrak{x},\mathfrak{y})$ holds. We argue by contradiction. Assume that there exist $(\mathfrak{x},\mathfrak{y})\in\partial S$ and $\omega\in(0,1)$ such that $(\mathfrak{x},\mathfrak{y})=\omega\Im(\mathfrak{x},\mathfrak{y})$. Then, we have

$$\begin{aligned}M=\|(\mathfrak{x},\mathfrak{y})\|_U=\omega\|\Im(\mathfrak{x},\mathfrak{y})\|_U&\leq\|\Im_1(\mathfrak{x},\mathfrak{y})\|_\infty+\|\Im_2(\mathfrak{x},\mathfrak{y})\|_\infty\\ &\leq(\Delta_1+\Delta_2c)M+\Delta_2bM^{p-1}+\frac{3}{2}b_5M^{q-1}+\Delta_2a<M,\end{aligned}$$

which is a contradiction. Therefore, by the LSNA, the operator $\Im$ admits a fixed point $(\mathfrak{x},\mathfrak{y})\in\bar{S}$. Consequently, Problem (3.1) has at least one solution on $[0,\iota]$. □

Finally, the existence of solutions to the BVP (3.1) can be established by applying the KFPT, which we present as our second result.

Theorem 3.2. Assume that $\mathfrak{f} : [0, \iota] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies Condition (C_1), together with the following condition:

(C_3) There exists a constant $\sigma > 0$ such that

$$\frac{\Delta_2 a + \Delta_2 b\sigma^{p-1} + 3 \times 2^{-1} b_5 \sigma^{q-1}}{(1 - \Delta_1 - \Delta_2 c)\sigma} \leq 1,$$

where a, b, c are defined as in Theorem 3.1. Then, the BVP (3.1) admits at least one solution on $[0, \iota]$, provided that

$$\Delta_2 c + \Delta_1 < 1.$$

Proof. Define the set B_σ as follows:

$$B_\sigma = \{(\mathfrak{x}, \mathfrak{y}) \in U : \|(\mathfrak{x}, \mathfrak{y})\|_U \leq \sigma\},$$

where

$$\sigma \geq \frac{\Delta_2(a + b\sigma^{p-1}) + 3 \times 2^{-1} b_5 \sigma^{q-1}}{(1 - \Delta_1 - \Delta_2 c)}.$$

In view of Condition (C_3), the set B_σ is well defined. Now, we introduce the operators $E,\ G : B_\sigma \to U$ defined by the following:

$$E(\mathfrak{x}, \mathfrak{y})(\zeta) = (E_1(\mathfrak{x}, \mathfrak{y})(\zeta), E_2(\mathfrak{x}, \mathfrak{y})(\zeta)),\ G(\mathfrak{x}, \mathfrak{y})(\zeta) = (G_1(\mathfrak{x}, \mathfrak{y})(\zeta), G_2(\mathfrak{x}, \mathfrak{y})(\zeta)),\ \zeta \in [0, \iota]$$

where

$$\begin{aligned}
E_1(\mathfrak{x}, \mathfrak{y})(\zeta) &\equiv 0,\\
E_2(\mathfrak{x}, \mathfrak{y})(\zeta) &= \frac{\zeta}{\Delta}\left[\frac{a_1(1-\xi)}{\xi\Gamma(\epsilon-\varepsilon-1)}\int_0^\iota (\iota-\varrho)^{\epsilon-\varepsilon-2}\mathfrak{y}(\varrho)d\varrho - \frac{a_2(1-\xi)}{\xi\Gamma(\epsilon-\varepsilon)}\int_0^\iota (\iota-\varrho)^{\epsilon-\varepsilon-1}\mathfrak{y}(\varrho)d\varrho\right]\\
&\quad + \frac{1}{\Delta}\left[1 + \frac{(1-\xi)\zeta^{\epsilon-\varepsilon}}{\xi\Gamma(\epsilon-\varepsilon+1)}\right]\left[\frac{2(1-\xi)}{\xi\Gamma(\epsilon-\varepsilon)}\int_0^\iota (\iota-\varrho)^{\epsilon-\varepsilon-1}\mathfrak{y}(\varrho)d\varrho\right.\\
&\quad \left. - \frac{\iota(1-\xi)}{\xi\Gamma(\epsilon-\varepsilon-1)}\int_0^\iota (\iota-\varrho)^{\epsilon-\varepsilon-2}\mathfrak{y}(\varrho)d\varrho\right] - \frac{1-\xi}{\xi\Gamma(\epsilon-\varepsilon)}\int_0^\zeta (\zeta-\varrho)^{\epsilon-\varepsilon-1}\mathfrak{y}(\varrho)d\varrho,\\
G_1(\mathfrak{x}, \mathfrak{y})(\zeta) &= \frac{1}{\Gamma(\gamma)}\int_0^\zeta (\zeta-\varrho)^{\gamma-1}\phi_q(\mathfrak{y}(\varrho))d\varrho - \frac{1}{2\Gamma(\gamma)}\int_0^\iota (\iota-\varrho)^{\gamma-1}\phi_q(\mathfrak{y}(\varrho))d\varrho,\\
G_2(\mathfrak{x}, \mathfrak{y})(\zeta) &= \frac{\zeta}{\Delta}\left[\frac{a_2}{\xi\Gamma(\epsilon)}\int_0^\iota (\iota-\varrho)^{\epsilon-1}\mathfrak{f}(\varrho, \mathfrak{x}(\varrho), \phi_q(\mathfrak{y}(\varrho)))d\varrho - \frac{a_1}{\xi\Gamma(\epsilon-1)}\int_0^\iota (\iota-\varrho)^{\epsilon-2}\mathfrak{f}(\varrho, \mathfrak{x}(\varrho), \phi_q(\mathfrak{y}(\varrho)))d\varrho\right]\\
&\quad + \frac{1}{\Delta}\left[1 + \frac{(1-\xi)\zeta^{\epsilon-\varepsilon}}{\xi\Gamma(\epsilon-\varepsilon+1)}\right]\left[\frac{\iota}{\xi\Gamma(\epsilon-1)}\int_0^\iota (\iota-\varrho)^{\epsilon-2}\mathfrak{f}(\varrho, \mathfrak{x}(\varrho), \phi_q(\mathfrak{y}(\varrho)))d\varrho\right.\\
&\quad \left. - \frac{2}{\xi\Gamma(\epsilon)}\int_0^\iota (\iota-\varrho)^{\epsilon-1}\mathfrak{f}(\varrho, \mathfrak{x}(\varrho), \phi_q(\mathfrak{y}(\varrho)))d\varrho\right] + \frac{1}{\xi\Gamma(\epsilon)}\int_0^\zeta (\zeta-\varrho)^{\epsilon-1}\mathfrak{f}(\varrho, \mathfrak{x}(\varrho), \phi_q(\mathfrak{y}(\varrho)))d\varrho.
\end{aligned}$$

Now, we proceed with the proof in three steps.

Step 1. We aim to show that for any $\mu = (\mathfrak{x}_1, \mathfrak{y}_1),\ \nu = (\mathfrak{x}_2, \mathfrak{y}_2) \in B_\sigma$, one has $E\mu + G\nu \in B_\sigma$. In fact, since $\mu, \nu \in B_\sigma$, it follows that $\|\mu\|_U \leq \sigma,\ \|\nu\|_U \leq \sigma$. Furthermore, by Condition (C_1), we deduce that

$$\begin{aligned}
|E_1(\mathfrak{x}_1, \mathfrak{y}_1)(\zeta) + G_1(\mathfrak{x}_2, \mathfrak{y}_2)(\zeta)| &= |G_1(\mathfrak{x}_2, \mathfrak{y}_2)(\zeta)|\\
&\leq \frac{1}{\Gamma(\gamma)}\int_0^\zeta (\zeta-\varrho)^{\gamma-1}|\phi_q(\mathfrak{y}_2(\varrho))|d\varrho + \frac{1}{2\Gamma(\gamma)}\int_0^\iota (\iota-\varrho)^{\gamma-1}|\phi_q(\mathfrak{y}_2(\varrho))|d\varrho\\
&\leq b_5\phi_q(\|\mathfrak{y}_2\|_\infty) + \frac{b_5}{2}\phi_q(\|\mathfrak{y}_2\|_\infty) \leq \frac{3}{2}b_5\sigma^{q-1},
\end{aligned}$$

and

$$\begin{aligned}
&|E_2(\mathfrak{x}_1,\mathfrak{y}_1)(\zeta)+G_2(\mathfrak{x}_2,\mathfrak{y}_2)(\zeta)|\\
&\le \frac{\iota}{|\Delta|}\left\{\frac{|a_1|}{|\xi|}\left[\frac{|1-\xi|}{\Gamma(\epsilon-\varepsilon-1)}\int_0^\iota(\iota-\varrho)^{\epsilon-\varepsilon-2}|\mathfrak{y}_1(\varrho)|d\varrho\ +\frac{1}{\Gamma(\epsilon-1)}\int_0^\iota(\iota-\varrho)^{\epsilon-2}|\mathfrak{f}(\varrho,\mathfrak{x}_2(\varrho),\phi_q(\mathfrak{y}_2(\varrho)))|d\varrho\right]\right.\\
&\quad\left.+\frac{|a_2|}{|\xi|}\left[\frac{|1-\xi|}{\Gamma(\epsilon-\varepsilon)}\int_0^\iota(\iota-\varrho)^{\epsilon-\varepsilon-1}|\mathfrak{y}_1(\varrho)|d\varrho\ +\frac{1}{\Gamma(\epsilon)}\int_0^\iota(\iota-\varrho)^{\epsilon-1}|\mathfrak{f}(\varrho,\mathfrak{x}_2(\varrho),\phi_q(\mathfrak{y}_2(\varrho)))|d\varrho\right]\right\}\\
&\quad+\frac{1}{|\Delta|}\left[1+\frac{|1-\xi|\iota^{\epsilon-\varepsilon}}{|\xi|\Gamma(\epsilon-\varepsilon+1)}\right]\left\{\frac{2}{|\xi|}\left[\frac{|1-\xi|}{\Gamma(\epsilon-\varepsilon)}\int_0^\iota(\iota-\varrho)^{\epsilon-\varepsilon-1}|\mathfrak{y}_1(\varrho)|d\varrho\right.\right.\\
&\quad\left.+\frac{1}{\Gamma(\epsilon)}\int_0^\iota(\iota-\varrho)^{\epsilon-1}|\mathfrak{f}(\varrho,\mathfrak{x}_2(\varrho),\phi_q(\mathfrak{y}_2(\varrho)))|d\varrho\right]+\frac{\iota}{|\xi|}\left[\frac{|1-\xi|}{\Gamma(\epsilon-\varepsilon-1)}\int_0^\iota(\iota-\varrho)^{\epsilon-\varepsilon-2}|\mathfrak{y}_1(\varrho)|d\varrho\right.\\
&\quad\left.\left.+\frac{1}{\Gamma(\epsilon-1)}\int_0^\iota(\iota-\varrho)^{\epsilon-2}|\mathfrak{f}(\varrho,\mathfrak{x}_2(\varrho),\phi_q(\mathfrak{y}_2(\varrho)))|d\varrho\right]\right\}+\frac{|1-\xi|}{|\xi|\Gamma(\epsilon-\varepsilon)}\int_0^\iota(\iota-\varrho)^{\epsilon-\varepsilon-1}|\mathfrak{y}_1(\varrho)|d\varrho\\
&\quad+\frac{1}{|\xi|\Gamma(\epsilon)}\int_0^\iota(\iota-\varrho)^{\epsilon-1}|\mathfrak{f}(\varrho,\mathfrak{x}_2(\varrho),\phi_q(\mathfrak{y}_2(\varrho)))|d\varrho\\
&\le\left\{\frac{1}{|\Delta|}[(1+b_1)(2b_1+b_3)+|a_1|b_3+\iota|a_2|b_1]+b_1\right\}\|\mathfrak{y}_1\|_\infty\\
&\quad+\left\{\frac{1}{|\Delta|}[(1+b_1)(2b_2+b_4)+|a_1|b_4+\iota|a_2|b_2]+b_2\right\}(a+b\|\mathfrak{x}_2\|_\infty^{p-1}+c\|\mathfrak{y}_2\|_\infty)\\
&\le(\Delta_1+\Delta_2c)\sigma+\Delta_2b\sigma^{p-1}+\Delta_2a.
\end{aligned}$$

Then,

$$\|E_1(\mathfrak{x}_1,\mathfrak{y}_1)(\zeta)+G_1(\mathfrak{x}_2,\mathfrak{y}_2)(\zeta)\|_\infty\le\frac{3}{2}b_5\sigma^{q-1},$$

and

$$\|E_2(\mathfrak{x}_1,\mathfrak{y}_1)(\zeta)+G_2(\mathfrak{x}_2,\mathfrak{y}_2)(\zeta)\|_\infty\le(\Delta_1+\Delta_2c)\sigma+\Delta_2b\sigma^{p-1}+\Delta_2a.$$

Hence, by combining the above inequalities, we obtain the following:

$$\begin{aligned}
\|E\mu+G\nu\|_U&=\|E_1(\mathfrak{x}_1,\mathfrak{y}_1)(\zeta)+G_1(\mathfrak{x}_2,\mathfrak{y}_2)(\zeta)\|_\infty+\|E_2(\mathfrak{x}_1,\mathfrak{y}_1)(\zeta)+G_2(\mathfrak{x}_2,\mathfrak{y}_2)(\zeta)\|_\infty\\
&\le(\Delta_1+\Delta_2c)\sigma+\Delta_2b\sigma^{p-1}+\frac{3}{2}\Delta_3\sigma^{q-1}+\Delta_2a.
\end{aligned}$$

Therefore, it follows that $E\mu+G\nu\in B_\sigma$.

Step 2. We show that E is a contraction mapping. Indeed, for any $\mu=(\mathfrak{x}_1,\mathfrak{y}_1)$, $\nu=(\mathfrak{x}_2,\mathfrak{y}_2)\in B_\sigma$, and $\zeta\in[0,\iota]$, one has

$$\begin{aligned}
&|E_2(\mathfrak{x}_1,\mathfrak{y}_1)(\zeta)-E_2(\mathfrak{x}_2,\mathfrak{y}_2)(\zeta)|\\
&\le\frac{\iota}{|\Delta|}\left[\frac{|a_1(1-\xi)|}{|\xi|\Gamma(\epsilon-\varepsilon-1)}\int_0^\iota(\iota-\varrho)^{\epsilon-\varepsilon-2}|\mathfrak{y}_1(\varrho)-\mathfrak{y}_2(\varrho)|d\varrho\ +\frac{|a_2(1-\xi)|}{|\xi|\Gamma(\epsilon-\varepsilon)}\int_0^\iota(\iota-\varrho)^{\epsilon-\varepsilon-1}|\mathfrak{y}_1(\varrho)-\mathfrak{y}_2(\varrho)|d\varrho\right]\\
&\quad+\frac{1}{|\Delta|}\left[1+\frac{|1-\xi|\zeta^{\epsilon-\varepsilon}}{|\xi|\Gamma(\epsilon-\varepsilon+1)}\right]\left[\frac{2|1-\xi|}{|\xi|\Gamma(\epsilon-\varepsilon)}\int_0^\iota(\iota-\varrho)^{\epsilon-\varepsilon-1}|\mathfrak{y}_1(\varrho)-\mathfrak{y}_2(\varrho)|d\varrho\right.\\
&\quad\left.+\frac{\iota|1-\xi|}{|\xi|\Gamma(\epsilon-\varepsilon-1)}\int_0^\iota(\iota-\varrho)^{\epsilon-\varepsilon-2}|\mathfrak{y}_1(\varrho)-\mathfrak{y}_2(\varrho)|d\varrho\right]+\frac{|1-\xi|}{|\xi|\Gamma(\epsilon-\varepsilon)}\int_0^\iota(\iota-\varrho)^{\epsilon-\varepsilon-1}|\mathfrak{y}_1(\varrho)-\mathfrak{y}_2(\varrho)|d\varrho\\
&\le\left\{\frac{1}{|\Delta|}[(1+b_1)(2b_1+b_3)+|a_1|b_3+\iota|a_2|b_1]+b_1\right\}\|\mathfrak{y}_1-\mathfrak{y}_2\|_\infty\\
&=\Delta_1\|\mathfrak{y}_1-\mathfrak{y}_2\|_\infty.
\end{aligned}$$

In view of the inequality $\Delta_2 c+\Delta_1 < 1$, we note that E_2 is a contraction. Moreover, E_1 is a zero operator, which is trivially a contraction. Thus,

$$\begin{aligned}
&\|E\mu - E\nu\|_U\\
&= \|E(\mathfrak{x}_1,\mathfrak{y}_1)(\zeta) - E(\mathfrak{x}_2,\mathfrak{y}_2)(\zeta)\|_U = \|E_1(\mathfrak{x}_1,\mathfrak{y}_1)(\zeta) - E_1(\mathfrak{x}_2,\mathfrak{y}_2)(\zeta)\|_\infty + \|E_2(\mathfrak{x}_1,\mathfrak{y}_1)(\zeta) - E_2(\mathfrak{x}_2,\mathfrak{y}_2)(\zeta)\|_\infty\\
&= \|E_2(\mathfrak{x}_1,\mathfrak{y}_1)(\zeta) - E_2(\mathfrak{x}_2,\mathfrak{y}_2)(\zeta)\|_\infty \le \Delta_1\|\mathfrak{y}_1 - \mathfrak{y}_2\|_\infty \le \Delta_1(\|\mathfrak{x}_1 - \mathfrak{x}_2\|_\infty + \|\mathfrak{y}_1 - \mathfrak{y}_2\|_\infty) = \Delta_1\|\mu - \nu\|_U.
\end{aligned}$$

Since $\Delta_2 c + \Delta_1 < 1$, we immediately obtain that $\Delta_1 < 1$. It follows that the operator E is a contraction mapping.

Step 3. We establish that G is completely continuous. In fact, in view of the continuity of $\mathfrak{f}$ and $\phi_q(\cdot)$, then the operator G is continuous. Furthermore, as demonstrated in Step 1, G is uniformly bounded on B_σ. It remains to verify that G is equicontinuous on B_σ. For any $(\mathfrak{x},\mathfrak{y}) \in B_\sigma$ and $\zeta_1,\zeta_2 \in [0,\iota]$, assume without a loss of generality that $0 \le \zeta_1 < \zeta_2 \le \iota$. Then, since $\zeta^{\epsilon-\varepsilon},\zeta^{\epsilon},\zeta^{\gamma}$, and ζ are uniformly continuous on $[\zeta_1,\zeta_2]$, we have

$$\begin{aligned}
|G_1(\mathfrak{x},\mathfrak{y})(\zeta_1) - G_1(\mathfrak{x},\mathfrak{y})(\zeta_2)| &\le \frac{1}{\Gamma(\gamma)}\int_0^{\zeta_1}[(\zeta_1-\varrho)^{\gamma-1} - (\zeta_2-\varrho)^{\gamma-1}]|\phi_q(\mathfrak{y}(\varrho))|d\varrho\\
&\quad + \frac{1}{\Gamma(\gamma)}\int_{\zeta_1}^{\zeta_2}(\zeta_2-\varrho)^{\gamma-1}|\phi_q(\mathfrak{y}(\varrho))|d\varrho\\
&\le \frac{\sigma^{q-1}}{\Gamma(\gamma+1)}\left[\zeta_1^{\gamma} - \zeta_2^{\gamma} + 2(\zeta_2-\zeta_1)^{\gamma}\right] \to 0, \text{ as } \zeta_1 \to \zeta_2,
\end{aligned}$$

and

$$\begin{aligned}
&|G_2(\mathfrak{x},\mathfrak{y})(\zeta_2) - G_2(\mathfrak{x},\mathfrak{y})(\zeta_1)|\\
&\le \frac{\zeta_2-\zeta_1}{|\Delta|}\left[\frac{|a_2|}{|\xi|\Gamma(\epsilon)}\int_0^{\iota}(\iota-\varrho)^{\epsilon-1}|\mathfrak{f}(\varrho,\mathfrak{x}(\varrho),\phi_q(\mathfrak{y}(\varrho)))|d\varrho\right.\\
&\quad\left.+\frac{|a_1|}{|\xi|\Gamma(\epsilon-1)}\int_0^{\iota}(\iota-\varrho)^{\epsilon-2}|\mathfrak{f}(\varrho,\mathfrak{x}(\varrho),\phi_q(\mathfrak{y}(\varrho)))|d\varrho\right]\\
&\quad+\frac{|1-\xi|(\zeta_2^{\epsilon-\varepsilon}-\zeta_1^{\epsilon-\varepsilon})}{|\xi\Delta|\Gamma(\epsilon-\varepsilon+1)}\left[\frac{2}{|\xi|\Gamma(\epsilon)}\int_0^{\iota}(\iota-\varrho)^{\epsilon-1}|\mathfrak{f}(\varrho,\mathfrak{x}(\varrho),\phi_q(\mathfrak{y}(\varrho)))|d\varrho\right.\\
&\quad\left.+\frac{\iota}{|\xi|\Gamma(\epsilon-1)}\int_0^{\iota}(\iota-\varrho)^{\epsilon-2}|\mathfrak{f}(\varrho,\mathfrak{x}(\varrho),\phi_q(\mathfrak{y}(\varrho)))|d\varrho\right]\\
&\quad+\frac{1}{|\xi|\Gamma(\epsilon)}\int_0^{\zeta_1}[(\zeta_2-\varrho)^{\epsilon-1}-(\zeta_1-\varrho)^{\epsilon-1}]|\mathfrak{f}(\varrho,\mathfrak{x}(\varrho),\phi_q(\mathfrak{y}(\varrho)))|d\varrho\\
&\quad+\frac{1}{|\xi|\Gamma(\epsilon)}\int_{\zeta_1}^{\zeta_2}(\zeta_2-\varrho)^{\epsilon-1}|\mathfrak{f}(\varrho,\mathfrak{x}(\varrho),\phi_q(\mathfrak{y}(\varrho)))|d\varrho\\
&\le\left[\frac{|1-\xi|(\zeta_2^{\epsilon-\varepsilon}-\zeta_1^{\epsilon-\varepsilon})}{|\xi\Delta|\Gamma(\epsilon-\varepsilon+1)}(2b_2+b_4)+\frac{\zeta_2-\zeta_1}{|\Delta|}(\iota^{-1}|a_1|b_4+|a_2|b_2)\right.\\
&\quad\left.+\frac{\zeta_2^{\epsilon}-\zeta_1^{\epsilon}}{|\xi|\Gamma(\epsilon+1)}\right](a+b\sigma^{p-1}+c\sigma) \to 0, \text{ as } \zeta_1 \to \zeta_2.
\end{aligned}$$

Therefore, both G_1 and G_2 are equicontinuous, and consequently G is equicontinuous as well. By the Arzelà-Ascoli theorem, it follows that G is compact on B_σ. Combined with the continuity of G, this implies that G is completely continuous. Hence, there exists a fixed point $(\mathfrak{x}, \mathfrak{y}) \in U$ such that $(\mathfrak{x}, \mathfrak{y}) = E(\mathfrak{x}, \mathfrak{y}) + G(\mathfrak{x}, \mathfrak{y})$. This shows that the BVP (3.1) admits at least one solution on $[0, \iota]$. □

Corollary 3.1. Assume that $\mathfrak{f} : [0, \iota] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies the following condition:

(C_4) There exists a function $\psi(\zeta) \in C([0, \iota], \mathbb{R}^+)$ such that for all $(\zeta, \mathfrak{x}, \phi_q(\mathfrak{y})) \in [0, \iota] \times \mathbb{R}^2$,

$$|\mathfrak{f}(\zeta, \mathfrak{x}(\zeta), \phi_q(\mathfrak{y}(\zeta)))| \leq \phi_p(\psi(\zeta)).$$

Then, the BVP (3.1) admits at least one solution on $[0, \iota]$, provided that

$$\Delta_1 < 1.$$

4. Examples

Example 4.1. We consider the multi-term p-Laplacian FDE given below, accompanied by anti-periodic BCs:

$$\begin{cases} (9/10)^C\mathscr{D}_{0+}^{7/4}\phi_3({}^C\mathscr{D}_{0+}^{4/5}\mathfrak{x}(\zeta)) + (1/10)^C\mathscr{D}_{0+}^{1/2}\phi_3({}^C\mathscr{D}_{0+}^{4/5}\mathfrak{x}(\zeta)) = \mathfrak{f}(\zeta, \mathfrak{x}(\zeta), {}^C\mathscr{D}_{0+}^{4/5}\mathfrak{x}(\zeta)), \quad \zeta \in (0, 53/100), \\ \mathfrak{x}(0) = -\mathfrak{x}(53/100), {}^C\mathscr{D}_{0+}^{4/5}\mathfrak{x}(0) = -{}^C\mathscr{D}_{0+}^{4/5}\mathfrak{x}(53/100), [\phi_3({}^C\mathscr{D}_{0+}^{4/5}\mathfrak{x}(0))]' = -[\phi_3({}^C\mathscr{D}_{0+}^{4/5}\mathfrak{x}(53/100))]'. \end{cases} \tag{4.1}$$

Let $\mathfrak{y}(\zeta) = \phi_3({}^C\mathscr{D}_{0+}^{4/5}\mathfrak{x}(\zeta))$; thus, Equation (4.1) is equivalent to the following:

$$\begin{cases} {}^C\mathscr{D}_{0+}^{4/5}\mathfrak{x}(\zeta) = \phi_{3/2}(\mathfrak{y}(\zeta)), \quad \zeta \in (0, 53/100), \\ (9/10)^C\mathscr{D}_{0+}^{7/4}\mathfrak{y}(\zeta) + (1/10)^C\mathscr{D}_{0+}^{1/2}\mathfrak{y}(\zeta) = \mathfrak{f}(\zeta, \mathfrak{x}(\zeta), \phi_{3/2}(\mathfrak{y}(\zeta))), \quad \zeta \in (0, 53/100), \\ \mathfrak{x}(0) = -\mathfrak{x}(53/100), \quad \mathfrak{y}(0) = -\mathfrak{y}(53/100), \quad \mathfrak{y}'(0) = -\mathfrak{y}'(53/100). \end{cases} \tag{4.2}$$

Here, corresponding to (3.1), the following holds:

$$p = 3, \ q = 3/2, \ \epsilon = 7/4, \ \varepsilon = 1/2,$$
$$\iota = 53/100, \ \gamma = 4/5, \ \xi = 9/10,$$

$$\mathfrak{f}(\zeta, \mathfrak{x}(\zeta), \phi_{3/2}(\mathfrak{y}(\zeta))) = \frac{1}{10}\cos\zeta + \frac{1}{10(1+\zeta^2)}|\mathfrak{x}(\zeta)|^2 + \frac{3}{20(1+\zeta^2)}|\mathfrak{y}(\zeta)|.$$

Let

$$a(\zeta) = \frac{1}{10}\cos\zeta, \ b(\zeta) = \frac{1}{10(1+\zeta^2)}, \ c(\zeta) = \frac{3}{20(1+\zeta^2)}.$$

It is easy to see that Condition (C_1) holds. By calculation, the following can be obtained:

$$a = 1/10, \ b = 1/10, \ c = 3/20, \ a_1 = 2 + \frac{(1-\xi)\iota^{\epsilon-\varepsilon}}{\xi\Gamma(\epsilon - \varepsilon + 1)} = 2.0443,$$
$$a_2 = \frac{(1-\xi)\iota^{\epsilon-\varepsilon-1}}{\xi\Gamma(\epsilon - \varepsilon)} = 0.1046, \ \Delta = 2a_1 - \iota a_2 = 4.0333,$$
$$\Delta_1 = 0.1104, \ \Delta_2 = 0.6531, \ \Delta_1 + \Delta_2 c = 0.2083 < 1.$$

Take M = 5; then,

$$\frac{\Delta_2(a+bM^{p-1})+1.5b_5M^{q-1}}{(1-\Delta_1-\Delta_2c)M}=0.9765<1.$$

Then, Condition (C_2) holds. By Theorem 3.1, it can be concluded that Problem (4.2) possesses at least one solution.

Example 4.2. Consider the following p-Laplacian multi-term FDE with anti-periodic BCs:

$$\begin{cases}(9/10)^C\mathscr{D}_{0+}^{39/20}\phi_3({}^C\mathscr{D}_{0+}^{9/10}\mathfrak{x}(\zeta))+(1/10)^C\mathscr{D}_{0+}^{2/5}\phi_3({}^C\mathscr{D}_{0+}^{9/10}\mathfrak{x}(\zeta))=\mathfrak{f}(\zeta,\mathfrak{x}(\zeta),{}^C\mathscr{D}_{0+}^{9/10}\mathfrak{x}(\zeta)),\quad \zeta\in(0,8/25),\\ \mathfrak{x}(0)=-\mathfrak{x}(8/25),{}^C\mathscr{D}_{0+}^{9/10}\mathfrak{x}(0)=-{}^C\mathscr{D}_{0+}^{9/10}\mathfrak{x}(8/25),[\phi_3({}^C\mathscr{D}_{0+}^{9/10}\mathfrak{x}(0))]'=-[\phi_3({}^C\mathscr{D}_{0+}^{9/10}\mathfrak{x}(8/25))]'.\end{cases}\tag{4.3}$$

Let $\mathfrak{y}(\zeta)=\phi_3({}^CD_{0+}^{9/10}\mathfrak{x}(\zeta))$; thus, Equation (4.3) is equivalent to the following:

$$\begin{cases}{}^C\mathscr{D}_{0+}^{9/10}\mathfrak{x}(\zeta)=\phi_{3/2}(\mathfrak{y}(\zeta)),\quad \zeta\in(0,8/25),\\ (9/10)^C\mathscr{D}_{0+}^{39/20}\mathfrak{y}(\zeta)+(1/10)^C\mathscr{D}_{0+}^{2/5}\mathfrak{y}(\zeta)=\mathfrak{f}(\zeta,\mathfrak{x}(\zeta),\phi_{3/2}(\mathfrak{y}(\zeta))),\quad \zeta\in(0,8/25),\\ \mathfrak{x}(0)=-\mathfrak{x}(1),\quad \mathfrak{y}(0)=-\mathfrak{y}(8/25),\quad \mathfrak{y}'(0)=-\mathfrak{y}'(8/25).\end{cases}\tag{4.4}$$

Here, corresponding to (3.1), the following holds:

$$\begin{aligned}&p=3,\ q=3/2,\ \epsilon=39/20,\ \varepsilon=2/5,\\&\iota=8/25,\ \gamma=9/10,\ \xi=9/10,\end{aligned}$$

$$\mathfrak{f}(\zeta,\mathfrak{x}(\zeta),\phi_{3/2}(\mathfrak{y}(\zeta)))=\cos\zeta+\frac{1}{1+e^{\zeta}}|\mathfrak{x}(\zeta)|^2+\frac{1}{10(1+e^{\zeta})}|\mathfrak{y}(\zeta)|.$$

Let

$$a(\zeta)=\cos\zeta,b(\zeta)=\frac{1}{1+e^{\zeta}},c(\zeta)=\frac{1}{10(1+e^{\zeta})}.$$

Through direct calculation, we can obtain the following:

$$\begin{aligned}&c=0.1,\ \Delta=2a_1-\iota a_2=4.0062,\ \Delta_1=0.0370,\\&\Delta_2=0.1811,\ \Delta_1+\Delta_2c=0.0558<1.\end{aligned}$$

Take $\sigma=1$; then,

$$\frac{\Delta_2a+\Delta_2b\sigma^{p-1}+3\times2^{-1}b_5\sigma^{q-1}}{(1-\Delta_1-\Delta_2c)\sigma}=0.9909\le1.$$

Then, Condition (C_3) holds. Theorem 3.2 ensures that Problem (4.4) has at least one solution.

5. Conclusions

This study addressed the existence of solutions for multi-term p-Laplacian FDEs subject to anti-periodic BCs. By applying the LSNA and KFPT, existence results were established under the assumption that the nonlinear term satisfies the $p-1$ growth condition. In addition, illustrative examples were provided to demonstrate the applicability of the main findings. This contribution enriches the research on multi-term FDEs that involve the p-Laplacian operator, offers new perspectives, and lays a solid foundation for further investigations into related BVPs. Future research may explore the existence of solutions for multi-term p-Laplacian FDEs under closed boundary conditions or nonlocal BCs. Such problems are of considerable importance in both theoretical analyses and applied modeling. Building upon the existence results, a further examination of stability properties, such as Ulam stability or other stability concepts, will provide deeper insights into the qualitative behavior of solutions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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