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Research article

Some properties for (K, K')-quasiconformal harmonic mappings to half-planes

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Abstract: Let $\mathbb{U}$ be the open unit disk, $\Omega = \{\omega \in \mathbb{C} : \text{Re } \omega > a, -\infty < a < a_0 < +\infty\}$ with $a, a_0 \in \mathbb{R}$, and $K \geq 1$ and $K' \geq 0$ as two given constants. In this paper, we study some properties of the class $\mathcal{H}(\mathbb{U}, \Omega)$ of mappings consisting of those sense-preserving Euclidean harmonic mappings from $\mathbb{U}$ onto Ω with the real, normalized conditions of $f(0) = a_0$, $f_z(0) > 0$ and $f_{\bar{z}}(0)$. We first show that a (K, K')-quasiconformal mapping $f \in \mathcal{H}(\mathbb{U}, \Omega)$ need not be Euclidean Lipschitz continuous. Subsequently, we give a sufficient and necessary condition for $f \in \mathcal{H}(\mathbb{U}, \Omega)$ to be (K, K')-quasiconformal. Additionally, coefficient estimates, distortion theorems, and area theorems are obtained.

Keywords: harmonic mapping; (K, K')-quasiconformal mapping; distortion theorem; coefficient estimate; area theorem

1. Introduction and main results

Let D and G be domains in $\mathbb{C}$. A real-value function $u : D \to \mathbb{R}$ is said to be absolutely continuous on line in D if for every closed rectangle $R \subset D$, whose sides are parallel to the axes x and y, the function u is absolutely continuous on almost every horizontal line and almost every vertical line in R. It is well known that those functions have partial derivatives u_x and u_y for almost every $z = x + iy \in D$. We say a topological mapping $f = u + iv : D \to G$ is (K, K')-quasiconformal [1] if f is absolutely continuous on lines in D and there exist two constants $K \geq 1$ and $K' \geq 0$ such that the inequality

$$(|f_z(z)| + |f_{\bar{z}}(z)|)^2 \leq K\big||f_z(z)|^2 - |f_{\bar{z}}(z)|^2\big| + K' \tag{1.1}$$

holds for almost every $z \in D$, where $f_z = (f_x - if_y)/2$ and $f_{\bar{z}} = (f_x + if_y)/2$. In particular, f is called a K-quasiconformal mapping if $K' = 0$.

Suppose that $\rho(z)|dz|^2$ is a conformal C^1 metric on D. Then, we say a mapping $f \in C^2(D, G)$ is ρ-harmonic if

$$f_{z\bar{z}} + (\log \rho)_\omega \circ f \cdot f_z f_{\bar{z}} = 0.$$

If we take ρ to be a Euclidean (hyperbolic) metric, the ρ-harmonic mapping is said to be Euclidean (hyperbolic) harmonic.

Suppose that the symbol $\mathbb{U}$ stands for the open unit disk in $\mathbb{C}$, and S_H stands for the class of Euclidean harmonic mappings f of $\mathbb{U}$, which are sense-preserving and univalent. Then, it is well known that every $f \in S_H$ has the representation of the form $f = h + \bar{g}$ with which h and g are holomorphic in $\mathbb{U}$; see [2]. By a result of Lewy [3], it can be seen that f is locally univalent and sense-preserving on $\mathbb{U}$ if and only if $J_f(z) > 0$. We say a mapping $f : \mathbb{U} \to \Omega$ is a Euclidean Lipschitz mapping if for any $x, y \in \mathbb{U}$, there exists a constant $L > 0$ such that $|f(x) - f(y)| \leq L|x - y|$.

It is well known that the composition of a Euclidean harmonic and a conformal mapping is still Euclidean harmonic. However, the composition of a conformal mapping and a Euclidean harmonic mapping is usually not a Euclidean harmonic mapping. This motivates many authors to study the class of those Euclidean harmonic functions, which are mapping $\mathbb{U}$ onto a specific simply connected domain (see, e.g. [4–8]). In particular, the distortion theorems, coefficient estimates, and area for their images of those Euclidean harmonic mappings from a unit disk onto certain simply connected domains are considered in [4–7].

A special kind of Euclidean harmonic mapping which requires that it is also quasiconformal have been studied by many researchers. It seems that Martio [9] was the first one to consider this kind of mappings on the unit disk. Recently, there are some references concerning the conditions which can guarantee a harmonic mapping to be K-quasiconformal (see, e.g. [10–13]). In particular, Fu and Huang [11] obtained results following Theorem 1.1, where the symbol $\mathcal{H}(\mathbb{U}, \Omega)$ stands for the class of mappings which consist of those sense-preserving Euclidean harmonic mappings from $\mathbb{U}$ onto Ω with the normalized conditions of $f(0) = a_0$, $f_z(0) > 0$ and $f_{\bar{z}}(0)$ being real, and where the symbol $\mathcal{H}_K(\mathbb{U}, \Omega)$ stands for the class consisting of function $f \in \mathcal{H}(\mathbb{U}, \Omega)$, which is also K-quasiconformal.

Theorem 1.1. [11, Theorem 1] *Suppose that $f \in \mathcal{H}(\mathbb{U}, \Omega)$; then, there is an analytic function ζ satisfying $\zeta(0) = 0$ and $\zeta'(0) \in \mathbb{R}$ such that*

$$f(z) = \frac{\varphi(z) + \zeta(z)}{2} + \overline{\frac{\varphi(z) - \zeta(z)}{2}}, z \in \mathbb{U},$$

where $\varphi(z) = ((a_0 - 2a)z + a_0)/(1 - z)$. Furthermore, $f \in \mathcal{H}_K(\mathbb{U}, \Omega)$ if and only if ζ satisfies

$$\left| \frac{(1 - z)^2 \zeta'(z)}{2(a_0 - a)} - \frac{1 + k^2}{1 - k^2} \right| \leq \frac{2k}{1 - k^2}, \tag{1.2}$$

where $K = (1 + k)/(1 - k)$ for some $0 \leq k < 1$.

On the other hand, some equivalent conditions for a Euclidean harmonic mapping to be (K, K')-quasiconformal were obtained in [1, 14, 15]. In particular, in [1, 14, 15], the authors showed that a Euclidean harmonic mapping f from the unit disk (or upper plane) onto itself is (K, K')-quasiconformal if and only if f is a Euclidean Lipschitz mapping. These results indicate that the following problem may have a positive answer.

Problem 1. For a given $K \geq 1$ and $K' \geq 0$, we suppose that the symbol $\mathcal{H}_{K,K'}(\mathbb{U}, \Omega)$ stands for the class of mapping $f \in \mathcal{H}(\mathbb{U}, \Omega)$, which is also (K, K')-quasiconformal. Then, the questions remains whether $f \in \mathcal{H}_{K,K'}(\mathbb{U}, \Omega)$ if and only if f is an Euclidean Lipschitz mapping.

The following result shows that this is not always the case.

Theorem 1.2. *For a given $K \geq 1$ and $K' \geq 0$, we have*

$$\mathcal{H}_K(\mathbb{U}, \Omega) \subsetneq \mathcal{H}_{K,K'}(\mathbb{U}, \Omega).$$

Moreover, there exists a mapping $f \in \mathcal{H}_{K,K'}(\mathbb{U}, \Omega)$ such that it is not Euclidean Lipschitz continuous.

Theorem 1.2 tells us that it is not work to establish the equivalent condition for a mapping $f \in \mathcal{H}(\mathbb{U}, \Omega)$ to be (K, K')-quasiconformal via the Euclidean Lipschitz property. In this paper, inspired by Theorem 1.1, we give equivalent conditions, which are similar to Inequality (1.2), for the mapping $f \in \mathcal{H}(\mathbb{U}, \Omega)$ to be (K, K')-quasiconformal. It is worth noting that, compared to the definition of K-quasiconformality, the definition of (K, K')-quasiconformality has an item of K', which may cause the main difficulty in obtaining Inequality (1.2) in the case for a (K, K')-quasiconformal mapping. Here, our main strategy is to replace k in Inequality (1.2) by a function. It is read as follows.

Theorem 1.3. *Suppose that $f \in \mathcal{H}(\mathbb{U}, \Omega)$. Then, there is an analytic mapping ζ satisfying $\zeta(0) = 0$ and $\zeta'(0) \in \mathbb{R}$ such that*

$$f(z) = \frac{\frac{(a_0-2a)z+a_0}{1-z} + \zeta(z)}{2} + \frac{\overline{\frac{(a_0-2a)z+a_0}{1-z} - \zeta(z)}}{2}, z \in \mathbb{U}. \tag{1.3}$$

Furthermore, if $|f_z(z)| > \sqrt{K'}/2$, then $f \in \mathcal{H}_{K,K'}(\mathbb{U}, \Omega)$ if and only if ζ satisfies

$$\left| \frac{(1-z^2)\zeta'(z)}{2(a_0-a)} - \frac{1+(Q(z))^2}{1-(Q(z))^2} \right| \leq \frac{2Q(z)}{1-(Q(z))^2}, z \in \mathbb{U}, \tag{1.4}$$

where $Q(z) = (\sqrt{K^2 + K'(K+1)|h(z)|^2} - 1)/(K+1)$ is a function from $\mathbb{U}$ into $[0, 1)$, and $h = 1/f_z$.

Remark 1. In Theorem 1.3, we require $|f_z(z)| > \sqrt{K'}/2$ to deduce that the function $Q(z) < 1$, which guarantees that the Inequalities (3.2) and (3.3), are equivalent. In addition, from the proof of Theorem 1.2, we see that the class $\mathcal{H}_{K,K'}(\mathbb{U}, \Omega)$ satisfying $|f_z(z)| > \sqrt{K'}/2$ is not empty. At last, if $K' = 0$, then the condition of $|f_z(z)| > \sqrt{K'}/2 = 0$ is automatically satisfied due to a result of Lewy [3]. Hence, Theorem 1.3 also gives a nontrivial generalization of Theorem 1.1.

As an application of Theorem 1.3, we obtain the coefficient estimates, distortion theorems, and area theorems for those mappings $f \in \mathcal{H}_{K,K'}(\mathbb{U}, \Omega)$. It is read as follows.

Theorem 1.4. *For given $K \geq 1$ and $K' \geq 0$, we suppose that $f \in \mathcal{H}_{K,K'}(\mathbb{U}, \Omega)$ and $|f_z(z)| > \sqrt{K'}/2$.*

(1) *If $f(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}$, then*

$$|a_1| + |b_1| \leq 2K(a_0 - a) + \frac{(K + \sqrt{K+1})\sqrt{K'}}{K+1},$$

and $||a_n| - |b_n|| \leq 2(a_0 - a)$ for $n = 2, 3, \cdots$.

(2) *If we denote $\theta = \arg z$, then we have*

$$|f(z) - a_0| \leq \frac{2K(a_0 - a)}{\sin\theta} \arctan\Big(\frac{|z|\sin\theta}{1 - |z|\cos\theta}\Big) + \frac{(K + \sqrt{K+1})\sqrt{K'}}{K+1}|z|$$

for every $z \in \mathbb{U}$.

(3) *If we denote* $U_r = \{z : |z| \le r < 1\}$, *then*

$$Areaf(U_r) \le \frac{4\pi K(a_0 - a)^2 r^2}{(1 - r^2)^2} + 2\pi(a_0 - a)\sqrt{K'} \log\left(\frac{1}{1 - r^2}\right).$$

Remark 2. It should be noted that the coefficient bound for $n \ge 2$ is simply $||a_n| - |b_n|| \le 2(a_0 - a)$. This is a very weak bound and follows directly from the representation, not from the (K, K') condition. For the part of (2) in Theorem 1.4, if $\sin\theta = 0$, then we define the calculation as follows:

$$\frac{1}{\sin\theta}\arctan\Big(\frac{|z|\sin\theta}{1 - |z|\cos\theta}\Big) := \lim_{\sin\theta\to 0}\frac{1}{\sin\theta}\arctan\Big(\frac{|z|\sin\theta}{1 - |z|\cos\theta}\Big) = \frac{|z|}{1 \mp |z|}.$$

The organization of the rest of this paper is as follows: The proofs of Theorems 1.2–1.4 are given in Sections 2, 3, and 4 respectively.

2. Proof of Theorem 1.2

In this section, we finish the proof of Theorem 1.2 by considering the following example, which states that a mapping $f \in \mathcal{H}_{K,K'}(\mathbb{U}, \Omega)$ and $|f_z(z)| > \sqrt{K'}/2$ is generally not $(K, 0)$-quasicomformal. In addition, it is not Euclidean Lipschitz continuous.

Example 1. *Let*

$$f(z) = \frac{(7a_0 - 9a)z - (a_0 - a)z^2 + 2a_0}{4(1 - z)} + \overline{\frac{2a_0 - (a_0 - a)z}{4}}, z \in \mathbb{U}.$$

Then, the mapping $f \in \mathcal{H}(\mathbb{U}, \Omega)$ *is a* (K, K')*-quasiconformal mapping with* $f(0) = a_0, f_z(0) = 9(a_0 - a)/4 > 0, f_{\bar{z}}(0) = -(a_0 - a)/4 \in \mathbb{R}$ *and* $|f_z(z)| > \sqrt{K'}/2$, *where* $K = 2, K' = (a_0 - a)^2/4$. *Moreover, f is not* $(K, 0)$*-quasiconformal and not Euclidean Lipschitz continuous.*

Proof. It is easy to verify that f is a Euclidean harmonic mapping from $\mathbb{U}$ onto the half-plane domain $\Omega = \{\omega : Re\omega > a, -\infty < a < a_0 < +\infty\}$ with $f(0) = a_0, f_z(0) = 9(a_0 - a)/4 > 0, f_{\bar{z}}(0) = -(a_0 - a)/4 \in \mathbb{R}$. Because $f_z(z) = 2(a_0 - a)/(1 - z)^2 + (a_0 - a)/4$, if we set $(a_0 - a)/(1 - z)^2 = \rho e^{i\theta}$ (and hence $\rho > (a_0 - a)/4$), then

$$\begin{aligned}|f_z(z)| &= \frac{\sqrt{16\rho^2 + 4(a_0 - a)\cos\theta\cdot\rho + \frac{(a_0-a)^2}{4}}}{2}\\ &> \frac{a_0 - a}{4} = \frac{\sqrt{K'}}{2} = |f_{\bar{z}}(z)|.\end{aligned}$$

This says that f is sense-preserving and univalent and satisfies $|f_z(z)| > \sqrt{K'}/2$ for every $z \in \mathbb{U}$. Because

$$|f_z(z)|^2 - 3|f_{\bar{z}}(z)|^2 + (\frac{a_0 - a}{2})^2 = |f_z(z)|^2 + |f_{\bar{z}}(z)|^2,$$

then

$$\Big(|f_z(z)|^2 - 3|f_{\bar{z}}(z)|^2 + (\frac{a_0 - a}{2})^2\Big) - 2|f_z(z)|\cdot|f_{\bar{z}}(z)|$$

$$=(|f_z(z)| - |f_{\bar{z}}(z)|)^2 \geq 0.$$

We see that

$$(|f_z(z)| + |f_{\bar{z}}(z)|)^2 \leq 2(|f_z(z)|^2 - |f_{\bar{z}}(z)|^2) + (\frac{a_0 - a}{2})^2,$$

which implies that f is a $(2, (a_0 - a)^2/4)$-quasiconformal mapping. In addition,

$$\begin{aligned}
&\lim_{\theta=\pi,\rho\to\frac{a_0-a}{4}} \left|\frac{f_{\bar{z}}(z)}{f_z(z)}\right| \\
=&\lim_{\theta=\pi,\rho\to\frac{a_0-a}{4}} \sqrt{\frac{(\frac{a_0-a}{2})^2}{16\rho^2 + 4(a_0 - a)\cos\theta\cdot\rho + (\frac{a_0-a}{2})^2}} = 1.
\end{aligned}$$

Hence, f is not $(K, 0)$-quuasiconformal. At last, because

$$\lim_{z\to 1^-} |f_z(z)| = \lim_{z\to 1^-} \left|\frac{\frac{4(a_0-a)}{(1-z)^2} + \frac{a_0-a}{2}}{2}\right| = \infty,$$

we see that f is not Euclidean Lipschitz continuous. This finishes the proof of Theorem 1.2. □

3. Proof of Theorem 1.3

In this section, we give the proof of Theorem 1.3.

Proof of Theorem 1.3. The existence of ζ with $\zeta(0) = 0$ and $\zeta'(0) \in \mathbb{R}$, which satisfies Eq (1.3), is provided by Theorem 1.1. Now, we will prove Inequality (1.4). As we assume that f is univalent and sense-preserving, by a result of Lewy [3], we see that $J_f(z) > 0$. This implies that $|f_z(z)| > |f_{\bar{z}}(z)| \geq 0$ for any $z \in \mathbb{D}$. If we let $h = 1/f_z$, then by the (K, K')-quasiconformality of f, we get that

$$(|f_z(z)| + |f_{\bar{z}}(z)|)^2 \leq K(|f_z(z)|^2 - |f_{\bar{z}}(z)|^2) + K'|f_z|^2 \cdot |h|^2,$$

which is equivalent to

$$\left|\frac{f_{\bar{z}}(z)}{f_z(z)}\right| \leq \frac{\sqrt{K^2 + K'(K+1)|h(z)|^2} - 1}{K+1} := Q(z), z \in \mathbb{U}. \tag{3.1}$$

Because the constants $K \geq 1$ and $K' \geq 0$ and $|f_z(z)| > \sqrt{K'}/2$, we see that

$$0 \leq Q(z) < 1.$$

Now, by the representation of

$$f(z) = \frac{\frac{(a_0-2a)z+a_0}{1-z} + \zeta(z)}{2} + \frac{\overline{\frac{(a_0-2a)z+a_0}{1-z} - \zeta(z)}}{2},$$

we get

$$f_z(z) = \frac{\frac{2(a_0-a)}{(1-z)^2} + \zeta'(z)}{2} \quad \textit{and} \quad \overline{f_{\bar{z}}}(z) = \frac{\frac{2(a_0-a)}{(1-z)^2} - \zeta'(z)}{2}.$$

Hence, the Inequality (3.1) becomes

$$\left|\frac{f_{\bar{z}}(z)}{f_z(z)}\right| = \left|\frac{1-\frac{(1-z)^2\zeta'(z)}{2(a_0-a)}}{1+\frac{(1-z)^2\zeta'(z)}{2(a_0-a)}}\right| \leq Q(z) < 1, \tag{3.2}$$

which is equivalent to

$$\left|\frac{(1-z^2)\zeta'(z)}{2(a_0-a)} - \frac{1+(Q(z))^2}{1-(Q(z))^2}\right| \leq \frac{2Q(z)}{1-(Q(z))^2}. \tag{3.3}$$

This completes the proof of Theorem 1.3. □

4. Proof of Theorem 1.4

In this section, we give the proof of Theorem 1.4.

Proof of Theorem 1.4. We first prove the coefficient estimates of part (1). Let $L_f := |f_z(z)| + |f_{\bar{z}}(z)|$ and $l_f := |f_z(z)| - |f_{\bar{z}}(z)|$; then, by the (K, K')-quasiconformality of f, we see that

$$L_f^2 \leq KL_f l_f + K',$$

which is equivalent to

$$L_f \leq \frac{Kl_f + \sqrt{K^2(l_f)^2 + 4K'}}{2}.$$

This gives us

$$|f_{\bar{z}}(z)| \leq \frac{(K-1)|f_z(z)| + \sqrt{K'}}{K+1}. \tag{4.1}$$

We set $N(z) = 2(a_0 - a)/(1-z)^2$, then from Inequality (4.1), we have

$$\frac{|\zeta'(z)| - |N(z)|}{2} \leq k\frac{|\zeta'(z)| + |N(z)|}{2} + \frac{\sqrt{K'}}{K+1},$$

where $k = (K-1)/(K+1)$. Namely,

$$|\zeta'(z)| \leq K|N(z)| + \sqrt{K'}. \tag{4.2}$$

Hence,

$$|f_z(z)| \leq \frac{|\zeta'(z)| + |N(z)|}{2} \leq \phi(z) := \frac{K+1}{2}|N(z)| + \frac{\sqrt{K'}}{2}. \tag{4.3}$$

Now, by combing Inequalities (3.2) and (4.3), we have

$$\begin{aligned}
&|f_z(z)| + |f_{\bar{z}}(z)| \\
\leq &|f_z(z)| + Q(z) \cdot |f_z(z)| \\
\leq &\frac{K\phi(z) + \sqrt{K^2\phi^2(z) + K'(K+1)}}{K+1} \\
\leq &\frac{2K\phi(z) + \sqrt{K'(K+1)}}{K+1} \\
\leq &\frac{2K(a_0-a)}{|1-z|^2} + \frac{(K+\sqrt{K+1})\sqrt{K'}}{K+1}.
\end{aligned} \tag{4.4}$$

Then, by letting $z = 0$ in (4.4), we get

$$|a_1| + |b_1| \le 2K(a_0 - a) + \frac{(K + \sqrt{K+1})\sqrt{K'}}{K+1}.$$

In addition, by letting $\zeta(z) = \sum_{n=1}^{\infty} c_n z^n$, we obtain

$$f(z) = a_0 + \sum_{n=1}^{\infty}(a_0 - a + \frac{c_n}{2})z^n + \overline{\sum_{n=1}^{\infty}(a_0 - a - \frac{c_n}{2})z^n};$$

thus, $||a_n| - |b_n|| = |\lvert(a_0 - a + c_n/2)\rvert - \lvert(a_0 - a - c_n/2)\rvert| \le 2(a_0 - a)$. This finishes part (1) of Theorem 1.4.

Next, we give the proof of the length distortion theorem in part (2) of Theorem 1.4. This is because by Inequality (4.4), we get that

$$\begin{aligned}
&|f(z) - a_0| \\
\le& \int_0^{|z|} (|f_z(z)| + |f_{\bar{z}}(z)|)|dz| \\
\le& \int_0^{|z|} \left(\frac{2K(a_0 - a)}{|1 - z|^2} + \frac{(K + \sqrt{K+1})\sqrt{K'}}{K+1} \right) |dz| \\
=& \frac{2K(a_0 - a)}{\sin\theta} \arctan\Big(\frac{|z|\sin\theta}{1 - |z|\cos\theta}\Big) + \frac{(K + \sqrt{K+1})\sqrt{K'}}{K+1}|z|.
\end{aligned}$$

At last, we give the proof of area theorem in part (3) of Theorem 1.4. By virtue of Theorem 1.3, there exists an analytic mapping ζ with $\zeta(0) = 0, \zeta'(0) \in \mathbb{R}$, such that

$$f(z) = \frac{\frac{(a_0-2a)z+a_0}{1-z} + \zeta(z)}{2} + \frac{\overline{\frac{(a_0-2a)z+a_0}{1-z} - \zeta(z)}}{2}, z \in \mathbb{U}.$$

Hence,

$$\begin{aligned}
Areaf(U_r) &= \frac{1}{4} \iint_{U_r} \left(\left|\zeta'(z) + \frac{2(a_0 - a)}{(1 - z)^2}\right|^2 - \left|\zeta'(z) - \frac{2(a_0 - a)}{(1 - z)^2}\right|^2 \right) dxdy \\
&= \iint_{U_r} Re \overline{\frac{2(a_0 - a)\zeta'(z)}{(1 - z)^2}} dxdy.
\end{aligned}$$

From Inequality (4.2), we have

$$\left| Re \overline{\frac{2(a_0 - a)\zeta'(z)}{(1 - z)^2}} \right| \le \left| \frac{2(a_0 - a)\zeta'(z)}{(1 - z)^2} \right| \le K|N(z)|^2 + \sqrt{K'}|N(z)|,$$

where $N(z) = 2(a_0 - a)/(1 - z)^2$. This leads to

$$\begin{aligned}
&Areaf(U_r) \\
\le& K \iint_{U_r} |N(z)|^2 dxdy + \sqrt{K'} \iint_{U_r} |N(z)| dxdy \\
=& \frac{4\pi K(a_0 - a)^2 r^2}{(1 - r^2)^2} + 2\pi\sqrt{K'}(a_0 - a)\log\left(\frac{1}{1 - r^2}\right).
\end{aligned}$$

So finishes the proof. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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