



Research article

Asymptotic convergence of solutions to elliptic hemivariational inequalities and optimal control problem

Lei Duan¹, Li-jun Zhu^{2,*} and Zhenhua He³

¹ School of Mathematics and Information Science, North Minzu University, Yinchuan 750021, China

² The Key Laboratory of Intelligent Information and Big Data Processing of NingXia Province, North Minzu University, Yinchuan 750021, China

³ College of Innovation and Entrepreneurship, Xinhua College of Ningxia University, Yinchuan 750021, China

* **Correspondence:** Email: zljmath@outlook.com.

Abstract: This paper investigates a class of elliptic hemivariational inequalities (HVIs) arising from steady-state heat conduction problems that involve linear and nonlinear mixed boundary effects and explores their applications to optimal control problems. First, we use nonsmooth analysis and pseudomonotone operator theory to establish the existence of solutions of the HVIs for the model. Under a relaxed monotonicity condition, we obtain uniqueness of the solution. Second, we prove that the solutions of the HVIs converge strongly to the solution of the corresponding Dirichlet boundary problem as the coefficient λ tends to infinity. Finally, a convergence analysis is provided for the solutions of the optimal control problem governed by HVIs. The results provide a theoretical framework for the study of elliptic problems with nonmonotone and multivalued boundary conditions, and they contribute to the mathematical foundation of related control and engineering applications.

Keywords: hemivariational inequality; Clarke generalized gradient; asymptotic analysis; optimal control; nonsmooth analysis

1. Introduction

Hemivariational inequalities (HVIs) were introduced to solve nonconvex and nonsmooth phenomena arising in mechanics. The theory was proposed by Panagiotopoulos in the 1980s [1,2] and overcomes the restriction of monotone operators in classical variational inequalities via the Clarke generalized gradient. In recent years, significant progress has been made in the well-posedness, equivalence, numerical convergence, and optimal control of hemivariational inequalities. This has gradually established a unified framework from theoretical analysis to applications and provided a solid foundation for their

further application in heat conduction, fluid mechanics, electrostatics, partial differential equations, foundational mathematics, and engineering control. Numerous scholars have extensively studied this theory, including Liu et al. [3,4], Han et al. [5,6], Migórski et al. [7,8], Gariboldi and Tarzia [9,10], and Sofonea et al. [11,12]. With the development of optimal control theory, the scope of study has expanded from systems governed by ordinary differential equations to infinite-dimensional problems constrained by partial differential equations and variational inequalities. In particular, when HVIs serve as state constraints, their inherent nonconvexity and nonsmoothness give rise to fundamental difficulties, such as nonuniqueness of state solutions, weak stability, and the absence of classical optimality conditions. This explains why related research has only gradually emerged in recent years. Although many results have been obtained on multivalued and optimal control problems under nonmonotone conditions (see [13–19]), substantial gaps remain in the existing theory. This paper is motivated by the work of Gariboldi et al., who studied elliptic models with nonmonotone multivalued boundary conditions and the associated optimal control problems governed by HVIs [20–22]. However, boundary conditions involving both linear and nonmonotone multivalued components were not considered in their studies. Building on this work, we investigate a class of elliptic equations governed by HVIs by introducing the boundary term $\beta\mu$, and we study the associated optimal control problems in this generalized framework. Under the boundary condition involving $\beta\mu$, the weak formulation of the elliptic problem (1.2) leads to the boundary hemivariational inequality (1.3). Its variational structure is a coupled form, involving a boundary linear term and a nonsmooth multivalued term, which differs from the structure considered in the work of Gariboldi et al. As a consequence, the uniqueness, convergence, and asymptotic behavior of solutions to HVIs in [21] are no longer direct consequences of existing results and must be reestablished within the new variational framework. These properties are analyzed and proved in Theorems 1–3. Moreover, this structural change also affects the optimal control problems governed by the associated HVIs. The introduction of $\beta\mu$ alters the variational structure of the state system, implying that the results in [22] on the existence, convergence, and asymptotic behavior of solutions for optimal control problems cannot be directly extended to the present setting. The corresponding results are revisited in Theorems 4 and 5. In this sense, the models considered here are more general, because they include previously studied models as particular cases (corresponding to $\beta = 0$), as well as problems involving linear and nonlinear mixed boundary effects.

We study steady-state heat conduction problems subject to mixed boundary conditions. Let $D \subset \mathbb{R}^d$ be a bounded domain whose boundary Σ satisfies the regularity assumptions. The boundary Σ consists of three measurable components Σ_i ($i = 1, 2, 3$) that are pairwise disjoint, each having a positive $(d - 1)$ -dimensional Hausdorff measure $|\Sigma_i|$. The vector \mathfrak{n} represents the exterior normal of Σ .

$$-\Delta\mu = f \text{ in } D, \mu|_{\Sigma_1} = 0, -\frac{\partial\mu}{\partial\mathfrak{n}}\Big|_{\Sigma_2} = r, \mu|_{\Sigma_3} = c, \quad (1.1)$$

$$-\Delta\mu = f \text{ in } D, \mu|_{\Sigma_1} = 0, -\frac{\partial\mu}{\partial\mathfrak{n}}\Big|_{\Sigma_2} = r, -\frac{\partial\mu}{\partial\mathfrak{n}}\Big|_{\Sigma_3} + \beta\mu \in \lambda\partial\psi(\mu). \quad (1.2)$$

Here, μ denotes the temperature in D , f represents the internal energy in D , c corresponds to the temperature on Σ_3 in (1.1) and to the temperature of the exterior neighborhood of Σ_3 in (1.2), r denotes the heat flux on Σ_2 , and $\lambda > 0$ is the heat transfer coefficient on Σ_3 . We assume that $\beta \in L^\infty(\Sigma_3)$, $f \in H = L^2(D)$, $r \in Q = L^2(\Sigma_2)$, and $c \in H^{\frac{1}{2}}(\Sigma_3)$. The function $\psi : \Sigma_3 \times \mathbb{R} \rightarrow \mathbb{R}$ is generally nonconvex, locally Lipschitz continuous, and nondifferentiable. The symbol ψ^0 denotes its Clarke generalized directional derivative.

The notational framework outlined below is applied consistently throughout the paper:

$$\begin{aligned} S &= H^1(D), \quad S_0 = \{s \in S \mid s = 0 \text{ on } \Sigma_1\}, \\ E &= \{s \in S \mid s = 0 \text{ on } \Sigma_1, s = c \text{ on } \Sigma_3\}, \\ E_0 &= \{s \in S \mid s = 0 \text{ on } \Sigma_1 \cup \Sigma_3\}, \\ A(\mu, s) &= \int_D \nabla \mu \cdot \nabla s \, dy, \quad F(s) = \int_D f s \, dy - \int_{\Sigma_2} r \gamma s \, d\Sigma. \end{aligned}$$

Here, $\gamma : S \rightarrow L^2(\Sigma)$ is the trace operator. In what follows, μ will also denote its trace on the boundary, for $\mu \in S$. The form A is bilinear, symmetric, continuous, and coercive; that is, for some constant $\tilde{m} > 0$, $A(s, s) = \|s\|_{S_0}^2 \geq \tilde{m} \|s\|_S^2$, $\forall s \in S_0$ [23]. We define the norms on S and S_0 as

$$\begin{aligned} \|s\|_S &= \left(\|s\|_{L^2(D)}^2 + \|\nabla s\|_{L^2(D; \mathbb{R}^d)}^2 \right)^{\frac{1}{2}}, \quad \forall s \in S; \\ \|s\|_{S_0} &= \|\nabla s\|_{L^2(D; \mathbb{R}^d)}, \quad \forall s \in S_0. \end{aligned}$$

It is a classical result that the two norms mentioned above are equivalent on S_0 . Moreover, the multivalued boundary condition on Σ_3 in problem (1.2) is characterized by nonmonotone relation through the Clarke generalized gradient. Together with Green's formula, the weak formulation of Problem (1.2) is given as follows: Find $\mu \in S_0$ such that

$$A(\mu, s) - \int_{\Sigma_3} \beta \mu s \, d\Sigma + \lambda \int_{\Sigma_3} \psi^0(\mu; s) \, d\Sigma \geq F(s), \quad \forall s \in S_0. \quad (1.3)$$

2. Preliminaries

Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ denote a Banach space, and let \mathcal{X}^* denote the dual space of \mathcal{X} . We use the notation $\langle \cdot, \cdot \rangle$ to represent the duality pairing between \mathcal{X}^* and \mathcal{X} . The following definitions are taken from Definitions 3.20, 3.21, 3.54, 3.57, 3.68 and Theorem 2.21 in [7].

Definition 1. Let $\phi_1 : \mathcal{X} \rightarrow \mathbb{R}$ be a locally Lipschitz function, that is, for each $y \in \mathcal{X}$, there exist a neighborhood y_ϵ of y and a constant $l_y > 0$ such that

$$|\phi_1(w) - \phi_1(v)| \leq l_y \|w - v\|_{\mathcal{X}} \quad \forall w, v \in y_\epsilon.$$

Definition 2. Let $\phi_1 : \mathcal{X} \rightarrow \mathbb{R}$ be a locally Lipschitz function. The Clarke generalized directional derivative of ϕ_1 at $y \in \mathcal{X}$ along the direction $s \in \mathcal{X}$ is given by

$$\phi_1^0(y; s) = \limsup_{w \rightarrow y, \alpha \rightarrow 0^+} \frac{\phi_1(w + \alpha s) - \phi_1(w)}{\alpha}.$$

The Clarke generalized gradient associated with ϕ_1 at y is a subset of \mathcal{X}^* , defined by

$$\partial \phi_1(y) = \{\zeta \in \mathcal{X}^* \mid \phi_1^0(y; s) \geq \langle \zeta, s \rangle, \quad \forall s \in \mathcal{X}\}.$$

Definition 3. Let $D \subset \mathbb{R}^d$ ($d \geq 2$) be a bounded domain having a Lipschitz continuous boundary. There is a bounded linear trace operator

$$\gamma : H^1(D) \rightarrow L^2(\Sigma)$$

which satisfies the following properties.

(1) *Boundedness: There is a constant $C_0 > 0$ such that*

$$\|\gamma w\|_{L^2(\Sigma)} \leq C_0 \|w\|_{H^1(D)} \quad \forall w \in H^1(D);$$

(2) *Compactness: The trace operator $\gamma : H^1(D) \rightarrow L^2(\Sigma)$ is compact.*

Definition 4. Let $B : X \rightarrow X^*$ be a hemicontinuous operator if the function

$$t \mapsto \langle B(y + tw), v \rangle_{X^* \times X} \quad \forall y, w, v \in X,$$

continuous on $[0, 1]$.

Definition 5. Let $T : X \rightarrow 2^{X^*}$ be an operator, and let $\text{Gr}(T)$ denote its graph. The operator T is said to be monotone if

$$\langle w^* - s^*, w - s \rangle_{X^* \times X} \geq 0 \quad \forall (w, w^*), (s, s^*) \in \text{Gr}(T).$$

Definition 6. Let X be a reflexive Banach space, and let $T : X \rightarrow 2^{X^*}$ be an operator. The operator T is pseudomonotone if the following conditions are satisfied:

- (1) *The values of T are subsets of X^* that are nonempty, closed, bounded, and convex.*
- (2) *T is upper semicontinuous from every finite-dimensional subspace of X to X^* , with the weak topology of X^* .*
- (3) *For $\{w_n\} \subset X$ with $w_n \rightarrow w$ in X and $w_n^* \in Tw_n$ satisfying $\limsup \langle w_n^*, w_n - w \rangle_{X^* \times X} \leq 0$, there exists an element $w^*(s) \in Tw$, $\forall s \in X$ such that*

$$\langle w^*(s), w - s \rangle_{X^* \times X} \leq \liminf \langle w_n^*, w_n - s \rangle_{X^* \times X}.$$

The operator T is said to be generalized pseudomonotone if $\{w_n\} \subset X$ and $\{w_n^*\} \subset X^*$ with $w_n^* \in Tw_n$, $w_n \rightarrow w$ in X , $w_n^* \rightarrow w^*$ in X^* , and $\limsup \langle w_n^*, w_n - w \rangle_{X^* \times X} \leq 0$; we obtain $w^* \in Tw$ and $\langle w_n^*, w_n \rangle_{X^* \times X} \rightarrow \langle w^*, w \rangle_{X^* \times X}$.

Proposition 1. [24] Let $\phi_1 : X \rightarrow \mathbb{R}$ be a function which is locally Lipschitz. Then, it satisfies the following properties:

- (1) *For each $y \in X$, the mapping $s \mapsto \phi_1^0(y; s)$ from X to \mathbb{R} is subadditive and positively homogeneous, that is,*

$$\begin{aligned} \phi_1^0(y; \alpha s) &= \alpha \phi_1^0(y; s) \quad \forall \alpha \geq 0, s \in X, \\ \phi_1^0(y; s_1 + s_2) &\leq \phi_1^0(y; s_1) + \phi_1^0(y; s_2) \quad \forall s_1, s_2 \in X; \end{aligned}$$

- (2) *For each $y \in X$, it holds that $\phi_1^0(y; s) = \max\{\langle \zeta, s \rangle \mid \zeta \in \partial\phi_1(y)\}$;*
- (3) *The mapping $(y, s) \mapsto \phi_1^0(y; s)$ from $X \times X$ to \mathbb{R} is upper semicontinuous;*
- (4) *For each $y \in X$, the $\partial\phi_1(y)$ is weak* compact, convex, and contains at least one element of X^* ;*
- (5) *The graph of $\partial\phi_1$ is closed in $X \times X^*$, with X^* in the weak* topology.*

Proposition 2. [25] Let X be a reflexive Banach space and $T : X \rightarrow 2^{X^*}$ be a generalized pseudomonotone operator. If T is bounded, and for all $y \in X$, the set $T(y) \subseteq X^*$ is convex, closed, and nonempty, then T is pseudomonotone.

We impose the following assumptions, denoted by A_1 . Let $\psi : \Sigma_3 \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy:

- (1) For every $e \in \mathbb{R}$, the mapping $\psi(\cdot, e)$ satisfies the measurability condition on Σ_3 ;
 (2) For almost every $y \in \Sigma_3$, $\psi(\cdot, e)$ is locally Lipschitz;
 (3) Nonnegative constants c_0, c_1 exist such that

$$|\partial\psi(y, e)| \leq c_0 + c_1|e| \quad \forall e \in \mathbb{R}, \text{ a.e. } y \in \Sigma_3;$$

- (4) A constant $c \in \mathbb{R}$ exists such that

$$\psi^0(y, e; c - e) \leq 0 \quad \forall e \in \mathbb{R}, \text{ a.e. } y \in \Sigma_3;$$

- (5) If $\psi^0(y, e; c - e) = 0 \quad \forall e \in \mathbb{R}, \text{ a.e. } y \in \Sigma_3$, then $e = c$.

Proposition 3. [21] Define the functional $L : L^2(\Sigma_3) \rightarrow \mathbb{R}$ by

$$L(z) = \int_{\Sigma_3} \psi(y, z(y)) d\Sigma, \quad \forall z \in L^2(\Sigma_3).$$

Under assumptions (1)–(3) of A_1 and by Corollary 4.15 in [7], the functional L satisfies the following properties:

- (1) The functional L is Lipschitz continuous on bounded subsets of $L^2(\Sigma_3)$;
 (2) For all $z, v \in L^2(\Sigma_3)$, $L^0(z; v) \leq \int_{\Sigma_3} \psi^0(y, z(y); v(y)) d\Sigma$;
 (3) For every $z \in L^2(\Sigma_3)$, the Clarke generalized gradient of L obeys the estimate

$$\|\partial L(z)\|_{L^2(\Sigma_3)} \leq \bar{c}_0 + \bar{c}_1 \|z\|_{L^2(\Sigma_3)},$$

where $\bar{c}_0, \bar{c}_1 \geq 0$ are constants.

Based on the preceding definitions, we now introduce some lemmas that provide the analytical framework for the investigation of the associated HVIs. Lemmas 1 and 2 are taken from Theorems 3.61 and 3.69 in [7], respectively.

Lemma 1. Let X be a reflexive Banach space. If $T : X \rightarrow 2^{X^*}$ is coercive and pseudomonotone, then T is a surjection.

Lemma 2. Let X be a reflexive Banach space. If $T : X \rightarrow X^*$ is bounded, monotone, and hemicontinuous, then T is pseudomonotone.

Lemma 3. [21] Let $\{f_n\} \subset L^2(D)$, $\{r_n\} \subset L^2(\Sigma_2)$, and $s_n \in S$ for $n \in \mathbb{N}$. Define $F_n \in S^*$ by

$$F_n(s) = \int_D f_n s dy - \int_{\Sigma_2} r_n s d\Sigma, \quad \forall s \in S.$$

Suppose that $f_n \rightharpoonup f$ in $L^2(D)$, $r_n \rightharpoonup r$ in $L^2(\Sigma_2)$, and $s_n \rightharpoonup s$ in S . Then,

$$\lim F_n(s_n) = F(s) \quad \text{and} \quad \|F_n\|_{S^*} \leq C \quad \forall n \in \mathbb{N},$$

where C is a positive constant independent of n .

Lemma 4. [20] The variational form of (1.1) is expressed as follows: Find $\mu \in E$ such that

$$A(\mu, s) = F(s), \quad \forall s \in E_0, \tag{2.1}$$

and the solution to this variational equation is unique.

3. Existence analysis of solutions to hemivariational inequalities

Theorem 1. Assume that A_1 holds. Let $\|\beta\|_{L^\infty} \leq 1/C_1^2$, where C_1^2 is the trace constant associated with the trace operator. Then, for any $F \in S_0^*$ and any $\lambda > 0$, the inequality (1.3) has a solution.

Proof. We denote by $\langle \cdot, \cdot \rangle$ the duality pairing between S_0^* and S_0 . Define the operator $B_1 : S_0 \rightarrow S_0^*$ by

$$\langle B_1\mu, s \rangle = A(\mu, s) - \int_{\Sigma_3} \beta\mu s \, d\Sigma, \quad \forall \mu, s \in S_0.$$

We show that the operator B_1 is hemicontinuous. To this end, define the function

$$g(t) = \langle B_1(\mu + ts), z \rangle, \quad \text{for } z \in S_0.$$

Substituting the operator B_1 , we obtain

$$g(t) = \left(A(\mu, z) - \int_{\Sigma_3} \beta\mu z \, d\Sigma \right) + t \left(A(s, z) - \int_{\Sigma_3} \beta s z \, d\Sigma \right). \quad (3.1)$$

By the definition of B_1 , we have

$$\langle B_1\mu, z \rangle = A(\mu, z) - \int_{\Sigma_3} \beta\mu z \, d\Sigma, \quad \langle B_1s, z \rangle = A(s, z) - \int_{\Sigma_3} \beta s z \, d\Sigma.$$

Hence, $g(t) = \langle B_1\mu, z \rangle + t\langle B_1s, z \rangle$, which is continuous on $[0, 1]$. It follows from Definition 4 that B_1 is hemicontinuous. By the definition of the operator norm, it holds that

$$\|B_1\mu\|_{S_0^*} = \sup_{\|s\|_{S_0} \leq 1} |\langle B_1\mu, s \rangle| = \sup_{\|s\|_{S_0} \leq 1} \left| A(\mu, s) - \int_{\Sigma_3} \beta\mu s \, d\Sigma \right|. \quad (3.2)$$

Applying the Cauchy–Schwarz inequality, we obtain

$$|A(\mu, s)| = \left| \int_D \nabla\mu \cdot \nabla s \, dy \right| \leq \|\nabla\mu\|_{L^2(D)} \|\nabla s\|_{L^2(D)} = \|\mu\|_{S_0} \|s\|_{S_0}. \quad (3.3)$$

The boundedness of the trace operator guarantees a constant $C_1 > 0$ for which $\|\gamma\mu\|_{L^2(\Sigma_3)} \leq C_1 \|\mu\|_{S_0}$, $\forall \mu \in S_0$. By the boundedness of β and the trace operator, we have

$$\left| \int_{\Sigma_3} \beta\mu s \, d\Sigma \right| \leq \|\beta\|_{L^\infty(\Sigma_3)} \|\mu\|_{L^2(\Sigma_3)} \|s\|_{L^2(\Sigma_3)} \leq C_1^2 \|\beta\|_{L^\infty(\Sigma_3)} \|\mu\|_{S_0} \|s\|_{S_0}. \quad (3.4)$$

Combining this with (3.3), we obtain

$$\begin{aligned} |\langle B_1\mu, s \rangle| &\leq \|\mu\|_{S_0} \|s\|_{S_0} + \|\beta\|_{L^\infty(\Sigma_3)} C_1^2 \|\mu\|_{S_0} \|s\|_{S_0} \\ &= (1 + \|\beta\|_{L^\infty(\Sigma_3)} C_1^2) \|\mu\|_{S_0} \|s\|_{S_0}. \end{aligned}$$

Taking the supremum for all $s \in S_0$ with $\|s\|_{S_0} \leq 1$, we get

$$\|B_1\mu\|_{S_0^*} \leq (1 + \|\beta\|_{L^\infty(\Sigma_3)} C_1^2) \|\mu\|_{S_0}.$$

To verify the coercivity of the operator B_1 , we compute

$$\begin{aligned} \langle B_1\mu, \mu \rangle &= A(\mu, \mu) - \int_{\Sigma_3} \beta \mu^2 d\Sigma \geq \|\mu\|_{S_0}^2 - \|\beta\|_{L^\infty(\Sigma_3)} \int_{\Sigma_3} \mu^2 d\Sigma \\ &\geq \|\mu\|_{S_0}^2 - \|\beta\|_{L^\infty(\Sigma_3)} C_1^2 \|\mu\|_{S_0}^2 = (1 - \|\beta\|_{L^\infty(\Sigma_3)} C_1^2) \|\mu\|_{S_0}^2. \end{aligned}$$

Because $\|\beta\|_{L^\infty} \leq 1/C_1^2$, the operator B_1 is coercive. By Definition 5, the operator B_1 is also monotone. In conclusion, the operator B_1 is hemicontinuous, bounded, and monotone. Therefore, B_1 satisfies Lemma 2 and is a pseudomonotone operator. Define the multivalued operator $B_2 : S_0 \rightarrow 2^{S_0^*}$ by

$$B_2(s) = \lambda \gamma^* \partial L(\gamma s), \quad \forall s \in S_0,$$

where $\gamma^* : L^2(\Sigma) \rightarrow S_0^*$ is the adjoint operator corresponding to the trace operator $\gamma : S_0 \rightarrow L^2(\Sigma)$. Proposition 1(4) ensures that, for every $s \in S_0$, the set $B_2(s)$ is convex, closed, and nonempty in S_0^* . Moreover, the operator B_2 satisfies the following boundedness estimate:

$$\begin{aligned} \|B_2 s\|_{S_0^*} &= \|\lambda \gamma^* \partial L(\gamma s)\|_{S_0^*} \leq \lambda \|\gamma^*\| \|\partial L(\gamma s)\|_{L^2(\Sigma_3)} \\ &\leq \lambda \|\gamma^*\| (\bar{c}_0 + \bar{c}_1 \|\gamma\| \|s\|_{S_0}), \quad \forall s \in S_0, \end{aligned} \quad (3.5)$$

where $\|\gamma\|$ stands for the operator norm corresponding to the mapping $\gamma : S_0 \rightarrow L^2(\Sigma)$. It will be shown that B_2 is generalized pseudomonotone. In accordance with Definition 6, let $\{s_n\} \subset S_0$ and $\{s_n^*\} \subset S_0^*$ be sequences such that

$$s_n \rightharpoonup s \text{ in } S_0, \quad s_n^* \rightharpoonup s^* \text{ in } S_0^*, \quad s_n^* \in B_2(s_n) \quad \forall n \in \mathbb{N},$$

and $\limsup \langle s_n^*, s_n - s \rangle_{S_0^* \times S_0} \leq 0$. To establish the generalized pseudomonotonicity of B_2 , it suffices to show that

$$s^* \in B_2(s) \quad \text{and} \quad \lim \langle s_n^*, s_n \rangle_{S_0^* \times S_0} = \langle s^*, s \rangle_{S_0^* \times S_0}.$$

By the definition of the operator B_2 , the inclusion $s_n^* \in B_2(s_n)$ guarantees the existence of $\rho_n \in \partial L(\gamma s_n)$ such that $s_n^* = \lambda \gamma^* \rho_n$. According to Proposition 3(3), the sequence $\{\rho_n\}$ is bounded in $L^2(\Sigma_3)$. There is a subsequence, which we continue to denote by $\{\rho_n\}$, and an element $\rho \in L^2(\Sigma_3)$ such that $\rho_n \rightharpoonup \rho$ in $L^2(\Sigma_3)$. Moreover, because the trace operator $\gamma : S_0 \rightarrow L^2(\Sigma_3)$ is compact, the weak convergence $s_n \rightharpoonup s$ in S_0 yields $\gamma s_n \rightarrow \gamma s$ in $L^2(\Sigma_3)$. By virtue of Proposition 1(5), the graph of the Clarke subdifferential ∂L is strongly-weakly closed. Hence, $\rho \in \partial L(\gamma s)$. We obtain $s^* = \lambda \gamma^* \rho$, and $s^* \in \lambda \gamma^* \partial L(\gamma s) = B_2(s)$. Next, we verify the convergence of the corresponding duality pairings. Indeed, for each $n \in \mathbb{N}$,

$$\langle s_n^*, s_n \rangle = \lambda \langle \rho_n, \gamma s_n \rangle_{L^2(\Sigma_3)} = \lambda \langle \rho_n, \gamma s \rangle_{L^2(\Sigma_3)} + \lambda \langle \rho_n, \gamma(s_n - s) \rangle_{L^2(\Sigma_3)}. \quad (3.6)$$

Because $\rho_n \rightharpoonup \rho$ in $L^2(\Sigma_3)$, and $\gamma s \in L^2(\Sigma_3)$ is fixed, we conclude that

$$\lim \langle \rho_n, \gamma s \rangle_{L^2(\Sigma_3)} = \langle \rho, \gamma s \rangle_{L^2(\Sigma_3)}.$$

Applying the Cauchy–Schwarz inequality together with the boundedness of the sequence $\{\rho_n\}$ in $L^2(\Sigma_3)$, we have

$$\left| \langle \rho_n, \gamma(s_n - s) \rangle_{L^2(\Sigma_3)} \right| \leq \|\rho_n\|_{L^2(\Sigma_3)} \|\gamma(s_n - s)\|_{L^2(\Sigma_3)}.$$

Consequently,

$$\lim \langle s_n^*, s_n \rangle = \lambda \langle \rho, \gamma s \rangle_{L^2(\Sigma_3)} = \langle \lambda \gamma^* \rho, s \rangle = \langle s^*, s \rangle.$$

Finally, by Proposition 2, the operator B_2 is pseudomonotone.

To prove the surjectivity of $B_1 + B_2$, it remains to show that it is coercive. Under assumption $A_1(4)$ and conditions (1) and (2) of Proposition 1, it holds that for all $e \in \mathbb{R}$ and for almost every $y \in \Sigma_3$,

$$\begin{aligned} \psi^0(y, e; -e) &= \psi^0(y, e; (c - e) - c) \leq \psi^0(y, e; c - e) + \psi^0(y, e; -c) \\ &\leq \psi^0(y, e; -c) \leq \sup_{\xi \in \partial \psi(y, e)} |\xi| |c| \leq |c|(c_0 + c_1|e|). \end{aligned} \quad (3.7)$$

Next, let $s \in S_0$ and $s^* \in B_2(s)$. Then, we get $\rho \in \partial L(\gamma s)$ such that $s^* = \lambda \gamma^* \rho$. According to Proposition 3(2), we derive

$$\begin{aligned} \lambda \langle \rho, -\gamma s \rangle_{L^2(\Sigma_3)} &\leq \lambda L^0(\gamma s; -\gamma s) \leq \lambda \int_{\Sigma_3} \psi^0(y, \gamma s(y); -\gamma s(y)) d\Sigma \\ &\leq \lambda |c| \int_{\Sigma_3} (c_0 + c_1 |\gamma s(y)|) d\Sigma \leq a_0 + a_1 \|s\|_{S_0}, \end{aligned}$$

where the constants a_0 and a_1 are defined by

$$a_0 = \lambda |c| c_0 |\Sigma_3| \geq 0, \quad a_1 = \lambda |c| c_1 \sqrt{|\Sigma_3|} \|\gamma\| \|s\|_{S_0} \geq 0.$$

By the relation $\lambda \langle \rho, \gamma s \rangle_{L^2(\Sigma_3)} = \langle s^*, s \rangle$, we infer that

$$\langle s^*, s \rangle \geq -a_0 - a_1 \|s\|_{S_0}, \quad \forall s \in S_0.$$

For the multivalued operator B_2 , it follows that

$$\inf_{s^* \in B_2 s} \langle s^*, s \rangle \geq -a_0 - a_1 \|s\|_{S_0}.$$

Combining this estimate with the coercivity of B_1 , we obtain

$$\langle (B_1 + B_2)s, s \rangle \geq \tilde{m} \|s\|_{S_0}^2 - a_1 \|s\|_{S_0} - a_0. \quad (3.8)$$

Consequently, the operator $B_1 + B_2$ is bounded, coercive, and pseudomonotone. By Lemma 1, it follows that $B_1 + B_2$ is surjective; hence, there exists $\mu \in S_0$ such that $F \in (B_1 + B_2)\mu$.

In the concluding step of the proof, we demonstrate that every element $\mu \in S_0$ satisfying the inclusion $F \in (B_1 + B_2)\mu$ also fulfills the hemivariational inequality (1.3). From the relation $B_1\mu + \lambda \gamma^* \rho = F$, $\rho \in \partial L(\gamma \mu)$, it follows that

$$\langle B_1\mu, s \rangle + \lambda \langle \rho, \gamma s \rangle_{L^2(\Sigma_3)} = \langle F, s \rangle, \quad \forall s \in S_0.$$

By combining the definition of $\partial L(\gamma \mu)$ with Proposition 3(2), we obtain

$$\begin{aligned} \langle F, s \rangle &= \langle B_1\mu, s \rangle + \lambda \langle \rho, \gamma s \rangle_{L^2(\Sigma_3)} \\ &\leq \langle B_1\mu, s \rangle + \lambda L^0(\gamma \mu; \gamma s) \\ &\leq A(\mu, s) - \int_{\Sigma_3} \beta \mu s d\Sigma + \lambda \int_{\Sigma_3} \psi^0(\gamma \mu; \gamma s) d\Sigma, \quad \forall s \in S_0. \end{aligned} \quad (3.9)$$

This shows that every $\mu \in S_0$ satisfying the inclusion $F \in (B_1 + B_2)\mu$ also constitutes a solution to the hemivariational inequality (1.3). \square

Suppose that the assumption $A_1(4)$ is replaced by the relaxed monotonicity condition

$$\psi^0(y, e; j - e) + \psi^0(y, j; e - j) \leq m_1 |e - j|^2, \quad \forall e, j \in \mathbb{R}, \text{ a.e. } y \in \Sigma_3,$$

where $m_1 \geq 0$. Here, the constant m_1 denotes the relaxed monotonicity constant associated with the locally Lipschitz function ψ ; when ψ is convex, the condition holds with $m_1 = 0$ [25]. Moreover, if the following smallness condition holds:

$$\tilde{m} - \|\beta\|_{L^\infty} \|\gamma\|^2 > \lambda m_1 \|\gamma\|^2, \quad \tilde{m} \geq 0,$$

then problem (1.3) has a unique solution.

Theorem 2. Assume that A_1 holds, and let $\lambda > 0$ be given. Let $F_n, F \in S^*$ be the associated linear functionals, and let $\mu_n \in S_0$ denote the solutions of the boundary hemivariational inequality

$$A(\mu_n, s) - \int_{\Sigma_3} \beta \mu_n s \, d\Sigma + \lambda \int_{\Sigma_3} \psi^0(\mu_n; s) \, d\Sigma \geq F_n(s), \quad \forall s \in S_0,$$

where the sequence $\{F_n\}$ satisfies

$$\lim F_n(v_n) = F(v) \quad \text{for any sequence } v_n \rightarrow v \text{ in } S. \quad (3.10)$$

Then, there exist a subsequence $\{\mu_n\}$ and an element $\xi \in S_0$ such that $\mu_n \rightarrow \xi$ in S , and ξ is a solution of (1.3). Moreover, if the relaxed monotonicity condition and the smallness condition are both satisfied, then (1.3) has a unique solution $\mu \in S_0$ associated with F . In this case, the entire sequence $\{\mu_n\}$ converges strongly to μ , namely,

$$\mu_n \rightarrow \mu \quad \text{in } S \quad \text{as } n \rightarrow \infty.$$

Proof. Let $\mu_n \in S_0$ denote the solution to problem (1.3) associated with F_n , and let $\mu^* \in E$ be the solution to problem (2.1). Choosing the test function $s_1 = \mu^* - \mu_n \in S_0$, we obtain

$$A(\mu_n, s_1) - \int_{\Sigma_3} \beta \mu_n s_1 \, d\Sigma + \lambda \int_{\Sigma_3} \psi^0(\mu_n; s_1) \, d\Sigma \geq F_n(s_1).$$

Consequently, we get

$$A(s_1, s_1) + \int_{\Sigma_3} \beta \mu_n s_1 \, d\Sigma \leq A(\mu^*, s_1) - F_n(s_1) + \lambda \int_{\Sigma_3} \psi^0(\mu_n; s_1) \, d\Sigma. \quad (3.11)$$

Because the bilinear form A satisfies the properties of boundedness and coercivity, it holds that

$$\begin{aligned} \tilde{m} \|s_1\|_S^2 &\leq A(\mu^*, \mu^* - \mu_n) + F_n(\mu_n - \mu^*) - \int_{\Sigma_3} \beta \mu_n (\mu^* - \mu_n) \, d\Sigma \\ &\leq M \|\mu^*\|_S \|s_1\|_S + \|\beta\|_{L^\infty} \|\gamma\|^2 \|\mu_n\|_S \|s_1\|_S + \|F_n\|_{S^*} \|s_1\|_S. \end{aligned}$$

Consequently, we have

$$\tilde{m} \|\mu^* - \mu_n\|_S \leq M \|\mu^*\|_S + \|\beta\|_{L^\infty} \|\gamma\|^2 \|\mu_n\|_S + \|F_n\|_{S^*}.$$

Applying the triangle inequality yields

$$(\bar{m} - \|\beta\|_{L^\infty} \|\gamma\|^2) \|s_1\|_S \leq (M + \|\beta\|_{L^\infty} \|\gamma\|^2) \|\mu^*\|_S + \|F_n\|_{S^*}.$$

Hence, we obtain

$$\|s_1\|_S \leq \frac{(M + \|\beta\|_{L^\infty} \|\gamma\|^2) \|\mu^*\|_S + \|F_n\|_{S^*}}{\bar{m} - \|\beta\|_{L^\infty} \|\gamma\|^2} =: k_a,$$

and by the triangle inequality,

$$\|\mu_n\|_S \leq \|\mu^* - \mu_n\|_S + \|\mu^*\|_S \leq k_a + \|\mu^*\|_S, \quad \forall n \in \mathbb{N},$$

where $k_a > 0$ is independent of n . The sequence $\{\mu_n\}$ is uniformly bounded in S and in S_0 . Because S_0 is reflexive, we can find $\xi \in S_0$ and a subsequence, which we continue to denote by $\{\mu_n\}$, such that

$$\mu_n \rightharpoonup \xi \quad \text{in } S_0 \quad \text{as } n \rightarrow \infty.$$

Next, we verify that $\xi \in S_0$ satisfies (2.1). Because $\mu_n \in S_0$, we have

$$A(\mu_n, s) - \int_{\Sigma_3} \beta \mu_n s \, d\Sigma + \lambda \int_{\Sigma_3} \psi^0(\mu_n; s) \, d\Sigma \geq F_n(s), \quad \forall s \in S_0.$$

By taking the limit superior as $n \rightarrow \infty$ and applying the weak continuity of $A(\cdot, s)$ together with (3.10), we obtain

$$A(\xi, s) - \int_{\Sigma_3} \beta \mu_n s \, d\Sigma + \lambda \limsup \int_{\Sigma_3} \psi^0(\mu_n; s) \, d\Sigma \geq F(s), \quad \forall s \in S_0. \quad (3.12)$$

The compactness of the trace operator from S into $L^2(\Sigma_3)$ ensures that

$$\mu_n|_{\Sigma_3} \rightarrow \xi|_{\Sigma_3} \quad \text{in } L^2(\Sigma_3) \quad \text{as } n \rightarrow \infty.$$

Furthermore, a subsequence, which we continue to denote by $\{\mu_n\}$, can be selected such that

$$\mu_n(y) \rightarrow \xi(y) \quad \text{a.e. } y \in \Sigma_3, \quad \text{and} \quad |\mu_n(y)| \leq h_1(y) \quad \text{a.e. } y \in \Sigma_3,$$

for some $h_1 \in L^2(\Sigma_3)$. Because the mapping $(e, j) \in \mathbb{R} \times \mathbb{R} \mapsto \psi^0(y, e, j) \in \mathbb{R}$ is upper semicontinuous almost everywhere on Σ_3 , we have

$$\limsup \psi^0(y, \mu_n(y); s(y)) \leq \psi^0(y, \xi(y); s(y)) \quad \text{a.e. } y \in \Sigma_3. \quad (3.13)$$

In view of the estimate

$$|\psi^0(y, \mu_n(y); s(y))| \leq (c_0 + c_1 |\mu_n(y)|) |s(y)| \leq k_1(y) \quad \text{a.e. } y \in \Sigma_3,$$

where $k_1 \in L^1(\Sigma_3)$ is defined by $k_1(y) = (c_0 + c_1 h_1(y)) |s(y)|$, we may apply the reverse Fatou lemma (see [26]) to get

$$\limsup \int_{\Sigma_3} \psi^0(y, \mu_n(y); s(y)) \, d\Sigma \leq \int_{\Sigma_3} \limsup \psi^0(y, \mu_n(y); s(y)) \, d\Sigma. \quad (3.14)$$

Combining (3.13) with (3.14), it follows that

$$\limsup \int_{\Sigma_3} \psi^0(y, \mu_n(y); s(y)) d\Sigma \leq \int_{\Sigma_3} \psi^0(y, \xi(y); s(y)) d\Sigma.$$

By inserting the result into (3.12), we obtain

$$A(\xi, s) - \int_{\Sigma_3} \beta \mu_n s d\Sigma + \lambda \int_{\Sigma_3} \psi^0(\xi; s) d\Sigma \geq F(s), \quad \forall s \in S_0. \quad (3.15)$$

Because $\mu_n \rightarrow \xi$ in S , and the trace operator $\gamma : S_0 \rightarrow L^2(\Sigma_3)$ is compact, it follows that $\mu_n|_{\Sigma_3} \rightarrow \xi|_{\Sigma_3}$ in $L^2(\Sigma_3)$. Consequently, we have

$$\lim \int_{\Sigma_3} \beta(y) \mu_n(y) s(y) d\Sigma = \int_{\Sigma_3} \beta(y) \xi(y) s(y) d\Sigma. \quad (3.16)$$

By taking the limit in the hemivariational inequality (3.12) satisfied by μ_n and substituting this result, we obtain

$$A(\xi, s) - \int_{\Sigma_3} \beta \xi s d\Sigma + \lambda \int_{\Sigma_3} \psi^0(\xi; s) d\Sigma \geq F(s), \quad \forall s \in S_0, \quad (3.17)$$

which confirms that $\xi \in S_0$ solves problem (1.3).

In addition, assuming the relaxed monotonicity and smallness conditions, the existence of a solution to (1.3) is ensured by the first part of the theorem. To verify uniqueness, assume that $\mu_1, \mu_2 \in S_0$ are two solutions of problem (1.3). For the inequality associated with μ_1 , we choose the test function $\mu_2 - \mu_1$, and for the inequality associated with μ_2 , we choose the test function $\mu_1 - \mu_2$. Adding the two resulting inequalities yields

$$\begin{aligned} & A(\mu_1 - \mu_2, \mu_2 - \mu_1) + \int_{\Sigma_3} \beta (\mu_2 - \mu_1)^2 d\Sigma \\ & + \lambda \int_{\Sigma_3} (\psi^0(\mu_1; \mu_2 - \mu_1) + \psi^0(\mu_2; \mu_1 - \mu_2)) d\Sigma \geq 0. \end{aligned}$$

Using the coercivity of the bilinear form A together with the relaxed monotonicity assumption, we derive

$$\begin{aligned} \tilde{m} \|\mu_1 - \mu_2\|_S^2 & \leq \int_{\Sigma_3} \beta (\mu_2 - \mu_1)^2 d\Sigma + \lambda m_1 \int_{\Sigma_3} |\mu_1(y) - \mu_2(y)|^2 d\Sigma \\ & \leq \|\beta\|_{L^\infty} \|\gamma\|^2 \|\mu_1 - \mu_2\|_S^2 + \lambda m_1 \|\gamma\|^2 \|\mu_1 - \mu_2\|_S^2. \end{aligned}$$

The smallness condition implies that $\mu_1 = \mu_2$. Therefore, the solution of problem (1.3) in S_0 is unique. In view of (3.17), it follows that $\xi = \mu$.

Finally, we prove the sequence $\mu_n \rightarrow \mu$ in S . By taking suitable test functions from S_0 in the hemivariational inequalities (1.3) and (3.17), we have

$$A(\mu_n, \mu - \mu_n) - \int_{\Sigma_3} \beta \mu_n (\mu - \mu_n) d\Sigma + \lambda \int_{\Sigma_3} \psi^0(\mu_n; \mu - \mu_n) d\Sigma \geq F_n(\mu - \mu_n), \quad (3.18)$$

$$A(\mu, \mu_n - \mu) - \int_{\Sigma_3} \beta \mu (\mu_n - \mu) d\Sigma + \lambda \int_{\Sigma_3} \psi^0(\mu; \mu_n - \mu) d\Sigma \geq F(\mu_n - \mu). \quad (3.19)$$

By summing the two inequalities, we deduce

$$A(\mu_n - \mu, \mu - \mu_n) + \lambda \int_{\Sigma_3} [\psi^0(\mu_n; \mu - \mu_n) + \psi^0(\mu; \mu_n - \mu)] d\Sigma \\ + \int_{\Sigma_3} \beta(\mu - \mu_n)^2 d\Sigma \geq F_n(\mu - \mu_n) + F(\mu_n - \mu).$$

Applying the coercivity of A together with the relaxed monotonicity condition, we obtain

$$\tilde{m} \|\mu_n - \mu\|_S^2 \leq \|\beta\|_{L^\infty} \|\gamma\|^2 \|\mu_n - \mu\|_S^2 + \lambda m_1 \|\gamma\|^2 \|\mu_n - \mu\|_S^2 \\ + F_n(\mu_n - \mu) + F(\mu - \mu_n), \quad (3.20)$$

which implies

$$(\tilde{m} - \|\beta\|_{L^\infty} \|\gamma\|^2 - \lambda m_1 \|\gamma\|^2) \|\mu_n - \mu\|_S^2 \leq F_n(\mu_n - \mu) + F(\mu - \mu_n).$$

Combining (3.10) with the smallness condition, we obtain

$$\|\mu_n - \mu\|_S \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Due to the uniqueness of $\mu \in S_0$, we get $\mu_n \rightarrow \mu$ in S . \square

4. Asymptotic analysis of solutions

In this section, we analyze the limiting behavior of the solutions to problem (1.3) as $\lambda \rightarrow \infty$.

Theorem 3. *Assume that A_1 and the smallness condition hold. Let $\mu_\lambda \in S_0$ consist of solutions to problem (1.3), and let $\mu \in E$ denote the unique solution corresponding to problem (2.1). As a consequence, as $\lambda \rightarrow \infty$, we have*

$$\mu_\lambda \rightarrow \mu \quad \text{in } S.$$

Proof. To prove the boundedness of the sequence $\{\mu_\lambda\}$, we take the test function $s_2 = \mu - \mu_\lambda \in S_0$ in problem (1.3), which gives

$$A(\mu_\lambda, s_2) - \int_{\Sigma_3} \beta \mu_\lambda s_2 d\Sigma + \lambda \int_{\Sigma_3} \psi^0(\mu_\lambda; s_2) d\Sigma \geq F(s_2).$$

Using the relation $A(\mu_\lambda, \mu - \mu_\lambda) = -A(\mu - \mu_\lambda, \mu - \mu_\lambda) + A(\mu, \mu - \mu_\lambda)$, we obtain

$$A(s_2, s_2) - \lambda \int_{\Sigma_3} \psi^0(\mu_\lambda; s_2) d\Sigma \leq A(\mu, s_2) - \int_{\Sigma_3} \beta \mu_\lambda s_2 d\Sigma - F(s_2). \quad (4.1)$$

Because we have $\psi^0(y, \mu_\lambda; \mu - \mu_\lambda) = \psi^0(y, \mu_\lambda; c - \mu_\lambda)$ on Σ_3 and by assumption $A_1(4)$, it follows that

$$\psi^0(y, \mu_\lambda; \mu - \mu_\lambda) \leq 0 \quad \text{on } \Sigma_3.$$

Therefore, the inequality reduces to

$$A(s_2, s_2) \leq A(\mu, s_2) - \int_{\Sigma_3} \beta \mu_\lambda s_2 d\Sigma - F(s_2). \quad (4.2)$$

Owing to the coercivity and continuity of the bilinear form A , we can find constants $\bar{m}, M > 0$ satisfying

$$\bar{m} \|s_2\|_S^2 \leq (M\|\mu\|_S + \|\beta\|_{L^\infty} \|\gamma\|^2 \|\mu_\lambda\|_S + \|F\|_{S^*}) \|s_2\|_S,$$

where $\|\gamma\|$ stands for the norm of the trace operator. Dividing both sides by $\|s_2\|_S$ yields

$$\bar{m} \|s_2\|_S \leq M\|\mu\|_S + \|\beta\|_{L^\infty} \|\gamma\|^2 \|\mu_\lambda\|_S + \|F\|_{S^*}. \quad (4.3)$$

Using the inequality $\|\mu_\lambda\|_S \leq \|s_2\|_S + \|\mu\|_S$ together with (4.3), we infer that

$$\|s_2\|_S \leq \frac{M\|\mu\|_S + \|\beta\|_{L^\infty} \|\gamma\|^2 \|\mu\|_S + \|F\|_{S^*}}{\bar{m} - \|\beta\|_{L^\infty} \|\gamma\|^2} =: k_b, \quad (4.4)$$

where $k_b > 0$ does not depend on λ . It follows that $\|\mu_\lambda\|_S \leq k_b + \|\mu\|_S$. The reflexivity of S ensures the existence of an element $\mu^* \in S$ and a subsequence, which we continue to denote by μ_λ , such that

$$\mu_\lambda \rightharpoonup \mu^* \quad \text{in } S \quad \text{as } \lambda \rightarrow \infty.$$

Next, we verify that the weak limit μ^* belongs to E . Because the sequence μ_λ converges weakly to μ^* in S , the compact embedding induced by the trace operator implies that $\mu_\lambda|_{\Sigma_3} \rightarrow \mu^*|_{\Sigma_3}$ as $\lambda \rightarrow \infty$. By extracting a subsequence, which we continue to denote by $\{\mu_\lambda\}$, we have $\mu_\lambda(y) \rightarrow \mu^*(y)$ for almost every $y \in \Sigma_3$, with a function $h_2 \in L^2(\Sigma_3)$ satisfying $|\mu_\lambda(y)| \leq h_2(y)$ for almost every $y \in \Sigma_3$. According to Proposition 1(3), the function ψ^0 is upper semicontinuous. Therefore, for almost every $y \in \Sigma_3$, we have

$$\limsup \psi^0(y, \mu_\lambda(y); c - \mu_\lambda(y)) \leq \psi^0(y, \mu^*(y); c - \mu^*(y)).$$

The following estimate holds for almost every $y \in \Sigma_3$:

$$|\psi^0(y, \mu_\lambda(y); c - \mu_\lambda(y))| \leq (c_0 + c_1 |\mu_\lambda(y)|) |c - \mu_\lambda(y)| \leq k_2(y),$$

where $k_2(y) = (c_0 + c_1 h_2(y))(c + h_2(y)) \in L^1(\Sigma_3)$. Applying the reverse Fatou lemma, we have

$$\limsup \int_{\Sigma_3} \psi^0(y, \mu_\lambda(y); c - \mu_\lambda(y)) d\Sigma \leq \int_{\Sigma_3} \limsup \psi^0(y, \mu_\lambda(y); c - \mu_\lambda(y)) d\Sigma. \quad (4.5)$$

Combining this with the upper semicontinuity property, we deduce

$$\limsup \int_{\Sigma_3} \psi^0(\mu_\lambda; c - \mu_\lambda) d\Sigma \leq \int_{\Sigma_3} \psi^0(\mu^*; c - \mu^*) d\Sigma.$$

Because the bilinear form A is nonnegative, Inequality (4.1) yields

$$-\lambda \int_{\Sigma_3} \psi^0(\mu_\lambda; c - \mu_\lambda) d\Sigma \leq \frac{1}{\bar{m}} (M\|\mu\|_S + \|\beta\|_{L^\infty} \|\gamma\|^2 \|\mu_\lambda\|_S + \|F\|_{S^*})^2 =: C_1,$$

where $C_1 > 0$ does not depend on λ . Hence, it holds that

$$-\int_{\Sigma_3} \psi^0(\mu_\lambda; c - \mu_\lambda) d\Sigma \leq \frac{C_1}{\lambda}. \quad (4.6)$$

Combining this estimate with assumption $A_1(4)$, we deduce

$$0 \leq - \int_{\Sigma_3} \psi^0(\mu^*; c - \mu^*) d\Sigma \leq \liminf \left(- \int_{\Sigma_3} \psi^0(\mu_\lambda; c - \mu_\lambda) d\Sigma \right) \leq 0.$$

This yields $\int_{\Sigma_3} \psi^0(\mu^*; c - \mu^*) d\Sigma = 0$. Applying assumption $A_1(4)$, we conclude that $\psi^0(y, \mu^*(y); c - \mu^*(y)) = 0$ for almost every $y \in \Sigma_3$. Then, by assumption $A_1(5)$, it follows that $\mu^*(y) = c$ for almost every $y \in \Sigma_3$.

Subsequently, we show that $\mu^* = \mu$. By the weak sequential closedness of S_0 in S , the weak limit μ^* of the sequence $\{\mu_\lambda\} \subset S_0$ belongs to S_0 . For any $z \in E$, we choose the test function $s = z - \mu_\lambda \in S_0$. Substituting this choice into the hemivariational inequality (1.3), we obtain

$$A(\mu_\lambda, z - \mu_\lambda) - \int_{\Sigma_3} \beta \mu_\lambda (z - \mu_\lambda) d\Sigma + \lambda \int_{\Sigma_3} \psi^0(\mu_\lambda; z - \mu_\lambda) d\Sigma \geq F(z - \mu_\lambda). \quad (4.7)$$

Because $z = c$ on Σ_3 , assumption $A_1(4)$ yields

$$\psi^0(\mu_\lambda; z - \mu_\lambda) = \psi^0(\mu_\lambda; c - \mu_\lambda) \leq 0 \quad \text{on } \Sigma_3.$$

As a consequence, we obtain

$$A(\mu_\lambda, z - \mu_\lambda) - \int_{\Sigma_3} \beta \mu_\lambda (c - \mu_\lambda) d\Sigma \geq F(z - \mu_\lambda).$$

As $\lambda \rightarrow \infty$, the sequence $\mu_\lambda \rightarrow \mu^*$ in S , and the compact embedding of the trace operator $\gamma : S_0 \rightarrow L^2(\Sigma_3)$ ensures that the strong convergence, that is, $\mu_\lambda|_{\Sigma_3} \rightarrow \mu^*|_{\Sigma_3}$ in $L^2(\Sigma_3)$. It follows that

$$\lim \int_{\Sigma_3} \mu_\lambda (c - \mu_\lambda) d\Sigma = \int_{\Sigma_3} \mu^* (c - \mu^*) d\Sigma. \quad (4.8)$$

Let us now define $v = z - \mu^*$. Noting that $z, \mu^* \in E$, we conclude that $v \in E_0$. Taking the limit inferior as $\lambda \rightarrow \infty$ and applying the fact that A is weakly lower semicontinuous, we obtain

$$A(\mu^*, v) \geq F(v) \quad \forall v \in E_0.$$

Noting that E_0 is a linear subspace, we have

$$A(\mu^*, v) = F(v) \quad \forall v \in E_0, \quad (4.9)$$

which shows that μ^* solves the problem (2.1). By uniqueness, we conclude that $\mu^* = \mu$.

Finally, we prove the sequence $\mu_\lambda \rightarrow \mu$ in S as $\lambda \rightarrow \infty$. Choosing test function $s_3 = \mu - \mu_\lambda \in S_0$ in the inequality (1.3) leads to

$$A(\mu_\lambda, s_3) - \int_{\Sigma_3} \beta \mu_\lambda s_3 d\Sigma + \lambda \int_{\Sigma_3} \psi^0(\mu_\lambda; s_3) d\Sigma \geq F(s_3).$$

By applying the relation

$$A(\mu_\lambda, \mu - \mu_\lambda) = A(\mu, \mu - \mu_\lambda) - A(\mu - \mu_\lambda, \mu - \mu_\lambda),$$

the inequality can be expressed as

$$A(s_3, s_3) \leq A(\mu, s_3) - F(s_3) - \int_{\Sigma_3} \beta \mu_\lambda s_3 d\Sigma + \lambda \int_{\Sigma_3} \psi^0(\mu_\lambda; s_3) d\Sigma.$$

Because $\mu = c$ on Σ_3 , and assumption $A_1(4)$ holds, we have

$$\psi^0(\mu_\lambda; \mu - \mu_\lambda) = \psi^0(\mu_\lambda; c - \mu_\lambda) \leq 0 \quad \text{a.e. on } \Sigma_3.$$

Using this property together with the coercivity of A , we obtain

$$\bar{m} \|s_3\|_S^2 \leq A(\mu, s_3) - F(s_3) - \int_{\Sigma_3} \beta \mu_\lambda s_3 d\Sigma. \quad (4.10)$$

The weak convergence $\mu_\lambda \rightharpoonup \mu$ in S as $\lambda \rightarrow \infty$ implies

$$\lim [A(\mu, \mu - \mu_\lambda) - F(\mu - \mu_\lambda)] = 0.$$

Because the trace operator $\Sigma : S_0 \rightarrow L^2(\Sigma_3)$ is compact, the sequence $\mu_\lambda|_{\Sigma_3}$ converges strongly to $\mu|_{\Sigma_3}$ in $L^2(\Sigma_3)$, which yields

$$\lim \int_{\Sigma_3} \mu_\lambda (\mu - \mu_\lambda) d\Sigma = 0.$$

Taking the limit $\lambda \rightarrow \infty$ in (4.10), we obtain $\lim \bar{m} \|\mu - \mu_\lambda\|_S^2 \leq 0$, and therefore $\|\mu - \mu_\lambda\|_S \rightarrow 0$ as $\lambda \rightarrow \infty$. This completes the proof. \square

5. Optimal control problem

We investigate an optimal control problem governed by HVIs. For each $\lambda > 0$, the problem aims to determine an optimal pair $(\bar{f}_\lambda, \bar{r}_\lambda) \in H \times Q$ satisfying

$$L_\lambda(\bar{f}_\lambda, \bar{r}_\lambda) = \min_{(f,r) \in H \times Q} L_\lambda(f, r).$$

The associated cost functional $L_\lambda : H \times Q \rightarrow \mathbb{R}$ is expressed as

$$L_\lambda(f, r) = \frac{1}{2} \|\mu_{\lambda fr} - v_d\|_H^2 + \frac{M_a}{2} \|f\|_H^2 + \frac{M_b}{2} \|r\|_Q^2, \quad (5.1)$$

where $\mu_{\lambda fr}$ denotes the state corresponding to the control variables (f, r) , defined as the solution of (1.3). Here, $v_d \in H$ is a prescribed desired state, and $M_a, M_b > 0$ are positive constants.

Theorem 4. *Assume that conditions (1)–(4) of A_1 and the smallness condition hold. For each parameter $\lambda > 0$, at least one solution exists for the optimal control problem governed by HVIs.*

Proof. We define $T_\lambda(f, r)$ as the set of solutions to the hemivariational inequality (1.3) corresponding to given controls $f \in H$ and $r \in Q$, for a given $\lambda > 0$, assuming that solutions exist. This allows us to define the value

$$m = \inf \{L_\lambda(f, r) : (f, r) \in H \times Q, \mu_{\lambda fr} \in T_\lambda(f, r)\} \geq 0. \quad (5.2)$$

For each $\lambda > 0$, assume that $\{(f_n^\lambda, r_n^\lambda)\}_{n \in \mathbb{N}} \subset H \times Q$ is a minimizing sequence satisfying

$$m \leq L_\lambda(f_n^\lambda, r_n^\lambda) \leq m + \frac{1}{n}, \quad (5.3)$$

with associated states $\mu_{\lambda f_n^\lambda r_n^\lambda}$ solving the hemivariational inequality (1.3). To verify boundedness, let $\mu \in E$ denote the solution of problem (2.1). Taking $s_4 = \mu - \mu_{\lambda f_n^\lambda r_n^\lambda} \in S_0$ in the inequality (1.3) yields

$$\begin{aligned} & A(\mu_{\lambda f_n^\lambda r_n^\lambda}, s_4) - \int_{\Sigma_3} \beta \mu_{\lambda f_n^\lambda r_n^\lambda} s_4 d\Sigma + \lambda \int_{\Sigma_3} \psi^0(\mu_{\lambda f_n^\lambda r_n^\lambda}; s_4) d\Sigma \\ & \geq \int_D f_n^\lambda s_4 dy - \int_{\Sigma_2} r_n^\lambda s_4 d\Sigma. \end{aligned}$$

Rewriting the inequality yields

$$\begin{aligned} A(s_4, s_4) & \leq A(\mu, s_4) - \int_{\Sigma_3} \beta \mu_{\lambda f_n^\lambda r_n^\lambda} s_4 d\Sigma + \lambda \int_{\Sigma_3} \psi^0(\mu_{\lambda f_n^\lambda r_n^\lambda}; s_4) d\Sigma \\ & \quad - \int_D f_n^\lambda s_4 dy + \int_{\Sigma_2} r_n^\lambda s_4 d\Sigma. \end{aligned} \quad (5.4)$$

Moreover, the coercivity of the functional L_λ ,

$$\lim_{\|(f,r)\|_{H \times Q} \rightarrow +\infty} L_\lambda(f, r) = +\infty, \quad \|(f, r)\|_{H \times Q}^2 = \|f\|_H^2 + \|r\|_Q^2,$$

guarantees that $\{(f_n^\lambda, r_n^\lambda)\}$ is bounded in $H \times Q$. Suppose that the sequence $\{(f_n^\lambda, r_n^\lambda)\}$ is unbounded. Then, the coercivity of L_λ would imply $L_\lambda(f_n^\lambda, r_n^\lambda) \rightarrow +\infty$, contradicting (5.3). This implies the existence of a λ -independent constant $C_2 > 0$ satisfying

$$\|f_n^\lambda\|_H \leq C_2, \quad \|r_n^\lambda\|_Q \leq C_2.$$

By assumption $A_1(4)$ together with the coercivity and boundedness of the $A(\cdot, \cdot)$, we have

$$\begin{aligned} \widetilde{m} \|s_4\|_S^2 & \leq A(\mu, s_4) - \int_{\Sigma_3} \beta \mu_{\lambda f_n^\lambda r_n^\lambda} s_4 d\Sigma - \int_D f_n^\lambda s_4 dy + \int_{\Sigma_2} r_n^\lambda s_4 d\Sigma \\ & \leq (M \|\mu\|_S + \|f_n^\lambda\|_H + \|r_n^\lambda\|_Q \|\gamma\| + \|\beta\|_{L^\infty} \|\gamma\|^2 \|\mu_{\lambda f_n^\lambda r_n^\lambda}\|_S) \|s_4\|_S. \end{aligned} \quad (5.5)$$

Observing that $\|\mu_{\lambda f_n^\lambda r_n^\lambda}\|_S \leq \|s_4\|_S + \|\mu\|_S$, we have

$$\|s_4\|_S \leq \frac{M \|\mu\|_S + \|f_n^\lambda\|_H + \|r_n^\lambda\|_Q \|\gamma\| + \|\beta\|_{L^\infty} \|\gamma\|^2 \|\mu\|_S}{\widetilde{m} - \|\beta\|_{L^\infty} \|\gamma\|^2}.$$

Consequently, the sequence of states satisfies

$$\|\mu_{\lambda f_n^\lambda r_n^\lambda}\|_S \leq \|s_4\|_S + \|\mu\|_S =: C_3,$$

where $\|\gamma\|$ stands for the norm of the trace operator, $\widetilde{m}, M > 0$ are constants, and $C_3 > 0$ is independent of λ . For every $\lambda > 0$, as $n \rightarrow \infty$,

$$\mu_{\lambda f_n^\lambda r_n^\lambda} \rightharpoonup \rho_\lambda \text{ in } S_0, \quad f_n^\lambda \rightharpoonup \tau_\lambda \text{ in } H, \quad r_n^\lambda \rightharpoonup \zeta_\lambda \text{ in } Q.$$

for some $\tau_\lambda \in H$, $\zeta_\lambda \in Q$, and $\rho_\lambda \in S_0$. Moreover, for every $s \in S_0$, the following hemivariational inequality is satisfied by each $(f_n^\lambda, r_n^\lambda)$:

$$\begin{aligned} A(\mu_{\lambda f_n^\lambda r_n^\lambda}, s) - \int_{\Sigma_3} \beta \mu_{\lambda f_n^\lambda r_n^\lambda} s \, d\Sigma + \lambda \int_{\Sigma_3} \psi^0(\mu_{\lambda f_n^\lambda r_n^\lambda}; s) \, d\Sigma \\ \geq \int_D f_n^\lambda \, dy - \int_{\Sigma_2} r_n^\lambda \, d\Sigma. \end{aligned} \quad (5.6)$$

Considering the limit superior $n \rightarrow \infty$, we deduce

$$\begin{aligned} A(\rho_\lambda, s) - \int_{\Sigma_3} \beta \rho_\lambda s \, d\Sigma + \lambda \limsup \int_{\Sigma_3} \psi^0(\mu_{\lambda f_n^\lambda r_n^\lambda}; s) \, d\Sigma \\ \geq \int_D \tau_\lambda \, dy - \int_{\Sigma_2} \zeta_\lambda \, d\Sigma, \quad \forall s \in S_0. \end{aligned} \quad (5.7)$$

By the compact embedding $S \hookrightarrow L^2(\Sigma_3)$, it follows that $\mu_{\lambda f_n^\lambda r_n^\lambda}|_{\Sigma_3} \rightarrow \rho_\lambda|_{\Sigma_3}$ in $L^2(\Sigma_3)$. By extracting a subsequence, which we continue to denote by $\mu_{\lambda f_n^\lambda r_n^\lambda}$, we have $\mu_{\lambda f_n^\lambda r_n^\lambda}(y) \rightarrow \rho_\lambda(y)$ for almost every $y \in \Sigma_3$, and there is a function $h_3 \in L^2(\Sigma_3)$ satisfying $|\mu_{\lambda f_n^\lambda r_n^\lambda}(y)| \leq h_3(y)$ for almost every $y \in \Sigma_3$. Then, by invoking the upper semicontinuity of the mapping $(e, j) \mapsto \psi^0(y, e; j)$, we obtain

$$\limsup \psi^0(y, \mu_{\lambda f_n^\lambda r_n^\lambda}(y); s(y)) \leq \psi^0(y, \rho_\lambda(y); s(y)) \quad \text{for a.e. } y \in \Sigma_3.$$

Under assumption $A_1(4)$, the following estimate holds:

$$|\psi^0(y, \mu_{\lambda f_n^\lambda r_n^\lambda}(y); s(y))| \leq (c_0 + c_1 |\mu_{\lambda f_n^\lambda r_n^\lambda}(y)|) |s(y)| \leq k_3(y),$$

where we set $k_3(y) = (c_0 + c_1 h_3(y)) |s(y)| \in L^1(\Sigma_3)$. Then, applying the reverse Fatou lemma, we deduce

$$\limsup \int_{\Sigma_3} \psi^0(\mu_{\lambda f_n^\lambda r_n^\lambda}; s) \, d\Sigma \leq \int_{\Sigma_3} \psi^0(\rho_\lambda; s) \, d\Sigma.$$

Substituting the inequality into (5.7) yields

$$\begin{aligned} A(\rho_\lambda, s) - \int_{\Sigma_3} \beta \rho_\lambda s \, d\Sigma + \lambda \int_{\Sigma_3} \psi^0(\rho_\lambda; s) \, d\Sigma \\ \geq \int_D \tau_\lambda \, dy - \int_{\Sigma_2} \zeta_\lambda \, d\Sigma, \quad \forall s \in S_0, \end{aligned} \quad (5.8)$$

which confirms that ρ_λ is a solution of the hemivariational inequality (1.3). Furthermore, the fact that L_λ is weakly lower semicontinuous yields

$$\begin{aligned} m &\geq \liminf L_\lambda(f_n^\lambda, r_n^\lambda) \\ &\geq \frac{1}{2} \|\rho_\lambda - v_d\|_H^2 + \frac{M_a}{2} \|\tau_\lambda\|_H^2 + \frac{M_b}{2} \|\zeta_\lambda\|_Q^2 \\ &= L_\lambda(\tau_\lambda, \zeta_\lambda), \end{aligned}$$

which demonstrates that $(\tau_\lambda, \zeta_\lambda)$ is an optimal control for the problem. \square

After confirming that a solution exists for the problem (5.1), we investigate its asymptotic behavior. The optimal control problem governed by the variational equality

$$\text{find } (\bar{f}, \bar{r}) \in H \times Q \text{ such that } L(\bar{f}, \bar{r}) = \min_{(f,r) \in H \times Q} L(f, r)$$

is known to have a unique solution [10]. The associated cost functional is defined by

$$L(f, r) = \frac{1}{2} \|\mu_{fr} - v_d\|_H^2 + \frac{M_a}{2} \|f\|_H^2 + \frac{M_b}{2} \|r\|_Q^2. \quad (5.9)$$

Here, μ_{fr} denotes the unique state corresponding to the control pair (f, r) , which solves the variational equality (2.1); $v_d \in H$ is a prescribed target state, and $M_a, M_b > 0$ are given constants.

Theorem 5. *Assume that A_1 holds. Let $(\bar{f}_\lambda, \bar{r}_\lambda)$ be the optimal pair for problem (5.1), and let (\bar{f}, \bar{r}) denote the unique optimal pair for problem (5.9). Then, as $\lambda \rightarrow \infty$, we have*

$$(\bar{f}_\lambda, \bar{r}_\lambda) \rightarrow (\bar{f}, \bar{r}) \quad \text{in } H \times Q, \quad \mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda} \rightarrow \mu_{\bar{f} \bar{r}} \quad \text{in } S.$$

Proof. To prove the boundedness, we observe that for any given $\lambda > 0$, the pair $(\bar{f}_\lambda, \bar{r}_\lambda)$ solves Problem (5.1). Consequently, by the optimality of $(\bar{f}_\lambda, \bar{r}_\lambda)$, the associated cost functional satisfies

$$\begin{aligned} & \frac{1}{2} \|\mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda} - v_d\|_H^2 + \frac{M_a}{2} \|\bar{f}_\lambda\|_H^2 + \frac{M_b}{2} \|\bar{r}_\lambda\|_Q^2 \\ & \leq \frac{1}{2} \|\mu_{fr} - v_d\|_H^2 + \frac{M_a}{2} \|f\|_H^2 + \frac{M_b}{2} \|r\|_Q^2, \end{aligned}$$

for all $(f, r) \in H \times Q$. By choosing $f = 0$ in D and $r = 0$ on Σ_2 , the optimality inequality yields a positive constant C_1 such that

$$\frac{1}{2} \|\mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda} - v_d\|_H^2 + \frac{M_a}{2} \|\bar{f}_\lambda\|_H^2 + \frac{M_b}{2} \|\bar{r}_\lambda\|_Q^2 \leq \frac{1}{2} \|\mu_{\lambda 00} - v_d\|_H^2 \leq C_1.$$

In view of Theorem 3, the sequence $\{\mu_{\lambda 00}\}$ converges as $\lambda \rightarrow \infty$. As a consequence, we have

$$\|\bar{f}_\lambda\|_H \leq C_2, \quad \|\bar{r}_\lambda\|_Q \leq C_3, \quad \|\mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda}\|_H \leq C_4, \quad (5.10)$$

where C_2, C_3 , and C_4 are positive constants that do not depend on λ . Assume the test function $s_5 = \mu_{\bar{f} \bar{r}} - \mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda} \in S_0$ and use it in the hemivariational inequality (1.3). It follows that

$$A(\mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda}, s_5) - \int_{\Sigma_3} \beta \mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda} s_5 \, d\Sigma + \lambda \int_{\Sigma_3} \psi^0(\mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda}; s_5) \, d\Sigma \geq F(s_5).$$

Using the bilinearity of the operator $A(\cdot, \cdot)$, we have

$$A(\mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda}, s_5) = A(\mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda}, s_5) - A(s_5, s_5).$$

Combining the above relations, we obtain

$$\begin{aligned} A(s_5, s_5) + \int_{\Sigma_3} \beta \mu_{\bar{f} \bar{r}} s_5 \, d\Sigma - \lambda \int_{\Sigma_3} \psi^0(\mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda}; s_5) \, d\Sigma \\ \leq A(\mu_{\bar{f} \bar{r}}, s_5) - F(s_5). \end{aligned} \quad (5.11)$$

Taking into account the fact that $\psi^0(\mu_{\lambda\bar{f}_\lambda\bar{r}_\lambda}; \mu_{\bar{f}} - \mu_{\lambda\bar{f}_\lambda\bar{r}_\lambda}) = \psi^0(\mu_{\lambda\bar{f}_\lambda\bar{r}_\lambda}; c - \mu_{\lambda\bar{f}_\lambda\bar{r}_\lambda})$ on Σ_3 , and invoking assumption $A_1(4)$, it follows that $\psi^0(\mu_{\lambda\bar{f}_\lambda\bar{r}_\lambda}; c - \mu_{\lambda\bar{f}_\lambda\bar{r}_\lambda}) \leq 0$ almost everywhere on Σ_3 . Consequently, we obtain

$$A(s_5, s_5) + \int_{\Sigma_3} \beta \mu_{\bar{f}} s_5 \, d\Sigma \leq A(\mu_{\bar{f}}, s_5) - F(s_5).$$

The coercivity and boundedness properties of the operator A imply that

$$\tilde{m} \|s_5\|_S^2 \leq \|s_5\|_S \left[M \|\mu_{\bar{f}}\|_S + \|\beta\|_{L^\infty} \|\gamma\|^2 \|\mu_{\bar{f}}\|_S + \|F\|_{S^*} \right],$$

for some positive constants \tilde{m} and M . It follows that

$$\begin{aligned} \|\mu_{\lambda\bar{f}_\lambda\bar{r}_\lambda}\|_S &\leq \|\mu_{\bar{f}} - \mu_{\lambda\bar{f}_\lambda\bar{r}_\lambda}\|_S + \|\mu_{\bar{f}}\|_S \\ &\leq \frac{1}{\tilde{m}} \left(M \|\mu_{\bar{f}}\|_S + \|\beta\|_{L^\infty} \|\gamma\|^2 \|\mu_{\bar{f}}\|_S + \|F\|_{S^*} \right) + \|\mu_{\bar{f}}\|_S \\ &=: C_5, \end{aligned} \tag{5.12}$$

where $C_5 > 0$ denotes a constant independent of λ . Note that $A(\mu_{\bar{f}} - \mu_{\lambda\bar{f}_\lambda\bar{r}_\lambda}, \mu_{\bar{f}} - \mu_{\lambda\bar{f}_\lambda\bar{r}_\lambda}) \geq 0$. Then, from (5.11), we have

$$\begin{aligned} -\lambda \int_{\Sigma_3} \psi^0(\mu_{\lambda\bar{f}_\lambda\bar{r}_\lambda}; \mu_{\bar{f}} - \mu_{\lambda\bar{f}_\lambda\bar{r}_\lambda}) \, d\Sigma &\leq \|\mu_{\bar{f}} - \mu_{\lambda\bar{f}_\lambda\bar{r}_\lambda}\|_S \left(M \|\mu_{\bar{f}}\|_S + \|\beta\|_{L^\infty} \|\gamma\|^2 \|\mu_{\bar{f}}\|_S + \|F\|_{S^*} \right) \\ &\leq \frac{1}{\tilde{m}} \left(M \|\mu_{\bar{f}}\|_S + \|\beta\|_{L^\infty} \|\gamma\|^2 \|\mu_{\bar{f}}\|_S + \|F\|_{S^*} \right)^2 \\ &=: C_6, \end{aligned} \tag{5.13}$$

where $C_6 > 0$ denotes a constant independent of λ . Consequently, we have

$$-\int_{\Sigma_3} \psi^0(\mu_{\lambda\bar{f}_\lambda\bar{r}_\lambda}; \mu_{\bar{f}} - \mu_{\lambda\bar{f}_\lambda\bar{r}_\lambda}) \, d\Sigma \leq \frac{C_6}{\lambda}. \tag{5.14}$$

It follows from (5.12) that the sequence $\{\mu_{\lambda\bar{f}_\lambda\bar{r}_\lambda}\}$ is bounded in S . After extracting a subsequence, which we continue to denote by $\mu_{\lambda\bar{f}_\lambda\bar{r}_\lambda}$, we have an element $p_1 \in S$ satisfying

$$\mu_{\lambda\bar{f}_\lambda\bar{r}_\lambda} \rightharpoonup p_1 \quad \text{weakly in } S \quad \text{as } \lambda \rightarrow \infty. \tag{5.15}$$

Furthermore, by virtue of (5.10), there exist elements $p_2 \in H$ and $p_3 \in Q$ such that, after selecting a suitable subsequence if necessary,

$$\bar{f}_\lambda \rightharpoonup p_2 \quad \text{weakly in } H \quad \text{as } \lambda \rightarrow \infty, \quad \text{and} \quad \bar{r}_\lambda \rightharpoonup p_3 \quad \text{weakly in } Q \quad \text{as } \lambda \rightarrow \infty.$$

Next, we prove that $p_1 = \mu_{\bar{f}}$, $p_2 = \bar{f}$, and $p_3 = \bar{r}$. We first show that $p_1 \in E$. Indeed, by (5.15) and the compactness of the trace operator, it follows that $\mu_{\lambda\bar{f}_\lambda\bar{r}_\lambda}|_{\Sigma_3} \rightarrow p_1|_{\Sigma_3}$ in $L^2(\Sigma_3)$ as $\lambda \rightarrow \infty$. By taking a suitable subsequence when necessary, we may assume that $\mu_{\lambda\bar{f}_\lambda\bar{r}_\lambda}(y) \rightarrow p_1(y)$ for almost every $y \in \Sigma_3$ and that there is $\tau \in L^2(\Sigma_3)$ satisfying $|\mu_{\lambda\bar{f}_\lambda\bar{r}_\lambda}(y)| \leq \tau(y)$ for almost every $y \in \Sigma_3$. By Proposition 1(3), it follows that

$$\limsup \psi^0(y, \mu_{\lambda\bar{f}_\lambda\bar{r}_\lambda}(y); c - \mu_{\lambda\bar{f}_\lambda\bar{r}_\lambda}(y)) \leq \psi^0(y, p_1(y); c - p_1(y))$$

for almost every $y \in \Sigma_3$. Using the following estimate,

$$\begin{aligned} \left| \psi^0(y, \mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda}(y); c - \mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda}(y)) \right| &\leq (c_0 + c_1 |\mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda}(y)|) |c - \mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda}(y)| \\ &\leq k_3(y) \quad \text{for a.e. } y \in \Sigma_3, \end{aligned}$$

where the function $k_3 \in L^1(\Sigma_3)$ is given by $k_3(y) = (c_0 + c_1 \tau(y))(|c| + \tau(y))$, we may apply the reverse Fatou lemma to get

$$\begin{aligned} \limsup \int_{\Sigma_3} \psi^0(y, \mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda}; c - \mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda}) d\Sigma \\ \leq \int_{\Sigma_3} \limsup \psi^0(y, \mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda}; c - \mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda}) d\Sigma. \end{aligned} \quad (5.16)$$

Putting together the above arguments, we conclude that

$$\limsup \int_{\Sigma_3} \psi^0(\mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda}; c - \mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda}) d\Sigma \leq \int_{\Sigma_3} \psi^0(p_1; c - p_1) d\Sigma.$$

From Assumption $A_1(4)$ and Relation (5.14), it follows that

$$0 \leq - \int_{\Sigma_3} \psi^0(p_1; c - p_1) d\Sigma \leq \liminf \left(- \int_{\Sigma_3} \psi^0(\mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda}; c - \mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda}) d\Sigma \right) \leq 0.$$

Therefore, we obtain $\int_{\Sigma_3} \psi^0(p_1; c - p_1) d\Sigma = 0$. By assumption $A_1(4)$, it follows that $\psi^0(y, p_1(y); c - p_1(y)) = 0$ for almost every $y \in \Sigma_3$. Moreover, assumption $A_1(5)$ implies that $p_1(y) = c$ for almost every $y \in \Sigma_3$, and consequently, $p_1 \in E$. We next show that $p_1 = \mu_{\bar{f}_\lambda}$. Clearly, $p_1 \in S_0$ because the sequence $\{\mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda}\} \subset S_0$ is sequentially weakly closed in S . For an arbitrary $z \in E$, we set $s_6 = z - \mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda} \in S_0$, and we use it in the hemivariational inequality (1.3). It follows that

$$A(\mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda}, s_6) - \int_{\Sigma_3} \beta \mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda} s_6 d\Sigma + \lambda \int_{\Sigma_3} \psi^0(\mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda}; s_6) d\Sigma \geq F(s_6). \quad (5.17)$$

Because $z = c$ on Σ_3 , the assumption $A_1(4)$ ensures that

$$\lambda \int_{\Sigma_3} \psi^0(\mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda}; z - \mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda}) d\Sigma = \lambda \int_{\Sigma_3} \psi^0(\mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda}; c - \mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda}) d\Sigma \leq 0.$$

Consequently, the hemivariational inequality yields

$$A(\mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda}, s_6) - \int_{\Sigma_3} \beta \mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda} s_6 d\Sigma \geq F(s_6).$$

Passing to the limit as $\lambda \rightarrow \infty$, the weak convergence $\mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda} \rightharpoonup p_1$ in S , together with the compact embedding induced by the trace operator $\Sigma : S_0 \rightarrow L^2(\Sigma_3)$, implies that $\mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda}|_{\Sigma_3} \rightarrow p_1|_{\Sigma_3}$ in $L^2(\Sigma_3)$. As a result, we obtain

$$\lim \int_{\Sigma_3} \mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda} (c - \mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda}) d\Sigma = \int_{\Sigma_3} p_1 (c - p_1) d\Sigma. \quad (5.18)$$

Let us define $j = z - p_1$. As both z and p_1 belong to E , it follows that $j \in E_0$. Taking the limit inferior as $\lambda \rightarrow \infty$ and applying the fact that A is weakly lower semicontinuous, we obtain

$$A(p_1, j) \geq F(j), \quad \forall j \in E_0.$$

As E_0 is a linear subspace, substituting j and $-j$ into the inequality yields

$$A(p_1, j) = F(j), \quad \forall j \in E_0.$$

This shows that p_1 solves Problem (2.1), and the uniqueness property immediately gives $p_1 = \mu_{p_2 p_3}$. As $\lambda \rightarrow \infty$, $\mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda} \rightarrow \mu_{p_2 p_3}$ in S . For any $(f, r) \in H \times Q$, we have

$$L_\lambda(\bar{f}_\lambda, \bar{r}_\lambda) \leq L_\lambda(f, r), \quad (5.19)$$

which implies

$$\begin{aligned} L(p_2, p_3) &= \frac{1}{2} \|\mu_{p_2 p_3} - v_d\|_H^2 + \frac{M_a}{2} \|p_2\|_H^2 + \frac{M_b}{2} \|p_3\|_Q^2 \\ &= \frac{1}{2} \|p_1 - v_d\|_H^2 + \frac{M_a}{2} \|p_2\|_H^2 + \frac{M_b}{2} \|p_3\|_Q^2 \\ &\leq \liminf L_\lambda(\bar{f}_\lambda, \bar{r}_\lambda) \leq \liminf L_\lambda(f, r) \\ &= \lim L_\lambda(f, r) = L(f, r), \quad \forall (f, r) \in H \times Q. \end{aligned}$$

Owing to the uniqueness of the solution of problem (5.9), it follows that $p_2 = \bar{f}$ and $p_3 = \bar{r}$. Consequently, $\mu_{p_2 p_3} = \mu_{\bar{f} \bar{r}}$.

Subsequently, we show that, as $\lambda \rightarrow \infty$,

$$\bar{f}_\lambda \rightarrow \bar{f} \text{ in } H, \quad \bar{r}_\lambda \rightarrow \bar{r} \text{ in } Q, \quad \text{and} \quad \mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda} \rightarrow \mu_{\bar{f} \bar{r}} \text{ in } S.$$

To prove the strong convergence of $\mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda}$ in S , we select $s_7 = \mu_{\bar{f} \bar{r}} - \mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda} \in S_0$ as a test function in Problem (1.3), which leads to

$$A(\mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda}, s_7) - \int_{\Sigma_3} \beta \mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda} s_7 \, d\Sigma + \lambda \int_{\Sigma_3} \psi^0(\mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda}; s_7) \, d\Sigma \geq F(s_7).$$

From this inequality, we deduce that

$$A(s_7, s_7) + \int_{\Sigma_3} \beta \mu_{\bar{f} \bar{r}} s_7 \, d\Sigma \leq A(\mu_{\bar{f} \bar{r}}, s_7) - F(s_7) + \lambda \int_{\Sigma_3} \psi^0(\mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda}; s_7) \, d\Sigma. \quad (5.20)$$

Noting that $\mu_{\bar{f} \bar{r}} = c$ on Σ_3 and applying Assumption $A_1(4)$ and Relation (5.18), together with coercivity of A , we obtain the estimate

$$\bar{m} \|\mu_{\bar{f} \bar{r}} - \mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda}\|_S^2 \leq A(\mu_{\bar{f} \bar{r}}, \mu_{\bar{f} \bar{r}} - \mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda}) + F(\mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda} - \mu_{\bar{f} \bar{r}}).$$

Utilizing the weak continuity of the operator $A(\mu_{\bar{f} \bar{r}}, \cdot)$, we infer that $\mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda} \rightarrow \mu_{\bar{f} \bar{r}}$ in S as $\lambda \rightarrow \infty$. We next demonstrate that the control sequences converge strongly, that is, $\bar{f}_\lambda \rightarrow \bar{f}$ in H and $\bar{r}_\lambda \rightarrow \bar{r}$ in Q as $\lambda \rightarrow \infty$. The convergence $\mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda} \rightarrow \mu_{\bar{f} \bar{r}}$ in H implies that

$$\lim \frac{1}{2} \|\mu_{\lambda \bar{f}_\lambda \bar{r}_\lambda} - v_d\|_H^2 = \frac{1}{2} \|\mu_{\bar{f} \bar{r}} - v_d\|_H^2. \quad (5.21)$$

Because the sequences $\{\bar{f}_\lambda\}$ and $\{\bar{r}_\lambda\}$ converge weakly to \bar{f} in H and to \bar{r} in Q , respectively, the weak lower semicontinuity of the norms yields

$$\|\bar{f}\|_H^2 \leq \liminf \|\bar{f}_\lambda\|_H^2, \quad \|\bar{r}\|_Q^2 \leq \liminf \|\bar{r}_\lambda\|_Q^2. \quad (5.22)$$

Combining these inequalities with Relations (5.21) and (5.22), we derive

$$\begin{aligned} & \frac{1}{2} \|\mu_{\bar{f}\bar{r}} - v_d\|_H^2 + \frac{M_a}{2} \|\bar{f}\|_H^2 + \frac{M_b}{2} \|\bar{r}\|_Q^2 \\ & \leq \liminf \left(\frac{1}{2} \|\mu_{\lambda\bar{f}_\lambda\bar{r}_\lambda} - v_d\|_H^2 + \frac{M_a}{2} \|\bar{f}_\lambda\|_H^2 + \frac{M_b}{2} \|\bar{r}_\lambda\|_Q^2 \right), \end{aligned}$$

which can be equivalently written as $L(\bar{f}, \bar{r}) \leq \liminf L_\lambda(\bar{f}_\lambda, \bar{r}_\lambda)$. By the definition of the optimal pair $(\bar{f}_\lambda, \bar{r}_\lambda)$, we have $L_\lambda(\bar{f}_\lambda, \bar{r}_\lambda) \leq L_\lambda(\bar{f}, \bar{r})$. Together with the strong convergence $\mu_{\lambda\bar{f}_\lambda\bar{r}_\lambda} \rightarrow \mu_{\bar{f}\bar{r}}$ in H , this implies $\limsup L_\lambda(\bar{f}_\lambda, \bar{r}_\lambda) \leq \limsup L_\lambda(\bar{f}, \bar{r}) = L(\bar{f}, \bar{r})$. Consequently, we conclude that $\lim L_\lambda(\bar{f}_\lambda, \bar{r}_\lambda) = L(\bar{f}, \bar{r})$, or, equivalently,

$$\begin{aligned} & \lim \left(\frac{1}{2} \|\mu_{\lambda\bar{f}_\lambda\bar{r}_\lambda} - v_d\|_H^2 + \frac{M_a}{2} \|\bar{f}_\lambda\|_H^2 + \frac{M_b}{2} \|\bar{r}_\lambda\|_Q^2 \right) \\ & = \frac{1}{2} \|\mu_{\bar{f}\bar{r}} - v_d\|_H^2 + \frac{M_a}{2} \|\bar{f}\|_H^2 + \frac{M_b}{2} \|\bar{r}\|_Q^2. \end{aligned} \quad (5.23)$$

Combining Relations (5.21) and (5.23), we infer that $\|\bar{f}_\lambda\|_H^2 \rightarrow \|\bar{f}\|_H^2$ and $\|\bar{r}_\lambda\|_Q^2 \rightarrow \|\bar{r}\|_Q^2$ as $\lambda \rightarrow \infty$. Together with the weak convergences $\bar{f}_\lambda \rightharpoonup \bar{f}$ in H and $\bar{r}_\lambda \rightharpoonup \bar{r}$ in Q , this implies that $\bar{f}_\lambda \rightarrow \bar{f}$ in H and $\bar{r}_\lambda \rightarrow \bar{r}$ in Q as $\lambda \rightarrow \infty$. This completes the proof. \square

6. Conclusions

This paper investigates a class of elliptic HVIs with nonmonotone and multivalued boundary conditions. Using nonsmooth analysis, Clarke's generalized gradient, and pseudomonotone operator theory, the existence of solutions is established, and uniqueness is obtained under a relaxed monotonicity condition together with a smallness condition. Moreover, we investigate the limiting behavior of the solutions and show strong convergence to the corresponding Dirichlet problem as $\lambda \rightarrow \infty$. On this basis, an optimal control problem governed by the HVIs is further studied, and the existence and asymptotic convergence of optimal solutions are demonstrated. Future research may focus on several directions: Considering more general nonlinear principal operators or time-dependent problems; investigating numerical discretization of the corresponding problem and the convergence of the associated algorithms; and extending the present theory to HVIs and optimal control problems under random perturbations or uncertainty frameworks, thereby enhancing the applicability of the model to complex real-world systems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported in part by the Natural Science Foundation of Ningxia Province (Grant Nos. 2023AAC03301 and 2023AAC03386), the National Natural Science Foundation of China (Grant No. 11861003), and the Postgraduate Innovation Project of North Minzu University (Grant No. CYX25128).

Conflict of interest

The authors declare there are no conflicts of interest.

References

1. P. D. Panagiotopoulos, Z. Nanievich, *Mathematical Theory of Hemivariational Inequalities and Applications*, CRC Press, 1994.
2. P. D. Panagiotopoulos, *Inequality Problems in Mechanics and Applications: Convex and Nonconvex Energy Functions*, Springer, 2012. <https://doi.org/10.1007/978-1-4612-5152-1>
3. Z. Liu, S. Migórski, A. Ochal, Homogenization of boundary hemivariational inequalities in linear elasticity, *J. Math. Anal. Appl.*, **340** (2008), 1347–1361. <https://doi.org/10.1016/j.jmaa.2007.09.050>
4. Z. Liu, M. Sofonea, Y. Xiao, Tykhonov well-posedness of a frictionless unilateral contact problem, *Math. Mech. Solids*, **25** (2020), 1294–1311. <https://doi.org/10.1177/1081286519884347>
5. W. Han, Z. Huang, C. Wang, W. Xu, Numerical analysis of elliptic hemivariational inequalities for semipermeable media, *J. Comput. Math.*, **37** (2019), 506–523. <https://doi.org/10.4208/jcm.1807-m2018-0035>
6. W. Han, M. Sofonea, Numerical analysis of hemivariational inequalities in contact mechanics, *Acta Numer.*, **28** (2019), 175–286. <https://doi.org/10.1017/S0962492919000023>
7. S. Migórski, A. Ochal, M. Sofonea, *Nonlinear Inclusions and Hemivariational Inequalities: Models and Analysis of Contact Problems*, Springer, (2012), 23–89. <https://doi.org/10.1007/978-1-4614-4232-5>
8. S. Migórski, A. Ochal, M. Sofonea, A class of variational-hemivariational inequalities in reflexive Banach spaces, *J. Elast.*, **127** (2017), 151–178. <https://doi.org/10.1007/s10659-016-9600-7>
9. C. M. Gariboldi, D. A. Tarzia, Convergence of boundary optimal control problems with restrictions in mixed elliptic Stefan-like problems, *Adv. Differ. Equations Control Processes*, **1** (2008), 113–132.
10. C. M. Gariboldi, D. A. Tarzia, Existence, uniqueness and convergence of simultaneous distributed-boundary optimal control problems, preprint, arXiv:1505.04154. <https://doi.org/10.48550/arXiv.1505.04154>
11. M. Sofonea, D. A. Tarzia, A penalty method for elliptic variational–hemivariational inequalities, *Axioms*, **13** (2024), 721. <https://doi.org/10.3390/axioms13100721>
12. M. Sofonea, Y. B. Xiao, S. D. Zeng, Generalized penalty method for history-dependent variational–hemivariational inequalities, *Nonlinear Anal.: Real World Appl.*, **61** (2021), 103329. <https://doi.org/10.1016/j.nonrwa.2021.103329>
13. M. Mohan Raja, V. Vijayakumar, C. W. Tsai, K. C. Veluvolu, Optimal control strategies and continuous dependence for stochastic hilfer fractional systems with delay: A volterra-fredholm integro-differential approach, *Optim. Control Appl. Methods*, **46** (2025), 2708–2726. <https://doi.org/10.1002/oca.70024>

14. M. Mohan Raja, V. Vijayakumar, K. C. Veluvolu, Improved order in Hilfer fractional differential systems: Solvability and optimal control problem for hemivariational inequalities, *Chaos Solitons Fractals*, **188** (2024), 115558. <https://doi.org/10.1016/j.chaos.2024.115558>
15. M. Mohan Raja, V. Vijayakumar, K. C. Veluvolu, An analysis concerning the existence of mild solution for Hilfer fractional neutral evolution system on infinite interval, *Math. Methods Appl. Sci.*, **46** (2023), 19277–19288. <https://doi.org/10.1002/mma.9626>
16. D. Han, W. Han, S. Migórski, J. Zhao, Convergence analysis of numerical solutions for optimal control of variational–hemivariational inequalities, *Appl. Math. Lett.*, **105** (2020), 106327. <https://doi.org/10.1016/j.aml.2020.106327>
17. Y. K. Ma, M. Mohan Raja, V. Vijayakumar, A. Shukla, W. Albalawi, K. S. Nisar, Existence and continuous dependence results for fractional evolution integrodifferential equations of order $r \in (1, 2)$, *Alexandria Eng. J.*, **61** (2022), 9929–9939. <https://doi.org/10.1016/j.aej.2022.03.010>
18. Y. K. Ma, M. Mohan Raja, K. S. Nisar, A. Shukla, V. Vijayakumar, Results on controllability for Sobolev type fractional differential equations of order $1 < r < 2$ with finite delay, *AIMS Math.*, **7** (2022), 10215–10233. <https://doi.org/10.3934/math.2022568>
19. S. Migórski, Evolution hemivariational inequalities in infinite dimension and their control, *Nonlinear Anal. Theory Methods Appl.*, **47** (2001), 101–112. [https://doi.org/10.1016/S0362-546X\(01\)00160-2](https://doi.org/10.1016/S0362-546X(01)00160-2)
20. C. M. Gariboldi, D. A. Tarzia, Convergence of distributed optimal controls on the internal energy in mixed elliptic problems when the heat transfer coefficient goes to infinity, *Appl. Math. Optim.*, **47** (2003), 213–230. <https://doi.org/10.1007/s00245-003-0761-y>
21. C. M. Gariboldi, S. Migórski, A. Ochal, D. A. Tarzia, Existence, comparison, and convergence results for a class of elliptic hemivariational inequalities, *Appl. Math. Optim.*, **84** (2021), 1453–1475. <https://doi.org/10.1007/s00245-021-09800-9>
22. C. M. Gariboldi, D. A. Tarzia, Distributed optimal control problems for a class of elliptic hemivariational inequalities with a parameter and its asymptotic behavior, *Commun. Nonlinear Sci. Numer. Simul.*, **104** (2022), 106027. <https://doi.org/10.1016/j.cnsns.2021.106027>
23. E. D. Tabacman, D. A. Tarzia, Sufficient and/or necessary condition for the heat transfer coefficient on Γ_1 and the heat flux on Γ_2 to obtain a steady-state two-phase Stefan problem, *J. Differ. Equations*, **77** (1989), 16–37. [https://doi.org/10.1016/0022-0396\(89\)90155-1](https://doi.org/10.1016/0022-0396(89)90155-1)
24. F. H. Clarke, Optimization and nonsmooth Analysis, *Soc. Ind. Appl. Math.*, (1990), 24–95.
25. S. Migórski, A. Ochal, M. Sofonea, A class of variational–hemivariational inequalities in reflexive Banach spaces, *J. Elast.*, **127** (2017), 151–178. <https://doi.org/10.1007/s10659-016-9600-7>
26. T. Kamihigashi, A generalization of Fatou’s lemma for extended real-valued functions on σ -finite measure spaces: with an application to infinite-horizon optimization in discrete time, *J. Inequal. Appl.*, **24** (2017), Article 24. <https://doi.org/10.1186/s13660-016-1288-5>



AIMS Press

©2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<https://creativecommons.org/licenses/by/4.0>)