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*Research article*

## Affine Yano and perturbed Yano–Ricci solitons on Lorentzian Lie groups

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**Abstract:** In this paper, we defined affine Yano–Ricci solitons and perturbed Yano–Ricci solitons on three-dimensional Lorentzian Lie groups and obtain their complete classification. The major results indicated that all groups  $G_3, \dots, G_7$  admit both affine Yano–Ricci solitons and their perturbed companions, whereas  $G_1$  supports neither. As a corollary, we found that the Lorentzian Lie groups  $G_3, G_5$ , and  $G_6$  are affine Einstein manifolds with respect to the Yano connection.

**Keywords:** affine Yano–Ricci solitons; affine perturbed Yano–Ricci solitons; Lorentzian Lie groups; affine Einstein manifold

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### 1. Introduction

In recent years, the rigid structure of Einstein metrics has been progressively relaxed through the introduction of several wider classes of canonical geometries. Chief among these is Hamilton’s Ricci soliton [1], a self-similar solution of the Ricci flow that generalizes Einstein manifolds by enabling the Ricci tensor to differ from a metric multiple through the Lie derivative of a complete vector field. Originally formulated in the Riemannian realm, the concept has been vigorously transplanted to the Lorentzian setting, where new dynamical phenomena appear. To enlarge the catalogue of Ricci-type solitons, researchers have shifted attention from the Levi–Civita connection to a variety of metric compatible affine connections. Wang gave the first systematic account of affine Ricci solitons on the seven simply connected three dimensional Lorentzian Lie groups when the underlying connection is taken to be the canonical, Kobayashi–Nomizu, or Bott connection attached to a natural product structure [2–4]. Building on these results, we studied algebraic Schouten solitons and Codazzi tensors governed by the Yano connection in [5, 6], while Jiang classified algebraic Schouten solitons with respect to the Bott connection [7]; we also classified the algebraic Schouten solitons that are associated with the perturbed Bott connection in [8]. Moreover, Azami generalized the picture by analyzing “generalized Ricci solitons” tied to the same family of connections [9, 10], and subsequently incorporated the Yano connection into the discussion [11]. These works provide a complete list of

coupling constants and derivation parameters that realize the affine Ricci soliton equation, highlighting the flexibility introduced by the non-Levi-Civita connection.

Yano connections first appeared in the 1950s as a distinguished class of affine connections adapted to an almost product structure on a pseudo Riemannian manifold. In contrast to the Levi–Civita connection, the Yano connection is not necessarily torsion free, as it is constructed so that the almost product structure is parallel, and the metric compatibility condition is weakened in a controlled way. This makes the Yano connection a natural tool for studying geometries in which a preferred splitting of the tangent bundle is present. Early work concentrated on the tangent bundle setting. Etayo and Santamaría gave a systematic description of Yano-type connections on manifolds endowed with an almost-product or complex structure, clarifying their torsion and curvature properties [12]. In particular, they showed how the Yano connection can be expressed in terms of the Levi–Civita connection and the covariant derivative of the almost product operator. In research on Yano connections on Lie groups, activity has shifted toward explicit classifications of Ricci solitons. Inspired by the systematic investigations of [2–4] on affine connections, we are devoted to studying Ricci-type solitons that arise from the Yano connection and their conformally perturbed counterparts on the seven three-dimensional Lorentzian Lie groups  $\{G_i\}_{i=1,\dots,7}$ . To single out the geometric data that distinguish the Yano connection from other affine connections, we fix an almost-product structure  $J$  compatible with the left-invariant Lorentzian metric. Concretely, we decompose the tangent bundle into the horizontal distribution  $H = \text{span}\{\tilde{f}_1, \tilde{f}_2\}$  and its Lorentz orthogonal complement  $H^\perp = \text{span}\{\tilde{f}_3\}$ , and prescribe  $J\tilde{f}_1 = \tilde{f}_1, J\tilde{f}_2 = \tilde{f}_2, J\tilde{f}_3 = -\tilde{f}_3$ . This splitting enables us to express the Yano connection and its one-parameter conformal perturbation in terms of the Levi–Civita connection and the covariant derivative of  $J$ . In [5, 6], we obtained explicit expressions for Yano connections and their associated curvatures on these Lorentzian Lie groups. In this paper, we define affine Yano–Ricci solitons and perturbed Yano–Ricci solitons by some algebraic systems. By solving the algebraic systems arising from these definitions, we obtain their complete classification, which indicates that all groups  $G_3, \dots, G_7$  admit affine Yano–Ricci solitons and their perturbed companions, whereas  $G_1$  support neither.

This paper is organized as follows: In Section 2, we present some basic notions on three-dimensional Lorentzian Lie groups. In Section 3, we complete the classification of affine Yano–Ricci solitons. In Section 4, we complete the classification of the affine perturbed Yano–Ricci solitons. In Section 5, we summarize this paper.

## 2. Preliminaries

In this section, we give a brief description of three-dimensional connected unimodular and non-unimodular Lorentzian Lie groups in [13].

**Theorem 2.1.** *Let  $(G, g)$  be a three-dimensional connected unimodular Lie group endowed with a left-invariant Lorentzian metric. Then there exists a pseudo-orthonormal basis  $\{\tilde{f}_1, \tilde{f}_2, \tilde{f}_3\}$  with  $\tilde{f}_3$  timelike in such a way that the corresponding Lie algebra  $\mathfrak{g}$  takes one of the following four mutually non-isomorphic forms:*

Type  $\mathfrak{g}_1$  :

$$[\tilde{f}_1, \tilde{f}_2] = \sigma\tilde{f}_1 - \varphi\tilde{f}_3, [\tilde{f}_1, \tilde{f}_3] = -\sigma\tilde{f}_1 - \varphi\tilde{f}_2,$$

$$[\tilde{f}_2, \tilde{f}_3] = \varphi\tilde{f}_1 + \sigma\tilde{f}_2 + \sigma\tilde{f}_3, \sigma \neq 0.$$

Type  $g_2$  :

$$[\tilde{f}_1, \tilde{f}_2] = \xi \tilde{f}_2 - \varphi \tilde{f}_3, [\tilde{f}_1, \tilde{f}_3] = -\varphi \tilde{f}_2 - \xi \tilde{f}_3, \\ [\tilde{f}_2, \tilde{f}_3] = \sigma \tilde{f}_1, \xi \neq 0.$$

Type  $g_3$  :

$$[\tilde{f}_1, \tilde{f}_2] = -\xi \tilde{f}_3, [\tilde{f}_1, \tilde{f}_3] = -\varphi \tilde{f}_2, [\tilde{f}_2, \tilde{f}_3] = \sigma \tilde{f}_1.$$

Type  $g_4$  :

$$[\tilde{f}_1, \tilde{f}_2] = -\tilde{f}_2 + (2\theta - \varphi)\tilde{f}_3, \theta = \pm 1, \\ [\tilde{f}_1, \tilde{f}_3] = -\varphi \tilde{f}_2 + \tilde{f}_3, [\tilde{f}_2, \tilde{f}_3] = \sigma \tilde{f}_1.$$

These four families exhaust all possible unimodular Lorentzian Lie algebras in dimension three.

**Theorem 2.2.** *Let  $(G, g)$  be a connected, non-unimodular, three-dimensional Lie group endowed with a left-invariant Lorentzian metric. One can always select a pseudo-orthonormal frame  $\{\tilde{f}_1, \tilde{f}_2, \tilde{f}_3\}$  with  $\tilde{f}_3$  timelike so that the associated Lie algebra  $\mathfrak{g}$  is represented by one of the following three non-isomorphic models:*

Type  $g_5$  :

$$[\tilde{f}_1, \tilde{f}_2] = 0, [\tilde{f}_1, \tilde{f}_3] = \sigma \tilde{f}_1 + \varphi \tilde{f}_2, [\tilde{f}_2, \tilde{f}_3] = \xi \tilde{f}_1 + \omega \tilde{f}_2, \\ \sigma + \omega \neq 0, \sigma \xi + \varphi \omega = 0.$$

Type  $g_6$  :

$$[\tilde{f}_1, \tilde{f}_2] = \sigma \tilde{f}_2 + \varphi \tilde{f}_3, [\tilde{f}_1, \tilde{f}_3] = \xi \tilde{f}_2 + \omega \tilde{f}_3, [\tilde{f}_2, \tilde{f}_3] = 0, \\ \sigma + \omega \neq 0, \sigma \xi - \varphi \omega = 0.$$

Type  $g_7$  :

$$[\tilde{f}_1, \tilde{f}_2] = -\sigma \tilde{f}_1 - \varphi \tilde{f}_2 - \varphi \tilde{f}_3, [\tilde{f}_1, \tilde{f}_3] = \sigma \tilde{f}_1 + \varphi \tilde{f}_2 + \varphi \tilde{f}_3, \\ [\tilde{f}_2, \tilde{f}_3] = \xi \tilde{f}_1 + \omega \tilde{f}_2 + \omega \tilde{f}_3, \sigma + \omega \neq 0, \sigma \xi = 0.$$

In what follows, the symbol  $\{G_i\}_{i=1,\dots,7}$  will stand for the complete set of simply connected, three-dimensional Lie groups whose Lie algebras  $\{\mathfrak{g}_i\}_{i=1,\dots,7}$  admit a left-invariant Lorentzian metric  $g$  of signature  $(+, +, -)$ . Each member of this family is uniquely determined by its structure constants, and the underlying manifold is diffeomorphic to  $R^3$  by the Lie group exponential map. Throughout our discussion, the Levi-Civita connection induced by  $g$  will be denoted by  $\nabla^L$ , which is the unique torsion-free, metric-compatible connection on  $G_i$ . We introduce the Yano connection  $\nabla^Y$  by the formula

$$\nabla_U^Y V = \nabla_U^L V - \frac{1}{2}(\nabla_V^L J)JU - \frac{1}{4}[(\nabla_U^L J)JV - (\nabla_{JU}^L J)V], \quad (2.1)$$

where  $\nabla_U^L J$  represents the covariant derivative of the structure  $J$  in the direction of  $U$  under the connection  $\nabla^L$ , for any vector fields  $U, V$ , the covariant derivative  $\nabla_U^L J$  is defined by the Leibniz rule, i.e.  $(\nabla_U^L J)(V) = \nabla_U^L(JV) - J(\nabla_U^L V)$ , and  $J$  is the left-invariant product structure on each  $\{G_i\}_{i=1,\dots,7}$  determined by  $J\tilde{f}_1 = \tilde{f}_1$ ,  $J\tilde{f}_2 = \tilde{f}_2$ ,  $J\tilde{f}_3 = -\tilde{f}_3$ . The curvature tensor of the Yano connection is defined in the usual way by

$$R^Y(U, V)W = \nabla_U^Y \nabla_V^Y W - \nabla_V^Y \nabla_U^Y W - \nabla_{[U, V]}^Y W. \quad (2.2)$$

This tensor has the symmetry property, namely,  $R^Y(U, V)W = -R^Y(V, U)W$ . Using a pseudo-orthonormal frame  $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3$  with  $\tilde{f}_3$  timelike, the Ricci tensor  $\rho^Y$  associated with  $\nabla^Y$  is

$$\rho^Y(U, V) = -g(R^Y(U, \tilde{f}_1)V, \tilde{f}_1) - g(R^Y(U, \tilde{f}_2)V, \tilde{f}_2) + g(R^Y(U, \tilde{f}_3)V, \tilde{f}_3). \quad (2.3)$$

Its symmetrization is denoted by

$$\tilde{\rho}^Y(U, V) = \frac{\rho^Y(U, V) + \rho^Y(V, U)}{2}. \quad (2.4)$$

Finally, for any vector field  $H$ , we set

$$(L_H^Y g)(U, V) := g(\nabla_U^Y H, V) + g(U, \nabla_V^Y H), \quad (2.5)$$

for all vector fields  $U$  and  $V$ .

**Definition 2.1** (Affine Yano–Ricci soliton). *A triple  $(G_i, g, \nabla^Y)$  is said to be an affine Ricci soliton with respect to the Yano connection  $\nabla^Y$  provided that there exist a constant  $\nu \in \mathbb{R}$  and a left-invariant field  $H = \nu_1 \tilde{f}_1 + \nu_2 \tilde{f}_2 + \nu_3 \tilde{f}_3$ , where  $\nu_1, \nu_2$ , and  $\nu_3$  are real numbers, such that*

$$(L_H^Y g)(U, V) + 2\tilde{\rho}^Y(U, V) + 2\nu g(U, V) = 0. \quad (2.6)$$

Furthermore, a triple  $(G_i, g, \nabla^Y)$  is said to be an affine Einstein manifold with respect to the Yano connection if the Yano–Ricci tensor is a constant multiple of the metric, i.e.,  $\tilde{\rho}^Y(U, V) = \nu g(U, V)$  for some constant  $\nu \in \mathbb{R}$  and for all vector fields  $U, V$ .

For brevity, an affine Ricci soliton with respect to the Yano connection will be abbreviated as an affine Yano–Ricci soliton in this paper.

### 3. Affine Yano–Ricci solitons

In this section, we will complete the classification of affine Yano–Ricci solitons.

First, we have computed the expressions of Yano connections and the associated curvatures in the seven three-dimensional Lorentzian Lie groups in [5, 6] based on [14]. We take only first case, for example, and other cases refer to [5, 6]. For  $G_1$ , the Yano connection is given by

$$\begin{aligned} \nabla_{\tilde{f}_1}^Y \tilde{f}_1 &= -\sigma \tilde{f}_2, \quad \nabla_{\tilde{f}_1}^Y \tilde{f}_2 = \sigma \tilde{f}_1 - \varphi \tilde{f}_3, \quad \nabla_{\tilde{f}_1}^Y \tilde{f}_3 = 0, \\ \nabla_{\tilde{f}_2}^Y \tilde{f}_1 &= \varphi \tilde{f}_3, \quad \nabla_{\tilde{f}_2}^Y \tilde{f}_2 = 0, \quad \nabla_{\tilde{f}_2}^Y \tilde{f}_3 = \sigma \tilde{f}_3, \\ \nabla_{\tilde{f}_3}^Y \tilde{f}_1 &= \sigma \tilde{f}_1 + \varphi \tilde{f}_2, \quad \nabla_{\tilde{f}_3}^Y \tilde{f}_2 = -\varphi \tilde{f}_1 - \sigma \tilde{f}_2, \quad \nabla_{\tilde{f}_3}^Y \tilde{f}_3 = 0. \end{aligned}$$

The ensuing curvature tensor  $R^Y$  reads

$$\begin{aligned} R^Y(\tilde{f}_1, \tilde{f}_2)\tilde{f}_1 &= \sigma\varphi\tilde{f}_1 + (\sigma^2 + \varphi^2)\tilde{f}_2, \quad R^Y(\tilde{f}_1, \tilde{f}_2)\tilde{f}_2 = -(\sigma^2 + \varphi^2)\tilde{f}_1 - \sigma\varphi\tilde{f}_2 + \sigma\varphi\tilde{f}_3, \\ R^Y(\tilde{f}_1, \tilde{f}_2)\tilde{f}_3 &= 0, \quad R^Y(\tilde{f}_1, \tilde{f}_3)\tilde{f}_1 = -3\sigma^2\tilde{f}_2, \quad R^Y(\tilde{f}_1, \tilde{f}_3)\tilde{f}_2 = -\sigma^2\tilde{f}_1, \\ R^Y(\tilde{f}_1, \tilde{f}_3)\tilde{f}_3 &= \sigma\varphi\tilde{f}_3, \quad R^Y(\tilde{f}_2, \tilde{f}_3)\tilde{f}_1 = -\sigma^2\tilde{f}_1, \quad R^Y(\tilde{f}_2, \tilde{f}_3)\tilde{f}_2 = \sigma^2\tilde{f}_2, \\ R^Y(\tilde{f}_2, \tilde{f}_3)\tilde{f}_3 &= -\sigma^2\tilde{f}_3. \end{aligned}$$

We can prove the following theorem on Yano–Ricci Ricci soliton on the first Lorentzian Lie group:

**Theorem 3.1.**  $(G_1, g, \nabla^Y)$  admits no affine Yano–Ricci soliton.

*Proof.* From (2.3), we extract the Ricci components

$$\begin{aligned}\rho^Y(\tilde{f}_1, \tilde{f}_1) &= -\sigma^2 - \varphi^2, \quad \rho^Y(\tilde{f}_1, \tilde{f}_2) = \sigma\varphi, \quad \rho^Y(\tilde{f}_1, \tilde{f}_3) = -\sigma\varphi, \\ \rho^Y(\tilde{f}_2, \tilde{f}_1) &= \sigma\varphi, \quad \rho^Y(\tilde{f}_2, \tilde{f}_2) = -(\sigma^2 + \varphi^2), \quad \rho^Y(\tilde{f}_2, \tilde{f}_3) = \sigma^2, \\ \rho^Y(\tilde{f}_3, \tilde{f}_1) &= 0, \quad \rho^Y(\tilde{f}_3, \tilde{f}_2) = 0, \quad \rho^Y(\tilde{f}_3, \tilde{f}_3) = 0.\end{aligned}$$

Symmetrizing gives the symmetric part

$$\begin{aligned}\tilde{\rho}^Y(\tilde{f}_1, \tilde{f}_1) &= -(\sigma^2 + \varphi^2), \quad \tilde{\rho}^Y(\tilde{f}_1, \tilde{f}_2) = \sigma\varphi, \quad \tilde{\rho}^Y(\tilde{f}_1, \tilde{f}_3) = -\frac{\sigma\varphi}{2}, \\ \tilde{\rho}^Y(\tilde{f}_2, \tilde{f}_2) &= -(\sigma^2 + \varphi^2), \quad \tilde{\rho}^Y(\tilde{f}_2, \tilde{f}_3) = \frac{\sigma^2}{2}, \quad \tilde{\rho}^Y(\tilde{f}_3, \tilde{f}_3) = 0.\end{aligned}$$

For the left-invariant vector field  $H = \nu_1\tilde{f}_1 + \nu_2\tilde{f}_2 + \nu_3\tilde{f}_3$ , Eq (2.5) gives

$$\begin{aligned}(L_{HG}^Y)(\tilde{f}_1, \tilde{f}_1) &= 2\nu_2\sigma, \quad (L_{HG}^Y)(\tilde{f}_1, \tilde{f}_2) = -\nu_1\sigma, \quad (L_{HG}^Y)(\tilde{f}_1, \tilde{f}_3) = \nu_1\sigma, \\ (L_{HG}^Y)(\tilde{f}_2, \tilde{f}_2) &= 0, \quad (L_{HG}^Y)(\tilde{f}_2, \tilde{f}_3) = -(\nu_2\sigma + \nu_3\sigma), \quad (L_{HG}^Y)(\tilde{f}_3, \tilde{f}_3) = 0.\end{aligned}$$

Inserting these pieces into the affine Yano–Ricci soliton condition (2.6) yields the closed system

$$\begin{cases} \nu_2\sigma - \sigma^2 - \varphi^2 + \nu = 0, \\ \nu_1\sigma - 2\sigma\varphi = 0, \\ \nu_1\sigma - \sigma\varphi = 0, \\ \sigma^2 + \varphi^2 - \nu = 0, \\ \nu_2\sigma + \nu_3\sigma - \sigma^2 = 0, \\ \nu = 0. \end{cases} \quad (3.1)$$

The last equation forces  $\nu = 0$ . The second and third imply  $\sigma\varphi = 0$ , while the fourth then yields  $\sigma^2 + \varphi^2 = 0$ . Hence,  $\sigma = \varphi = 0$ , violating the geometric assumption  $\sigma \neq 0$ . Consequently, no parameter set admits an affine Yano–Ricci soliton on  $(G_1, g, \nabla^Y)$ .

Second, by analog computation on the second Lorentzian Lie group, we can prove the following classification result for affine Yano–Ricci solitons:

**Theorem 3.2.**  $(G_2, g, \nabla^Y)$  admits no affine Yano–Ricci soliton.

*Proof.* A direct computation via (2.3) gives

$$\begin{aligned}\rho^Y(\tilde{f}_1, \tilde{f}_1) &= -(\varphi^2 + \xi^2), \quad \rho^Y(\tilde{f}_2, \tilde{f}_2) = -(\xi^2 + \sigma\varphi), \quad \rho^Y(\tilde{f}_2, \tilde{f}_3) = -\sigma\xi, \\ \rho^Y(\tilde{f}_3, \tilde{f}_2) &= 0, \quad \text{other components} = 0.\end{aligned}$$

Symmetrizing yields

$$\tilde{\rho}^Y(\tilde{f}_1, \tilde{f}_1) = -(\varphi^2 + \xi^2), \quad \tilde{\rho}^Y(\tilde{f}_2, \tilde{f}_2) = -(\xi^2 + \sigma\varphi), \quad \tilde{\rho}^Y(\tilde{f}_2, \tilde{f}_3) = -\frac{\sigma\xi}{2},$$

with all remaining symmetric components vanishing. For the left-invariant vector field  $H = \nu_1 \tilde{f}_1 + \nu_2 \tilde{f}_2 + \nu_3 \tilde{f}_3$ , Eq (2.5) produces

$$\begin{aligned} (L_H^Y g)(\tilde{f}_1, \tilde{f}_1) &= 0, (L_H^Y g)(\tilde{f}_1, \tilde{f}_2) = \nu_2 \xi, (L_H^Y g)(\tilde{f}_1, \tilde{f}_3) = \nu_2 \varphi - \nu_2 \sigma + \nu_3 \xi, \\ (L_H^Y g)(\tilde{f}_2, \tilde{f}_2) &= -2\nu_1 \xi, (L_H^Y g)(\tilde{f}_2, \tilde{f}_3) = 0, (L_H^Y g)(\tilde{f}_3, \tilde{f}_3) = 0. \end{aligned}$$

Inserting the above expressions into the affine Yano–Ricci soliton Eq (2.6), it gives the closed algebraic system

$$\begin{cases} \varphi^2 + \xi^2 - \nu = 0, \\ \nu_2 \xi = 0, \\ \nu_2 \varphi - \nu_2 \sigma + \nu_3 \xi = 0, \\ \nu_1 \xi + \xi^2 + \varphi^2 - \nu = 0, \\ \sigma \xi = 0, \\ \nu = 0. \end{cases} \quad (3.2)$$

Solving the last equation first ( $\nu = 0$ ) and substituting into the first relation, we obtain  $\varphi^2 + \xi^2 = 0$ ; thus,  $\varphi = 0$  and  $\xi = 0$ . This is a contradiction to  $\xi \neq 0$ . Consequently, no parameter set admits an affine Yano–Ricci soliton on  $(G_2, g, \nabla^Y)$ .

For the third Lorentzian Lie group, we can prove the following theorem:

**Theorem 3.3.** *The triple  $(G_3, g, \nabla^Y)$  is an affine Yano–Ricci soliton precisely when the parameters satisfy either*

- (i)  $\nu = \varphi = \xi = \nu_2 \sigma = 0$ ,
- (ii)  $\nu = \sigma = \varphi = \nu_1 = 0, \xi \neq 0$ ,
- (iii)  $\nu = \xi = \nu_2 \sigma = \nu_1 = 0, \varphi \neq 0$ .

*Proof.* The Yano–Ricci components, extracted from Eq (2.3), are

$$\begin{aligned} \rho^Y(\tilde{f}_1, \tilde{f}_1) &= -\varphi \xi, \rho^Y(\tilde{f}_1, \tilde{f}_2) = 0, \rho^Y(\tilde{f}_1, \tilde{f}_3) = 0, \\ \rho^Y(\tilde{f}_2, \tilde{f}_1) &= 0, \rho^Y(\tilde{f}_2, \tilde{f}_2) = -\sigma \xi, \rho^Y(\tilde{f}_2, \tilde{f}_3) = 0, \\ \rho^Y(\tilde{f}_3, \tilde{f}_1) &= 0, \rho^Y(\tilde{f}_3, \tilde{f}_2) = 0, \rho^Y(\tilde{f}_3, \tilde{f}_3) = 0. \end{aligned}$$

Symmetrizing gives

$$\begin{aligned} \tilde{\rho}^Y(\tilde{f}_1, \tilde{f}_1) &= -\varphi \xi, \tilde{\rho}^Y(\tilde{f}_1, \tilde{f}_2) = 0, \tilde{\rho}^Y(\tilde{f}_1, \tilde{f}_3) = 0, \\ \tilde{\rho}^Y(\tilde{f}_2, \tilde{f}_2) &= -\sigma \xi, \tilde{\rho}^Y(\tilde{f}_2, \tilde{f}_3) = 0, \tilde{\rho}^Y(\tilde{f}_3, \tilde{f}_3) = 0. \end{aligned}$$

For  $H = \nu_1 \tilde{f}_1 + \nu_2 \tilde{f}_2 + \nu_3 \tilde{f}_3$ , Eq (2.5) yields

$$\begin{aligned} (L_H^Y g)(\tilde{f}_1, \tilde{f}_1) &= 0, (L_H^Y g)(\tilde{f}_1, \tilde{f}_2) = 0, (L_H^Y g)(\tilde{f}_1, \tilde{f}_3) = \nu_2 \xi - \nu_2 \sigma, \\ (L_H^Y g)(\tilde{f}_2, \tilde{f}_2) &= 0, (L_H^Y g)(\tilde{f}_2, \tilde{f}_3) = \nu_1 \varphi - \nu_1 \xi, (L_H^Y g)(\tilde{f}_3, \tilde{f}_3) = 0. \end{aligned}$$

Plugging the above into (2.6) gives the closed algebraic system

$$\begin{cases} \varphi \xi - \nu = 0, \\ \nu_2 \xi - \nu_2 \sigma = 0, \\ \sigma \xi - \nu = 0, \\ \nu_1 \varphi - \nu_1 \xi = 0, \\ \nu = 0. \end{cases} \quad (3.3)$$

We begin by assuming  $\xi = 0$  and  $\varphi = 0$ , from which  $\nu_2\sigma = 0$  is derived. Under the condition  $\xi \neq 0$ ,  $\sigma = 0$ , and  $\nu_1 = 0$  can be inferred from the third and fourth equations in the system above. Finally, the assumption  $\varphi \neq 0$  and  $\xi = 0$  leads to  $\nu_2\sigma = 0$  and  $\nu_1 = 0$ . Then, we get Theorem 3.3.

For the fourth Lorentzian Lie group, we can prove the following theorem on affine Ricci soliton:

**Theorem 3.4.**  $(G_4, g, \nabla^Y)$  is an affine Yano–Ricci soliton if and only if  $\theta = \pm 1$ ,  $\nu_1 = -1$ ,  $\nu = \nu_2 = \nu_3 = 0$ .

*Proof.* By (2.3), we have

$$\begin{aligned}\rho^Y(\tilde{f}_1, \tilde{f}_1) &= 2\varphi\theta - \varphi^2 - 1, \rho^Y(\tilde{f}_1, \tilde{f}_2) = 0, \rho^Y(\tilde{f}_1, \tilde{f}_3) = 0, \\ \rho^Y(\tilde{f}_2, \tilde{f}_1) &= 0, \rho^Y(\tilde{f}_2, \tilde{f}_2) = 2\sigma\theta - \sigma\varphi - 1, \rho^Y(\tilde{f}_2, \tilde{f}_3) = \sigma, \\ \rho^Y(\tilde{f}_3, \tilde{f}_1) &= 0, \rho^Y(\tilde{f}_3, \tilde{f}_2) = 0, \rho^Y(\tilde{f}_3, \tilde{f}_3) = 0.\end{aligned}$$

Then

$$\begin{aligned}\tilde{\rho}^Y(\tilde{f}_1, \tilde{f}_1) &= 2\varphi\theta - \varphi^2 - 1, \tilde{\rho}^Y(\tilde{f}_1, \tilde{f}_2) = 0, \tilde{\rho}^Y(\tilde{f}_1, \tilde{f}_3) = 0, \\ \tilde{\rho}^Y(\tilde{f}_2, \tilde{f}_2) &= 2\sigma\theta - \sigma\varphi - 1, \tilde{\rho}^Y(\tilde{f}_2, \tilde{f}_3) = \frac{\sigma}{2}, \tilde{\rho}^Y(\tilde{f}_3, \tilde{f}_3) = 0.\end{aligned}$$

By (2.5), we have

$$\begin{aligned}(L_{HG}^Y)(\tilde{f}_1, \tilde{f}_1) &= 0, (L_{HG}^Y)(\tilde{f}_1, \tilde{f}_2) = -\nu_2, (L_{HG}^Y)(\tilde{f}_1, \tilde{f}_3) = -\nu_2(2\theta - \varphi + \sigma) - \nu_3, \\ (L_{HG}^Y)(\tilde{f}_2, \tilde{f}_2) &= 2\nu_1, (L_{HG}^Y)(\tilde{f}_2, \tilde{f}_3) = 2\nu_1\theta, (L_{HG}^Y)(\tilde{f}_3, \tilde{f}_3) = 0.\end{aligned}$$

Then, if  $(G_4, g, \nabla^Y)$  is an affine Yano–Ricci soliton, by (2.6), we have the following six equations:

$$\begin{cases} 2\varphi\theta - \varphi^2 + \nu - 1 = 0, \\ \nu_2 = 0, \\ 2\nu_2\theta - \nu_2\varphi + \nu_2\sigma + \nu_3 = 0, \\ \nu_1 + 2\sigma\theta - \sigma\varphi - 1 + \nu = 0, \\ 2\nu_1\theta + \sigma = 0, \\ \nu = 0. \end{cases} \quad (3.4)$$

We consider the cases where  $\theta = \pm 1$ . First, assume  $\theta = 1$ . Substituting this into the first equation of the system, we obtain  $\varphi = 1$ . Similarly, substituting it into the fourth and fifth equations of a system, we get  $\nu_1 = -1$ ,  $\sigma = 2$ . Next, assume  $\theta = -1$ . Substituting this into the first equation of the system, we obtain  $\varphi = -1$ . Similarly, substituting it into the fourth and fifth equations of the system, we get  $\nu_1 = -1$ ,  $\sigma = -2$ . This completes the proof of Theorem 3.4.

For the fifth Lorentzian Lie group, we prove the following theorem on the affine Ricci soliton:

**Theorem 3.5.** The triple  $(G_5, g, \nabla^Y)$  is an affine Yano–Ricci soliton if and only if its parameters satisfy exactly one of the following conditions

- (i)  $\nu = \nu_1 = \nu_2 = 0, \sigma + \omega \neq 0, \sigma\xi + \varphi\omega = 0$ ,
- (ii)  $\nu = \nu_1 = \xi = \omega = 0, \sigma \neq 0$ ,
- (iii)  $\nu = \nu_2 = \sigma = \varphi = 0, \omega \neq 0$ .

*Proof.* A direct computation via (2.3) shows that every component of the Yano–Ricci tensor vanishes:  $\rho^Y(\tilde{f}_i, \tilde{f}_j) = 0$ , where  $1 \leq i, j \leq 3$ . Hence, the symmetric part  $\tilde{\rho}^Y$  is identically zero. Equation (2.5) gives

$$\begin{aligned} (L_{HG}^Y)(\tilde{f}_1, \tilde{f}_1) &= 0, (L_{HG}^Y)(\tilde{f}_1, \tilde{f}_2) = 0, (L_{HG}^Y)(\tilde{f}_1, \tilde{f}_3) = -\nu_1\sigma - \nu_2\xi, \\ (L_{HG}^Y)(\tilde{f}_2, \tilde{f}_2) &= 0, (L_{HG}^Y)(\tilde{f}_2, \tilde{f}_3) = -\nu_1\varphi - \nu_2\omega, (L_{HG}^Y)(\tilde{f}_3, \tilde{f}_3) = 0. \end{aligned}$$

Plugging the above into (2.6) produces the reduced system

$$\begin{cases} \nu = 0, \\ \nu_2\xi + \nu_1\sigma = 0, \\ \nu_1\varphi + \nu_2\omega = 0. \end{cases} \quad (3.5)$$

The first equation forces  $\nu = 0$ . The remaining second and third equations form a homogeneous linear system

$$\begin{pmatrix} \sigma & \xi \\ \varphi + \xi & -\omega \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Let  $\Delta = -\sigma\omega - \xi(\varphi + \xi)$  be its determinant. If  $\Delta \neq 0$ , then the unique solution is  $\nu_1 = \nu_2 = 0$ . If  $\Delta = 0$ , i.e.,  $-\sigma\omega - \xi(\varphi + \xi) = 0$ . For the case  $\xi = \omega = 0$ , the second and third equations reduce to  $\nu_1\sigma = 0, \nu_1\varphi = 0$ . Since  $\sigma + \omega = \sigma \neq 0$  (from the Lie-algebra condition), we obtain  $\nu_1 = 0$  and  $\nu_2$  is free. For the case  $\xi \neq 0$  and  $\omega \neq 0$ , the determinant condition together with the Lie-algebra identity  $\sigma\xi + \varphi\omega = 0$  forces  $\sigma = \varphi = \xi = 0$ . Then the second and third equations become  $0 = 0, -\nu_2\omega = 0$ . This indicates  $\nu_2 = 0$ , with  $\nu_1$  arbitrary, so  $\nu_1$  is free and  $\nu_2 = 0$ . Summarizing the above solution, we complete the proof of Theorem 3.5.

For the sixth Lorentzian Lie group, we can prove the following theorem:

**Theorem 3.6.**  $(G_6, g, \nabla^Y)$  is an affine Yano–Ricci soliton if and only if  $\nu = \sigma = \varphi = \nu_3 = \nu_1\xi = 0, \omega \neq 0$ .

*Proof.* By (2.3), we have

$$\begin{aligned} \rho^Y(\tilde{f}_1, \tilde{f}_1) &= -(\varphi\xi + \sigma^2), \rho^Y(\tilde{f}_1, \tilde{f}_2) = 0, \rho^Y(\tilde{f}_1, \tilde{f}_3) = 0, \\ \rho^Y(\tilde{f}_2, \tilde{f}_1) &= 0, \rho^Y(\tilde{f}_2, \tilde{f}_2) = -\sigma^2, \rho^Y(\tilde{f}_2, \tilde{f}_3) = 0, \\ \rho^Y(\tilde{f}_3, \tilde{f}_1) &= 0, \rho^Y(\tilde{f}_3, \tilde{f}_2) = 0, \rho^Y(\tilde{f}_3, \tilde{f}_3) = 0. \end{aligned}$$

Then

$$\begin{aligned} \tilde{\rho}^Y(\tilde{f}_1, \tilde{f}_1) &= -(\varphi\xi + \sigma^2), \tilde{\rho}^Y(\tilde{f}_1, \tilde{f}_2) = 0, \tilde{\rho}^Y(\tilde{f}_1, \tilde{f}_3) = 0, \\ \tilde{\rho}^Y(\tilde{f}_2, \tilde{f}_2) &= -\sigma^2, \tilde{\rho}^Y(\tilde{f}_2, \tilde{f}_3) = 0, \tilde{\rho}^Y(\tilde{f}_3, \tilde{f}_3) = 0. \end{aligned}$$

By (2.5), we have

$$\begin{aligned} (L_{HG}^Y)(\tilde{f}_1, \tilde{f}_1) &= 0, (L_{HG}^Y)(\tilde{f}_1, \tilde{f}_2) = \nu_2\sigma, (L_{HG}^Y)(\tilde{f}_1, \tilde{f}_3) = -\nu_2\varphi - \nu_3\omega, \\ (L_{HG}^Y)(\tilde{f}_2, \tilde{f}_2) &= -2\nu_1\sigma, (L_{HG}^Y)(\tilde{f}_2, \tilde{f}_3) = \nu_1\varphi - \nu_1\xi, (L_{HG}^Y)(\tilde{f}_3, \tilde{f}_3) = 0. \end{aligned}$$

Then, if  $(G_6, g, \nabla^Y)$  is an affine Yano–Ricci soliton, by (2.6), we have the following six equations:

$$\begin{cases} v - \varphi\xi - \sigma^2 = 0, \\ v_2\sigma = 0, \\ v_2\varphi + v_3\omega = 0, \\ v - v_1\sigma - \sigma^2 = 0, \\ v_1\varphi - v_1\xi = 0, \\ v = 0. \end{cases} \quad (3.6)$$

From the sixth equation of the system, we substitute  $v = 0$  into the system and consider the cases where  $\sigma + \omega = 0$  and  $\sigma\xi - \varphi\omega = 0$ . First, assume  $\omega = 0$  and  $\sigma = 0$ . Then it follows that  $\varphi = 0$ . Substituting these into the third equation of the system, we obtain  $v_3 = 0$ . Similarly, substituting them into the fifth equation of the system, we get  $v_1\xi = 0$ . Next, assume  $\sigma = 0$  and  $\omega = 0$ . Then we have  $\xi = 0$ . Substituting these into the first equation of the system, we obtain  $\sigma^2 = -\varphi\xi$ . This implies  $\xi = 0$ , which means the system has no solution in this case. This completes the proof of Theorem 3.6.

Finally, for the seventh Lorentzian Lie group, we can prove the following theorem on the affine Ricci soliton:

**Theorem 3.7.**  $(G_7, g, \nabla^Y)$  is an affine Yano–Ricci soliton if and only if

- (i)  $v = \sigma = \varphi = \xi = 0, \omega \neq 0, v_2 + v_3 - 2\omega = 0,$
- (ii)  $v = \sigma = \varphi = v_2 = 0, \omega \neq 0, \xi \neq 0, v_3 - 2\omega = 0,$
- (iii)  $v = \sigma = 0, \varphi \neq 0, \omega \neq 0, v_1 = \varphi + \xi, v_2 = \omega, v_3 = \frac{3\varphi\omega - \xi\omega}{\varphi}, \xi = \frac{2\varphi^3 + 2\omega^2\varphi}{\omega^2 - \varphi^2}.$

*Proof.* By (2.3), we have

$$\begin{aligned} \rho^Y(\tilde{f}_1, \tilde{f}_1) &= -\sigma^2, \quad \rho^Y(\tilde{f}_1, \tilde{f}_2) = -\sigma\varphi, \quad \rho^Y(\tilde{f}_1, \tilde{f}_3) = \sigma\varphi + \varphi\omega, \\ \rho^Y(\tilde{f}_2, \tilde{f}_1) &= \varphi\omega, \quad \rho^Y(\tilde{f}_2, \tilde{f}_2) = -\sigma^2 - \varphi^2 - \varphi\xi, \quad \rho^Y(\tilde{f}_2, \tilde{f}_3) = \varphi\xi + \omega^2, \\ \rho^Y(\tilde{f}_3, \tilde{f}_1) &= \sigma\varphi + \varphi\omega, \quad \rho^Y(\tilde{f}_3, \tilde{f}_2) = \sigma\omega + \omega^2, \quad \rho^Y(\tilde{f}_3, \tilde{f}_3) = 0. \end{aligned}$$

Then

$$\begin{aligned} \tilde{\rho}^Y(\tilde{f}_1, \tilde{f}_1) &= -\sigma^2, \quad \tilde{\rho}^Y(\tilde{f}_1, \tilde{f}_2) = \frac{\varphi\omega - \sigma\varphi}{2}, \quad \tilde{\rho}^Y(\tilde{f}_1, \tilde{f}_3) = \sigma\varphi + \varphi\omega, \\ \tilde{\rho}^Y(\tilde{f}_2, \tilde{f}_2) &= -\sigma^2 - \varphi^2 - \varphi\xi, \quad \tilde{\rho}^Y(\tilde{f}_2, \tilde{f}_3) = \frac{2\omega^2 + \sigma\omega + \varphi\xi}{2}, \quad \tilde{\rho}^Y(\tilde{f}_3, \tilde{f}_3) = 0. \end{aligned}$$

By (2.5), we have

$$\begin{aligned} (L_{Hg}^Y)(\tilde{f}_1, \tilde{f}_1) &= -2v_2\sigma, \quad (L_{Hg}^Y)(\tilde{f}_1, \tilde{f}_2) = v_1\sigma - v_2\varphi, \\ (L_{Hg}^Y)(\tilde{f}_1, \tilde{f}_3) &= -v_1\sigma + v_2\varphi - v_2\xi - v_3\varphi, \quad (L_{Hg}^Y)(\tilde{f}_2, \tilde{f}_2) = 2v_1\varphi, \\ (L_{Hg}^Y)(\tilde{f}_2, \tilde{f}_3) &= -2v_1\varphi - v_2\omega - v_3\omega, \quad (L_{Hg}^Y)(\tilde{f}_3, \tilde{f}_3) = 0. \end{aligned}$$

Then, if  $(G_7, g, \nabla^Y)$  is an affine Yano–Ricci soliton, by (2.6), we have the following six equations:

$$\begin{cases} v_2\sigma + \sigma^2 - \nu = 0, \\ v_1\sigma - v_2\varphi + \varphi\omega - \sigma\varphi = 0, \\ v_2\varphi - v_3\varphi - v_1\sigma - v_2\xi + 2\sigma\varphi + 2\varphi\omega = 0, \\ v_1\varphi - \sigma^2 - \varphi^2 - \varphi\xi + \nu = 0, \\ 2v_1\varphi + v_2\omega + v_3\omega - 2\omega^2 - \varphi\xi - \sigma\omega = 0, \\ \nu = 0. \end{cases} \quad (3.7)$$

Considering the case where  $\sigma + \omega \neq 0$  and  $\sigma\xi = 0$ , we first assume  $\sigma = 0$ , which implies  $\omega \neq 0$ . If  $\xi = 0$ , then the system of equations reduces to

$$\begin{cases} -v_2\varphi + \varphi\omega = 0, \\ v_2\varphi - v_3\varphi + 2\varphi\omega = 0, \\ v_1\varphi - \varphi^2 = 0, \\ 2v_1\varphi + v_2\omega + v_3\omega - 2\omega^2 = 0, \\ \nu = 0. \end{cases}$$

From the preceding equation, we obtain  $\varphi = 0$ . Substituting this into the fourth equation of the preceding equation yields  $v_2 + v_3 - 2\omega = 0$ . If  $\xi \neq 0$ , then system (3.7) reduces to

$$\begin{cases} -v_2\varphi + \varphi\omega = 0, \\ v_2\varphi - v_3\varphi - v_2\xi + 2\varphi\omega = 0, \\ v_1\varphi - \varphi^2 - \varphi\xi = 0, \\ 2v_1\varphi + v_2\omega + v_3\omega - 2\omega^2 - \varphi\xi = 0, \\ \nu = 0. \end{cases}$$

On the one hand, assume  $\varphi = 0$ . Substituting this into the second equation of the preceding equation, we obtain  $v_2 = 0$ . Substituting this into the fourth equation of the preceding equation yields  $v_3 - 2\omega = 0$ . On the other hand, assume  $\varphi \neq 0$ . Substituting this into the first equation of the preceding equation, we obtain  $\omega = v_2$ . Substituting this into the third equation of the preceding yields  $v_3 = \frac{3\omega\varphi - \omega\xi}{\xi}$ . Furthermore, substituting equations into the fifth equation, we get  $\xi = \frac{2\varphi^3 + 2\varphi\omega^2}{\omega^2 - \varphi^2}$ . Next, assume  $\omega = 0$ , which implies  $\sigma \neq 0$  and thus  $\xi = 0$ . In this case, system (3.7) reduces to  $\sigma^2 + \varphi^2 = 0$  and thus  $\sigma = \varphi = 0$ , which is a contradiction. By solving (3.7), Theorem 3.7 is established.

**Corollary 3.1** (Affine Einstein structures with vanishing potential). *Taking the potential vector field  $H \equiv 0$  in the affine Yano–Ricci soliton equation reduces the soliton condition to the standard affine Einstein equation*

$$\tilde{\rho}^Y = \nu g.$$

*For the seven simply-connected three-dimensional Lorentzian Lie groups, the complete classification is as follows:*

- (a)  $(G_1, g, \nabla^Y)$  admits no affine Einstein structure with respect to the Yano connection.
- (b)  $(G_2, g, \nabla^Y)$  admits no affine Einstein structure with respect to the Yano connection.

(c)  $(G_3, g, \nabla^Y)$  is an affine Einstein manifold if and only if either

$$(i) \xi = \nu = 0 \quad \text{or} \quad (ii) \nu = \varphi = \sigma = 0, \xi \neq 0.$$

(d)  $(G_4, g, \nabla^Y)$  admits no affine Einstein structure with respect to the Yano connection.

(e)  $(G_5, g, \nabla^Y)$  is an affine Einstein manifold precisely when

$$\nu = 0, \quad \sigma + \omega \neq 0, \quad \sigma\xi + \varphi\omega = 0.$$

(f)  $(G_6, g, \nabla^Y)$  is an affine Einstein manifold if and only if

$$\nu = \sigma = \varphi = 0, \quad \omega \neq 0.$$

(g)  $(G_7, g, \nabla^Y)$  admits no affine Einstein structure with respect to the Yano connection.

#### 4. Affine perturbed Yano–Ricci solitons

In this section, we investigate how to realize affine Ricci solitons with non-zero scaling parameter  $\nu$  by deforming the Yano connection. To prepare for this construction, we define Lorentz dual 1-forms for the basis  $\tilde{f}_i$  by  $\tilde{f}_i^* := g(\tilde{f}_i, \cdot)$ , where  $i = 1, 2, 3$ . Since  $\tilde{f}_3$  is timelike, we get  $\tilde{f}_3^*(\tilde{f}_3) = g(\tilde{f}_3, \tilde{f}_3) = -1$  and  $\tilde{f}_3^*(\tilde{f}_i) = g(\tilde{f}_3, \tilde{f}_i) = 0$  for  $i = 1, 2$ . Then we define the perturbed Yano connection for each  $G_i$  by

$$\tilde{\nabla}_U^Y V = \nabla_U^Y V + a_0 \tilde{f}_3^*(U) \tilde{f}_3^*(V) \tilde{f}_3, \quad (4.1)$$

where  $i = 1, \dots, 7$  and  $a_0$  is a nonzero real number. This modification affects only the  $\tilde{f}_3$  component:

$$\tilde{\nabla}_{\tilde{f}_3}^Y \tilde{f}_3 = a_0 \tilde{f}_3, \quad \tilde{\nabla}_{\tilde{f}_s}^Y \tilde{f}_t = \nabla_{\tilde{f}_s}^Y \tilde{f}_t, \quad (4.2)$$

where  $s$  and  $t$  does not equal 3. We now define the perturbed Lie-derivative term by

$$(\tilde{L}_H^Y g)(U, V) := g(\tilde{\nabla}_U^Y H, V) + g(U, \tilde{\nabla}_V^Y H), \quad (4.3)$$

for vector fields  $U, V$ , and  $W$ , satisfying

$$(\tilde{L}_H^Y g)(\tilde{f}_3, \tilde{f}_3) = -2a_0\nu_3, (\tilde{L}_H^Y g)(\tilde{f}_s, \tilde{f}_t) = (L_H^Y g)(\tilde{f}_s, \tilde{f}_t), \quad (s, t \neq 3). \quad (4.4)$$

**Definition 4.1** (Perturbed Affine Yano–Ricci Soliton). *For each  $i = 1, \dots, 7$ , the triple  $(G_i, g, \tilde{\nabla}^Y)$  is said to be an affine Ricci soliton associated with the perturbed Yano connection  $\tilde{\nabla}^Y$  if there exists a real number  $\nu$  and a left-invariant vector field  $H = \nu_1 \tilde{f}_1 + \nu_2 \tilde{f}_2 + \nu_3 \tilde{f}_3$  with  $\nu_1, \nu_2, \nu_3 \in \mathbb{R}$  such that*

$$(\tilde{L}_H^Y g)(U, V) + 2\tilde{\rho}^Y(U, V) + 2\nu g(U, V) = 0, \quad (4.5)$$

for every pair of vector fields  $U, V$ .

**Theorem 4.1.** *The triple  $(G_1, g, \tilde{\nabla}^Y)$  supports no perturbed affine Yano–Ricci soliton.*

*Proof.* We first compute the required curvature data and then solve the resulting algebraic system. Using the perturbed connection, we obtain the non-zero curvature components

$$\begin{aligned}\tilde{R}^Y(\tilde{f}_1, \tilde{f}_2)\tilde{f}_3 &= \varphi a_0 \tilde{f}_3, \quad \tilde{R}^Y(\tilde{f}_1, \tilde{f}_3)\tilde{f}_2 = -\sigma^2 \tilde{f}_1 + \varphi a_0 \tilde{f}_3, \\ \tilde{R}^Y(\tilde{f}_2, \tilde{f}_3)\tilde{f}_1 &= -\sigma^2 \tilde{f}_1 - a_0 \varphi \tilde{f}_3, \quad \tilde{R}^Y(\tilde{f}_2, \tilde{f}_3)\tilde{f}_3 = -\sigma^2 \tilde{f}_3 - \sigma a_0 \tilde{f}_3, \\ \tilde{R}^Y(\tilde{f}_s, \tilde{f}_t)\tilde{f}_p &= R^Y(\tilde{f}_s, \tilde{f}_t)\tilde{f}_p,\end{aligned}$$

whenever  $(s, t, p) \neq (1, 2, 3), (1, 3, 2), (2, 3, 1),$  and  $(2, 3, 3)$ . From the above, the essential components of the symmetric Ricci tensor are

$$\begin{aligned}\bar{\rho}^Y(\tilde{f}_1, \tilde{f}_2) &= \sigma\varphi - a_0\varphi, \quad \bar{\rho}^Y(\tilde{f}_2, \tilde{f}_1) = \sigma\varphi + a_0\varphi, \\ \bar{\rho}^Y(\tilde{f}_2, \tilde{f}_3) &= \sigma^2 + \sigma a_0, \quad \bar{\rho}^Y(\tilde{f}_s, \tilde{f}_t) = \rho^Y(\tilde{f}_s, \tilde{f}_t),\end{aligned}$$

for  $(s, t) \neq (1, 2), (2, 1), (2, 3)$ . Then

$$\tilde{\rho}^Y(\tilde{f}_2, \tilde{f}_3) = \frac{\sigma^2 + \sigma a_0}{2}, \quad \tilde{\rho}^Y(\tilde{f}_s, \tilde{f}_t) = \bar{\rho}^Y(\tilde{f}_s, \tilde{f}_t),$$

for the pair  $(s, t) \neq (1, 2), (2, 3)$ . Inserting these expressions into the perturbed soliton condition (4.5) yields

$$\begin{cases} \nu_2\sigma - \sigma^2 - \varphi^2 + \nu = 0, \\ \nu_1\sigma - 2\sigma\varphi = 0, \\ \nu_1\sigma - \sigma\varphi = 0, \\ \sigma^2 + \varphi^2 - \nu = 0, \\ \nu_2\sigma + \nu_3\sigma - \sigma^2 - \sigma a_0 = 0, \\ a_0\nu_3 + \nu = 0. \end{cases} \quad (4.6)$$

By the first, third, and fourth equations in (4.6) and  $\sigma \neq 0$ , we get  $\nu_2 = 0$ ,  $\nu_1 = \varphi$ ,  $\nu = \sigma^2 + \varphi^2$ . By the second equation in (4.6), we have  $\varphi = 0$ . By the fifth equation in (4.6), we get  $\nu_3 = \sigma + a_0$ . By the sixth equation in (4.6), we have  $a_0^2 + a_0\sigma + \sigma^2 = 0$ . Then we get  $\sigma = a_0 = 0$ , which is a contradiction. Theorem 4.1 is established.

**Theorem 4.2.** *The triple  $(G_2, g, \tilde{\nabla}^Y)$  is a perturbed affine Yano–Ricci soliton if and only if the following six equalities hold simultaneously:  $\nu_2 = \sigma = 0$ ,  $\xi \neq 0$ ,  $\nu_3 = a_0$ ,  $\nu = -a_0^2$ , and  $\nu_1\xi + \varphi^2 = 0$ .*

*Proof.* For the perturbed Yano connection, we obtain

$$\begin{aligned}\tilde{R}^Y(\tilde{f}_1, \tilde{f}_2)\tilde{f}_3 &= \varphi a_0 \tilde{f}_3, \quad \tilde{R}^Y(\tilde{f}_1, \tilde{f}_3)\tilde{f}_2 = (\varphi\xi - \sigma\xi)\tilde{f}_1 + \varphi a_0 \tilde{f}_3, \\ \tilde{R}^Y(\tilde{f}_1, \tilde{f}_3)\tilde{f}_3 &= \xi a_0 \tilde{f}_3, \quad \tilde{R}^Y(\tilde{f}_2, \tilde{f}_3)\tilde{f}_1 = (\varphi\xi - \sigma\xi)\tilde{f}_1 - \varphi a_0 \tilde{f}_3, \\ \tilde{R}^Y(\tilde{f}_s, \tilde{f}_t)\tilde{f}_p &= R^Y(\tilde{f}_s, \tilde{f}_t)\tilde{f}_p,\end{aligned}$$

for  $(s, t, p) \neq (1, 2, 3), (1, 3, 2), (1, 3, 3), (2, 3, 1)$ . The symmetric Ricci tensor components are

$$\begin{aligned}\bar{\rho}^Y(\tilde{f}_1, \tilde{f}_2) &= -\varphi a_0, \quad \bar{\rho}^Y(\tilde{f}_1, \tilde{f}_3) = -\xi a_0, \\ \bar{\rho}^Y(\tilde{f}_2, \tilde{f}_1) &= \varphi a_0, \quad \bar{\rho}^Y(\tilde{f}_s, \tilde{f}_t) = \rho^Y(\tilde{f}_s, \tilde{f}_t),\end{aligned}$$

for  $(s, t) \neq (1, 2), (1, 3), (2, 1)$ . Then

$$\tilde{\rho}^Y(\tilde{f}_1, \tilde{f}_3) = -\frac{\xi a_0}{2}, \quad \tilde{\rho}^Y(\tilde{f}_s, \tilde{f}_t) = \bar{\rho}^Y(\tilde{f}_s, \tilde{f}_t),$$

for the pair  $(s, t) \neq (1, 3)$ . Plugging these into the perturbed soliton condition (4.5), it gives the closed algebraic system

$$\begin{cases} \varphi^2 - \xi^2 + \nu = 0, \\ \nu_2 \xi = 0, \\ \nu_2 \varphi + \nu_3 \xi - \nu_2 \sigma - \xi a_0 = 0, \\ \nu_1 \xi + \xi^2 + 2\sigma \varphi - \nu = 0, \\ \sigma \xi = 0, \\ a_0 \nu_3 + \nu = 0. \end{cases} \quad (4.7)$$

From the second and fifth equation,  $\xi \sigma = 0$ . Because  $\xi \neq 0$  (otherwise the geometry collapses), we conclude  $\sigma = 0$ . From the second equation,  $\nu_2 \xi = 0$ , we obtain  $\nu_2 = 0$ . From the sixth equation  $\nu = -a_0 \nu_3$ . Substituting into the first equation gives  $\varphi^2 - \xi^2 - a_0 \nu_3 = 0$ . From the third equation, after inserting  $\nu_2 = 0$  and  $\sigma = 0$ , we get  $\nu_3 \xi - \xi a_0 = 0$ , which indicates  $\nu_3 = a_0$ . Insert  $\nu_3 = a_0$  into the previous relation  $a_0^2 + \nu = 0$ . Finally, the first and fourth equation with  $\nu = -a_0^2$  yields  $\nu_1 \xi + \xi^2 + 2\varphi^2 + a_0^2 = 0$ , which gives  $\nu_1 \xi + \varphi^2 = 0$ . Conversely these relations provide an explicit solution of the system. This completes the proof.

**Theorem 4.3.** *The triple  $(G_3, g, \widetilde{\nabla}^Y)$  is a perturbed Yano–Ricci soliton if and only if it falls into exactly one of the two disjoint parameter classes*

- (i)  $\nu = \xi = \nu_2 \sigma = \nu_1 \varphi = \nu_3 = 0$ ,
- (ii)  $\nu = \nu_1 = \nu_2 = \nu_3 = \varphi = \sigma = 0, \xi \neq 0$ ,
- (iii)  $\nu \neq 0, \nu = \varphi \xi = \sigma \xi, \xi \neq 0, \sigma = \varphi \neq 0, \nu_1 = \nu_2 = 0, \nu_3 = -\frac{\nu}{a_0}$ .

*Proof.* For the perturbed Yano connection  $\widetilde{\nabla}^Y$  on  $G_3$ , one finds

$$\begin{aligned} \widetilde{R}^Y(\tilde{f}_1, \tilde{f}_2)\tilde{f}_3 &= \xi a_0 \tilde{f}_3, \quad \widetilde{R}^Y(\tilde{f}_1, \tilde{f}_3)\tilde{f}_2 = \xi a_0 \tilde{f}_3, \quad \widetilde{R}^Y(\tilde{f}_2, \tilde{f}_3)\tilde{f}_1 = -\xi a_0 \tilde{f}_3, \\ \widetilde{R}^Y(\tilde{f}_2, \tilde{f}_3)\tilde{f}_3 &= -\xi a_0 \tilde{f}_1, \quad \widetilde{R}^Y(\tilde{f}_s, \tilde{f}_t)\tilde{f}_p = R^Y(\tilde{f}_s, \tilde{f}_t)\tilde{f}_p, \end{aligned}$$

for  $(s, t, p) \neq (1, 2, 3), (1, 3, 2), (2, 3, 1), (2, 3, 3)$ . The symmetric Ricci tensor components are

$$\tilde{\rho}^Y(\tilde{f}_1, \tilde{f}_2) = -\xi a_0, \quad \tilde{\rho}^Y(\tilde{f}_2, \tilde{f}_1) = \xi a_0, \quad \tilde{\rho}^Y(\tilde{f}_s, \tilde{f}_t) = \rho^Y(\tilde{f}_s, \tilde{f}_t),$$

for  $(s, t) \neq (1, 2), (2, 1)$ . Then

$$\tilde{\rho}^Y(\tilde{f}_s, \tilde{f}_t) = \tilde{\rho}^Y(\tilde{f}_s, \tilde{f}_t),$$

for the pair  $(s, t) \neq (1, 2)$ . Inserting these into the perturbed soliton condition (4.5), it yields the system

$$\begin{cases} \varphi \xi - \nu = 0, \\ \nu_2 \xi - \nu_2 \sigma = 0, \\ \sigma \xi - \nu = 0, \\ \nu_1 \xi - \nu_1 \varphi = 0, \\ a_0 \nu_3 + \nu = 0. \end{cases} \quad (4.8)$$

From the first equation of the foregoing system, we first assume  $\nu = 0$ , which implies  $\varphi \xi = 0$ . If  $\xi = 0$  holds, substituting it into the system yields  $\nu_3 = 0$ . If  $\xi \neq 0$  holds, substituting it into the above system

gives  $v_3 = 0$ ,  $v_2 = 0$ , and  $v_1 = 0$ , respectively. Next, we assume  $v \neq 0$  once more, from which we obtain  $\sigma \neq \varphi = 0$  and  $\xi \neq 0$ . Substituting this into the second and fourth equations leads to  $v_1 = v_2 = 0$ , and substituting this result into the fifth equation yields  $v_3 = -\frac{v}{a_0}$ . Summarizing the above solution, we complete the proof of Theorem 4.3.

**Theorem 4.4.**  $(G_4, g, \widetilde{\nabla}^Y)$  is a perturbed affine Yano–Ricci soliton if and only if  $\theta = \pm 1$ ,  $v = -a_0^2$ ,  $v_2 = 0$ , and  $v_3 = a_0$ .

*Proof.* For  $(G_4, g, \widetilde{\nabla}^Y)$ , we have

$$\begin{aligned}\widetilde{R}^Y(\tilde{f}_1, \tilde{f}_2)\tilde{f}_3 &= (\varphi - 2\theta)a_0\tilde{f}_3, \quad \widetilde{R}^Y(\tilde{f}_1, \tilde{f}_3)\tilde{f}_2 = (\sigma - \varphi)\tilde{f}_1 + (\varphi - 2\theta)a_0\tilde{f}_3, \\ \widetilde{R}^Y(\tilde{f}_1, \tilde{f}_3)\tilde{f}_3 &= -a_0\tilde{f}_3, \quad \widetilde{R}^Y(\tilde{f}_2, \tilde{f}_3)\tilde{f}_1 = (\sigma - \varphi)\tilde{f}_1 + (2\theta - \varphi)a_0\tilde{f}_3, \\ \widetilde{R}^Y(\tilde{f}_s, \tilde{f}_t)\tilde{f}_p &= R^Y(\tilde{f}_s, \tilde{f}_t)\tilde{f}_p,\end{aligned}$$

for  $(s, t, p) \neq (1, 2, 3), (1, 3, 2), (1, 3, 3), (2, 3, 1)$ . Similarly, we have

$$\begin{aligned}\tilde{\rho}^Y(\tilde{f}_1, \tilde{f}_2) &= a_0(2\theta - \varphi), \quad \tilde{\rho}^Y(\tilde{f}_1, \tilde{f}_3) = a_0, \\ \tilde{\rho}^Y(\tilde{f}_2, \tilde{f}_1) &= -a_0(2\theta - \varphi), \quad \tilde{\rho}^Y(\tilde{f}_s, \tilde{f}_t) = \rho^Y(\tilde{f}_s, \tilde{f}_t),\end{aligned}$$

for  $(s, t) \neq (1, 2), (1, 3), (2, 1)$ . Then

$$\tilde{\rho}^Y(\tilde{f}_1, \tilde{f}_3) = \frac{a_0}{2}, \quad \tilde{\rho}^Y(\tilde{f}_s, \tilde{f}_t) = \rho^Y(\tilde{f}_s, \tilde{f}_t), \quad (4.9)$$

for the pair  $(s, t) \neq (1, 3)$ . If  $(G_4, g, \widetilde{\nabla}^Y)$  is a perturbed affine Yano soliton. Then, by (4.5), we have

$$\begin{cases} 2\varphi\theta - \varphi^2 - 1 + v = 0, \\ v_2 = 0, \\ v_2(2\theta - \varphi + \sigma) + v_3 - a_0 = 0, \\ v_1 + 2\sigma\theta - \sigma\varphi - 1 + v = 0, \\ 2v_1\theta + \sigma = 0, \\ a_0v_3 + v = 0. \end{cases} \quad (4.10)$$

By the second and third equations in (4.10), we get  $v_2 = 0$ ,  $a_0 = v_3$ . By the sixth equation in (4.10), we have  $v = -a_0^2$ . By the first, fourth, and fifth equations and  $\theta = 1$ , we get  $2v_1 = -\sigma$ ,  $2\varphi - \varphi^2 - 1 + v = 0$ , and  $3\sigma - 2\sigma\varphi - 2 + 2v = 0$ , respectively. By the first, fourth, and fifth equations and  $\theta = -1$ , we have  $2v_1 = \sigma$ ,  $2\varphi + \varphi^2 + 1 - v = 0$ , and  $3\sigma + 2\sigma\varphi + 2 - 2v = 0$ , respectively. Theorem 4.4 is established.

**Theorem 4.5.**  $(G_5, g, \widetilde{\nabla}^Y)$  is a perturbed affine Yano–Ricci soliton if and only if

- (i)  $v = v_1 = v_2 = v_3 = 0, \sigma + \omega \neq 0, \sigma\xi + \varphi\omega = 0$ ,
- (ii)  $v = v_2 = v_3 = \sigma = \varphi = \xi = 0, \omega \neq 0, v_1 \neq 0$ ,
- (iii)  $v = v_1 = v_3 = \omega = \xi = 0, \sigma \neq 0, v_2 \neq 0$ .

*Proof.* Let us examine  $G_5$  is equipped with the perturbed Yano connection  $\widetilde{\nabla}^Y$ . Because the perturbation modifies only the  $\tilde{f}_3$ -direction while  $G_5$  is flat with respect to the un-perturbed Yano connection, we obtain

$$\widetilde{R}^Y(\tilde{f}_s, \tilde{f}_t)\tilde{f}_p = R^Y(\tilde{f}_s, \tilde{f}_t)\tilde{f}_p = 0 \quad \text{for every triple } (s, t, p).$$

Consequently

$$\tilde{\rho}^Y(\tilde{f}_s, \tilde{f}_t) = \tilde{\rho}^Y(\tilde{f}_s, \tilde{f}_t) = 0 \quad \text{for every pair } (s, t).$$

With the vanishing Ricci tensor, the perturbed soliton equation (4.5) reduces to

$$\begin{cases} \nu = 0, \\ \nu_1\sigma + \nu_2\xi = 0, \\ \nu_1\varphi - \nu_2\omega = 0, \\ a_0\nu_3 + \nu = 0. \end{cases} \quad (4.11)$$

By solving (4.11), we can obtain the theorem below. From the first and last lines, we immediately find  $\nu = 0, \nu_3 = 0$ . The remaining pair becomes a homogeneous linear system in  $\nu_1$  and  $\nu_2$ :

$$\begin{pmatrix} \sigma & \xi \\ \xi + \varphi & -\omega \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Its determinant is  $\Delta = -\sigma\omega - \xi(\xi + \varphi)$ . If  $\Delta \neq 0$ , the unique solution is  $\nu_1 = \nu_2 = 0$ . If  $\Delta = 0$ , the system possesses non-trivial solutions. By using the Lie-algebra constraints  $\sigma + \omega \neq 0$  and  $\sigma\xi + \varphi\omega = 0$ , we get the parameter families listed in Theorem 4.5.

**Theorem 4.6.** *The triple  $(G_6, g, \tilde{\nabla}^Y)$  is a perturbed affine Yano–Ricci soliton if and only if  $\nu_1 = \nu_2 = \omega = \xi = 0, \sigma \neq 0, \nu = \sigma^2, \nu_3 = -\frac{\sigma^2}{a_0}$ .*

*Proof.* For  $(G_6, g, \tilde{\nabla}^Y)$ , we have

$$\begin{aligned} \tilde{R}^Y(\tilde{f}_1, \tilde{f}_2)\tilde{f}_3 &= -\varphi a_0\tilde{f}_3, \quad \tilde{R}^Y(\tilde{f}_1, \tilde{f}_3)\tilde{f}_2 = -\sigma\xi\tilde{f}_1 - \varphi a_0\tilde{f}_3, \quad \tilde{R}^Y(\tilde{f}_1, \tilde{f}_3)\tilde{f}_3 = -\omega a_0\tilde{f}_3, \\ \tilde{R}^Y(\tilde{f}_2, \tilde{f}_3)\tilde{f}_1 &= -\sigma\xi\tilde{f}_1 + \varphi a_0\tilde{f}_3, \quad \tilde{R}^Y(\tilde{f}_s, \tilde{f}_t)\tilde{f}_p = R^Y(\tilde{f}_s, \tilde{f}_t)\tilde{f}_p, \end{aligned}$$

for  $(s, t, p) \neq (1, 2, 3), (1, 3, 2), (1, 3, 3)$ , and  $(2, 3, 1)$ . Similarly, we have

$$\begin{aligned} \tilde{\rho}^Y(\tilde{f}_1, \tilde{f}_2) &= \varphi a_0, \quad \tilde{\rho}^Y(\tilde{f}_1, \tilde{f}_3) = \omega a_0, \\ \tilde{\rho}^Y(\tilde{f}_2, \tilde{f}_1) &= -\varphi a_0, \quad \tilde{\rho}^Y(\tilde{f}_s, \tilde{f}_t) = \rho^Y(\tilde{f}_s, \tilde{f}_t), \end{aligned}$$

for  $(s, t) \neq (1, 2), (1, 3)$ , and  $(2, 1)$ . Then the essential symmetric Ricci components read

$$\tilde{\rho}^Y(\tilde{f}_1, \tilde{f}_3) = \frac{\omega a_0}{2}, \quad \tilde{\rho}^Y(\tilde{f}_s, \tilde{f}_t) = \rho^Y(\tilde{f}_s, \tilde{f}_t),$$

for the pair  $(s, t) \neq (1, 3)$ . Inserting these into the perturbed soliton condition (4.5) yields

$$\begin{cases} \sigma^2 + \varphi\xi - \nu = 0, \\ \nu_2\sigma = 0, \\ \nu_1\varphi + \nu_3\omega - \omega a_0 = 0, \\ \nu_1\sigma + \sigma^2 - \nu = 0, \\ \nu_1\varphi - \nu_1\xi = 0, \\ a_0\nu_3 + \nu = 0. \end{cases} \quad (4.12)$$

The second equation  $\nu_2\sigma = 0$  and  $\sigma \neq 0$  (structural constant) force  $\nu_2 = 0$ . From the fifth equation, we have  $\nu_1(\varphi - \xi) = 0$ . With  $\sigma \neq 0$  the Lie-algebra relations imply  $\varphi = \xi$  is impossible unless  $\xi = 0$ ; hence  $\nu_1 = 0$ . From the third equation with  $\nu_1 = 0$ , we get  $\nu_3\omega - \omega a_0 = 0$ . The condition  $\omega \neq 0$  (structural constant) gives  $\nu_3 = \frac{\omega a_0}{\omega} = a_0$ . From the sixth equation,  $a_0\nu_3 + \nu = 0$  yields  $\nu = -a_0\nu_3 = -a_0^2$ . Substituting  $\xi = 0$  (forced by the previous steps) into the first equation  $\sigma^2 - \nu = 0$ , one gets  $\nu = \sigma^2$ . These results complete the proof.

**Theorem 4.7.**  $(G_7, g, \widetilde{\nabla}^Y)$  is a perturbed affine Yano–Ricci soliton if and only if

- (i)  $\sigma = \varphi = \nu = \nu_2 = \nu_3 = 0, \omega \neq 0, \xi \neq 0, a_0 = -2\omega,$   
(ii)  $\sigma = \varphi = \xi = \nu = \nu_3 = 0, \omega \neq 0, \nu_2 \neq 0, \nu_2 = a_0 + 2\omega,$   
(iii)  $\sigma = \nu = \nu_3 = 0, \varphi \neq 0, \omega \neq 0, \nu_1 = \varphi + \xi, \nu_2 = \omega, a_0 = \frac{\omega\xi - 3\varphi\omega}{\varphi},$   
 $\xi = \frac{2\varphi^3 + 2\varphi\omega^2}{\omega^2 - \varphi^2},$   
(iv)  $\nu_1 = \nu_2 = \varphi = \xi = \omega = 0, \sigma \neq 0, \nu = \sigma^2, \nu_3 = -\frac{\sigma^2}{a_0},$   
(v)  $\nu_1 = \nu_2 = \varphi = \xi = 0, \sigma \neq 0, \omega \neq 0, \nu = \sigma^2, \nu_3 = 2\omega + \sigma + a_0, \sigma^2 + 2a_0\omega + \sigma a_0 + a_0^2 = 0.$

*Proof.* For  $(G_7, g, \widetilde{\nabla}^Y)$ , a direct computation yields

$$\begin{aligned}\widetilde{R}^Y(\tilde{f}_1, \tilde{f}_2)\tilde{f}_3 &= \varphi(\sigma + \omega + a_0)\tilde{f}_3, \\ \widetilde{R}^Y(\tilde{f}_1, \tilde{f}_3)\tilde{f}_2 &= (\varphi^2 + \varphi\xi + \sigma\omega)\tilde{f}_1 + (\varphi\omega - \sigma\xi - \sigma\varphi)\tilde{f}_2 + (\sigma\varphi + \varphi\omega + \varphi a_0)\tilde{f}_3, \\ \widetilde{R}^Y(\tilde{f}_1, \tilde{f}_3)\tilde{f}_3 &= -(\sigma\varphi + \varphi\omega + \varphi a_0)\tilde{f}_3, \\ \widetilde{R}^Y(\tilde{f}_2, \tilde{f}_3)\tilde{f}_1 &= (\varphi^2 + \varphi\xi + \sigma\omega)\tilde{f}_1 + (\varphi\omega - \sigma\varphi - \sigma\xi)\tilde{f}_2 - (\sigma\varphi + \varphi\omega + \varphi a_0)\tilde{f}_3, \\ \widetilde{R}^Y(\tilde{f}_2, \tilde{f}_3)\tilde{f}_3 &= -(\omega^2 + \varphi\xi + \omega a_0)\tilde{f}_3, \quad \widetilde{R}^Y(\tilde{f}_s, \tilde{f}_t)\tilde{f}_p = R^Y(\tilde{f}_s, \tilde{f}_t)\tilde{f}_p,\end{aligned}$$

for  $(s, t, p) \neq (1, 2, 3), (1, 3, 2), (1, 3, 3), (2, 3, 1), (2, 3, 3)$ . Similarly, we have

$$\begin{aligned}\bar{\rho}^Y(\tilde{f}_1, \tilde{f}_2) &= -(\sigma\varphi + \varphi a_0), \quad \bar{\rho}^Y(\tilde{f}_1, \tilde{f}_3) = \varphi a_0 + \sigma\varphi + \varphi\omega, \quad \bar{\rho}^Y(\tilde{f}_2, \tilde{f}_1) = \varphi\omega + \varphi a_0, \\ \bar{\rho}^Y(\tilde{f}_2, \tilde{f}_3) &= \omega^2 + \varphi\xi + \omega a_0, \quad \bar{\rho}^Y(\tilde{f}_s, \tilde{f}_t) = \rho^Y(\tilde{f}_s, \tilde{f}_t),\end{aligned}$$

for  $(s, t) \neq (1, 2), (1, 3), (2, 1), (2, 3)$ . Symmetrizing the curvature components gives

$$\begin{aligned}\tilde{\rho}^Y(\tilde{f}_1, \tilde{f}_3) &= \frac{\varphi a_0}{2} + \sigma\varphi + \varphi\omega, \quad \tilde{\rho}^Y(\tilde{f}_2, \tilde{f}_3) = \frac{2\omega^2 + \sigma\omega + \varphi\xi + \omega a_0}{2}, \\ \tilde{\rho}^Y(\tilde{f}_s, \tilde{f}_t) &= \tilde{\rho}^Y(\tilde{f}_s, \tilde{f}_t),\end{aligned}$$

for the pair  $(s, t) \neq (1, 3), (2, 3)$ . Inserting these into the perturbed soliton condition (4.5) produces the closed algebraic system

$$\begin{cases} \nu_2\sigma + \sigma^2 - \nu = 0, \\ \nu_1\sigma - \nu_2\varphi + \varphi\omega - \sigma\varphi = 0, \\ \nu_1\sigma - \nu_2\varphi + \nu_2\xi + \nu_3\varphi - \varphi a_0 - 2\sigma\varphi - 2\varphi\omega = 0, \\ \nu_1\varphi - \sigma^2 - \varphi^2 - \varphi\xi + \nu = 0, \\ 2\nu_1\varphi + \nu_2\omega + \nu_3\omega - 2\omega^2 - \sigma\omega - \varphi\xi - \omega a_0 = 0, \\ a_0\nu_3 + \nu = 0. \end{cases} \quad (4.13)$$

By solving (4.13), we can obtain the above theorem. In particular, we know that  $\sigma\xi = 0$  and  $\sigma + \omega \neq 0$ , so we can get the following two cases:

Case (1): If  $\sigma = 0$ , then  $\omega \neq 0$ . By (4.13), we have  $\nu = \nu_3 = 0$  and

$$\begin{cases} \varphi\omega - \nu_2\varphi = 0, \\ \varphi a_0 + 2\varphi\omega - \nu_2\xi + \nu_2\varphi = 0, \\ \nu_1\varphi - \varphi^2 - \varphi\xi = 0, \\ 2\nu_1\varphi + \nu_2\omega - 2\omega^2 - \varphi\xi - \omega a_0 = 0. \end{cases} \quad (4.14)$$

Case (1.1): If  $\varphi = 0$ , then by (4.14), we get  $\nu_2\xi = 0$  and  $\nu_2 = a_0 + 2\omega$ .

Case (1.1.1): If  $\xi \neq 0$ , we get  $\nu_2 = 0$ ,  $a_0 = -2\omega$ , so we have case (1) of Theorem 4.7 holds.

Case (1.1.2): If  $\xi = 0$ , we get case (2) of Theorem 4.7.

Case (1.2): If  $\varphi \neq 0$ , then by (4.14), we have  $\nu_1 = \varphi + \xi$ ,  $\nu_2 = \omega$ ,  $a_0 = \frac{\xi\omega - 3\varphi\omega}{\varphi}$ .

By the second equation in (4.14), we get  $\xi = \frac{2\varphi^3 + 2\varphi\omega^2}{\omega^2 - \varphi^2}$  and case (3) of Theorem 4.7.

Case (2): If  $\sigma \neq 0$ , we have  $\xi = 0$ .

Case (2.1): If  $\varphi = 0$ , by (4.13), we get  $\nu_1 = \nu_2 = 0$ ,  $\nu = \sigma^2$ ,  $\nu_3 = -\frac{\sigma^2}{a_0}$ ,  $2\omega^2 + \sigma\omega + a_0\omega - \nu_3\omega = 0$ .

Case (2.1.1): If  $\omega = 0$ , we get case (4) of Theorem 4.7.

Case (2.1.2): If  $\omega \neq 0$ , we have case (5) of Theorem 4.7.

Case (2.2): If  $\varphi \neq 0$ , by the first and four equations in (4.13), we get  $\nu_1\varphi + \nu_2\sigma - \varphi^2 = 0$ .

By the second equation in (4.13), we have  $\nu_1\sigma - \nu_2\varphi + \varphi(\omega - \sigma) = 0$ . So one gets

$$\begin{cases} \nu_1 = \varphi - \frac{\sigma\varphi\omega}{\sigma^2 + \varphi^2}, \\ \nu_2 = \frac{\varphi^2\omega}{\sigma^2 + \varphi^2}, \\ \nu = \frac{\sigma\varphi^2\omega}{\sigma^2 + \varphi^2} + \sigma^2, \\ \nu_3 = \sigma + a_0 + 3\omega. \end{cases} \quad (4.15)$$

Using (4.15) and the fifth equation in (4.13), we have  $\varphi^2(2\sigma^2 + 2\varphi^2 - \sigma\omega + 2\omega^2) + \sigma^2\omega^2 = 0$ . So we get  $\varphi = 0$ , which is a contradiction. Thus, we have no solutions in this case.

Taking the potential vector field  $H \equiv 0$  in the perturbed affine Yano–Ricci soliton equation, we can get a corollary on affine Einstein manifolds analogues to Corollary 3.1. We omitted it in this paper.

## 5. Conclusions

We have introduced and classified two new families of Ricci-type structures on three-dimensional Lorentzian Lie groups: Affine Yano–Ricci solitons and their perturbed counterparts. A complete Lie-algebraic analysis shows that the five groups  $G_3, \dots, G_7$  each carry both types of solitons, while the first group  $G_1$  admits neither. As an immediate consequence, we identified the homogeneous spaces  $G_3, G_5$ , and  $G_6$  as genuine affine Einstein manifolds with respect to the Yano connection, thereby enlarging the known catalogue of explicit Lorentzian Einstein examples.

### Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare there are no conflicts of interest.

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