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Research article

Logarithmic decay rate for a structural-acoustic interaction model with supercritical source and nonlinear damping

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Abstract: In this paper, we study a structural acoustics model with supercritical source and nonlinear damping. The model consists of a wave equation defined on a bounded domain which is strongly coupled with a Berger plate equation acting on the elastic wall. The aim of the paper is to remove the strong condition $u \in L^\infty(\mathbb{R}^+; L^{\frac{3}{2}}(\Omega))$ for $m_1 > 5$ in the work [1] to obtain a weaker energy decay, and give a logarithmic decay rate under a weak assumption on the initial data.

Keywords: structural acoustics model; energy decay; supercritical sources; nonlinear damping

1. Introduction

This paper is concerned with a structural acoustic model with supercritical source and nonlinear damping:

$$\begin{cases} u_{tt} - \Delta u + g_1(u_t) = f_1(u) & \text{in } \Omega \times (0,T), \\ w_{tt} + \Delta^2 w + u_t|_\Gamma + g_2(w_t) = f_2(w) & \text{in } \Gamma \times (0,T), \\ u = 0 & \text{on } \Gamma_0 \times (0,T), \\ \partial_\nu u = w_t & \text{on } \Gamma \times (0,T), \\ w = \partial_{\nu_\Gamma} w = 0 & \text{on } \partial\Gamma \times (0,T), \\ (u(0), u_t(0)) = (u_0, u_1), \quad (w(0), w_t(0)) = (w_0, w_1), \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is an open, bounded, and connected domain with smooth boundary $\partial\Omega = \overline{\Gamma_0 \cup \Gamma}$. The sets Γ_0 and Γ are disjoint, open, connected subsets of $\partial\Omega$ with positive Lebesgue measure, and Γ is assumed to be a *flat* portion of $\partial\Omega$. The boundary part Γ_0 represents a rigid wall, whereas the coupling occurs through the flexible wall Γ. The functions f_1 and f_2 are source terms and f_1 is of supercritical order. Finally, $g_1(u_t)$ and $g_2(w_t)$ are nonlinear frictional damping terms.

Models such as (1.1) arise in the modeling of gas pressure in an acoustic cavity whose boundary is composed of both rigid and flexible walls. The pressure inside the cavity is governed by a wave equation, while the vibration of the flexible wall is described by a coupled Berger plate equation.

The study of structural–acoustic interactions dates back to the pioneering works [2–6]. The underlying physical setting is that, in a domain with smooth and compact boundary, the gas undergoes a small irrotational perturbation from a stationary state. In [4], Beale and Rosencrans considered a wave equation with acoustic boundary conditions and proved global existence and regularity for the corresponding nonlinear problem. Beale [3] subsequently discussed spectral properties of this model. In [2], the authors introduced a two-dimensional model to describe acoustic–structure interaction in a rectangular cavity in which the boundary consists of three rigid walls and one vibrating wall, the latter being modeled by the Euler-Bernoulli beam equation. With respect to the stability and controllability of structural-acoustic models, Lasiecka provided a comprehensive overview in the book [7]. For further results on the asymptotic behavior of related structural-acoustic models, we refer the reader to [8–12] and the references therein.

Note that, in (1.1), the nonlinear source competes with the nonlinear damping. In [13], the authors investigated a wave equation with both a source and a damping term, and proved global existence of solutions provided that the damping term dominates the source term. In contrast, solutions may blow-up when the source dominates the damping. Analogous results for wave equations with nonlinear boundary damping and nonlinear sources were obtained in [14, 15]. Bociu and Lasiecka [16] studied a wave equation with supercritical interior and boundary sources, together with damping terms, and proved local Hadamard well-posedness of finite-energy (weak) solutions by allowing for more singular sources. Moreover, uniqueness of weak solutions for semilinear wave equations with supercritical boundary/interior sources and damping was established in [17], while blow-up results were obtained in [18]. Guo and Rammaha studied a coupled wave equation with damping and a supercritical source; Hadamard well-posedness, energy decay for potential well solutions, and blow-up of weak solutions were established in [19–21]. For further contributions on wave equations with supercritical source terms, we refer to [22–30], to name but a few.

For system (1.1), Hadamard well-posedness was established in [31]. More specifically, existence of local and global weak solutions was obtained by means of a semigroup approach for nonlinear operators. Uniqueness of solutions and continuous dependence on the initial data were also proved therein. In [1], Feng et al. proved the global existence of potential well solutions and derived their energy decay rates of such solutions. The blow-up of weak solutions was proved in [32].

However, in their study of the energy decay rates for (1.1) in [1], the authors assumed that $u \in L^\infty(\mathbb{R}^+; L^{\frac{3}{2}}(\Omega))$ for $m_1 > 5$, which requires a smoother initial datum u_0. The main goal of this paper is to remove the above strong assumptions on u required in [1] to obtain a weaker energy decay. In particular, we establish a logarithmic decay rate.

The paper is organized as follows. In Section 2, we present the preliminaries, recall some known results, and state our main theorem on energy decay. In Section 3, we provide the proof.

2. Preliminaries and main result

2.1. Notations and assumptions

We adopt the notation of [1]. For $1 \leq p \leq \infty$, we write $\|u\|_p := \|u\|_{L^p(\Omega)}$ and $|u|_p := \|u\|_{L^p(\Gamma)}$. Moreover, $(u,v)_\Omega := (u,v)_{L^2(\Omega)}$ and $(u,v)_\Gamma := (u,v)_{L^2(\Gamma)}$ denote the usual L^2 inner products on Ω and on Γ, respectively.

We define the space

$$H^1_{\Gamma_0}(\Omega) := \{u \in H^1(\Omega) : u|_{\Gamma_0} = 0\}. \tag{2.1}$$

By the Poincaré inequality, for $u \in H^1_{\Gamma_0}(\Omega)$, the H^1-norm is equivalent to $\|\nabla u\|_2$. Throughout the paper, $C > 0$ denotes a generic constant whose value may vary from line to line.

Similar to that [1], we impose the following assumptions.

Assumption 2.1. *(I) For $i = 1, 2$, the functions $g_i : \mathbb{R} \to \mathbb{R}$ are continuous and monotone increasing, with $g_i(0) = 0$. Moreover, we assume that there exist positive constants α and β such that, for $|s| \geq 1$,*

$$\alpha |s|^{m_i+1} \leq g_i(s)s \leq \beta |s|^{m_i+1}, \ \text{with } m_i \geq 1.$$

(II) The functions f_i are of class $C^1(\mathbb{R})$ and satisfy

$$|f_i'(s)| \leq C(|s|^{p_i-1} + 1), \ \text{with } 1 \leq p_1 < 6, \ \ p_2 \geq 1.$$

(III) The exponents satisfy

$$p_1 \frac{m_1 + 1}{m_1} < 6.$$

To study the potential well solutions, we should impose additional assumptions on the source terms f_1 and f_2 as in [1].

Assumption 2.2. *For $i = 1, 2$, there exist nonnegative functions $F_i(s) \in C^1(\mathbb{R})$ such that $F_i'(s) = f_i(s)$, and F_i are homogeneous of order $p_i + 1$, that is, $F_i(\lambda s) = \lambda^{p_i+1} F_i(s)$, for $\lambda > 0$, $s \in \mathbb{R}$.*

Remark 2.3. *By the Euler theorem on homogeneous functions, the homogeneity of F_i yields*

$$s f_i(s) = (p_i + 1) F_i(s), \ i = 1, 2. \tag{2.2}$$

Moreover, since $F_i(s)$ are homogeneous, there exists $M > 0$ such that

$$F_i(s) \leq M |s|^{p_i+1}, \ i = 1, 2. \tag{2.3}$$

2.2. Some known results

The quadratic energy associated with (1.1) is defined by

$$E(t) := \frac{1}{2}\left(\|u_t(t)\|_2^2 + |w_t(t)|_2^2 + \|\nabla u(t)\|_2^2 + |\Delta w(t)|_2^2\right). \tag{2.4}$$

In particular, one has the energy identity

$$\begin{aligned}E(t) &+ \int_0^t \int_\Omega g_1(u_t)u_t dx d\tau + \int_0^t \int_\Gamma g_2(w_t)w_t d\Gamma d\tau \\ &= E(0) + \int_0^t \int_\Omega f_1(u)u_t dx d\tau + \int_0^t \int_\Gamma f_2(w)w_t d\Gamma d\tau. \qquad (2.5)\end{aligned}$$

We define total energy of (1.1) by

$$\begin{aligned}\mathcal{E}(t) : &= E(t) - \int_\Omega F_1(u)dx - \int_\Gamma F_2(w)d\Gamma \\ &= \frac{1}{2}\left(\|u_t(t)\|_2^2 + |w_t(t)|_2^2 + \|\nabla u(t)\|_2^2 + |\Delta w(t)|_2^2\right) - \int_\Omega F_1(u)dx - \int_\Gamma F_2(w)d\Gamma, \qquad (2.6)\end{aligned}$$

which implies that

$$\mathcal{E}(t) + \int_0^t \int_\Omega g_1(u_t)u_t dx d\tau + \int_0^t \int_\Gamma g_2(w_t)w_t d\Gamma d\tau = \mathcal{E}(0), \quad \text{for all } t \in [0, T). \qquad (2.7)$$

Now we define the potential energy functional $J(t)$ by

$$J(u, w) = \frac{1}{2}(\|\nabla u\|_2^2 + |\Delta w|_2^2) - \int_\Omega F_1(u)dx - \int_\Gamma F_2(w)d\Gamma,$$

and the Nehari manifold by

$$N = \{(u, w) \in \mathbb{H}\backslash\{(0, 0)\} : \langle J'(u, w), (u, w)\rangle = 0\}, \qquad (2.8)$$

where $\mathbb{H} = H^1_{\Gamma_0}(\Omega) \times H^2_0(\Gamma)$, and the Fréchet derivative of J at $(u, w) \in \mathbb{H}$ is given by

$$\langle J'(u, w), (\phi, \psi)\rangle = \int_\Omega \nabla u \cdot \nabla\phi dx + \int_\Gamma \Delta w \cdot \Delta\psi d\Gamma - \int_\Omega f_1(u)\phi dx - \int_\Gamma f_2(w)\psi d\Gamma,$$

for $(\phi, \psi) \in \mathbb{H}$. This together with (2.2) yields

$$N = \left\{(u, w) \in \mathbb{H}\backslash\{(0, 0)\} : \|\nabla u\|_2^2 + |\Delta w|_2^2 = (p_1 + 1)\int_\Omega F_1(u)dx + (p_2 + 1)\int_\Gamma F_2(w)d\Gamma\right\}, \qquad (2.9)$$

The potential well is defined by

$$W := \{(u, w) \in \mathbb{H} : J(u, w) < d\}, \qquad (2.10)$$

where the *depth* d of the well W is given by

$$d := \inf_{(u,w)\in N} J(u, w). \qquad (2.11)$$

We further decompose W into two disjoint subsets W_1 and W_2:

$$W_1 = \left\{(u, w) \in W : \|\nabla u\|_2^2 + |\Delta w|_2^2 > (p_1 + 1)\int_\Omega F_1(u)dx + (p_2 + 1)\int_\Gamma F_2(w)d\Gamma\right\} \cup \{(0, 0)\},$$

$$W_2 = \left\{(u, w) \in W : \|\nabla u\|_2^2 + |\Delta w|_2^2 < (p_1 + 1) \int_\Omega F_1(u)dx + (p_2 + 1) \int_\Gamma F_2(w)d\Gamma\right\}.$$

We refer to W_1 as the *stable* part and to W_2 as the *unstable* part. Clearly, $W_1 \cap W_2 = \varnothing$ and $W_1 \cup W_2 = W$.

We now give the definition of a weak solution of (1.1).

Definition 2.4. *A pair* (u, w) *is said to be a weak solution to* (1.1) *on* $[0, T]$ *if*

$$\begin{aligned} &u \in C([0, T]; H^1_{\Gamma_0}(\Omega)), \qquad u_t \in C([0, T]; L^2(\Omega)) \cap L^{m_1+1}(\Omega \times (0, T)),\\ &w \in C([0, T]; H^2_0(\Gamma)), \qquad w_t \in C([0, T]; L^2(\Gamma)) \cap L^{m_2+1}(\Gamma \times (0, T)),\end{aligned}$$

the initial data satisfy

$$(u(0), u_t(0)) = (u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega), \qquad (w(0), w_t(0)) = (w_0, w_1) \in H^2_0(\Gamma) \times L^2(\Gamma),$$

and the following variational identities hold for all $t \in [0, T]$. *For every test function*

$$v \in C([0, T]; H^1_{\Gamma_0}(\Omega)) \cap L^{m_1+1}(\Omega \times (0, T)), \qquad v_t \in L^1(0, T; L^2(\Omega)),$$

we have

$$\begin{aligned} &(u_t(t), v(t))_\Omega - (u_1, v(0))_\Omega - \int_0^t (u_t(s), v_t(s))_\Omega\, ds + \int_0^t (\nabla u(s), \nabla v(s))_\Omega\, ds\\ &\quad - \int_0^t (w_t(s), \gamma v(s))_\Gamma\, ds + \int_0^t\int_\Omega g_1(u_t(s))\, v(s)\, dx\, ds = \int_0^t\int_\Omega f_1(u(s))\, v(s)\, dx\, ds, \end{aligned} \tag{2.12}$$

where γu *denotes the trace of* u *on* Γ. *For every test function*

$$z \in C([0, T]; H^2_0(\Gamma)), \qquad z_t \in L^1(0, T; L^2(\Gamma)),$$

we have

$$\begin{aligned} &(w_t(t) + \gamma u(t), z(t))_\Gamma - (w_1 + \gamma u(0), z(0))_\Gamma - \int_0^t (w_t(s), z_t(s))_\Gamma\, ds - \int_0^t (\gamma u(s), z_t(s))_\Gamma\, ds\\ &\quad + \int_0^t (\Delta w(s), \Delta z(s))_\Gamma\, ds + \int_0^t\int_\Gamma g_2(w_t(s))\, z(s)\, d\Gamma\, ds = \int_0^t\int_\Gamma f_2(w(s))\, z(s)\, d\Gamma\, ds. \end{aligned} \tag{2.13}$$

The existence of global potential well solutions is summarized in the following theorem, proved in [1].

Theorem 2.5. *([1]) Let Assumptions 2.1 and 2.2 hold. Assume that* $1 < p_1 \le 5$ *and* $p_2 > 1$. *Assume further that* $(u_0, w_0) \in W_1$ *and* $\mathcal{E}(0) < d$. *Then, problem* (1.1) *admits a global weak solution* (u, w). *Moreover, for any* $t \ge 0$,

- *(i)* $J(u(t), w(t)) \le \mathcal{E}(t) \le \mathcal{E}(0) < d$;
- *(ii)* $(u(t), w(t)) \in W_1$;

(iii) $E(t) < \dfrac{c\,d}{c-2}$;

(iv) $\dfrac{c-2}{c}\,E(t) \le \mathcal{E}(t) \le E(t)$,

where $c = \min\{p_1+1, p_2+1\} > 2$.

Since $H^1_{\Gamma_0}(\Omega) \hookrightarrow L^{p_1+1}(\Omega)$ for $1 < p_1 \le 5$ and $H^2_0(\Gamma) \hookrightarrow L^{p_2+1}(\Gamma)$ for $p_2 > 1$, we introduce the optimal embedding constants:

$$K_1 := \sup_{u\in H^1_{\Gamma_0}(\Omega)\setminus\{0\}} \frac{\|u\|^{p_1+1}_{p_1+1}}{\|\nabla u\|_2^{p_1+1}}, \quad K_2 := \sup_{w\in H^2_0(\Gamma)\setminus\{0\}} \frac{|w|^{p_2+1}_{p_2+1}}{|\Delta w|_2^{p_2+1}}. \tag{2.14}$$

Combining (2.3) and (2.14), we obtain

$$\begin{aligned} J(u,w) &\ge \frac{1}{2}(\|\nabla u\|_2^2 + |\Delta w|_2^2) - M(\|u\|^{p_1+1}_{p_1+1} + |w|^{p_2+1}_{p_2+1}) \\ &\ge \frac{1}{2}(\|\nabla u\|_2^2 + |\Delta w|_2^2) - MK_1\|\nabla u\|_2^{p_1+1} - MK_2|\Delta w|_2^{p_2+1} \\ &\ge \frac{1}{2}\,\|(u,w)\|^2_{\mathbb{H}} - MK_1\|(u,w)\|^{p_1+1}_{\mathbb{H}} - MK_2\|(u,w)\|^{p_2+1}_{\mathbb{H}}. \end{aligned} \tag{2.15}$$

By introducing the function

$$G(s) := \frac{1}{2}s^2 - MK_1 s^{p_1+1} - MK_2 s^{p_2+1}, \tag{2.16}$$

we get from (2.15) that

$$J(u,w) \ge G(\|(u,w)\|_{\mathbb{H}}), \quad \text{for any } (u,v) \in \mathbb{H}. \tag{2.17}$$

Since $p_1, p_2 > 1$, we have

$$G'(s) = s[1 - MK_1(p_1+1)s^{p-1} - MK_2(p_2+1)s^{p_2-1}].$$

Thus, G' has a unique positive zero s_0, where s_0 is the unique root of

$$MK_1(p_1+1)s^{p_1-1} + MK_2(p_2+1)s^{p_2-1} = 1. \tag{2.18}$$

It is easy to check that $G(s)$ has a maximum value at s_0 on $[0,\infty)$, that is,

$$\sup_{s\in[0,\infty)} G(s) = G(s_0) > 0.$$

Let

$$\tilde{W}_1 := \{(u,w) \in \mathbb{H} : \|(u,w)\|_{\mathbb{H}} < s_0, J(u,w) < G(s_0)\}. \tag{2.19}$$

For any small $\theta > 0$, a closed subset of $\tilde{W}_1$ is defined by

$$\tilde{W}_1^{\theta} := \{(u,w) \in \mathbb{H} : \|(u,w)\|_{\mathbb{H}} \le s_0 - \theta,\ J(u,w) \le G(s_0-\theta)\}. \tag{2.20}$$

Clearly,

$$\tilde{W}_1^{\theta} \subset \tilde{W}_1 \subset W_1.$$

2.3. Main result

We now state the main result of the paper.

Theorem 2.6. *Let $1 < p_1 \leq 5$ and assume that Assumptions 2.1 and 2.2 hold. Suppose moreover that $g_i(s)$ $(i = 1, 2)$ are linearly bounded near the origin, i.e., there exist constants $\alpha_i > 0$ and $\beta_i > 0$ such that*

$$\alpha_i|s| \leq g_i(s) \leq \beta_i|s|, \quad i = 1, 2, \quad \forall\, |s| < 1.$$

Fix $\theta > 0$ sufficiently small satisfying (3.31), *and let $u_0 \in W_1^\theta$ with $\mathcal{E}(0) \leq G(s_0 - \theta)$.*
If $m_1 > 5$, assume in addition that $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing C^1-function and that

$$\alpha(t) = \int_0^t \xi(s)ds \;\; with \;\; \lim_{t\to+\infty} \alpha(t) = +\infty, \tag{2.21}$$

where $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-increasing differentiable function with $\xi(0) > 0$, and $(1+t)\xi(t)$ is uniformly bounded for $t \geq 0$. Then, there exist positive constants C and δ such that

$$E(t) \leq \frac{c}{c-2}\mathcal{E}(t) \leq C\mathcal{E}(0)\left(\frac{1+\mu}{1+\mu\delta\alpha(t)}\right)^{\frac{1}{\mu}}, \;\; \forall\, t \geq 0, \tag{2.22}$$

where $\mu = \frac{1}{m_1}$. In particular, we get the logarithmic energy decay

$$\frac{c-2}{c}E(t) \leq \mathcal{E}(t) \leq C\mathcal{E}(0)\Big[\log(2+t)\Big]^{-\frac{1}{\mu}}, \;\; \forall\, t \geq 0.$$

3. Proof of main result

In this section, we prove Theorem 2.6. To this end, we recall the following lemma from [33], which will be crucial in the proof.

Lemma 3.1. *([33]) Suppose that $h : \mathbb{R}^+ \to \mathbb{R}^+$ is a decreasing function and $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing C^1-function satisfying $\beta(0) = 0$ and $\lim_{t\to+\infty}\beta(t) = +\infty$. Assume that there exist constants $\zeta > 0$ and $\rho > 0$ such that*

$$\int_t^{+\infty} \beta'(s)\Big[h(s)\Big]^{1+\zeta}ds \leq \frac{1}{\rho}\Big[h(0)\Big]^{\zeta}h(t).$$

Then, h satisfies the following decay estimate:

$$h(t) \leq h(0)\left(\frac{1+\zeta}{1+\rho\zeta\beta(t)}\right)^{\frac{1}{\zeta}}, \;\; \forall\, t \geq 0.$$

Proof of Theorem 2.6. By the regularity of weak solutions in Definition 2.4, we have $u_t \in L^{m_1+1}(\Omega \times (0, T))$. Noting $u_0 \in L^{m_1+1}(\Omega)$, it follows that

$$\begin{aligned}\int_0^T \int_\Omega |u|^{m_1+1}dxdt &= \int_0^T \int_\Omega \left|\int_0^t u_t(\tau)d\tau + u_0\right|^{m_1+1} dxdt \\ &\leq C(T^{m_1+1}\|u_t(t)\|^{m_1+1}_{L^{m_1+1}(\Omega\times(0,T))} + T\|u_0\|^{m_1+1}_{m_1+1}) < \infty,\end{aligned} \tag{3.1}$$

which gives us $u \in L^{m_1+1}(\Omega\times(0,T))$. Hence, replacing v by $\xi(t)u(t)\mathcal{E}^{\mu}(t)$ in (2.12) and z by $\xi(t)w(t)\mathcal{E}^{\mu}(t)$ in (2.13), where $\mu \geq 0$ will be determined later, we deduce that

$$\begin{aligned}
&\xi(T)\mathcal{E}^{\mu}(T)\int_{\Omega} u_t(T)u(T)dx - \xi(S)\mathcal{E}^{\mu}(S)\int_{\Omega} u_t(S)u(S)dx - \int_S^T \xi(t)\mathcal{E}^{\mu}(t)\|u_t\|_2^2 dt\\
&-\int_S^T [\xi(t)\mathcal{E}^{\mu}(t)]_t\int_{\Omega} u_t u dxdt + \int_S^T \xi(t)\mathcal{E}^{\mu}(t)\|\nabla u\|_2^2 dt - \int_S^T \xi(t)\mathcal{E}^{\mu}(t)\int_{\Gamma} uw_t d\Gamma dt\\
&+\int_S^T \xi(t)\mathcal{E}^{\mu}(t)\int_{\Omega} |u_t|^{m_1-1}u_t u dxdt + \xi(T)\mathcal{E}^{\mu}(T)\int_{\Gamma} w_t(T)w(T)d\Gamma\\
&-\xi(S)\mathcal{E}^{\mu}(S)\int_{\Gamma} w_t(S)w(S)d\Gamma - \int_S^T \xi(t)\mathcal{E}^{\mu}(t)|w_t|_2^2 dt - \int_S^T [\xi(t)\mathcal{E}^{\mu}(t)]_t\int_{\Gamma} w_t w d\Gamma dt\\
&+\int_S^T \xi(t)\mathcal{E}^{\mu}(t)|\Delta w|_2^2 dt + \xi(T)\mathcal{E}^{\mu}(T)\int_{\Gamma} w(T)u(T)d\Gamma - \xi(S)\mathcal{E}^{\mu}(S)\int_{\Gamma} w(S)u(S)d\Gamma\\
&-\int_S^T \xi(t)\mathcal{E}^{\mu}(t)\int_{\Gamma} uw_t d\Gamma dt - \int_S^T [\xi(t)\mathcal{E}^{\mu}(t)]_t\int_{\Gamma} uw d\Gamma dt\\
&+\int_S^T \xi(t)\mathcal{E}^{\mu}(t)\int_{\Gamma} |w_t|^{m_2-1}w_t w d\Gamma dt = \int_S^T \xi(t)\mathcal{E}^{\mu}(t)\left(\int_{\Omega} f_1(u)u dx + \int_{\Gamma} f_2(w)w d\Gamma\right)dt,
\end{aligned}$$

which along with (2.6) yields

$$\begin{aligned}
2\int_S^T \xi(t)\mathcal{E}^{\mu+1}(t)dt = &-\xi(T)\mathcal{E}^{\mu}(T)\int_{\Omega} u_t(T)u(T)dx + \xi(S)\mathcal{E}^{\mu}(S)\int_{\Omega} u_t(S)u(S)dx\\
&-\xi(T)\mathcal{E}^{\mu}(T)\int_{\Gamma} w_t(T)w(T)d\Gamma + \xi(S)\mathcal{E}^{\mu}(S)\int_{\Gamma} w_t(S)w(S)d\Gamma\\
&-\xi(T)\mathcal{E}^{\mu}(T)\int_{\Gamma} w(T)u(T)d\Gamma + \xi(S)\mathcal{E}^{\mu}(S)\int_{\Gamma} w(S)u(S)d\Gamma\\
&+\int_S^T [\xi(t)\mathcal{E}^{\mu}(t)]_t\int_{\Omega} u_t u dxdt + \int_S^T [\xi(t)\mathcal{E}^{\mu}(t)]_t\int_{\Gamma} w_t w d\Gamma dt\\
&+\int_S^T [\xi(t)\mathcal{E}^{\mu}(t)]_t\int_{\Gamma} uw d\Gamma dt + 2\int_S^T \xi(t)\mathcal{E}^{\mu}(t)\|u_t\|_2^2 dt + 2\int_S^T \xi(t)\mathcal{E}^{\mu}(t)|w_t|_2^2 d\Gamma\\
&+2\int_S^T \xi(t)\mathcal{E}^{\mu}(t)\int_{\Gamma} uw_t d\Gamma dt - \int_S^T \xi(t)\mathcal{E}^{\mu}(t)\int_{\Omega} g_1(u_t)u dxdt\\
&-\int_S^T \xi(t)\mathcal{E}^{\mu}(t)\int_{\Gamma} g_2(w_t)w d\Gamma dt\\
&+\int_S^T \xi(t)\mathcal{E}^{\mu}(t)\left((p-1)\int_{\Omega} F_1(u)dx + (q-1)\int_{\Gamma} F_2(w)d\Gamma\right)dt. \qquad (3.2)
\end{aligned}$$

In what follows, we shall estimate each term on the right-hand side of (3.2) so that we can use Lemma 3.1 to prove our main result.

Using that $\mathcal{E}(t)$ and $\xi(t)$ are nonincreasing, and applying the embedding inequality together with (2.5), we obtain that

$$
\begin{aligned}
&-\xi(T)\mathcal{E}^{\mu}(T)\int_{\Omega}u_t(T)u(T)dx+\xi(S)\mathcal{E}^{\mu}(S)\int_{\Omega}u_t(S)u(S)dx\\
&\le\frac{1}{2}\xi(0)\mathcal{E}^{\mu}(0)(\|u_t(T)\|_2^2+\|u(T)\|_2^2+\|u_t(S)\|_2^2+\|u(S)\|_2^2)\\
&\le\frac{1}{2}\xi(0)\mathcal{E}^{\mu}(0)(\|u_t(T)\|_2^2+c_0\|\nabla u(T)\|_2^2+\|u_t(S)\|_2^2+c_0\|\nabla u(S)\|_2^2)\\
&\le\frac{1}{2}\xi(0)\mathcal{E}^{\mu}(0)(2c_0+2)(E(T)+E(S))\le\xi(0)\mathcal{E}^{\mu}(0)(2c_0+2)\frac{c}{c-2}\mathcal{E}(S),
\end{aligned}
\tag{3.3}
$$

where $c_0>0$ is the constant such that $\|u\|_2^2\le c_0\|\nabla u\|_2^2$.

Similarly, we can get

$$
\begin{aligned}
&-\xi(T)\mathcal{E}^{\mu}(T)\int_{\Gamma}w_t(T)w(T)d\Gamma+\xi(S)\mathcal{E}^{\mu}(S)\int_{\Gamma}w_t(S)w(S)d\Gamma\\
&\le\xi(0)\mathcal{E}^{\mu}(0)(2c_1+2)\frac{c}{c-2}\mathcal{E}(S),
\end{aligned}
\tag{3.4}
$$

and

$$
\begin{aligned}
&-\xi(T)\mathcal{E}^{\mu}(T)\int_{\Gamma}w(T)u(T)d\Gamma+\xi(S)\mathcal{E}^{\mu}(S)\int_{\Gamma}w(S)u(S)d\Gamma\\
&\le\xi(0)\mathcal{E}^{\mu}(0)(2c_1+2c_*)\frac{c}{c-2}\mathcal{E}(S),
\end{aligned}
\tag{3.5}
$$

where $c_1>0$ and $c_*>0$ are constants such that $|w|_2^2\le c_1|\Delta w|_2^2$ and $|\gamma u|_2^2\le c_*\|\nabla u\|_2^2$, respectively.

Note that

$$
\begin{aligned}
&\int_S^T[\xi(t)\mathcal{E}^{\mu}(t)]_t\int_{\Omega}u_tudxdt\\
&=\int_S^T\xi'(t)\mathcal{E}^{\mu}(t)\int_{\Omega}u_tudxdt+\mu\int_S^T\xi(t)\mathcal{E}^{\mu-1}(t)\mathcal{E}'(t)\int_{\Omega}u_tudxdt.
\end{aligned}
$$

By Young's inequality and (2.5), we obtain

$$
\begin{aligned}
&\left|\int_S^T\xi'(t)\mathcal{E}^{\mu}(t)\int_{\Omega}u_tudxdt\right|=-\int_S^T\xi'(t)\mathcal{E}^{\mu}(t)\int_{\Omega}u_tudxdt\\
&\le-\frac{1}{2}\int_S^T\xi'(t)\mathcal{E}^{\mu}(t)(c_0\|\nabla u\|_2^2+\|u_t\|_2^2)dt\\
&\le-(1+c_0)\int_S^T\xi'(t)\mathcal{E}^{\mu}(t)E(t)dt\le-(1+c_0)\frac{c}{c-2}\int_S^T\xi'(t)\mathcal{E}^{\mu+1}(t)dt\\
&=-(1+c_0)\frac{c}{c-2}\left(\xi(t)\mathcal{E}^{\mu+1}(t)\Big|_S^T-\int_S^T(\mu+1)\mathcal{E}^{\mu}(t)\mathcal{E}'(t)\xi(t)\right)dt\\
&\le(1+c_0)\frac{c}{c-2}\mathcal{E}^{\mu}(0)\xi(0)\mathcal{E}(S).
\end{aligned}
\tag{3.6}
$$

and

$$
\begin{aligned}
&\mu\int_S^T \xi(t)\mathcal{E}^{\mu-1}(t)\mathcal{E}'(t)\int_\Omega u_t u\,dxdt\\
&\leq -\frac{\mu}{2}\int_S^T \xi(t)\mathcal{E}^{\mu-1}(t)\mathcal{E}'(t)(c_0\|\nabla u\|_2^2+\|u_t\|_2^2)dt\\
&\leq -\frac{\mu}{2}(2+2c_0)\int_S^T \xi(t)\mathcal{E}^{\mu-1}(t)\mathcal{E}'(t)E(t)dt \leq -\frac{\mu}{2}(2+2c_0)\frac{c}{c-2}\int_S^T \xi(t)\mathcal{E}^{\mu}(t)\mathcal{E}'(t)dt\\
&\leq \frac{\mu}{\mu+1}(1+c_0)\frac{c}{c-2}\mathcal{E}^{\mu}(0)\xi(0)\mathcal{E}(S).
\end{aligned}
\tag{3.7}
$$

Hence, combining (3.6) and (3.7) we obtain

$$
\int_S^T [\xi(t)\mathcal{E}^{\mu}(t)]_t\int_\Omega u_t u\,dxdt \leq \frac{c(1+c_0)}{c-2}\cdot\frac{2\mu+1}{\mu+1}\mathcal{E}^{\mu}(0)\xi(0)\mathcal{E}(S).
\tag{3.8}
$$

Similarly, we have

$$
\int_S^T [\xi(t)\mathcal{E}^{\mu}(t)]_t\int_\Gamma w_t w\,d\Gamma dt \leq \frac{c(1+c_1)}{c-2}\cdot\frac{2\mu+1}{\mu+1}\mathcal{E}^{\mu}(0)\xi(0)\mathcal{E}(S),
\tag{3.9}
$$

and

$$
\int_S^T [\xi(t)\mathcal{E}^{\mu}(t)]_t\int_\Gamma uw\,d\Gamma dt \leq \frac{c(c_*+c_1)}{c-2}\cdot\frac{2\mu+1}{\mu+1}\mathcal{E}^{\mu}(0)\xi(0)\mathcal{E}(S).
\tag{3.10}
$$

Let

$$
\Omega_1=\{x\in\Omega : |u_t(x,t)|<1\},\ \ \Omega_2=\{x\in\Omega : |u_t(x,t)|\geq 1\}.
$$

Then,

$$
2\int_S^T \xi(t)\mathcal{E}^{\mu}(t)\|u_t\|_2^2dt = 2\int_S^T \xi(t)\mathcal{E}^{\mu}(t)\int_{\Omega_1} u_t^2dxdt + 2\int_S^T \xi(t)\mathcal{E}^{\mu}(t)\int_{\Omega_2} u_t^2dxdt.
\tag{3.11}
$$

Now, Assumption 2.1 and (2.7) imply

$$
\begin{aligned}
&2\int_S^T \xi(t)\mathcal{E}^{\mu}(t)\int_{\Omega_1} u_t^2dxdt \leq \frac{2}{\alpha_1}\int_S^T \xi(t)\mathcal{E}^{\mu}(t)\int_{\Omega_1} g_1(u_t)u_t\,dxdt\\
&\leq \frac{2}{\alpha_1}\int_S^T \xi(t)\mathcal{E}^{\mu}(t)\int_{\Omega} g_1(u_t)u_t\,dxdt \leq \frac{2}{\alpha_1}\int_S^T \xi(t)\mathcal{E}^{\mu}(t)(-\mathcal{E}'(t))dt\\
&\leq \frac{2}{\alpha_1(\mu+1)}\xi(0)\mathcal{E}^{\mu}(0)\mathcal{E}(S).
\end{aligned}
\tag{3.12}
$$

Since $|u_t|^2 \leq |u_t|^{m_1+1}$ on the set $|u_t|>1$ and $m_1>5$, we obtain

$$
\begin{aligned}
&2\int_S^T \xi(t)\mathcal{E}^\mu(t)\int_{\Omega_2} u_t^2 dxdt \leq 2\int_S^T \xi(t)\mathcal{E}^\mu(t)\int_{\Omega_2} |u_t|^{m_1+1} dxdt\\
&\leq \frac{2}{\alpha}\int_S^T \xi(t)\mathcal{E}^\mu(t)\int_{\Omega_2} g_1(u_t)u_t dxdt \leq \frac{2}{\alpha}\int_S^T \xi(t)\mathcal{E}^\mu(t)\int_{\Omega} g_1(u_t)u_t dxdt\\
&\leq \frac{2}{\alpha(\mu+1)}\xi(0)\mathcal{E}^\mu(0)\mathcal{E}(S).
\end{aligned} \tag{3.13}
$$

Substituting (3.12) and (3.13) into (3.11), it follows that

$$
2\int_S^T \xi(t)\mathcal{E}^\mu(t)\|u_t\|^2 dt \leq \frac{2\alpha+2\alpha_1}{\alpha\alpha_1(\mu+1)}\xi(0)\mathcal{E}^\mu(0)\mathcal{E}(S). \tag{3.14}
$$

Similarly, we can get

$$
2\int_S^T \xi(t)\mathcal{E}^\mu(t)|w_t|^2 d\Gamma \leq \frac{2\alpha+2\alpha_1}{\alpha\alpha_1(\mu+1)}\xi(0)\mathcal{E}^\mu(0)\mathcal{E}(S). \tag{3.15}
$$

By using Young's inequality, we get for any $\varepsilon > 0$,

$$
\begin{aligned}
&2\int_S^T \xi(t)\mathcal{E}^\mu(t)\int_\Gamma u w_t d\Gamma dt\\
&\leq \int_S^T \xi(t)\mathcal{E}^\mu(t)(\varepsilon\|\nabla u\|_2^2 + C(\varepsilon)|w_t|_2^2)dt\\
&= \varepsilon\int_S^T \xi(t)\mathcal{E}^\mu(t)\|\nabla u\|^2 dt + C(\varepsilon)\int_S^T \xi(t)\mathcal{E}^\mu(t)|w_t|^2 dt\\
&\leq \frac{2c}{c-2}\varepsilon\int_S^T \xi(t)\mathcal{E}^{\mu+1}(t)dt + C(\varepsilon)\int_S^T \xi(t)\mathcal{E}^\mu(t)|w_t|^2 dt,
\end{aligned}
$$

which together with (3.15) implies

$$
2\int_S^T \xi(t)\mathcal{E}^\mu(t)\int_\Gamma u w_t d\Gamma dt \leq \frac{2c}{c-2}\varepsilon\int_S^T \xi(t)\mathcal{E}^{\mu+1}(t)dt + C(\varepsilon)\xi(0)\mathcal{E}^\mu(0)\mathcal{E}(S). \tag{3.16}
$$

We denote

$$
A_\Gamma = \{x \in \Gamma : |w_t(x,t)| < 1\}, \quad B_\Gamma = \{x \in \Gamma : |w_t(x,t)| \geq 1\}.
$$

Then,

$$
\begin{aligned}
&-\int_S^T \xi(t)\mathcal{E}^\mu(t)\int_\Gamma g_2(w_t)w d\Gamma dt\\
&= -\int_S^T \xi(t)\mathcal{E}^\mu(t)\int_{A_\Gamma} g_2(w_t)w d\Gamma dt - \int_S^T \xi(t)\mathcal{E}^\mu(t)\int_{B_\Gamma} g_2(w_t)w d\Gamma dt.
\end{aligned}
$$

Since $g_2(s)$ is linearly bounded near the origin, then by using Young's inequality and noting (2.4), we have for any $\varepsilon > 0$,

$$
\begin{aligned}
-\int_S^T \xi(t)\mathcal{E}^{\mu}(t)\int_{A_\Gamma} g_2(w_t)w\,d\Gamma dt &\le \int_S^T \xi(t)\mathcal{E}^{\mu}(t)\int_\Gamma |w_t||w|d\Gamma dt\\
&\le \varepsilon\int_S^T \xi(t)\mathcal{E}^{\mu}(t)E(t)dt + C(\varepsilon)\int_S^T \xi(t)\mathcal{E}^{\mu}(t)|w_t|_2^2dt,
\end{aligned}
$$

which along with Theorem 2.5 and (3.15) implies

$$
-\int_S^T \xi(t)\mathcal{E}^{\mu}(t)\int_{A_\Gamma} g_2(w_t)w\,d\Gamma dt \le \frac{2c}{c-2}\varepsilon\int_S^T \xi(t)\mathcal{E}^{\mu+1}(t)dt + C(\varepsilon)\xi(0)\mathcal{E}^{\mu}(0)\mathcal{E}(S). \tag{3.17}
$$

From Assumption 2.1 and Hölder's inequality, it is inferred that

$$
\begin{aligned}
&-\int_S^T \xi(t)\mathcal{E}^{\mu}(t)\int_{B_\Gamma} g_2(w_t)w\,d\Gamma dt \le \int_S^T \xi(t)\mathcal{E}^{\mu}(t)\int_{B_\Gamma} |g_2(w_t)w|d\Gamma dt\\
&\le \int_S^T \xi(t)\mathcal{E}^{\mu}(t)\left(\int_{B_\Gamma}|w|^{m_2+1}d\Gamma\right)^{\frac{1}{m_2+1}}\left(\int_{B_\Gamma}|g_2(w_t)|^{\frac{m_2+1}{m_2}}d\Gamma\right)^{\frac{m_2}{m_2+1}}dt\\
&\le \beta^{\frac{1}{m_2+1}}\int_S^T \xi(t)\mathcal{E}^{\mu}(t)\left(\int_\Gamma|w|^{m_2+1}d\Gamma\right)^{\frac{1}{m_2+1}}\left(\int_\Gamma|g_2(w_t)|w_t d\Gamma\right)^{\frac{m_2}{m_2+1}}dt,
\end{aligned}
\tag{3.18}
$$

where we used the fact that $|g_2(s)| \le \beta|s|^{m_2}$ for all $|s| \ge 1$. Noting that $E(t) < \frac{c}{c-2}d$ for all $t \ge 0$, then it follows from the embedding inequality that

$$
\int_\Gamma |w|^{m_2+1}d\Gamma \le C|\Delta w|^{m_2+1} \le CE^{\frac{m_2+1}{2}}(t) \le C(d,m_2)E(t). \tag{3.19}
$$

By Young's inequality, it is concluded from (3.18) and (3.19) that for any $\varepsilon > 0$,

$$
\begin{aligned}
&-\int_S^T \xi(t)\mathcal{E}^{\mu}(t)\int_{B_\Gamma} g_2(w_t)w\,d\Gamma dt\\
&\le C(d,m_2)\int_S^T \xi(t)\mathcal{E}^{\mu}(t)\,(E(t))^{\frac{1}{m_2+1}}\left(\int_\Gamma|g_2(w_t)|w_t d\Gamma\right)^{\frac{m_2}{m_2+1}}dt\\
&\le \int_S^T \xi(t)\mathcal{E}^{\mu}(t)E(t)dt + C(\varepsilon)\int_S^T \xi(t)\mathcal{E}^{\mu}(t)\left(\int_\Gamma|g_2(w_t)|w_t d\Gamma\right)dt\\
&\le \frac{2c}{c-2}\varepsilon\int_S^T \xi(t)\mathcal{E}^{\mu+1}(t) + C(\varepsilon)\int_S^T \xi(t)\mathcal{E}^{\mu}(t)(-\mathcal{E}'(t))dt\\
&\le \frac{2c}{c-2}\varepsilon\int_S^T \xi(t)\mathcal{E}^{\mu+1}(t) + C(\varepsilon)\xi(0)\mathcal{E}^{\mu}(0)\mathcal{E}(S).
\end{aligned}
\tag{3.20}
$$

Then, (3.17) and (3.20) imply that for any $\varepsilon > 0$,

$$
-\int_S^T \xi(t)\mathcal{E}^{\mu}(t)\int_\Gamma g_2(w_t)w\,d\Gamma dt \le \frac{4c}{c-2}\varepsilon\int_S^T \xi(t)\mathcal{E}^{\mu+1}(t) + C(\varepsilon)\xi(0)\mathcal{E}^{\mu}(0)\mathcal{E}(S). \tag{3.21}
$$

As in (3.11), we have

$$\begin{aligned}&-\int_S^T \xi(t)\mathcal{E}^\mu(t)\int_\Omega g_1(u_t)udxdt\\&=-\int_S^T \xi(t)\mathcal{E}^\mu(t)\int_{\Omega_1} g_1(u_t)udxdt-\int_S^T \xi(t)\mathcal{E}^\mu(t)\int_{\Omega_2} g_1(u_t)udxdt.\end{aligned}$$

By Young's inequality and Assumption 2.1, we get for any $\varepsilon > 0$,

$$\begin{aligned}&-\int_S^T \xi(t)\mathcal{E}^\mu(t)\int_{\Omega_1} g_1(u_t)udxdt\\&\le \varepsilon\int_S^T \xi(t)\mathcal{E}^\mu(t)\|u\|_2^2dt+C(\varepsilon)\int_S^T \xi(t)\mathcal{E}^\mu(t)\int_{\Omega_1}|g_1(u_t)|^2dxdt\\&\le c_0\varepsilon\int_S^T \xi(t)\mathcal{E}^\mu(t)\|\nabla u\|_2^2dt+C(\varepsilon)\int_S^T \xi(t)\mathcal{E}^\mu(t)\int_{\Omega_1} g_1(u_t)u_tdxdt\\&\le c_0\varepsilon\int_S^T \xi(t)\mathcal{E}^\mu(t)\|\nabla u\|_2^2dt+C(\varepsilon)\int_S^T \xi(t)\mathcal{E}^\mu(t)\int_{\Omega} g_1(u_t)u_tdxdt,\end{aligned}$$

which along with (2.6) and (2.7) yields

$$\begin{aligned}-\int_S^T \xi(t)\mathcal{E}^\mu(t)\int_{\Omega_1} g_1(u_t)udxdt&\le \frac{2c}{c-2}\varepsilon\int_S^T \xi(t)\mathcal{E}^{\mu+1}(t)dt-C(\varepsilon)\int_S^T \xi(t)\mathcal{E}^\mu(t)\mathcal{E}'(t)dt\\&\le \frac{2c}{c-2}\varepsilon\int_S^T \xi(t)\mathcal{E}^{\mu+1}(t)dt+C(\varepsilon)\xi(0)\mathcal{E}^\mu(0)\mathcal{E}(S).\end{aligned}\tag{3.22}$$

In the following, we no longer assume $u \in L^\infty(\mathbb{R}^+; L^{\frac{3}{2}}(\Omega))$. Following the approach in [34], and using that $u \in C^1([0,T]; L^2(\Omega))$, we obtain that for all $t > 0$,

$$u(x,t) = u_0(x) + \int_0^t u_t(s)ds,\quad a.e.\ x \in \Omega.$$

This leads to, for all $t > 0$,

$$|u(x,t)|^{m_1+1} \le 2^m\left(|u_0|^{m_1+1} + \left|\int_0^t u_t(s)ds\right|^{m_1+1}\right),\quad a.e.\ x \in \Omega.$$

It is inferred from Hölder's inequality that

$$\|u(x,t)\|_{m_1+1}^{m_1+1} \le 2^m\left(\|u_0\|_{m_1+1}^{m_1+1} + t^{m_1}\int_0^t \|u_t(s)\|_{m_1+1}^{m_1+1}ds\right).$$

For $m_1 > 5$ and $|u| < 1$, we have $|u|^{m_1+1} < |u|^2$. Then, by Assumption 2.1 and (2.7), we conclude that

$$\begin{aligned}
\int_0^t \|u_t(s)\|_{m_1+1}^{m_1+1} ds &= \int_0^t \int_{\Omega_1} |u_t(s)|^{m_1+1} dxds + \int_0^t \int_{\Omega_2} |u_t(s)|^{m_1+1} dxds \\
&\le \int_0^t \int_{\Omega_1} |u_t(s)|^2 dxds + \int_0^t \int_{\Omega_2} |u_t(s)|^{m_1+1} dxds \\
&\le \beta_1 \int_0^t \int_{\Omega_1} g_1(u_t)u_t dxds + \beta \int_0^t \int_{\Omega_2} g_1(u_t)u_t dxds \\
&\le (\beta_1 + \beta) \int_0^t \int_{\Omega} g_1(u_t)u_t dxds \le (\beta_1 + \beta)\mathcal{E}(0).
\end{aligned}$$

Therefore, for all $t > 0$,

$$\|u(x,t)\|_{m_1+1} \le 2^{\frac{m_1}{m_1+1}} \left(\|u_0\|_{m_1+1} + (\mathcal{E}(0))^{\frac{1}{m_1+1}} t^{\frac{m_1}{m_1+1}} \right). \tag{3.23}$$

Then, it follows from (2.7) that

$$\begin{aligned}
& - \int_S^T \xi(t)\mathcal{E}^{\mu}(t) \int_{\Omega_2} g_1(u_t) u dxdt \\
& \le \int_S^T \xi(t)\mathcal{E}^{\mu}(t) \left(\int_{\Omega_2} |g_1(u_t)|^{\frac{m_1+1}{m_1}} dx \right)^{\frac{m_1}{m_1+1}} \left(\int_{\Omega_2} |u|^{m_1+1} dx \right)^{\frac{1}{m_1+1}} dt \\
& \le \beta^{\frac{1}{m_1}} \int_S^T \xi(t)\mathcal{E}^{\mu}(t) \left(\int_{\Omega_2} g_1(u_t)u_t dx \right)^{\frac{m_1}{m_1+1}} \left(\int_{\Omega_2} |u|^{m_1+1} dx \right)^{\frac{1}{m_1+1}} dt \\
& \le \beta^{\frac{1}{m_1}} \int_S^T \xi(t)\mathcal{E}^{\mu}(t) \left(\int_{\Omega} g_1(u_t)u_t dx \right)^{\frac{m_1}{m_1+1}} \left(\int_{\Omega} |u|^{m_1+1} dx \right)^{\frac{1}{m_1+1}} dt \\
& \le \beta^{\frac{1}{m_1}+1} \int_S^T \xi(t)\mathcal{E}^{\mu}(t) \Big[-\mathcal{E}'(t) \Big]^{\frac{m_1}{m_1+1}} \|u\|_{m_1+1} dt.
\end{aligned}$$

This together with (3.23) implies

$$\begin{aligned}
& - \int_S^T \xi(t)\mathcal{E}^{\mu}(t) \int_B g_1(u_t) u dxdt \\
& \le 2^{\frac{m_1}{m_1+1}} \beta^{\frac{1}{m_1}+1} \|u_0\|_{m+1} \int_S^T \xi(t)\mathcal{E}^{\mu}(t) \Big[-\mathcal{E}'(t) \Big]^{\frac{m_1}{m_1+1}} dt \\
& \quad + 2^{\frac{m_1}{m_1+1}} \beta^{\frac{1}{m_1}+1} (\mathcal{E}(0))^{\frac{1}{m_1+1}} \int_S^T \xi(t)\mathcal{E}^{\mu}(t) \Big[-\mathcal{E}'(t) \Big]^{\frac{m_1}{m_1+1}} t^{\frac{m_1}{m_1+1}} dt.
\end{aligned} \tag{3.24}$$

In view of

$$\mathcal{E}^{\mu}(t) = \mathcal{E}^{\frac{\mu+1}{m+1}}(t) \cdot \mathcal{E}^{\frac{m_1\mu-1}{m_1+1}}(t),$$

we get

$$
\begin{aligned}
&2^{\frac{m_1}{m_1+1}}\beta^{\frac{1}{m_1}+1}\|u_0\|_{m_1+1}\int_S^T \xi(t)\mathcal{E}^{\mu}(t)\Big[-\mathcal{E}'(t)\Big]^{\frac{m_1}{m_1+1}}dt\\
&\quad = 2^{\frac{m_1}{m_1+1}}\beta^{\frac{1}{m_1}+1}\|u_0\|_{m_1+1}\int_S^T \xi(t)\mathcal{E}^{\frac{\mu+1}{m+1}}(t)\cdot\mathcal{E}^{\frac{m_1\mu-1}{m_1+1}}(t)\Big[-\mathcal{E}'(t)\Big]^{\frac{m_1}{m_1+1}}dt\\
&\quad \le \varepsilon\int_S^T \xi(t)\mathcal{E}^{\mu+1}(t)dt + C(\varepsilon)\int_S^T \xi(t)\mathcal{E}^{\frac{m_1\mu-1}{m_1}}(t)\Big[-\mathcal{E}'(t)\Big]dt,
\end{aligned}
\tag{3.25}
$$

and

$$
\begin{aligned}
&2^{\frac{m_1}{m_1+1}}\beta^{\frac{1}{m_1}+1}(\mathcal{E}(0))^{\frac{1}{m_1+1}}\int_S^T \xi(t)\mathcal{E}^{\mu}(t)\Big[-\mathcal{E}'(t)\Big]^{\frac{m_1}{m_1+1}}t^{\frac{m_1}{m_1+1}}dt\\
&\quad \le \varepsilon\int_S^T \xi(t)\mathcal{E}^{\mu+1}(t)dt + C(\varepsilon)\int_S^T t\xi(t)\mathcal{E}^{\frac{m_1\mu-1}{m_1}}(t)\Big[-\mathcal{E}'(t)\Big]dt,
\end{aligned}
\tag{3.26}
$$

for any $\varepsilon > 0$.

We replace (3.25) and (3.26) in (3.24) and make the assumption $\mu \ge \frac{1}{m_1}$ to derive

$$
\begin{aligned}
&-\int_S^T \xi(t)\mathcal{E}^{\mu}(t)\int_B g_1(u_t)u\,dxdt\\
&\le 2\varepsilon\int_S^T \xi(t)\mathcal{E}^{\mu+1}(t)dt + C(\varepsilon)\mathcal{E}^{\frac{m_1\mu-1}{m_1}}(0)\int_S^T (1+t)\xi(t)\Big[-\mathcal{E}'(t)\Big]dt,
\end{aligned}
$$

for any $\varepsilon > 0$. Since $(1+t)\xi(t)$ is uniformly bounded, then we obtain that for any $\varepsilon > 0$ and $\mu \ge \frac{1}{m_1}$,

$$
-\int_S^T \xi(t)\mathcal{E}^{\mu}(t)\int_B g_1(u_t)u\,dxdt \le 2\varepsilon\int_S^T \xi(t)\mathcal{E}^{\mu+1}(t)dt + C(\varepsilon)\mathcal{E}^{\frac{m_1\mu-1}{m_1}}(0)\mathcal{E}(S).
\tag{3.27}
$$

Combining (3.22) and (3.27), we deduce that, for any $\varepsilon > 0$ and any $\mu \ge \frac{1}{m_1}$, that

$$
\begin{aligned}
&-\int_S^T \xi(t)\mathcal{E}^{\mu}(t)\int_\Omega g_1(u_t)u\,dxdt\\
&\le \left(\frac{2c}{c-2}+2\right)\varepsilon\int_S^T \xi(t)\mathcal{E}^{\mu+1}(t)dt + C(\varepsilon)\xi(0)\mathcal{E}^{\mu}(0)\mathcal{E}(S)\\
&\quad + C(\varepsilon)\mathcal{E}^{\frac{m_1\mu-1}{m}}(0)\mathcal{E}(S).
\end{aligned}
\tag{3.28}
$$

By Remark 2.3, we can estimate as follows:

$$\begin{aligned}
&\int_S^T \xi(t)\mathcal{E}^\mu(t)\left((p_1-1)\int_\Omega F_1(u)dx + (p_2-1)\int_\Gamma F_2(w)d\Gamma\right)dt \\
&\le \int_S^T \xi(t)\mathcal{E}^\mu(t)[M(p-1)\|u\|_{p_1+1}^{p_1+1} + M(p_2-1)|w|_{p_2+1}^{p_2+1}]dt \\
&= \int_S^T \xi(t)\mathcal{E}^\mu(t)\left[\frac{p-1}{p+1}M(p+1)\|u\|_{p_1+1}^{p_1+1} + \frac{p_2-1}{p_2+1}M(p_2+1)|w|_{p_2+1}^{p_2+1}\right]dt \\
&\le \max\left\{\frac{p_1-1}{p_1+1}, \frac{p_2-1}{p_2+1}\right\}\int_S^T \xi(t)\mathcal{E}^\mu(t)\left[M(p_1+1)\|u\|_{p_1+1}^{p_1+1} + M(p_2+1)|w|_{p_2+1}^{p_2+1}\right]dt \\
&\le \max\left\{\frac{p_1-1}{p_1+1}, \frac{p_2-1}{p_2+1}\right\}\int_S^T \xi(t)\mathcal{E}^\mu(t)\left[M(p_1+1)K_1\|\nabla u\|_2^{p_1+1} + M(p_2+1)K_2|\Delta w|_2^{p_2+1}\right]dt \\
&\le \max\left\{\frac{p_1-1}{p_1+1}, \frac{p_2-1}{p_2+1}\right\}\int_S^T \xi(t)\mathcal{E}^\mu(t)(\|\nabla u\|_2^2 + |\Delta w|_2^2) \\
&\qquad\qquad \cdot\left[M(p_1+1)K_1\|(u,w)\|_{\mathbb{H}}^{p_1-1} + M(p_2+1)K_2\|(u,w)\|_{\mathbb{H}}^{p_2-1}\right]dt \\
&\le \frac{2c}{c-2}\cdot\max\left\{\frac{p_1-1}{p_1+1}, \frac{p_2-1}{p_2+1}\right\}\int_S^T \xi(t)\mathcal{E}^{\mu+1}(t) \\
&\qquad \cdot\left[M(p_1+1)K_1\|(u,w)\|_{\mathbb{H}}^{p_1-1} + M(p_2+1)K_2\|(u,w)\|_{\mathbb{H}}^{p_2-1}\right]dt.
\end{aligned} \tag{3.29}$$

Because $(u_0, w_0) \in \tilde{W}_1^\theta$ and $\mathcal{E}(0) \le G(s_0 - \theta)$, we have

$$\|(u(t), w(t))\|_{\mathbb{H}} \le s_0 - \theta, \quad \text{for all } t \ge 0. \tag{3.30}$$

Then, it follows from (3.30), (3.29), and (2.5) that

$$\begin{aligned}
&\int_S^T \xi(t)\mathcal{E}^\mu(t)\left((p_1-1)\int_\Omega F_1(u)dx + (p_2-1)\int_\Gamma F_2(w)d\Gamma\right)dt \\
&\le \frac{2c}{c-2}\cdot\max\left\{\frac{p_1-1}{p_1+1}, \frac{p_2-1}{p_2+1}\right\}\int_S^T \xi(t)\mathcal{E}^{\mu+1}(t) \\
&\qquad \cdot[M(p_1+1)K_1(s_0-\theta)^{p_1-1} + M(p_2+1)K_2(s_0-\theta)^{p_2-1}]dt.
\end{aligned}$$

Thanks to (2.18), we can take $\theta > 0$ such that

$$\begin{aligned}
\nu_1 := \frac{2c}{c-2}\cdot\max\left\{\frac{p_1-1}{p_1+1}, \frac{p_2-1}{p_2+1}\right\}&\cdot[M(p_1+1)K_1(s_0-\theta)^{p_1-1} \\
&+ M(p_2+1)K_2(s_0-\theta)^{p_2-1}] < 2,
\end{aligned} \tag{3.31}$$

which leads to

$$\int_S^T \xi(t)\mathcal{E}^\mu(t)\left((p_1-1)\int_\Omega F_1(u)dx + (p_2-1)\int_\Gamma F_2(w)d\Gamma\right)dt \le \nu_1\int_S^T \xi(t)\mathcal{E}^{\mu+1}(t)dt. \tag{3.32}$$

Finally, substituting (3.3)–(3.5), (3.8)–(3.10), (3.14)–(3.16), (3.21), (3.28), and (3.32) into (3.2), we deduce that for any $\varepsilon > 0$,

$$(2-\nu_1)\int_S^T \xi(t)\mathcal{E}^{\mu+1}(t)dt \le C\varepsilon\int_S^T \mathcal{E}^{\mu+1}(t)dt + C(\varepsilon, \xi(0), \mathcal{E}(0))\mathcal{E}^\mu(0)\mathcal{E}(S).$$

At this point, we select $\varepsilon > 0$ small enough such that $2 - \nu_1 - C\varepsilon = 1 - \nu_1/2 > 0$ and let $T \to +\infty$ to obtain

$$\int_S^{+\infty} \xi(t)\mathcal{E}^{\mu+1}(t)dt \leq \frac{C(\varepsilon, \xi(0), \mathcal{E}(0))}{1 - \nu_1/2}\mathcal{E}^{\mu}(0)\mathcal{E}(S).$$

Now we apply Lemma 3.1 to the above inequality and take $\beta(t) = \displaystyle\int_0^t \xi(s)ds$ to conclude

$$\mathcal{E}(t) \leq \mathcal{E}(0)\left(\frac{1+\mu}{1+\rho\mu\beta(t)}\right)^{\frac{1}{\mu}},$$

where $\rho = \frac{1-\nu_1/2}{C(\varepsilon,\xi(0),\mathcal{E}(0))}$.

In particular, we take

$$\xi(t) = \frac{1}{2+t}.$$

It is straightforward to verify that $\xi(t)$ satisfies the required assumptions. Moreover,

$$\alpha(t) = \int_0^t \frac{1}{2+s}ds = \log(2+t) - \log 2.$$

Therefore, the logarithmic energy decay

$$\frac{c-2}{c}E(t) \leq \mathcal{E}(t) \leq C\mathcal{E}(0)\Big[\log(2+t)\Big]^{-\frac{1}{\mu}}, \quad \forall\, t \geq 0,$$

follows. This completes the proof.

4. Conclusions

In this article, we studied a structural acoustics model with supercritical source and nonlinear damping. The model consists of a wave equation defined on a bounded domain, which is strongly coupled with a Berger plate equation acting on the elastic wall. In the previous paper [1], the authors have obtained some energy decay results of potential well solutions. However, in their results [1, Theorem 2.14], for the case $m_1 > 5$, they need the assumption $u \in L^{\infty}(\mathbb{R}^+; L^{\frac{3}{2}}(\Omega))$, which requires a smoother initial datum u_0.

In this paper we remove the above assumption for $m_1 > 5$ to obtain a weaker energy decay by using the approach in [34], and give a logarithmic decay rate under a weak assumption on the initial data.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

References

1. B. Feng, Y. Guo, M. A. Rammaha, On the asymptotic behavior of solutions to a structural acoustics model, *J. Differ. Equations*, **372** (2023), 315–347. https://doi.org/10.1016/j.jde.2023.06.047
2. H. T. Banks, W. Fang, R. J. Sicox, R. C. Smith, *Approximation Methods for Control of Acoustics/Structure Models with Piezoceramic Actuators*, Contract Report 189578, NASA, 1991.
3. J. T. Beale, Spectral properties of an acoustic boundary condition, *Indiana Univ. Math. J.*, **25** (1976), 895–917. https://www.jstor.org/stable/24891055
4. J. T. Beale, S. I. Rosencrans, Acoustic boundary conditions, *Bull. Am. Math. Soc.*, **80** (1974), 1276–1278. https://doi.org/10.1090/S0002-9904-1974-13714-6
5. M. S. Howe, *Acoustics of Fluid-Structure Interactions*, Cambridge Monographs on Mechanics, Cambridge University Press, Cambridge, 1998. https://doi.org/10.1017/CBO9780511662898
6. P. M. Morse, K. U. Ingard, *Theoretical Acoustics*, McGraw-Hill, New York, 1968.
7. I. Lasiecka, *Mathematical Control Theory of Coupled PDEs*, Society for Industrial and Applied Mathematics (SIAM), 2002. https://doi.org/10.1137/1.9780898717099
8. G. Avalos, I. Lasiecka, Uniform decay rates for solutions to a structural acoustics model with nonlinear dissipation, *Appl. Math. Comput. Sci.*, **8** (1998), 287–312.
9. G. Avalos, I. Lasiecka, Exact controllability of structural acoustic interactions, *J. Math. Pures Appl.*, **82** (2003), 1047–1073. https://doi.org/10.1016/S0021-7824(03)00016-3
10. G. Avalos, I. Lasiecka, Exact controllability of finite energy states for an acoustic wave/plate interaction under the influence of boundary and localized controls, *Adv. Differ. Equations*, **10** (2005), 901–930. https://doi.org/10.57262/ade/1355867823
11. J. Cagnol, I. Lasiecka, C. Lebiedzik, J. P. Zolésio, Uniform stability in structural acoustic models with flexible curved walls, *J. Differ. Equations*, **186** (2002), 88–121. https://doi.org/10.1016/S0022-0396(02)00029-3
12. I. Lasiecka, Boundary stabilization of a 3-dimensional structural acoustic model, *J. Math. Pures Appl.*, **78** (1999), 203–232. https://doi.org/10.1016/S0021-7824(01)80009-X
13. V. Georgiev, G. Todorova, Existence of a solution of the wave equation with nonlinear damping and source terms, *J. Differ. Equations*, **109** (1994), 295–308. https://doi.org/10.1006/jdeq.1994.1051
14. E. Vitillaro, Global existence for the wave equation with nonlinear boundary damping and source terms, *J. Differ. Equations*, **186** (2002), 259–298. https://doi.org/10.1016/S0022-0396(02)00023-2
15. M. M. Cavalcanti,V. N. D. Cavalcanti, I. Lasiecka, Well-posedness and optimal decay rates for the wave equation with nonlinear boundary damping-source interaction, *J. Differ. Equations*, **236** (2007), 407–459. https://doi.org/10.1016/j.jde.2007.02.004

16. L. Bociu, I. Lasiecka, Local Hadamard well-posedness for nonlinear wave equations with supercritical sources and damping, *J. Differ. Equations*, **249** (2010), 654–683. https://doi.org/10.1016/j.jde.2010.03.009

17. L. Bociu, I. Lasiecka, Uniqueness of weak solutions for the semilinear wave equations with supercritical boundary/interior sources and damping, *Discrete Contin. Dyn. Syst.*, **22** (2008), 835–860. https://doi.org/10.3934/dcds.2008.22.835

18. L. Bociu, I. Lasiecka, Blow-up of weak solutions for the semilinear wave equations with nonlinear boundary and interior sources and damping, *Appl. Math.*, **35** (2008), 281–304. https://doi.org/10.4064/am35-3-3

19. Y. Guo, M. A. Rammaha, Systems of nonlinear wave equations with damping and supercritical boundary and interior sources, *Trans. Amer. Math. Soc.*, **366** (2014), 2265–2325. https://www.jstor.org/stable/23813106

20. Y. Guo, M. A. Rammaha, Blow-up of solutions to systems of nonlinear wave equations with supercritical sources, *Appl. Anal.*, **92** (2013), 1101–1115. https://doi.org/10.1080/00036811.2011.649734

21. Y. Guo, M. A. Rammaha, Global existence and decay of energy to systems of wave equations with damping and supercritical sources, *Z. Angew. Math. Phys.*, **64** (2013), 621–658. https://doi.org/10.1007/s00033-012-0252-6

22. A. R. Becklin, M. A. Rammaha, Global solutions to a structure acoustic interaction model with nonlinear sources, *J. Math. Anal. Appl.*, **487** (2020), 123977. https://doi.org/10.1016/j.jmaa.2020.123977

23. V. N. D. Cavalcanti, M. M. Cavalcanti, T. D. Marchiori, C. M. Webler, Asymptotic behaviour of the energy to the viscoelastic wave equation with localized hereditary memory and supercritical source term, *J. Dyn. Differ. Equations*, **35** (2023), 3381–3431. https://doi.org/10.1007/s10884-022-10142-5

24. Y. Guo, M. A. Rammaha, S. Sakuntasathien, E. S. Titi, D. Toundykov, Hadamard well-posedness for a hyperbolic equation of viscoelasticity with supercritical sources and damping, *J. Differ. Equations*, **257** (2014), 3778–3812. https://doi.org/10.1016/j.jde.2014.07.009

25. Y. Guo, M. A. Rammaha, S. Sakuntasathien, Energy decay of a viscoelastic wave equation with supercritical nonlinearities, *Z. Angew. Math. Phys.*, **69** (2018), 65. https://doi.org/10.1007/s00033-018-0961-6

26. Y. Guo, M. A. Rammaha, S. Sakuntasathien, Blow-up of a hyperbolic equation of viscoelasticity with supercritical nonlinearities, *J. Differ. Equations*, **262** (2017), 1956–1979. https://doi.org/10.1016/j.jde.2016.10.037

27. M. Liao, Energy decay of wave equations with infinite memory effects versus supercritical frictional dampings, *Ric. Mat.*, **74** (2025), 1037–1050. https://doi.org/10.1007/s11587-023-00832-7

28. M. Liao, B. Feng, A piezoelectric beam model with nonlinear interior dampings and supercritical sources, *Discrete Contin. Dyna. Syst.*, **52** (2026), 293–341. http://doi.org/10.3934/dcds.2026053

29. P. Pei, M. A. Rammaha, D. Toundykov, Weak solutions and blow-up for wave equations of $p-$Laplacian type with supercritical sources, *J. Math. Phys.*, **56** (2015), 081503. https://doi.org/10.1063/1.4927688

30. M. A. Rammaha, Z. Wilstein, Hadamard well-posedness for wave equations with p-Laplacian damping and supercritical sources, *Adv. Differ. Equations*, **17** (2012), 105–150. https://doi.org/10.57262/ade/1355703099

31. A. R. Becklin, M. A. Rammaha, Hadamard well-posedness for a structure acoustic model with a supercritical source and damping terms, *Evol. Equations Control Theory*, **10** (2021), 797–836. https://doi.org/10.3934/eect.2020093

32. B. Feng, Y. Guo, M. A. Rammaha, Blow-up theorems for a structural acoustics model, *J. Math. Anal. Appl.*, **529** (2024), 127600. https://doi.org/10.1016/j.jmaa.2023.127600

33. P. Martinez, A new method to obtain decay rate estimates for dissipative systems, *ESAIM: Control Optim. Calc. Var.*, **4** (1999), 419–444. https://doi.org/10.1051/cocv:1999116

34. A. Haraux, L. Tebou, Energy decay estimates for the wave equation with supercritical nonlinear damping, *J. Math. Anal. Appl.*, **550** (2025), 129622. https://doi.org/10.1016/j.jmaa.2025.129622

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