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https://www.aimspress.com/journal/era

ERA, 34(4): 2483–2510.
DOI: 10.3934/era.2026115
Received: 26 November 2025
Revised: 04 February 2026
Accepted: 04 March 2026
Published: 23 March 2026

Research article

Blow-up for coupled systems of semilinear damped wave equations on the Heisenberg group

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Abstract: The main purpose of this paper is to study behaviors of solutions to the Cauchy problem for coupled systems of semilinear damped wave equations with power nonlinearities on the Heisenberg group, where the Fujita critical exponent is determined. By means of the Bihari inequality as well as the Gagliardo-Nirenberg-type inequality and the contraction mapping principle, the local well-posedness of the Cauchy problem is successfully proved. Combining energy estimates and decay estimates together with a contradiction argument, the global existence of solutions is also demonstrated. At the same time, the blow-up results and upper bound estimation of the solutions is derived by using the test function technique. Our main new contribution is the derivation of a fundamental inequality utilized in weighted energy estimates, as well as the selection of exponents in the scaling of two convex functions for constructing test functions.

Keywords: coupled damped wave equations; Heisenberg group; well-posedness; test function method; blow-up

1. Introduction

In this paper, we aim to study the following Cauchy problem for coupled system of wave equations with weak damping terms:

$$\begin{cases} u_{tt} - \Delta_{\mathbb{H}} u + u_t = |v|^p, & t > 0, \eta \in \mathbb{H}^n, \\ v_{tt} - \Delta_{\mathbb{H}} v + v_t = |u|^q, & t > 0, \eta \in \mathbb{H}^n, \\ u(0,\eta) = u_0(\eta), v(0,\eta) = v_0(\eta), & \eta \in \mathbb{H}^n, \\ u_t(0,\eta) = u_1(\eta), v_t(0,\eta) = v_1(\eta), & \eta \in \mathbb{H}^n, \end{cases} \tag{1.1}$$

where the exponents $p, q > 1$, $\Delta_{\mathbb{H}}$ denotes the sub-Laplacian on $\mathbb{H}^n$.

First, we recall several previous research results regarding the existence and nonexistence of global solutions to the Cauchy problem of nonlinear wave equation as follows (see [1–5]):

$$\begin{cases} u_{tt} - \Delta u = f(u, u_t), & t > 0, x \in \mathbb{R}^n, \\ u(0, x) = u_0(x),\ u_t(0, x) = u_1(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.2}$$

Zhou and Han [6] Found that the solution to Problem (1.2) with $f(u, u_t) = |u|^p$ has the Strauss critical exponent. When $n = 1$, $p_S(1) = \infty$. When $n \geq 2$, $p_S(n)$ is the positive root of the quadratic equation $-(n-1)p^2 + (n+1)p + 2 = 0$. John [7] studied Problem (1.2) with the power nonlinearity $f(u, u_t) = |u|^p$. When the spatial dimension is $n = 3$ and $1 < p < p_S(3)$, it is seen that the problem has no global solution (in time). In the case, the existence of a global solution to the initial value problem was obtained. For Problem (1.2) with $f(u, u_t) = |u|^p$ in the critical case $p = p_S(n)(n = 2, 3)$, it was shown that lifespan of the solution to this problem exhibits an exponential-type estimation result (see [8]). In addition, an upper bound estimate of the solution's lifespan has been derived (see [9, 10]).

Many researchers have concentrated on investigating the semilinear damped wave equation with a power-type nonlinearity on $\mathbb{R}^n$, that is

$$\begin{cases} u_{tt} - \Delta u + u_t = |u|^p, & t > 0, x \in \mathbb{R}^n, \\ u(0, x) = u_0(x),\ u_t(0, x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.3}$$

where Δ denotes the Laplacian operator on $\mathbb{R}^n$ (see the detailed illustrations in [11–15]). If the initial data satisfy the conventional conditions and in are L^1 space, depending on the critical exponent, the existence of a global solution and the blow-up behavior of solution to Problem (1.3) are thoroughly investigated in [16–19] and the references cited therein. The so-called critical exponent is actually a threshold condition that the index p satisfies. It also critically determines the existence of global Sobolev solutions and the blow-up behavior of the local (in time) weak solution with small initial data. The critical exponent of the solution to Problem (1.3) is the Fujita exponent $p_{Fuj}(n) = 1 + \frac{2}{n}$ (see [16]). For $n = 1, 2$, Matsumura [17] proves that the small data solution exists globally when $p > p_{Fuj}(n)$. For $n \geq 1$, Todorova and Yordanov [18] found that the problem has a global solution when $p > p_{Fuj}(n)$ and the initial data have compact support. However, in the subcritical case $1 < p < p_{Fuj}(n)$, the local solution will blow up in finite time. When $p = p_{Fuj}(n)$, Zhang successfully obtained the blow-up result of the solution in [19]. We refer the readers to the works in [20–24] for more details on the qualitative theory of nonlinear wave equations on $\mathbb{R}^n$.

Recently, Ikehata and Tanizawa [16] studied the supercritical case $p > p_{Fuj}(n)$. The restriction that the initial data need to have compact support is removed. The interested readers can refer to [25–29]. In addition, after imposing the additional assumption that the initial data belong to L^1 space, the exact lifespan of the solution of the Cauchy problem (1.3) can be estimated by following formula:

$$T_\varepsilon = \begin{cases} \infty, & p > p_{Fuj}(n), \\ \exp(C\varepsilon^{-(p-1)}), & p = p_{Fuj}(n), \\ C\varepsilon^{-\frac{2-n(p-1)}{2(p-1)}}, & p < p_{Fuj}(n), \end{cases}$$

where C is a positive constant independent of ε. For specific research on the lifespan estimation of the Cauchy problem (1.3). The global small data solution of the semilinear damped wave equation in the Euclidean framework is studied in [30]. For the initial value problem of the wave equation with a weak damping term on the Heisenberg group, Georgiev and Palmieri [31] dealt with the weak damping term by defining the function $\psi(t,\eta)$ to derive the energy inequality, combining the Duhamel principle with a fundamental solution of the linear Cauchy problem to represent the solution and conducting an analysis in Sobolev spaces with exponential weights. It is shown that when the integrals of the initial values satisfy certain sign assumptions, the solution blows up in the subcritical and critical cases $1 < p \le p_{Fuj}(Q)$, whereas the global solution exists in the supercritical case $p > p_{Fuj}(Q)$.

In recent years, numerous researchers have paid substantial attention to semilinear heat equations on the Heisenberg group (see [31]). Ruzhansky and Yessirkegenov [32] studied the Cauchy problem related to the semilinear heat equation under a unimodular Lie group with polynomial volume growth. This approach and main results heavily rely on the theory and property of the heat semigroup. Notably, the Fujita-type critical exponent does not address aspects such as the lifespan estimate of the local solution. In contrast, Georgiev and Palmieri [33] focused on the study of wave equations related to the Heisenberg group. To prove the local and global existence results, different appropriate weighted $L^\infty(\mathbb{H}^n)$ spaces and nonlinear integral operators and fixed point theorem are defined. In order to obtain the blow-up results, specific test functions are constructed to derive key estimates and prove blow-up. The authors not only proved criticality of the Fujita exponent $1+\frac{2}{Q}$ (here, $Q = 2n+2$ is a homogeneous dimension of the Heisenberg group) but also provided precise lower and upper bound estimates for the lifespan of the local solution in the subcritical and critical cases.

The Cauchy problem of semilinear heat equation on the Heisenberg group

$$\begin{cases} u_t - \Delta_{\mathbb{H}} u = |u|^p, & t>0, \eta \in \mathbb{H}^n, \\ u(0,\eta) = u_0(\eta), & \eta \in \mathbb{H}^n \end{cases} \tag{1.4}$$

attracts more attention (see the detailed illustrations in [33,34]). In the Case $1 < p \le p_{Fuj} = 1+\frac{2}{Q}$, the existence of a global positive solution to Problem (1.4) is studied in [14,35], where an appropriate test function is constructed and the properties of the sub-Laplacian operator together with growth of the function are utilized. In the supercritical case $p > p_{Fuj} = 1+\frac{2}{Q}$, Pascucci [35] used the fixed point theorem to investigate the existence of positive global solution to problem (1.4) with sufficiently small initial data. The situation on general locally compact group for the problem is studied in [32].

The Cauchy problem of semilinear subelliptic heat equation with a forcing term which depends only on the group elements

$$\begin{cases} u_t(t,x) - \Delta_{\mathbb{G}} u(t,x) = |u(t,x)|^p + f(x), & t>0, x \in \mathbb{G}, \\ u(0,x) = u_0(x), & x \in \mathbb{G} \end{cases}$$

is investigated. Suragan and Talwar [36] applied the structure of the stratified Lie group, the test function method, and the Banach fixed point theorem to prove that $\frac{Q}{Q-2}$ is the Fujita exponent for semilinear heat equation on an arbitrary stratified Lie group with a homogeneous dimension Q in the case of the Euclidean space. Suragan and Talwar [37] mainly focused on the semilinear heat equation

$$u_t(t,x) - \Delta_{\mathbb{G}} u(t,x) - \frac{\lambda |\nabla_{\mathbb{G}} d(x)|^2 u}{d(x)^2} = u(t,x)^p + f(x)$$

with Hardy potential on the stratified Lie group. It aims to find the conditions for the scalar λ and exponent p to determine the existence and nonexistence of local and global weak solutions. When $p \ge 1+\frac{2}{\alpha}$, a local solution to the problem does not exist. When $p \le \frac{Q-\alpha}{Q-2-\alpha}$, the problem does not admit a global solution. Meanwhile, under the conditions $1 < p < 1 + \frac{2}{\alpha}$ and $|\nabla_{\mathbb{G}} d| = 1$ almost everywhere, the existence of a local solution is proved. When $\frac{Q-\alpha}{Q-2-\alpha} < p < 1 + \frac{2}{\alpha}$, the existence of a global solution is presented. Moreover, the critical exponent for threshold of existence of local and global solutions are predicted on the basis of these results.

We recall several results related to the Cauchy problem of the semilinear damped wave equation $u_{tt}(t) - \mathcal{L}u(t) + bu_t(t) + mu(t) = |u|^p$ for the sub-Laplacian operator on the Heisenberg group $\mathbb{H}^n$. In the case of small initial data, Ruzhansky and Tokmagambetov [38] showed that the problem is globally well-posed. In addition, similar results for the wave equation related to Rockland operators on the general graded Lie group are also obtained. In particular, the results cover the high-order operator on $\mathbb{R}^n$ and the Heisenberg group (see [35, 38, 39]). Dasgupta et al. [39] conducted a thorough investigation into the Cauchy problem of semilinear damped wave equations with the fractional sub-Laplacian operator $(-\mathcal{L}_{\mathbb{H}})^\alpha$ $(\alpha > 0)$ as well as a power-type nonlinearity on the Heisenberg group $\mathbb{H}^n$.

Nishihara and Wakasugi [40] studied the Cauchy problem of a weakly coupled system of damped wave equations

$$\begin{cases} u_{tt} - \Delta u + u_t = |v|^p, & t > 0, x \in \mathbb{R}^n, \\ v_{tt} - \Delta v + v_t = |u|^q, & t > 0, x \in \mathbb{R}^n, \\ (u, u_t, v, v_t)(0, x) = \varepsilon(u_0, u_1, v_0, v_1)(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.5}$$

The critical exponent of the Cauchy problem in all spatial dimensions is determined. For the supercritical nonlinear terms, the global existence of the solutions based on the weighted energy method is proved. When the exponent is less than the so-called Fujita critical exponent, the multiplier in the proof is appropriately modified. Moreover, for the subcritical nonlinear terms, an upper bound estimate of the lifespan of the solutions is provided. For the weakly coupled system of semilinear wave equations with two types of damping terms

$$\begin{cases} u_{tt} - \Delta u + u_t - \Delta u_t = f_1(v, v_t), & t > 0,\ x \in \Omega^c, \\ v_{tt} - \Delta v + v_t - \Delta v_t = f_1(u, u_t), & t > 0,\ x \in \Omega^c, \\ (u, u_t, v, v_t)(0, x) = (u_0, u_1, v_0, v_1)(x), & x \in \Omega^c, \\ u|_{\partial\Omega^c} = 0,\ v|_{\partial\Omega^c} = 0, & t > 0, \end{cases}$$

Fan et al. [41] focused their research on the formation of the singularity of solutions on the exterior domain in n dimensions. This system involves two types of damping terms and combined nonlinearities $|v_t|^{p_1} + |v|^{q_1}$ and $|u_t|^{p_2} + |u|^{q_2}$. The local existence and uniqueness of mild solutions to this problem are successfully determined. Meanwhile, in the case of the coupled system with power-type nonlinearities $|v|^p$ and $|u|^q$ or in the case of a single equation with the power-type nonlinearity $|u|^p$, there are no global solutions to the problem.

Inspired by the previous works in [16–19], we study the Cauchy problem of a coupled system of semilinear damped wave equations on the Heisenberg group $\mathbb{H}^n$. For the initial value problem of the single wave equation with a weak damping term on $\mathbb{H}^n$, it has been proved that a global solution exists in the supercritical case ($p > p_{Fuj}(Q)$) (see [31]). However, the solution will blow up in the subcritical and critical cases ($1 < p \leq p_{Fuj}(Q)$) when the integrals of the initial values satisfy certain sign assumptions. It is noteworthy that in the existing literature, there are no studies about the existence of global solutions and blow-up results for the Cauchy problem of coupled damped wave equations (1.1) on the Heisenberg group $\mathbb{H}^n$. To fill this gap, we establish the local well-posedness of Problem (1.1) by taking advantage of the contraction mapping principle (see Theorem 1.1). We deduce that global solutions to Problem (1.1) exist in the supercritical case ($p > p_{Fuj}(Q)$) by making use of decay estimates and the existence of local solutions (see Theorem 1.2). In the subcritical and critical cases ($1 < p \leq p_{Fuj}(Q)$), we then prove the blow-up results of solutions to Problem (1.1) by applying the test function technique and deriving contradictions in this paper (see Theorem 1.3). Thus we extend the existence of a local solution, the blow-up results of local solution, and the existence of a global solution for the initial value problem of a single wave equation with a weak damping term and power nonlinearity on the Heisenberg group studied in [31] to the case of the corresponding coupled system (see Theorems 1.1–1.3). Moreover, we also upgrade the Laplacian operator on the Euclidean space $\mathbb{R}^n$ in the problems studied in [17–19] to the sub-Laplacian operator on the Heisenberg group $\mathbb{H}^n$ and conduct research in the context of the corresponding coupled system (see Theorems 1.1 and 1.3). To our knowledge, the lifespan estimates of the solutions to Problem (1.1) on the Heisenberg group $\mathbb{H}^n$ has not been derived. We obtain the upper bound lifespan estimates of the solutions by using a test function method, which is different from the test functions constructed in [12] (see Theorem 1.4). As far as we know, the results in Theorems 1.1–1.4 are new.

In this paper, we use the following notation: $f \lesssim g$ means that a positive constant C exists such that $f \leq Cg$. Similarly, the definition of $f \gtrsim g$ follows the same pattern. In addition, $f \approx g$ means that $f \lesssim g$ and $f \gtrsim g$.

Throughout this paper, we set

$$\psi(t, \eta) = \frac{|x|^2 + |y|^2 + 4|\tau|}{8(1 + t)}$$

for $\eta = (x, y, \tau) \in \mathbb{H}^n$. We define the Sobolev spaces L^2 and H^1 with an exponential weight $e^{\psi(t,\cdot)}$ as follows:

$$L^2_{\psi(t,\cdot)}(\mathbb{H}^n) = \{v \in L^2(\mathbb{H}^n) \big| \, \|e^{\psi(t,\cdot)} v\|_{L^2(\mathbb{H}^n)} < \infty\},$$
$$H^1_{\psi(t,\cdot)}(\mathbb{H}^n) = \{v \in H^1(\mathbb{H}^n) \big| \, \|e^{\psi(t,\cdot)} v\|_{L^2(\mathbb{H}^n)} + \|e^{\psi(t,\cdot)} \nabla_{\mathbb{H}} v\|_{L^2(\mathbb{H}^n)} < \infty\},$$

with the norms

$$\|v\|_{L^2_{\psi(t,\cdot)}(\mathbb{H}^n)} = \|e^{\psi(t,\cdot)} v\|_{L^2(\mathbb{H}^n)},$$
$$\|v\|_{H^1_{\psi(t,\cdot)}(\mathbb{H}^n)} = \|e^{\psi(t,\cdot)} v\|_{L^2(\mathbb{H}^n)} + \|e^{\psi(t,\cdot)} \nabla_{\mathbb{H}} v\|_{L^2(\mathbb{H}^n)}.$$

The space $A(\mathbb{H}^n)$ represents

$$A(\mathbb{H}^n) = H^1_{\psi(0,)}(\mathbb{H}^n) \times L^2_{\psi(0,)}(\mathbb{H}^n),$$

which the initial data will be required to belong to.

We now present the main results of this work.

Theorem 1.1. *Let $n \geq 1$, $1 < p$ and $q \leq p_{GN}(Q) = \frac{Q}{Q-2}$. The initial values satisfy $(u_0, u_1), (v_0, v_1) \in A(\mathbb{H}^n)$. The Cauchy problem* (1.1) *then admits unique solutions*

$$(u, v) \in (C([0, T_{max}), H^1(\mathbb{H}^n)) \cap C^1([0, T_{max}), L^2(\mathbb{H}^n)))$$

$$\times(C([0,T_{max}),H^1(\mathbb{H}^n))\cap C^1([0,T_{max}),L^2(\mathbb{H}^n))),$$

where T_{max} is a positive constant ($0 < T_{max} \le \infty$). Furthermore, for all $T \in (0, T_{max})$, it holds that

$$\sup_{t\in[0,T]}(\|e^{\psi(t,\cdot)}(u,v)(t,\cdot)\|_{L^2(\mathbb{H}^n)}+\|e^{\psi(t,\cdot)}(\nabla_{\mathbb{H}}u,\nabla_{\mathbb{H}}v)(t,\cdot)\|_{L^2(\mathbb{H}^n)}+\|e^{\psi(t,\cdot)}(u_t,v_t)(t,\cdot)\|_{L^2(\mathbb{H}^n)})<\infty.$$

If $T_{max} < \infty$, then

$$\limsup_{T\to T_{max}^-}(\|e^{\psi(t,\cdot)}(u,v)(t,\cdot)\|_{L^2(\mathbb{H}^n)}+\|e^{\psi(t,\cdot)}(\nabla_{\mathbb{H}}u,\nabla_{\mathbb{H}}v)(t,\cdot)\|_{L^2(\mathbb{H}^n)}+\|e^{\psi(t,\cdot)}(u_t,v_t)(t,\cdot)\|_{L^2(\mathbb{H}^n)})=\infty.$$

Theorem 1.2. *Let $n \ge 1$. Assume that $1 < p, q \le p_{GN}(Q)$ such that $p > p_{Fuj}(Q)$ and $q > p_{Fuj}(Q)$. Suppose that the initial data $(u_0, u_1), (v_0, v_1) \in A(\mathbb{H}^n)$ satisfy*

$$\|(u_0,u_1)\|_{A(\mathbb{H}^n)}\le\varepsilon_0,\quad \|(v_0,v_1)\|_{A(\mathbb{H}^n)}\le\varepsilon_0, \tag{1.6}$$

where $\varepsilon_0 > 0$ is a positive constant. Then unique solutions $(u, v) \in (C([0,\infty), H^1_{\psi(t,\cdot)}(\mathbb{H}^n)) \cap C^1([0,\infty), L^2_{\psi(t,\cdot)}(\mathbb{H}^n)))^2$ to the Cauchy problem (1.1) exist. Moreover, u satisfies the following estimates:

$$\begin{aligned}
&\|u(t,\cdot)\|_{L^2(\mathbb{H}^n)}\lesssim(1+t)^{-\frac{Q}{4}}\|(u_0,u_1)\|_{A(\mathbb{H}^n)},\\
&\|\nabla_{\mathbb{H}}u(t,\cdot)\|_{L^2(\mathbb{H}^n)}\lesssim(1+t)^{-\frac{Q}{4}-\frac{1}{2}}\|(u_0,u_1)\|_{A(\mathbb{H}^n)},\\
&\|u_t(t,\cdot)\|_{L^2(\mathbb{H}^n)}\lesssim(1+t)^{-\frac{Q}{4}-1}\|(u_0,u_1)\|_{A(\mathbb{H}^n)},\\
&\|e^{\psi(t,\cdot)}\nabla_{\mathbb{H}}u(t,\cdot)\|_{L^2(\mathbb{H}^n)}\lesssim\|(u_0,u_1)\|_{A(\mathbb{H}^n)},\\
&\|e^{\psi(t,\cdot)}u_t(t,\cdot)\|_{L^2(\mathbb{H}^n)}\lesssim\|(u_0,u_1)\|_{A(\mathbb{H}^n)}
\end{aligned}$$

for $t \ge 0$. In an analogous way, v has the same estimates.

Remark 1.1. We should note that the condition $(u_0, u_1) \in A(\mathbb{H}^n)$ in Theorem 1.2 is more stringent than the supposition $(u_0, u_1) \in (H^1(\mathbb{H}^n) \cap L^2(\mathbb{H}^n)) \times (L^1(\mathbb{H}^n) \cap L^1(\mathbb{H}^n))$. Indeed, the embedding $L^2_{\sigma\psi(t,\cdot)}(\mathbb{H}^n) \hookrightarrow L^1(\mathbb{H}^n) \times L^2(\mathbb{H}^n)$ holds for $\sigma > 0$ and $t \ge 0$. Using the Cauchy-Schwarz inequality and non-negativity of ψ results in

$$\begin{aligned}
&\|w\|_{L^2(\mathbb{H}^n)}\lesssim\|e^{\sigma\psi(t,\cdot)}w\|_{L^2(\mathbb{H}^n)},\\
&\|w\|_{L^1(\mathbb{H}^n)}\lesssim(1+t)^{\frac{Q}{4}}\|e^{\sigma\psi(t,\cdot)}w\|_{L^2(\mathbb{H}^n)}.
\end{aligned} \tag{1.7}$$

Theorem 1.3. *Let $n \ge 1$ and $u_0, u_1, v_0, v_1 \in L^1(\mathbb{H}^n)$ such that*

$$\liminf_{R\to\infty}\int_{D_R}((u_0,v_0)(\eta)+(u_1,v_1)(\eta))d\eta>0, \tag{1.8}$$

where $D_R = B^n(R)\times B^n(R)\times[-R^2, R^2]$, and $R > 2$ is a constant. Let us assume that $(u, v) \in (L^q_{loc}([0,T)\times\mathbb{R}^n)\times L^p_{loc}([0,T)\times\mathbb{R}^n))$ are solutions to Problem (1.1) with the lifespan $T > 0$. If $1 < p, q \le p_{Fuj}(Q)$, then $T < \infty$. That is, the solutions u and v will blow up in finite time.

Remark 1.2. We extend the blow-up results of the solutions and the existence of global solutions to the initial value problem of a single wave equation with a weak damping term and power nonlinearity on the Heisenberg group studied in [31] to the case of the corresponding coupled system. Moreover, we extend the results for the single wave equation on $\mathbb{R}^n$ in [17–19] to the case of the corresponding coupled system on the Heisenberg group $\mathbb{H}^n$. The replacement of the Euclidean dimension n with the homogeneous dimension $Q = 2n+2$ is primarily intended to accommodate the non-Euclidean geometric structure of the Heisenberg group, which is stratified and features quadratic coupling of the vertical components. This substitution ensures that the critical exponent can accurately characterize balance nonlinear growth, diffusion on the group, and damping effects. It also matches the scaling properties of the Haar measure and the diffusion law of the sub-Laplacian operator on the Heisenberg group.

Theorem 1.4. *Let the initial data satisfy* $(u(0,x), u_t(0,x), v(0,x), v_t(0,x)) = \varepsilon(u_0, u_1, v_0, v_1)$ *in Problem* (1.1) *, where* $\varepsilon > 0$ *is a small parameter describing the size of the Cauchy data. Let* $1 < p, q \leq p_{Fuj}(Q)$ *and* $u_0, u_1, v_0, v_1 \in L^1(\mathbb{H}^n)$ *satisfy*

$$\int_{\mathbb{H}^n} ((u_0, v_0)(\eta) + (u_1, v_1)(\eta))d\eta > 0.$$

The initial values have compact supports, which satisfy supp $(u_0, u_1, v_0, v_1) \subset \{(x, y, \tau) \in \mathbb{H}^n \big| \, |x|^2 + |y|^2 + |\tau| < R_0\}$ *for some* $R_0 > 0$. *Then* $\varepsilon_0 > 0$, *for* $\varepsilon \in (0, \varepsilon_0]$ *and constant* $C > 0$ *exists such that*

$$T(\varepsilon) \leq \begin{cases} C\varepsilon^{-\frac{pq-1}{(p-1)(\frac{1}{p-1}-\frac{Q}{2})+(q-1)(\frac{1}{q-1}-\frac{Q}{2})}}, & p, q \in (1, p_{Fuj}(Q)), \\ exp(C\varepsilon^{-(pq-1)}), & p = q = p_{Fuj}(Q), \\ C\varepsilon^{-\frac{pq-1}{(q-1)(\frac{1}{q-1}-\frac{Q}{2})}}, & p = p_{Fuj}(Q), q \in (1, p_{Fuj}(Q)), \\ C\varepsilon^{-\frac{pq-1}{(p-1)(\frac{1}{p-1}-\frac{Q}{2})}}, & q = p_{Fuj}(Q), p \in (1, p_{Fuj}(Q)). \end{cases}$$

Remark 1.3. For the special case $p = q$ in Problem (1.1), the second lifespan estimate in Theorem 1.4 is exactly the same as the upper bound lifespan estimate of the corresponding small initial value problem of a single wave equation with weak damping (see [31]). We need to point out that the authors of [31] only provide the lifespan estimate of the solution without specific proof. It is worth mentioning that we present the detailed derivation of the upper bound lifespan estimates of the solutions to the case of a coupled system with small initial values.

2. Proof of Theorem 1.1

2.1. Preliminary

The Heisenberg group viewed as a Lie group $\mathbb{H}^n = \mathbb{R}^{2n+1}$ has the following multiplication rule

$$(x, y, \tau) \circ (x', y', \tau') = (x + x', y + y', \tau + \tau' + \frac{1}{2}(x \cdot y' - x' \cdot y)),$$

where the symbol $\cdot$ represents the standard scalar product in $\mathbb{R}^n$. A set of left-invariant vector fields that generate the Lie algebra is

$$X_j = \partial_{x_j} - \frac{y_j}{2}\partial_\tau, \quad Y_j = \partial_{y_j} + \frac{x_j}{2}\partial_\tau, \quad \partial_\tau,$$

where $1 \leq j \leq n$. The sub-Laplacian on $\mathbb{H}^n$ is defined by

$$\Delta_{\mathbb{H}} = \sum_{j=1}^{n}(X_j^2 + Y_j^2) = \sum_{j=1}^{n}(\partial_{x_j}^2 + \partial_{y_j}^2) + \frac{1}{4}\sum_{j=1}^{n}(x_j^2 + y_j^2)\partial_\tau^2 + \sum_{j=1}^{n}(x_j\partial_{y_j\tau}^2 - y_j\partial_{x_j\tau}^2).$$

For a function $v : \mathbb{H}^n \to \mathbb{R}$, the horizontal gradient of v is

$$\nabla_{\mathbb{H}} v = (X_1 v, \cdots, X_n v, Y_1 v, \cdots, Y_n v) = \sum_{j=1}^{n}((X_j v)X_j + (Y_j v)Y_j).$$

In particular, the sub-Laplacian can also be written as $\Delta_{\mathbb{H}} v = div(\nabla_{\mathbb{H}} v)$.

It is necessary for us to obtain decay estimates for the linear Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + u_t = 0, & t > 0, \eta \in \mathbb{H}^n, \\ v_{tt} - \Delta v + v_t = 0, & t > 0, \eta \in \mathbb{H}^n, \\ u(0,\eta) = u_0(\eta),\ v(0,\eta) = v_0(\eta), & \eta \in \mathbb{H}^n, \\ u_t(0,\eta) = u_1(\eta),\ v_t(0,\eta) = v_1(\eta), & \eta \in \mathbb{H}^n. \end{cases} \tag{2.1}$$

Let $E_0(t,\eta), E_1(t,\eta), \bar{E}_0(t,\eta)$, and $\bar{E}_1(t,\eta)$ be the fundamental solutions to the Cauchy problem (2.1). That is to say, these are the distributional solutions corresponding to the initial data $(u_0, u_1) = (\delta_0, 0)$ and $(u_0, u_1) = (0, \delta_0)$, and $(v_0, v_1) = (\delta_1, 0)$ and $(v_0, v_1) = (0, \delta_1)$, respectively, where δ_0, δ_1 are the Dirac distribution in the η variable. Moreover, if we take advantage of $*_{(\eta)}$ to represent the group convolution with respect to the variable η, then the solutions of the Cauchy problem (2.1) can be expressed as follows:

$$u(t,\eta) = u_0(\eta) *_\eta E_0(t,\eta) + u_1(\eta) *_\eta E_1(t,\eta),$$
$$v(t,\eta) = v_0(\eta) *_\eta \bar{E}_0(t,\eta) + v_1(\eta) *_\eta \bar{E}_1(t,\eta).$$

In accordance with the Duhamel principle tailored to the context of Lie groups, we find that

$$u(t,\eta) = \int_0^t F(s,\eta) *_{(\eta)} E_1(t-s,\eta)ds, \tag{2.2}$$

$$v(t,\eta) = \int_0^t G(s,\eta) *_{(\eta)} \bar{E}_1(t-s,\eta)ds \tag{2.3}$$

are mild solutions to the inhomogeneous Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + u_t = F(t,\eta), & t > 0, \eta \in \mathbb{H}^n, \\ v_{tt} - \Delta v + v_t = G(t,\eta), & t > 0, \eta \in \mathbb{H}^n, \\ u(0,\eta) = 0,\ v(0,\eta) = 0, & \eta \in \mathbb{H}^n, \\ u_t(0,\eta) = 0,\ v_t(0,\eta) = 0, & \eta \in \mathbb{H}^n. \end{cases}$$

Using the fact that the identity $L(w *_{(\eta)} E_1) = w *_{(\eta)} L(E_1)$ holds for the left invariant differential operator L on $\mathbb{H}^n$. Consequently, as mild solutions to Problem (1.1) on $(0,T) \times \mathbb{H}^n$, we consider any fixed point of the nonlinear integral operator N defined as follows

$$u \in X(T) \to Nu(t,\eta) = u_0(\eta) *_{(\eta)} E_0(t,\eta) + u_1(\eta) *_{(\eta)} E_1(t,\eta)$$

$$+\int_0^t |v(s,\eta)|^p *_{(\eta)} E_1(t-s,\eta)ds, \tag{2.4}$$

$$\begin{aligned} v\in X(T)\to Nv(t,\eta) &= v_0(\eta) *_{(\eta)} \bar{E}_0(t,\eta) + v_1(\eta) *_{(\eta)} \bar{E}_1(t,\eta) \\ &\quad + \int_0^t |u(s,\eta)|^q *_{(\eta)} \bar{E}_1(t-s,\eta)ds \end{aligned}$$

for a suitably chosen space $X(T)$.

In this paper, the global solutions we focus on are the solutions in $X(T)$ of the integral equations, namely

$$u(t,\eta) = u_0(\eta) *_{(\eta)} E_0(t,\eta) + u_1(\eta) *_{(\eta)} E_1(t,\eta) + \int_0^t |v(s,\eta)|^p *_{(\eta)} E_1(t-s,\eta)ds,$$
$$v(t,\eta) = v_0(\eta) *_{(\eta)} \bar{E}_0(t,\eta) + v_1(\eta) *_{(\eta)} \bar{E}_1(t,\eta) + \int_0^t |u(s,\eta)|^q *_{(\eta)} \bar{E}_1(t-s,\eta)ds,$$

which can be extended for all positive times.

We focus our deliberation on the weighted energy space

$$X(T) = C([0,T], H^1_{\psi(t,\cdot)}(\mathbb{H}^n)) \cap C^1([0,T], L^2_{\psi(t,\cdot)}(\mathbb{H}^n))$$

in the proof. We set the function

$$\psi(t,\eta) = \frac{|x|^2+|y|^2+4|\tau|}{8(1+t)},$$

where $\eta=(x,y,\tau)\in\mathbb{H}^n$. Straightforward computations lead to

$$\psi_t(t,\eta) = -\frac{|x|^2+|y|^2+4|\tau|}{8(1+t)^2}, \quad X_j\psi(t,\eta) = \frac{x_j - sign(\tau)y_j}{4(1+t)}, \quad Y_j\psi(t,\eta) = \frac{y_j + sign(\tau)x_j}{4(1+t)}$$

for $j=1,\cdots,n$. Subsequently, we consider weak derivatives. Then the previous expressions for $X_j\psi$ and $Y_j\psi$ are in the sense of distributions. Consequently, the following inequalities are satisfied:

$$|\nabla_{\mathbb{H}}\psi(t,\eta)|^2 + \psi_t(t,\eta) = -\frac{|\tau|}{2(1+t)^2} \le 0, \tag{2.5}$$

$$\Delta_{\mathbb{H}}\psi(t,\eta) = \frac{n}{2(1+t)} + \frac{|x|^2+|y|^2}{4(1+t)}\delta_0(\tau) \tag{2.6}$$

for $t\ge 0$ and $\eta\in\mathbb{H}^n$, where $\delta_0(\tau)$ represents the Dirac delta at 0 corresponding to the τ variable. Making use of the fact that the sub-Laplacian can be expressed as divergence of the horizontal gradient and identity

$$div(\alpha X) = \alpha div X + X(\alpha)$$

for $\alpha\in C^1(\mathbb{H}^n)$ and the horizontal vector field X on $\mathbb{H}^n$, we get

$$e^{2\psi}u_t\Delta_{\mathbb{H}}u = e^{2\psi}u_t div(\nabla_{\mathbb{H}}u)$$

$$= div(e^{2\psi}u_t\nabla_{\mathbb{H}}u) - (\nabla_{\mathbb{H}}u)(e^{2\psi}u_t)$$
$$= div(e^{2\psi}u_t\nabla_{\mathbb{H}}u) - \sum_{j=1}^{n} X_j(u)(e^{2\psi}X_j(u_t) + 2e^{2\psi}u_tX_j(\psi))$$
$$- \sum_{j=1}^{n} Y_j(u)(e^{2\psi}Y_j(u_t) + 2e^{2\psi}u_tY_j(\psi)), \tag{2.7}$$

where $|\nabla_{\mathbb{H}}v|^2 = \sum_{j=1}^{n}(|X_jv|^2 + |Y_jv|^2)$ is the Euclidean norm of $\nabla_{\mathbb{H}}v$ provided the identification $\nabla_{\mathbb{H}}v \simeq (X_1u, \cdots, X_nu, Y_1u, \cdots, Y_nu) : \mathbb{H}^n \to \mathbb{R}^{2n}$. Equivalently, we consider on each fiber of the horizontal sub-bundle $\mathrm{H}_\eta\mathbb{H}^n$, where the norm is induced by the scalar product $\langle\cdot,\cdot\rangle_\eta$. Since

$$\sum_{j=1}^{n} X_j(u)X_j(u_t)e^{2\psi} = \sum_{j=1}^{n}(\frac{\partial}{\partial t}(\frac{e^{2\psi}}{2}|X_j(u)|^2) - \psi_te^{2\psi}|X_j(u)|^2).$$

Analogously, we acquire

$$\sum_{j=1}^{n} Y_j(u)Y_j(u_t)e^{2\psi} = \sum_{j=1}^{n}(\frac{\partial}{\partial t}(\frac{e^{2\psi}}{2}|Y_j(u)|^2) - \psi_te^{2\psi}|Y_j(u)|^2).$$

It follows that

$$\sum_{j=1}^{n}(X_j(u)X_j(u_t)e^{2\psi} + Y_j(u)Y_j(u_t)e^{2\psi}) = \frac{\partial}{\partial t}(\frac{e^{2\psi}}{2}|\nabla_{\mathbb{H}}u|^2) - \psi_te^{2\psi}|\nabla_{\mathbb{H}}u|^2. \tag{2.8}$$

Applying the $(0, 2)$ symmetric tensor $\langle\cdot,\cdot\rangle$ on $\mathbb{H}^n$, whose restriction to each fiber $\mathrm{H}_\eta\mathbb{H}^n$ of the horizontal sub-bundle is the scalar product $\langle\cdot,\cdot\rangle_\eta$ with an orthonormal basis given by canonical generators of the horizontal layer, we find that

$$\langle u_t\nabla_{\mathbb{H}}\psi, \psi_t\nabla_{\mathbb{H}}u\rangle = \sum_{j=1}^{n}(u_t\psi_tX_j(\psi)X_j(u) + u_t\psi_tY_j(\psi)Y_j(u)).$$

Therefore, we have

$$2\sum_{j=1}^{n}(e^{2\psi}u_tX_j(\psi)X_j(u) + e^{2\psi}u_tY_j(\psi)Y_j(u))$$
$$= 2\frac{e^{2\psi}}{\psi_t}\langle u_t\nabla_{\mathbb{H}}\psi, \psi_t\nabla_{\mathbb{H}}u\rangle$$
$$= \frac{e^{2\psi}}{\psi_t}(u_t^2|\nabla_{\mathbb{H}}\psi|^2 + \psi_t^2|\nabla_{\mathbb{H}}u|^2 - |u_t\nabla_{\mathbb{H}}\psi - \psi_t\nabla_{\mathbb{H}}u|^2). \tag{2.9}$$

Combining (2.7)–(2.9), we obtain

$$e^{\psi}u_t\Delta_{\mathbb{H}}u = div(e^{2\psi}u_t\nabla_{\mathbb{H}}u) - \frac{\partial}{\partial t}(\frac{e^{2\psi}}{2}|\nabla_{\mathbb{H}}u|^2) + \frac{e^{2\psi}}{\psi_t}(|u_t\nabla_{\mathbb{H}}\psi - \psi_t\nabla_{\mathbb{H}}u|^2 - u_t^2|\nabla_{\mathbb{H}}\psi|^2). \tag{2.10}$$

In addition, we arrive at

$$e^{2\psi}u_tu_{tt}=\frac{e^{2\psi}}{2}\partial_t|u_t|^2=\frac{\partial}{\partial t}(\frac{e^{2\psi}}{2}|u_t|^2)-\psi_te^{2\psi}u_t^2. \tag{2.11}$$

Applying (2.10) and (2.11), we find

$$\begin{aligned}&e^{2\psi}u_t(u_{tt}-\Delta_{\mathbb{H}}u+u_t)\\&=\frac{\partial}{\partial t}(\frac{e^{2\psi}}{2}(|u_t|^2+|\nabla_{\mathbb{H}}u|^2))-div(e^{2\psi}u_t\nabla_{\mathbb{H}}u)\\&\quad+\frac{e^{2\psi}}{\psi_t}u_t^2(|\nabla_{\mathbb{H}}\psi|^2+\psi_t)-\psi_te^{2\psi}u_t^2-\frac{e^{2\psi}}{\psi_t}|u_t\nabla_{\mathbb{H}}\psi-\psi_t\nabla_{\mathbb{H}}u|^2.\end{aligned} \tag{2.12}$$

Similarly, we obtain

$$\begin{aligned}&e^{2\psi}v_t(v_{tt}-\Delta_{\mathbb{H}}v+v_t)\\&=\frac{\partial}{\partial t}(\frac{e^{2\psi}}{2}(|v_t|^2+|\nabla_{\mathbb{H}}v|^2))-div(e^{2\psi}v_t\nabla_{\mathbb{H}}v)\\&\quad+\frac{e^{2\psi}}{\psi_t}v_t^2(|\nabla_{\mathbb{H}}\psi|^2+\psi_t)-\psi_te^{2\psi}v_t^2-\frac{e^{2\psi}}{\psi_t}|v_t\nabla_{\mathbb{H}}\psi-\psi_t\nabla_{\mathbb{H}}v|^2.\end{aligned} \tag{2.13}$$

To prove Theorem 1.1, we need several indispensable lemmas.

Lemma 2.1. *[31] Suppose that k is a non-negative and continuous function, M is a real constant, and g is a continuous, nondecreasing and non-negative function such that*

$$G(u)=\int_0^u\frac{ds}{g(s)}. \tag{2.14}$$

Let $y(t)$ be a continuous function such that

$$y(t)\le M+\int_0^t k(s)g(y(s))ds \tag{2.15}$$

for $t\ge 0$. Then,

$$G(y(t))\le G(M)+\int_0^t k(s)ds \tag{2.16}$$

for $t\ge 0$.

Lemma 2.2. *[31] Let $n\ge 1$ and $2\le q\le 2+\frac{2}{n}=\frac{2Q}{Q-2}$. Then the following weighted Gagliardo-Nirenberg inequality holds:*

$$||v||_{L^q(\mathbb{H}^n)}\le C||\nabla_{\mathbb{H}}v||^{\theta(q)}_{L^2(\mathbb{H}^n)}||v||^{1-\theta(q)}_{L^2(\mathbb{H}^n)}$$

for $v\in H^1(\mathbb{H}^n)$, where C is a non-negative constant, and $\theta(q)\in[0,1]$ satisfies

$$\theta(q)=Q(\frac{1}{2}-\frac{1}{q}).$$

Lemma 2.3. *[31] Let $n\ge 1$, $\sigma\in(0,1]$, and $t\ge 0$. Suppose $2\le q\le 2+\frac{2}{n}=\frac{2Q}{Q-2}$. Then the weighted Gagliardo-Nirenberg inequality*

$$||e^{\sigma\psi(t,\cdot)}v||_{L^q(\mathbb{H}^n)}\le C(1+t)^{(1-\theta(q))/2}||\nabla_{\mathbb{H}}v||^{1-\sigma}_{L^2(\mathbb{H}^n)}||e^{\psi(t,\cdot)}\nabla_{\mathbb{H}}v||^{\sigma}_{L^2(\mathbb{H}^n)} \tag{2.17}$$

holds for $v\in H^1_{\psi(t,\cdot)}(\mathbb{H}^n)$, where C is a non-negative constant.

2.2. Proof of Theorem 1.1

Let T and K be positive constants. We define

$$B_{T,K,\psi} = \{(z_1, z_2) \in (C([0,T], H^1(\mathbb{H}^n)) \cap C^1([0,T], L^2(\mathbb{H}^n)))^2 \Big| \|(z_1, z_2)\|_{T,\psi} \leq K\},$$

where the norm $\|\cdot\|_{T,\psi}$ is given by

$$\|(z_1, z_2)\|_{T,\psi} = \sup_{t\in[0,T]} (\|e^{\psi(t,\cdot)}(z_1, z_2)(t,\cdot)\|_{L^2(\mathbb{H}^n)} + \|e^{\psi(t,\cdot)}\nabla_{\mathbb{H}}(z_1, z_2)(t,\cdot)\|_{L^2(\mathbb{H}^n)} \\ + \|e^{\psi(t,\cdot)}(\partial_t z_1, \partial_t z_2)(t,\cdot)\|_{L^2(\mathbb{H}^n)}).$$

We introduce the map

$$\Phi : B_{T,K,\psi} \to (C([0,T], H^1(\mathbb{H}^n)) \cap C^1([0,T], L^2(\mathbb{H}^n)))^2, \\ (z_1, z_2) \mapsto (v, u) = \Phi(z_1, z_2),$$

where (u, v) solves the following Cauchy problem:

$$\begin{cases} u_{tt} - \Delta_{\mathbb{H}} u + u_t = |z_2|^p, & t > 0, \eta \in \mathbb{H}^n, \\ v_{tt} - \Delta_{\mathbb{H}} v + v_t = |z_1|^q, & t > 0, \eta \in \mathbb{H}^n, \\ u(0,\eta) = u_0(\eta),\ v(0,\eta) = v_0(\eta), & \eta \in \mathbb{H}^n, \\ u_t(0,\eta) = u_1(\eta),\ v_t(0,\eta) = v_1(\eta), & \eta \in \mathbb{H}^n. \end{cases} \tag{2.18}$$

Using (2.12), it results in

$$e^{2\psi} u_t |z_2|^p \geq \frac{\partial}{\partial t}(\frac{e^{2\psi}}{2}(u_t^2 + |\nabla_{\mathbb{H}} u|^2)) - div(e^{2\psi} u_t \nabla_{\mathbb{H}} u).$$

Hence, we introduce the weighted energies of the functions u and v

$$\xi_\psi[u](t) = \frac{1}{2}\int_{\mathbb{H}^n} e^{2\psi(t,\eta)}(|u_t(t,\eta)|^2 + |\nabla_{\mathbb{H}} u(t,\eta)|^2) d\eta, \\ \xi_\psi[v](t) = \frac{1}{2}\int_{\mathbb{H}^n} e^{2\psi(t,\eta)}(|v_t(t,\eta)|^2 + |\nabla_{\mathbb{H}} v(t,\eta)|^2) d\eta.$$

Straightforward computations lead to

$$\xi_\psi[u](t) \leq \xi_\psi[u](0) + \int_0^t (\int_{\mathbb{H}^n} e^{2\psi(s,\eta)}|z_2(s,\eta)|^{2p} d\eta)^{\frac{1}{2}} (\int_{\mathbb{H}^n} e^{2\psi(s,\eta)}|u_t(s,\eta)|^2 d\eta)^{\frac{1}{2}} ds \\ \leq \xi_\psi[u](0) + \sqrt{2}\int_0^t (\int_{\mathbb{H}^n} e^{2\psi(s,\eta)}|z_2(s,\eta)|^{2p} d\eta)^{\frac{1}{2}} \xi_\psi[u](s)^{\frac{1}{2}} ds.$$

Applying Lemma 2.1 yields

$$\xi_\psi[u](t)^{\frac{1}{2}} \leq \xi_\psi[u](0)^{\frac{1}{2}} + \frac{1}{\sqrt{2}}\int_0^t (\int_{\mathbb{H}^n} e^{2\psi(s,\eta)}|z_2(s,\eta)|^{2p} d\eta)^{\frac{1}{2}} ds. \tag{2.19}$$

Since $z_1, z_2 \in B_{T,K,\psi}$, we have $z_1(t,\cdot), z_2(t,\cdot) \in H^1_{\psi(t,\cdot)}$ for $t \in [0,T]$. From Lemma 2.3, we conclude that

$$\int_{\mathbb{H}^n} e^{2\psi(s,\eta)}|z_2(s,\eta)|^{2p} d\eta \lesssim (1+s)^{p(1-\theta(2p))} K^{2p}.$$

Thus, applying (2.19) leads to

$$\xi_\psi[u](t)^{\frac{1}{2}} \le \xi_\psi[u](0)^{\frac{1}{2}} + C_p T(1+T)^{p(1-\theta(2p))/2} K^p,$$

where $C_p > 0$ is a multiplicative constant independent of T and K. Consequently, we get

$$\|e^{\psi(t,\cdot)} u_t(t,\cdot)\|_{L^2(\mathbb{H}^n)} + \|e^{\psi(t,\cdot)} \nabla_{\mathbb{H}} u(t,\cdot)\|_{L^2(\mathbb{H}^n)} \le C_p \xi_\psi[u](0)^{\frac{1}{2}} + C_p T(1+T)^{p(1-\theta(2p))/2} K^p. \quad (2.20)$$

Since

$$e^{\psi(t,\eta)} u(t,\eta) = e^{\psi(t,\eta)} u_0(\eta) + \int_0^t e^{\psi(t,\eta)} u_t(s,\eta) ds$$

and ψ is decreasing with respect to t and (2.20), we have

$$\|e^{\psi(t,\cdot)} u(t,\cdot)\|_{L^2(\mathbb{H}^n)} \le \|e^{\psi(t,\eta)} u_0\|_{L^2(\mathbb{H}^n)} + C_p \xi_\psi[u](0)^{\frac{1}{2}} T + C_p T^2 (1+T)^{p(1-\theta(2p))/2} K^p.$$

Therefore, we obtain

$$\begin{aligned}
&\|e^{\psi(t,\cdot)} u(t,\cdot)\|_{L^2(\mathbb{H}^n)} + \|e^{\psi(t,\cdot)} u_t(t,\cdot)\|_{L^2(\mathbb{H}^n)} + \|e^{\psi(t,\cdot)} \nabla_{\mathbb{H}} u(t,\cdot)\|_{L^2(\mathbb{H}^n)} \\
&\le \|e^{\psi(t,\cdot)} u_0\|_{L^2(\mathbb{H}^n)} + C_p(1+T)\xi_\psi[u](0)^{\frac{1}{2}} + C_p T^2(1+T)^{p(1-\theta(2p))/2} K^p \\
&\le \|e^{\psi(t,\cdot)} u_0\|_{L^2(\mathbb{H}^n)} + C_p(1+T)(\|e^{\psi(t,\cdot)} \nabla_{\mathbb{H}} u_0\|_{L^2(\mathbb{H}^n)} + \|e^{\psi(t,\cdot)} u_1\|_{L^2(\mathbb{H}^n)}) \\
&\quad + C_p T^2 (1+T)^{p(1-\theta(2p))/2} K^p.
\end{aligned}$$

Clearly, we can choose K to be large enough such that

$$\|e^{\psi(t,\cdot)} u_0\|_{L^2(\mathbb{H}^n)} + C_p(\|e^{\psi(t,\cdot)} \nabla_{\mathbb{H}} u_0\|_{L^2(\mathbb{H}^n)} + \|e^{\psi(t,\cdot)} u_1\|_{L^2(\mathbb{H}^n)}) < \frac{K}{2}.$$

We choose $T > 0$ to be sufficiently small that

$$\frac{K}{2} T + C_p T^2 (1+T)^{p(1-\theta(2p))/2} K^p < \frac{K}{2}.$$

As a consequence, $\|u\|_{T,\psi} \le K$. In a similar way, we obtain $\|v\|_{T,\psi} \le K$. That is, Φ maps $B_{T,K,\psi}$ to itself.

Finally, we need to show that Φ is a contraction map, provided that T is small enough. Let $z_1, \overline{z_1}, z_2, \overline{z_2}, \in B_{T,K,\psi}$. We write $u = \Phi(z_2), \overline{u} = \Phi(\overline{z_2}), v = \Phi(z_1), \overline{v} = \Phi(\overline{z_1})$. Then $f = u - \overline{u}$ and $g = v - \overline{v}$ solve the Cauchy problem

$$\begin{cases}
f_{tt} - \Delta_{\mathbb{H}} f + f_t = |z_2|^p - |\overline{z_2}|^p, & t > 0, \eta \in \mathbb{H}^n, \\
g_{tt} - \Delta_{\mathbb{H}} g + g_t = |z_1|^q - |\overline{z_1}|^q, & t > 0, \eta \in \mathbb{H}^n, \\
u(0,\eta) = u_t(\eta) = 0,\ v(0,\eta) = v_t(\eta) = 0, & \eta \in \mathbb{H}^n.
\end{cases}$$

By means of (2.12) and (2.13) as well as the divergence theorem and integrating over the domain $[0,t] \times \mathbb{H}^n$, we arrive at

$$\xi_\psi[f](t) \le \int_0^t \int_{\mathbb{H}^n} e^{2\psi(s,\eta)} (|z_2(s,\eta)|^p - |\overline{z_2}(s,\eta)|^p) f_t(s,\eta) d\eta ds,$$

$$\xi_\psi[g](t) \leq \int_0^t \int_{\mathbb{H}^n} e^{2\psi(s,\eta)}(|z_1(s,\eta)|^q - |\overline{z_1}(s,\eta)|^q) g_t(s,\eta) d\eta ds.$$

Applying $||z_2|^p - |\overline{z_2}|^p| \leq p|z_2 - \overline{z_2}|(|z_2| + |\overline{z_2}|)^{p-1}$ and the Cauchy-Schwarz inequality, we derive

$$\xi_\psi[f](t) \lesssim \int_0^t \xi_\psi[f](s)^{\frac{1}{2}} (\int_{\mathbb{H}^n} e^{2\psi(s,\eta)} |z_2(s,\eta) - \overline{z_2}(s,\eta)|^2 (|z_2(s,\eta)| + |\overline{z_2}(s,\eta)|)^{2(p-1)} d\eta)^{\frac{1}{2}} ds.$$

Using Lemma 2.1, we obtain

$$\xi_\psi[f](t)^{\frac{1}{2}} \lesssim \int_0^t (\int_{\mathbb{H}^n} e^{2\psi(s,\eta)} |z_2(s,\eta) - \overline{z_2}(s,\eta)|^2 (|z_2(s,\eta)| + |\overline{z_2}(s,\eta)|)^{2(p-1)} d\eta)^{\frac{1}{2}} ds. \tag{2.21}$$

Making use of the Hölder inequality, we obtain

$$\begin{aligned}
&\|e^{\psi(s,\cdot)}|z_2(s,\cdot) - \overline{z_2}(s,\cdot)|(|z_2(s,\cdot)| + |\overline{z_2}(s,\cdot)|)^{p-1}\|_{L^2(\mathbb{H}^n)} \\
&\leq \|e^{(2-p)\psi(s,\cdot)}|z_2(s,\cdot) - \overline{z_2}(s,\cdot)|\|_{L^{2p}(\mathbb{H}^n)} \|e^{(p-1)\psi(s,\cdot)}(|z_2(s,\cdot)| + |\overline{z_2}(s,\cdot)|)^{p-1}\|_{L^{\frac{2p}{p-1}}(\mathbb{H}^n)}.
\end{aligned}$$

Taking Lemma 2.3 into account and the fact $\psi \geq 0$, we obtain

$$\|e^{(2-p)\psi(s,\cdot)}|z_2(s,\cdot) - \overline{z_2}(s,\cdot)|\|_{L^{2p}(\mathbb{H}^n)} \lesssim (1+s)^{(1-\theta(2p))/2} \|e^{\psi(s,\cdot)} \nabla_{\mathbb{H}}(z_2(s,\cdot) - \overline{z_2}(s,\cdot))\|_{L^2(\mathbb{H}^n)} \tag{2.22}$$

and

$$\begin{aligned}
&\|e^{(p-1)\psi(s,\cdot)}(|z_2(s,\cdot)| + |\overline{z_2}(s,\cdot)|)^{p-1}\|_{L^{\frac{2p}{p-1}}(\mathbb{H}^n)} \\
&\lesssim (\|e^{\psi(s,\cdot)} z_2(s,\cdot)\|_{L^{2p}(\mathbb{H}^n)} + \|e^{\psi(s,\cdot)} \overline{z_2}(s,\cdot)\|_{L^{2p}(\mathbb{H}^n)})^{p-1} \\
&\lesssim (1+s)^{(1-\theta(2p))(p-1)/2} (\|e^{\psi(s,\cdot)} \nabla_{\mathbb{H}} z_2(s,\cdot)\|_{L^2(\mathbb{H}^n)} + \|e^{\psi(s,\cdot)} \nabla_{\mathbb{H}} \overline{z_2}(s,\cdot)\|_{L^2(\mathbb{H}^n)})^{p-1}.
\end{aligned} \tag{2.23}$$

Taking advantage of (2.21)–(2.23), we have

$$\begin{aligned}
&\|e^{\psi(t,\cdot)} f_t(t,\cdot)\|_{L^2(\mathbb{H}^n)} + \|e^{\psi(t,\cdot)} \nabla_{\mathbb{H}} f(t,\cdot)\|_{L^2(\mathbb{H}^n)} \\
&\leq C_p \int_0^t (1+s)^{p(1-\theta(2p))/2} \|e^{\psi(s,\cdot)} \nabla_{\mathbb{H}}(z_2(s,\cdot) - \overline{z_2}(s,\cdot))\|_{L^2(\mathbb{H}^n)} \\
&\quad \times (\|e^{\psi(s,\cdot)} \nabla_{\mathbb{H}} z_2(s,\cdot)\|_{L^2(\mathbb{H}^n)} + \|e^{\psi(s,\cdot)} \nabla_{\mathbb{H}} \overline{z_2}(s,\cdot)\|_{L^2(\mathbb{H}^n)})^{p-1} ds \\
&\leq C_p T (1+T)^{p(1-\theta(2p))/2} K^{p-1} \|z_2 - \overline{z_2}\|_{T,\psi}.
\end{aligned} \tag{2.24}$$

Moreover, since

$$e^{\psi(t,\eta)} f(t,\eta) = \int_0^t e^{\psi(t,\eta)} f_t(s,\eta) ds$$

and ψ is decreasing with respect to t and (2.24), it follows that

$$\begin{aligned}
\|e^{\psi(t,\cdot)} f(t,\cdot)\|_{L^2(\mathbb{H}^n)} &\leq \int_0^t \|e^{\psi(s,\eta)} f_t(s,\eta)\|_{L^2(\mathbb{H}^n)} ds \\
&\leq C_p T^2 (1+T)^{p(1-\theta(2p))/2} K^{p-1} \|z_2 - \overline{z_2}\|_{T,\psi}.
\end{aligned} \tag{2.25}$$

By using (2.24) and (2.25), we get

$$\|\Phi(z_2) - \Phi(\overline{z_2})\|_{T,\psi} = \|f\|_{T,\psi} \le C_p T(1+T)^{1+p(1-\theta(2p))/2} K^{p-1} \|z_2 - \overline{z_2}\|_{T,\psi}.$$

Therefore, when $T > 0$ is small enough, we find that Φ is a contraction mapping. According to Banach's fixed point theorem, the problem (1.1) has unique solutions $(u, v) \in (C([0, T_{max}), H^1(\mathbb{H}^n)) \cap C^1([0, T_{max}),$ and $L^2(\mathbb{H}^n)))^2$. For $t \in [0, T_{max})$, the energy $E_\psi[u](t)$ corresponding to the solution u is finite. Moreover, $T_{max} < \infty$ implies that the energy will blow up as $T \to T_{max}^-$. Otherwise, in the left neighborhood of T_{max}, the energy of the solution u will still remain finite. When we take the time $t = 0$ as the initial condition and apply the same reasoning process again, we will be able to extend this solution. However, this contradicts with the maximality of T_{max}. This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

3.1. Preliminary

If u is a solution of the problem (1.1), since

$$e^{2\psi} u_t |v|^p = \frac{\partial}{\partial t}(e^{2\psi}|v|^p u) - 2\psi_t e^{2\psi}|v|^p u - e^{2\psi} p|v|^{p-1} u v_t,$$

from (2.5), (2.12), and $\psi_t \le 0$, we find

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big(\frac{e^{2\psi}}{2}(|u_t|^2 + |\nabla_{\mathbb{H}} u|^2) - e^{2\psi}|v|^p u\Big) \\
&= div(e^{2\psi} u_t \nabla_{\mathbb{H}} u) - \frac{e^{2\psi}}{\psi_t} u_t^2 (|\nabla_{\mathbb{H}}\psi|^2 + \psi_t) + \psi_t e^{2\psi} u_t^2 \\
&\quad + \frac{e^{2\psi}}{\psi_t} |u_t \nabla_{\mathbb{H}}\psi - \psi_t \nabla_{\mathbb{H}} u|^2 - 2\psi_t e^{2\psi}|v|^p u - e^{2\psi} p|v|^{p-1} u v_t \\
&\le div(e^{2\psi} u_t \nabla_{\mathbb{H}} u) - 2\psi_t e^{2\psi}|v|^p u. \qquad (3.1)
\end{aligned}$$

Lemma 3.1. *Let $n \ge 1$ and $1 < p, q \le \frac{Q}{Q-2}$. Assume that $(u_0, u_1), (v_0, v_1) \in A(\mathbb{H}^n)$. If (u, v) solve*

$$\begin{cases}
u_{tt} - \Delta_{\mathbb{H}} u + u_t = |v|^p, & t > 0, \eta \in \mathbb{H}^n, \\
v_{tt} - \Delta_{\mathbb{H}} v + v_t = |u|^q, & t > 0, \eta \in \mathbb{H}^n, \\
u(0,\eta) = u_0(\eta),\ v(0,\eta) = v_0(\eta), & \eta \in \mathbb{H}^n, \\
u_t(0,\eta) = u_1(\eta),\ v_t(0,\eta) = v_1(\eta), & \eta \in \mathbb{H}^n,
\end{cases}$$

then, for $t \in [0, T)$ and an arbitrarily small $\delta > 0$, the following energy estimates hold:

$$\begin{aligned}
\xi_\psi[u](t) \lesssim I_0^2 + I_0^{p+1} &+ \big(\sup_{s\in[0,t]} (1+s)^\delta \|e^{(\frac{2}{p+1}+\delta)\psi(s,\cdot)} u(s,\cdot)\|_{L^{p+1}(\mathbb{H}^n)}\big)^{p+1} \\
&+ J_0^{p+1} + \big(\sup_{s\in[0,t]} (1+s)^\delta \|e^{(\frac{2}{p+1}+\delta)\psi(s,\cdot)} v(s,\cdot)\|_{L^{p+1}(\mathbb{H}^n)}\big)^{p+1}, \qquad (3.2)
\end{aligned}$$

$$\xi_\psi[v](t) \lesssim J_0^2 + J_0^{p+1} + \big(\sup_{s\in[0,t]} (1+s)^\delta \|e^{(\frac{2}{p+1}+\delta)\psi(s,\cdot)} v(s,\cdot)\|_{L^{p+1}(\mathbb{H}^n)}\big)^{p+1}$$

$$+I_0^{p+1}+(\sup_{s\in[0,t]}(1+s)^{\delta}\|e^{(\frac{2}{p+1}+\delta)\psi(s,\cdot)}u(s,\cdot)\|_{L^{p+1}(\mathbb{H}^n)})^{p+1}, \tag{3.3}$$

where

$$I_0^2=\int_{\mathbb{H}^n}e^{2\psi(0,\eta)}(|u_1(\eta)|^2+|\nabla_{\mathbb{H}}u_0(\eta)|^2)d\eta,$$
$$J_0^2=\int_{\mathbb{H}^n}e^{2\psi(0,\eta)}(|v_1(\eta)|^2+|\nabla_{\mathbb{H}}v_0(\eta)|^2)d\eta.$$

Proof of Lemma 3.1. Integrating the relation (3.1) over $[0,t]\times\mathbb{H}^n$, we obtain

$$\zeta_\psi[u](t)\le\zeta_\psi[u](0)-2\int_0^t\int_{\mathbb{H}^n}\psi_t(s,\eta)e^{2\psi(s,\eta)}|v(s,\eta)|^pu(s,\eta)d\eta ds,$$

where

$$\zeta_\psi[u](t)=\xi_\psi[u](t)-\int_{\mathbb{H}^n}e^{2\psi(t,\eta)}|v(t,\eta)|^pu(t,\eta)d\eta.$$

Hence, we derive

$$\xi_\psi[u](t)\lesssim\zeta_\psi[u](0)+\|e^{\frac{2}{p+1}\psi(t,\cdot)}v(t,\cdot)^{\frac{p}{p+1}}u(t,\cdot)^{\frac{1}{p+1}}\|^{p+1}_{L^{p+1}(\mathbb{H}^n)}$$
$$+\int_0^t\int_{\mathbb{H}^n}|\psi_t(s,\eta)|e^{2\psi(s,\eta)}|v(s,\eta)|^pu(s,\eta)d\eta ds.$$

Direct computation gives rise to

$$\zeta_\psi[u](0)=\xi_\psi[u](0)-\int_{\mathbb{H}^n}e^{\psi(0,\eta)}|v_0(\eta)|^pu_0(\eta)d\eta$$
$$\lesssim I_0^2+\int_{\mathbb{H}^n}e^{\psi(0,\eta)}|v_0(\eta)|^pu_0(\eta)d\eta.$$

Because of $p+1<\frac{Q}{Q-2}+1<\frac{2Q}{Q-2}$, we have the Sobolev embedding

$$H^1(\mathbb{H}^n)\hookrightarrow L^{p+1}(\mathbb{H}^n).$$

Using the fact $(1+(|x|^2+|y|^2))e^{\frac{2}{p+1}\psi(0,\eta)}\lesssim e^{2\psi(0,\eta)}$, we achieve

$$\int_{\mathbb{H}^n}e^{\psi(0,\eta)}|v_0(\eta)|^pu_0(\eta)d\eta\lesssim\|e^{\frac{1}{p+1}\psi(0,\cdot)}v_0^{\frac{p}{p+1}}u_0^{\frac{1}{p+1}}\|^{p+1}_{H^1(\mathbb{H}^n)}$$
$$\lesssim\Big(\int_{\mathbb{H}^n}e^{\frac{2}{p+1}\psi(0,\eta)}(|v_0^{\frac{p}{p+1}}(\eta)u_0^{\frac{1}{p+1}}(\eta)|^2+|\nabla_{\mathbb{H}}(v_0^{\frac{p}{p+1}}(\eta)u_0^{\frac{1}{p+1}}(\eta))|^2$$
$$+(|x|^2+|y|^2)|v_0^{\frac{p}{p+1}}(\eta)u_0^{\frac{1}{p+1}}(\eta)|^2)d\eta\Big)^{\frac{p+1}{2}}$$
$$\lesssim J_0^{p+1}+I_0^{p+1}.$$

Consequently, we deduce that

$$\xi_\psi[u](t)\lesssim I_0^2+I_0^{p+1}+J_0^{p+1}+\|e^{\frac{2}{p+1}\psi(t,\cdot)}v(t,\cdot)^{\frac{p}{p+1}}u(t,\cdot)^{\frac{1}{p+1}}\|^{p+1}_{L^{p+1}(\mathbb{H}^n)}$$

$$+\int_0^t \int_{\mathbb{H}^n} |\psi_t(s,\eta)| e^{2\psi(s,\eta)} |v(s,\eta)|^p u(s,\eta) d\eta ds. \tag{3.4}$$

Similarly, we obtain

$$\xi_\psi[v](t) \lesssim J_0^2 + J_0^{p+1} + I_0^{p+1} + \| e^{\frac{2}{p+1}\psi(t,\cdot)} v(t,\cdot)^{\frac{p}{p+1}} u(t,\cdot)^{\frac{1}{p+1}} \|_{L^{p+1}(\mathbb{H}^n)}^{p+1}$$
$$+\int_0^t \int_{\mathbb{H}^n} |\psi_t(s,\eta)| e^{2\psi(s,\eta)} |u(s,\eta)|^p v(s,\eta) d\eta ds.$$

On the basis of the relation $\psi_t(s,\eta) = -(1+s)^{-1}\psi(s,\eta)$, we acquire

$$|\psi_t(s,\eta)| e^{(2-r(p+1))\psi(s,\eta)} = \frac{1}{1+s}\psi(s,\eta) e^{-\delta(p+1)\psi(s,\eta)} \lesssim (1+s)^{-1},$$

where $r = \frac{2}{p+1} + \delta\,(\delta > 0)$. Therefore

$$\begin{aligned} &\int_0^t \int_{\mathbb{H}^n} |\psi_t(s,\eta)| e^{2\psi(s,\eta)} |v(s,\eta)|^p u(s,\eta) d\eta ds \\ &\lesssim \int_0^t (1+s)^{-1} \int_{\mathbb{H}^n} e^{r(p+1)\psi(s,\eta)} |v(s,\eta)|^p u(s,\eta) d\eta ds \\ &\lesssim (\sup_{s\in[0,t]} (1+s)^\delta \| e^{r\psi(s,\cdot)} (v(s,\cdot))^{\frac{p}{p+1}} (u(s,\cdot))^{\frac{1}{p+1}} \|_{L^{p+1}(\mathbb{H}^n)})^{p+1}. \end{aligned} \tag{3.5}$$

Since $r > \frac{2}{p+1}$, we find

$$\begin{aligned} &\| e^{\frac{2}{p+1}\psi(t,\cdot)} (v(t,\cdot))^{\frac{p}{p+1}} (u(t,\cdot))^{\frac{1}{p+1}} \|_{L^{p+1}(\mathbb{H}^n)}^{p+1} \\ &\le ((1+t)^\delta \| e^{r\psi(s,\cdot)} (v(t,\cdot))^{\frac{p}{p+1}} (u(t,\cdot))^{\frac{1}{p+1}} \|_{L^{p+1}(\mathbb{H}^n)})^{p+1}. \end{aligned} \tag{3.6}$$

Combining (3.4), (3.5), and (3.6), we derive (3.2). Similarly, we get (3.3).

3.2. Proof of Theorem 1.2

By contradiction, let us suppose that for $\varepsilon_0 > 0$, some initial data satisfying (1.6) exist such that $(u,v) \in (C([0,T_{max}), H^1_{\psi(t,)}(\mathbb{H}^n)) \cap C^1([0,T_{max}), L^2_{\psi(t,)}(\mathbb{H}^n)))^2$ are the solutions of Problem (1.1), whose local existence is guaranteed by Theorem 1.1. This implies that $T_{max} < \infty$.

For $T \in (0, T_{\max})$, we can define the Banach space

$$X(T) = C([0,T], H^1_{\psi(t,)}(\mathbb{H}^n)) \cap C^1([0,T], L^2_{\psi(t,)}(\mathbb{H}^n)),$$

which is equipped with the norm

$$\begin{aligned} \|(u,v)\|_{X(T)} = \sup_{t\in[0,T]} [&\| e^{\psi(t,\cdot)} \nabla_{\mathbb{H}} (u,v)(t,\cdot)\|_{L^2(\mathbb{H}^n)} + (1+t)^{\frac{Q}{4}+\frac{1}{2}} \\ &\times \|\nabla_{\mathbb{H}}(u,v)(t,\cdot)\|_{L^2(\mathbb{H}^n)} + (1+t)^{\frac{Q}{4}+1} \|(u_t,v_t)(t,\cdot)\|_{L^2(\mathbb{H}^n)} \\ &+ \| e^{\psi(t,\cdot)} (u_t,v_t)(t,\cdot)\|_{L^2(\mathbb{H}^n)} + (1+t)^{\frac{Q}{4}} \|(u,v)(t,\cdot)\|_{L^2(\mathbb{H}^n)}]. \end{aligned}$$

Making use of Lemma 3.1, we acquire

$$\begin{aligned}&\|e^{\psi(t,\cdot)}u_t(t,\cdot)\|_{L^2(\mathbb{H}^n)}+\|e^{\psi(t,\cdot)}\nabla_{\mathbb{H}}u(t,\cdot)\|_{L^2(\mathbb{H}^n)}\\&\lesssim \varepsilon_0+\varepsilon_0^{\frac{p+1}{2}}+\varepsilon_0^{\frac{p+1}{2}}+(\sup_{s\in[0,t]}(1+s)^{\delta}\|e^{(\frac{2}{p+1}+\delta)\psi(s,\cdot)}u(s,\cdot)\|_{L^{p+1}(\mathbb{H}^n)})^{\frac{p+1}{2}}\\&\quad+(\sup_{s\in[0,t]}(1+s)^{\delta}\|e^{(\frac{2}{p+1}+\delta)\psi(s,\cdot)}v(s,\cdot)\|_{L^{p+1}(\mathbb{H}^n)})^{\frac{p+1}{2}}.\end{aligned}\tag{3.7}$$

Due to $p>1$ and $p+1<2p\leqslant\frac{2Q}{Q-2}$, we conclude that $\theta(p+1)\in(0,1]$. Moreover, we take $\delta>0$ to be sufficiently small that $\delta+\frac{2}{p+1}<1$. Applying Lemma 2.3, for $s\in[0,t]$, we get

$$\begin{aligned}&\|e^{(\frac{2}{p+1}+\delta)\psi(s,\cdot)}u(s,\cdot)\|_{L^{p+1}(\mathbb{H}^n)}\\&\lesssim(1+s)^{\frac{1}{2}(1-\theta(p+1))}\|\nabla_{\mathbb{H}}u(s,\cdot)\|_{L^2(\mathbb{H}^n)}^{1-(\delta+\frac{2}{p+1})}\|e^{\psi(s,\cdot)}\nabla_{\mathbb{H}}u(s,\cdot)\|_{L^2(\mathbb{H}^n)}^{\delta+\frac{2}{p+1}}\\&\lesssim(1+s)^{\frac{Q+1}{p+1}-\frac{Q}{2}+\delta(\frac{Q}{4}+\frac{1}{2})}\|u\|_{X(t)}.\end{aligned}$$

Analogously, we find

$$\|e^{(\frac{2}{p+1}+\delta)\psi(s,\cdot)}v(s,\cdot)\|_{L^{p+1}(\mathbb{H}^n)}\lesssim(1+s)^{\frac{Q+1}{p+1}-\frac{Q}{2}+\delta(\frac{Q}{4}+\frac{1}{2})}\|v\|_{X(t)}.$$

We assume that $p>p_{Fuj}(Q)$ such that

$$\frac{Q+1}{p+1}-\frac{Q}{2}+\delta(\frac{Q}{4}+\frac{1}{2}+1)<0.$$

Consequently, taking (3.7) into account, we arrive at

$$\begin{aligned}&\|e^{\psi(t,\cdot)}u_t(t,\cdot)\|_{L^2(\mathbb{H}^n)}+\|e^{\psi(t,\cdot)}\nabla_{\mathbb{H}}u(t,\cdot)\|_{L^2(\mathbb{H}^n)}\\&\lesssim\varepsilon_0+\varepsilon_0^{\frac{p+1}{2}}+\|u\|_{X(t)}^{\frac{p+1}{2}}+\|v\|_{X(t)}^{\frac{p+1}{2}}.\end{aligned}\tag{3.8}$$

Similarly, we obtain

$$\|e^{\psi(t,\cdot)}v_t(t,\cdot)\|_{L^2(\mathbb{H}^n)}+\|e^{\psi(t,\cdot)}\nabla_{\mathbb{H}}v(t,\cdot)\|_{L^2(\mathbb{H}^n)}\lesssim\varepsilon_0+\varepsilon_0^{\frac{p+1}{2}}+\|v\|_{X(t)}^{\frac{p+1}{2}}+\|u\|_{X(t)}^{\frac{p+1}{2}}.$$

Let us estimate the $L^2(\mathbb{H}^n)$ norm of the solutions (see [31]). Therefore, for $k+l=0,1$, we achieve

$$\begin{aligned}&\|\partial_t^l\nabla_{\mathbb{H}}^k u(t,\cdot)\|_{L^2(\mathbb{H}^n)}\\&\lesssim\varepsilon_0(1+t)^{-\frac{Q}{4}-\frac{k}{2}-l}+\int_{t/2}^t(1+t-s)^{-\frac{k}{2}-l}\|u(s,\cdot)\|_{L^{2p}(\mathbb{H}^n)}^p ds\\&\quad+\int_0^{t/2}(1+t-s)^{-\frac{Q}{4}-\frac{k}{2}-l}(\|u(s,\cdot)\|_{L^p(\mathbb{H}^n)}^p+\|u(s,\cdot)\|_{L^{2p}(\mathbb{H}^n)}^p)ds.\end{aligned}\tag{3.9}$$

For $\sigma=\delta p$, using (1.7) and (2.17) together with the definition of the norm $\|\cdot\|_{X(t)}$, we find

$$\|u(s,\cdot)\|_{L^p(\mathbb{H}^n)}^p\lesssim(1+s)^{\frac{Q}{4}}\|e^{\delta\psi(t,\cdot)}u(s,\cdot)\|_{L^{2p}(\mathbb{H}^n)}^p=(1+s)^{-\frac{Qp}{2}+\frac{Q}{2}+\delta p(\frac{Q}{4}+\frac{1}{2})}\|u\|_{X(t)}^p$$

and

$$\|u(s,\cdot)\|^p_{L^{2p}(\mathbb{H}^n)} \lesssim \|e^{\delta\psi(t,\cdot)}u(s,\cdot)\|^p_{L^{2p}(\mathbb{H}^n)} = (1+s)^{-\frac{Qp}{2}+\frac{Q}{4}+\delta p(\frac{Q}{4}+\frac{1}{2})}\|u\|^p_{X(t)}.$$

Since $p > p_{Fuj}(Q)$ is equivalent to $-\frac{Qp}{2}+\frac{Q}{2} < -1$, we obtain $\delta > 0$ such that

$$-\frac{Qp}{2}+\frac{Q}{2}+\delta p(\frac{Q}{4}+\frac{1}{2}) < -1. \tag{3.10}$$

Taking advantage of (3.10), for the integral over the interval $[0,\frac{t}{2}]$, we get

$$\begin{aligned}
&\int_0^{t/2}(1+t-s)^{-\frac{Q}{4}-\frac{k}{2}-l}(\|u(s,\cdot)\|^p_{L^p(\mathbb{H}^n)}+\|u(s,\cdot)\|^p_{L^{2p}(\mathbb{H}^n)})ds\\
&\lesssim (1+t)^{-\frac{Q}{4}-\frac{k}{2}-l}\int_0^{t/2}(1+s)^{-\frac{Qp}{2}+\frac{Q}{2}+\delta p(\frac{Q}{4}+\frac{1}{2})}ds\|u\|^p_{X(t)}\\
&\lesssim (1+t)^{-\frac{Q}{4}-\frac{k}{2}-l}\|u\|^p_{X(t)}.
\end{aligned}$$

Applying (3.10), for the integral over $[\frac{t}{2},t]$, we obtain

$$\int_{t/2}^{t}(1+t-s)^{-\frac{k}{2}-l}\|u(s,\cdot)\|^p_{L^{2p}(\mathbb{H}^n)}ds \lesssim (1+t)^{-\frac{Q}{4}-\frac{k}{2}-l}\|u\|^p_{X(t)}.$$

Using (3.9) gives rise to

$$(1+t)^{\frac{Q}{4}+\frac{k}{2}+l}\|\partial_t^l\nabla_{\mathbb{H}}^k u(t,\cdot)\|_{L^2(\mathbb{H}^n)} \lesssim \varepsilon_0+\|u\|^p_{X(t)}. \tag{3.11}$$

For $T\in(0,T_{\max})$, combining (3.8) and (3.11), we obtain

$$\|u\|_{X(T)} \lesssim \varepsilon_0+\varepsilon_0^{\frac{p+1}{2}}+\|u\|^{\frac{p+1}{2}}_{X(T)}+\|v\|^{\frac{p+1}{2}}_{X(T)}+\|u\|^p_{X(T)} \lesssim \varepsilon_0, \tag{3.12}$$

where $\varepsilon_0 > 0$ is small enough. Similarly, we find that $\|(u,v)\|_{X(T)}$ is uniformly bounded.

Using the fact $e^{\psi(t,\eta)}u(t,\eta) = e^{\psi(t,\eta)}u_0(\eta)+\int_0^t e^{\psi(t,\eta)}u_t(s,\eta)ds$, the monotonicity of ψ with respect to t and applying (3.12), we find

$$\|e^{\psi(t,\cdot)}u(t,\cdot)\|_{L^2(\mathbb{H}^n)} \lesssim \varepsilon_0+\int_0^t\|e^{\psi(s,\cdot)}u_t(s,\cdot)\|_{L^2(\mathbb{H}^n)}ds \lesssim \varepsilon_0(1+T).$$

If $T_{\max}<\infty$, it follows that

$$\lim_{T\to T_{\max}^-}\sup(\|e^{\psi(t,\cdot)}u(t,\cdot)\|_{L^2(\mathbb{H}^n)}+\|e^{\psi(t,\cdot)}\nabla_{\mathbb{H}}u(t,\cdot)\|_{L^2(\mathbb{H}^n)}\|e^{\psi(t,\cdot)}u_t(t,\cdot)\|_{L^2(\mathbb{H}^n)}) \lesssim \varepsilon_0(1+T)<\infty.$$

However, according to the proof of Theorem 1.1, this is impossible. Therefore $T_{\max}=\infty$, which means that u is a global solution. The decay estimates for u and its first-order derivatives can be derived from the relation (3.12), which holds uniformly with respect to T. This finishes the proof of Theorem 1.2.

4. Proof of Theorem 1.3

Let us start with the definition of weak solutions of Problem (1.1).

Definition 4.1. *Let* $(u, v) \in L^p_{loc}((0,T)\times\mathbb{H}^n)\times L^p_{loc}((0,T)\times\mathbb{H}^n)$ *be weak solutions of the Cauchy problem* (1.1) *on* $[0,T)\times\mathbb{H}^n$ *which satisfy*

$$\int_0^T\int_{\mathbb{H}^n}|v(t,\eta)|^p\varphi(t,\eta)d\eta dt+\int_{\mathbb{H}^n}(u_0(\eta)+u_1(\eta))\varphi(0,\eta)d\eta-\int_{\mathbb{H}^n}u_0(\eta)\partial_t\varphi(0,\eta)d\eta$$
$$=\int_0^T\int_{\mathbb{H}^n}u(t,\eta)(\partial_t^2\varphi(t,\eta)-\Delta_{\mathbb{H}}\varphi(t,\eta)-\partial_t\varphi(t,\eta))d\eta dt, \tag{4.1}$$

$$\int_0^T\int_{\mathbb{H}^n}|u(t,\eta)|^q\varphi(t,\eta)d\eta dt+\int_{\mathbb{H}^n}(v_0(\eta)+v_1(\eta))\varphi(0,\eta)d\eta-\int_{\mathbb{H}^n}v_0(\eta)\partial_t\varphi(0,\eta)d\eta$$
$$=\int_0^T\int_{\mathbb{H}^n}v(t,\eta)(\partial_t^2\varphi(t,\eta)-\Delta_{\mathbb{H}}\varphi(t,\eta)-\partial_t\varphi(t,\eta))d\eta dt \tag{4.2}$$

for $\varphi\in C_0^\infty([0,T)\times\mathbb{H}^n)$, φ *is a test function that vanishes near* $t=T$. *If* $T=\infty$, *we call* (u,v) *are global weak solutions to Problem* (1.1)*; otherwise, we say that* (u,v) *are local weak solutions.*

Proof of Theorem 1.3

The proof is provided by contradiction. Suppose that there are global weak solutions for Problem (1.1). Let $\alpha\in C_0^\infty(\mathbb{R}^n)$ and $\beta\in C_0^\infty(\mathbb{R})$ be two bump functions. In addition, we stipulate that α and β are radial symmetric and decreasing with respect to the radial variable. We have $\alpha=1$ on $B_n(\frac{1}{2})$ and $\beta=1$ on $[-\frac{1}{4},\frac{1}{4}]$, $\operatorname{supp}\alpha\subset B_n(1)$, and $\operatorname{supp}\beta\subset(-1,1)$. For a parameter $R>2$ and $(t,x,y,\tau)\in[0,\infty)\times\mathbb{R}^{2n+1}$, we define the test function $\varphi_R\in C_0^\infty([0,\infty)\times\mathbb{R}^{2n+1})$ with separate variables as follows:

$$\varphi_R(t,x,y,\tau)=\beta(\frac{t}{R^2})\alpha(\frac{x}{R})\alpha(\frac{y}{R})\beta(\frac{\tau}{R^2}). \tag{4.3}$$

The core reason for selecting (4.3) as the test function is that its structure is highly consistent with the topological and algebraic properties of the Heisenberg group, the nonlinear term, and the damping coupling characteristics of equation. The variables' separable form fits the component structure of the Heisenberg group. The scaling of the horizontal component is R^{-1} and the vertical component R^{-2} match the homogeneous dimension $Q=2n+2$ of the group. The test function has compact support, smoothness, and radial decreasing properties. We have

$$|\partial_j\alpha|\lesssim\alpha^{max\{\frac{1}{p},\frac{1}{q}\}}\ (1\le j\le n),$$
$$|\partial_j\partial_k\alpha|\lesssim\alpha^{max\{\frac{1}{p},\frac{1}{q}\}}\ (1\le j,k\le n),$$
$$|\beta'|\lesssim\beta^{max\{\frac{1}{p},\frac{1}{q}\}},\qquad |\beta''|\lesssim\beta^{max\{\frac{1}{p},\frac{1}{q}\}}.$$

In addition, $0\le\alpha,\beta\le1$ implies $\alpha\le\alpha^{max\{\frac{1}{p},\frac{1}{q}\}}$ and $\beta\le\beta^{max\{\frac{1}{p},\frac{1}{q}\}}$. Hence, from the relations

$$\partial_t\varphi_R(t,x,y,\tau)=R^{-2}\beta'(\frac{t}{R^2})\alpha(\frac{x}{R})\alpha(\frac{y}{R})\beta(\frac{\tau}{R^2}),$$
$$\partial_t^2\varphi_R(t,x,y,\tau)=R^{-4}\beta''(\frac{t}{R^2})\alpha(\frac{x}{R})\alpha(\frac{y}{R})\beta(\frac{\tau}{R^2}),$$

$$\begin{aligned}\Delta_{\mathbb{H}}\varphi_R(t,x,y,\tau) = {} & R^{-2}\beta(\frac{t}{R^2})\Delta\alpha(\frac{x}{R})\alpha(\frac{y}{R})\beta(\frac{\tau}{R^2}) + R^{-2}\beta(\frac{t}{R^2})\alpha(\frac{x}{R})\Delta\alpha(\frac{y}{R})\beta(\frac{\tau}{R^2}) \\ & + R^{-3}\sum_{j=1}^{n} x_j\beta(\frac{t}{R^2})\alpha(\frac{x}{R})\partial_j\alpha(\frac{y}{R})\beta'(\frac{\tau}{R^2}) \\ & - R^{-3}\sum_{j=1}^{n} y_j\beta(\frac{t}{R^2})\partial_j\alpha(\frac{x}{R})\alpha(\frac{y}{R})\beta'(\frac{\tau}{R^2}) \\ & + \frac{1}{4}R^{-4}(|x|^2+|y|^2)\beta(\frac{t}{R^2})\alpha(\frac{x}{R})\alpha(\frac{y}{R})\beta''(\frac{\tau}{R^2}),\end{aligned}$$

where Δ denotes the Laplace operator on $\mathbb{R}^n$, we find

$$\begin{aligned}&|\partial_t\varphi_R| \lesssim R^{-2}(\varphi_R)^{max\{\frac{1}{p},\frac{1}{q}\}}, \\ &|\partial_t^2\varphi_R| \lesssim R^{-4}(\varphi_R)^{max\{\frac{1}{p},\frac{1}{q}\}} \lesssim R^{-2}(\varphi_R)^{max\{\frac{1}{p},\frac{1}{q}\}}, \\ &|\Delta_{\mathbb{H}}\varphi_R| \lesssim R^{-2}(\varphi_R)^{max\{\frac{1}{p},\frac{1}{q}\}}.\end{aligned} \tag{4.4}$$

We have utilized the condition $\operatorname{supp}\varphi_R \subset [0,R^2]\times B^n(R)\times B^n(R)\times[-R^2,R^2]$ in order to estimate the polynomial terms in the estimation of $|\Delta_{\mathbb{H}}\varphi_R|$.

Combining (4.1) and (4.4), we derive

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{H}^n}|v(t,\eta)|^p\varphi_R(t,\eta)d\eta dt + \int_{\mathbb{H}^n}(u_0(\eta)+u_1(\eta))\varphi_R(0,\eta)d\eta \\ &\leq \int_0^\infty\int_{\mathbb{H}^n}|u(t,\eta)|(|\partial_t^2\varphi_R(t,\eta)| + |\Delta_{\mathbb{H}}\varphi_R(t,\eta)| + |\partial_t\varphi_R(t,\eta)|)d\eta dt \\ &\lesssim R^{-2}\int_0^\infty\int_{\mathbb{H}^n}|u(t,\eta)|(\varphi_R(t,\eta))^{max\{\frac{1}{p},\frac{1}{q}\}}d\eta dt \\ &\leq R^{-2}\left(\int_0^\infty\int_{\mathbb{H}^n}|u(t,\eta)|^q\varphi_R(t,\eta)d\eta dt\right)^{\frac{1}{q}}\left(\int_0^{R^2}\int_{D_R}\mathrm{d}\eta dt\right)^{\frac{1}{q'}}.\end{aligned} \tag{4.5}$$

Let

$$I_R = \int_0^\infty\int_{\mathbb{H}^n}|v(t,\eta)|^p\varphi_R(t,\eta)d\eta dt,\quad J_R = \int_{\mathbb{H}^n}(u_0(\eta)+u_1(\eta))\varphi_R(0,\eta)d\eta. \tag{4.6}$$

In fact, since $\operatorname{supp}\varphi_R(0,\cdot)\subset D_R$ and $\varphi_R(0,\cdot)=1$ on $D_{R/2}$, we obtain

$$J_R = \int_{D_R}(u_0(\eta)+u_1(\eta))\varphi_R(0,\eta)d\eta \geq \int_{D_{R/2}}(u_0(\eta)+u_1(\eta))d\eta.$$

Analogously, we have

$$\begin{aligned}M_R + N_R &= \int_0^\infty\int_{\mathbb{H}^n}|u(t,\eta)|^q\varphi_R(t,\eta)d\eta dt + \int_{\mathbb{H}^n}(v_0(\eta)+v_1(\eta))\varphi_R(0,\eta)d\eta \\ &\lesssim R^{-2}\left(\int_0^\infty\int_{\mathbb{H}^n}|v(t,\eta)|^p\varphi_R(t,\eta)d\eta dt\right)^{\frac{1}{p}}\left(\int_0^{R^2}\int_{D_R}\mathrm{d}\eta dt\right)^{\frac{1}{p'}}\end{aligned}$$

and

$$N_R = \int_{D_R} (v_0(\eta) + v_1(\eta))\varphi_R(0,\eta)d\eta \geq \int_{D_{R/2}} (v_0(\eta) + v_1(\eta))d\eta.$$

Because of the assumptions on initial data in (1.8), we derive $\lim_{R\to\infty} \inf J_R > 0$ and $\lim_{R\to\infty} \inf N_R > 0$. Consequently, it follows that $J_R > 0$ and $N_R > 0$ for $R \geq R_0$, where R_0 is a proper positive real valued number. Using the fact that $R \geq R_0$ and $\text{meas}(D_R) \approx R^Q$, we get

$$I_R \leq I_R + J_R \lesssim R^{-2+\frac{2n+4}{q'}} M_R^{\frac{1}{q}} = R^{Q-\frac{Q+2}{q}} M_R^{\frac{1}{q}}. \tag{4.7}$$

Similarly, we find

$$M_R \leq M_R + N_R \lesssim R^{Q-\frac{Q+2}{p}} I_R^{\frac{1}{p}}.$$

Thus, we arrive at

$$I_R \lesssim R^{Q-\frac{Q+2}{q}} R^{\frac{Q}{q}-\frac{Q+2}{pq}} I_R^{\frac{1}{pq}} = R^{Q-\frac{2}{q}-\frac{Q+2}{pq}} I_R^{\frac{1}{pq}} \leq R^{Q-\frac{Q+2}{p}} I_R^{\frac{1}{p}}.$$

For $p, q < p_{Fuj}(Q)$, we derive

$$0 \leq I_R^{1-\frac{1}{p}} \lesssim R^{Q-\frac{Q+2}{p}} \to 0 \qquad \text{as} \quad R \to \infty.$$

Analogously, we obtain

$$0 \leq M_R^{1-\frac{1}{q}} \lesssim R^{Q-\frac{Q+2}{q}} \to 0 \qquad \text{as} \quad R \to \infty.$$

Therefore, $\lim_{R\to\infty} I_R = 0$ and $\lim_{R\to\infty} M_R = 0$. We acquire a contradiction.

For $p, q = p_{Fuj}(Q)$, we utilize the fact that the support of $\partial_t\varphi_R$ is in $\widehat{P_R} = [\frac{R^4}{2}, R^2] \times D_R$ and the support of $\Delta_{\mathbb{H}}\varphi_R$ is in $\widetilde{P_R} = [0, R^2] \times (D_{1,R} \cup D_{2,R} \cup D_{3,R})$, where

$$\begin{aligned}
D_{1,R} &= (B_n(R) \setminus B_n(R/2)) \times B_n(R) \times [-R^2, R^2], \\
D_{2,R} &= B_n(R) \times (B_n(R) \setminus B_n(R/2)) \times [-R^2, R^2], \\
D_{3,R} &= B_n(R) \times (B_n(R)) \times ([-R^2, R^2] \setminus [-R^2/4, R^2/4]).
\end{aligned}$$

Thus, for $R \geq R_0$, we arrive at

$$I_R \leq I_R + J_R \lesssim \widehat{M}_R^{\frac{1}{q}} + \widetilde{M}_R^{\frac{1}{q}}, \quad M_R \leq M_R + N_R \lesssim \widehat{I}_R^{\frac{1}{p}} + \widetilde{I}_R^{\frac{1}{p}}, \tag{4.8}$$

where

$$\begin{aligned}
\widehat{I}_R &= \int_{\widehat{P_R}} |v(t,\eta)|^p \varphi_R(t,\eta)d\eta dt, \quad & \widehat{M}_R &= \int_{\widehat{P_R}} |u(t,\eta)|^q \varphi_R(t,\eta)d\eta dt, \\
\widetilde{I}_R &= \int_{\widetilde{P_R}} |v(t,\eta)|^p \varphi_R(t,\eta)d\eta dt, \quad & \widetilde{M}_R &= \int_{\widetilde{P_R}} |u(t,\eta)|^q \varphi_R(t,\eta)d\eta dt.
\end{aligned}$$

It follows from (4.8) that M_R is uniformly bounded as $R \to \infty$. By using the Lebesgue monotone convergence theorem, we then get

$$\lim_{R\to\infty} M_R = \lim_{R\to\infty} \int_0^\infty \int_{\mathbb{H}^n} |u(t,\eta)|^q \varphi_R(t,\eta) d\eta dt = \int_0^\infty \int_{\mathbb{H}^n} |u(t,\eta)|^q d\eta dt \lesssim 1.$$

This means that $u \in L^q([0,\infty) \times \mathbb{H}^n)$. Now, applying the Lebesgue dominated convergence theorem, the characteristic functions of the sets $\widehat{P_R}$ and $\widetilde{P_R}$ tend to the zero function as $R \to \infty$. We find

$$\lim_{R\to\infty} \widehat{I_R} = \lim_{R\to\infty} \int_{\widehat{P_R}} |v(t,\eta)|^p \varphi_R(t,\eta) d\eta dt = 0,$$
$$\lim_{R\to\infty} \widetilde{I_R} = \lim_{R\to\infty} \int_{\widetilde{P_R}} |v(t,\eta)|^p \varphi_R(t,\eta) d\eta dt = 0.$$

Letting $R \to \infty$, using (4.8) leads to $\lim\limits_{R\to\infty} M_R = 0$. Similarly, we get $\lim\limits_{R\to\infty} I_R = 0$. This yields a contradiction. The proof of Theorem 1.3 is finished.

5. Proof of Theorem 1.4

Here, we present a lemma which plays an important role in the proof.

Lemma 5.1. *[33] If $g = g(s)$ is a measurable function satisfying the property that g is supported in $[\frac{1}{2}, 1]$, and $g(s)$ is a decreasing function for $s > \frac{1}{2}$, then for $R > 2, A > 0$, we have*

$$\int_0^R g(\frac{A}{r^2}) \frac{dr}{r} \le \frac{\log 2}{2} g(\frac{A}{R^2}). \tag{5.1}$$

Proof of Theorem 1.4

If $T(\varepsilon) \le 2R_0$, for an arbitrarily sufficiently small positive constant ε, the estimate (1.9) holds. Let $2R_0 \le T(\varepsilon)$ and $\phi \in C_0^\infty([0,\infty))$ be a decreasing bump function such that $\phi = 1$ on $[0, \frac{1}{2}]$ and $\operatorname{supp} \phi \subset [0,1)$. Moreover, we write

$$\phi^*(r) = \begin{cases} 0, & r \in [0, \frac{1}{2}), \\ \phi(r), & r \in [\frac{1}{2}, \infty), \end{cases}$$

which is not smooth. We set the two test functions $\psi_R(t,\eta)$ and $\psi_R^*(t,\eta)$ which are used to derive the lifespan estimates of the solutions to Problem (1.1). Namely,

$$\psi_R(t,\eta) = [\phi(s_R(t,\eta))]^{2p_0'}, \quad \psi_R^*(t,\eta) = [\phi^*(s_R(t,\eta))]^{2p_0'}, \tag{5.2}$$

where the constant $R > 2$, $\frac{1}{p_0} = max\{\frac{1}{p}, \frac{1}{q}\}$ and

$$s_R(t,\eta) = \frac{t^2 + |x|^4 + |y|^4 + |\tau|^2}{R^2}, \qquad t \ge 0, \eta = (x,y,\tau) \in \mathbb{H}^n.$$

We see

$$\operatorname{supp} \psi_R \subset Q_R = [0,R] \times B^n(R^{\frac{1}{2}}) \times B^n(R^{\frac{1}{2}}) \times [-R,R].$$

Furthermore, we obtain the estimates

$$\begin{aligned}
|\partial_t\psi_R(t,\eta)| &\lesssim R^{-1}[\psi_R^*(t,\eta)]^{\frac{1}{p_0}},\\
|\partial_t^2\psi_R(t,\eta)| &\lesssim R^{-1}[\psi_R^*(t,\eta)]^{\frac{1}{p_0}},\\
|\Delta_{\mathbb{H}}\psi_R(t,\eta)| &\lesssim R^{-1}[\psi_R^*(t,\eta)]^{\frac{1}{p_0}}.
\end{aligned} \tag{5.3}$$

Hence, combined with (5.3), we find

$$\begin{aligned}
&\int_0^T\int_{\mathbb{H}^n}|v(t,\eta)|^p\psi_R(t,\eta)d\eta dt+\varepsilon\int_{\mathbb{H}^n}(u_0(\eta)+u_1(\eta))d\eta\\
&\lesssim R^{-1}\int_0^T\int_{\mathbb{H}^n}|u(t,\eta)|[\psi_R^*(t,\eta)]^{\frac{1}{p_0}}d\eta dt\\
&=R^{-\frac{q-1}{q}(\frac{1}{q-1}-\frac{Q}{2})}(\int_0^T\int_{\mathbb{H}^n}|u(t,\eta)|^q\psi_R^*(t,\eta)d\eta dt)^{\frac{1}{q}}.
\end{aligned}$$

Setting

$$\begin{aligned}
&X(r)=\int_0^T\int_{\mathbb{H}^n}|u(t,\eta)|^q\psi_r(t,\eta)d\eta dt,\ \ Y(r)=\int_0^T\int_{\mathbb{H}^n}|u(t,\eta)|^q\psi_r^*(t,\eta)d\eta dt,\\
&M(r)=\int_0^T\int_{\mathbb{H}^n}|v(t,\eta)|^p\psi_r(t,\eta)d\eta dt,\ \ N(r)=\int_0^T\int_{\mathbb{H}^n}|v(t,\eta)|^p\psi_r^*(t,\eta)d\eta dt
\end{aligned}$$

and using $\psi_R^*\le\psi_R$, we get

$$\begin{aligned}
X(R)+\varepsilon I[v]&\lesssim R^{-\frac{p-1}{p}(\frac{1}{p-1}-\frac{Q}{2})}M(R)^{\frac{1}{p}},\\
M(R)+\varepsilon I[u]&\lesssim R^{-\frac{q-1}{q}(\frac{1}{q-1}-\frac{Q}{2})}X(R)^{\frac{1}{q}},
\end{aligned}$$

where $I[v]=\int_{\mathbb{H}^n}(v_0(\eta)+v_1(\eta))d\eta$ and $I[u]=\int_{\mathbb{H}^n}(u_0(\eta)+u_1(\eta))d\eta$. This implies

$$X(R)+\varepsilon I[v]\lesssim R^{-\frac{p-1}{pq}(\frac{1}{p-1}-\frac{Q}{2})-\frac{q-1}{pq}(\frac{1}{q-1}-\frac{Q}{2})}X(R)^{\frac{1}{pq}}. \tag{5.4}$$

Applying Lemma 5.1 with $g=[\phi]^{2p'}$ and $A=s_1(t,\eta)$, we obtain

$$\frac{2}{\log 2}\int_0^R X(r)\frac{dr}{r}\le X(R). \tag{5.5}$$

Let

$$W(R)=\int_0^R X(r)\frac{dr}{r}.$$

Using the fact $RW'(R)=X(R)$ and combining (5.4) and (5.5), we get

$$\frac{2W(R)}{\log 2}+\varepsilon I[v]\lesssim R^{-\frac{p-1}{pq}(\frac{1}{p-1}-\frac{Q}{2})-\frac{q-1}{pq}(\frac{1}{q-1}-\frac{Q}{2})+\frac{1}{pq}}(W'(R))^{\frac{1}{pq}}. \tag{5.6}$$

Straightforward computations lead to

$$CR^{(p-1)(\frac{1}{p-1}-\frac{Q}{2})+(q-1)(\frac{1}{q-1}-\frac{Q}{2})-1}\le W'(R)(\varepsilon I[v]+\frac{2W(R)}{\log 2})^{-pq}, \tag{5.7}$$

where C is a positive constant which may change from line to line in the next estimate.

Integrating (5.7) over the interval $[2R_0, T(\varepsilon)]$, we find

$$\int_{2R_0}^{T} W'(R)(\varepsilon I[v] + \frac{2W(R)}{\log 2})^{-pq} dR \leq \frac{\log 2}{2(pq-1)}(\varepsilon I[v] + \frac{2W(2R_0)}{\log 2})^{1-pq} \lesssim \varepsilon^{1-pq} \tag{5.8}$$

and

$$\begin{aligned}&\int_{2R_0}^{T} R^{(p-1)(\frac{1}{p-1}-\frac{Q}{2})+(q-1)(\frac{1}{q-1}-\frac{Q}{2})-1} dR \\ &= \begin{cases} T^{(p-1)(\frac{1}{p-1}-\frac{Q}{2})+(q-1)(\frac{1}{q-1}-\frac{Q}{2})} - (2R_0)^{(p-1)(\frac{1}{p-1}-\frac{Q}{2})+(q-1)(\frac{1}{q-1}-\frac{Q}{2})}, & p, q \in (1, p_{Fuj}(Q)), \\ \log(\frac{T}{2R_0}), & p = q = p_{Fuj}(Q), \\ T^{(q-1)(\frac{1}{q-1}-\frac{Q}{2})} - (2R_0)^{(q-1)(\frac{1}{q-1}-\frac{Q}{2})}, & p = p_{Fuj}(Q), q \in (1, p_{Fuj}(Q)), \\ T^{(p-1)(\frac{1}{p-1}-\frac{Q}{2})} - (2R_0)^{(p-1)(\frac{1}{p-1}-\frac{Q}{2})}, & q = p_{Fuj}(Q), p \in (1, p_{Fuj}(Q)). \end{cases}\end{aligned}$$

It follows from some calculations that

$$T(\varepsilon) \leq \begin{cases} C\varepsilon^{-\frac{pq-1}{(p-1)(\frac{1}{p-1}-\frac{Q}{2})+(q-1)(\frac{1}{q-1}-\frac{Q}{2})}}, & p, q \in (1, p_{Fuj}(Q)), \\ \exp(C\varepsilon^{-(pq-1)}), & p = q = p_{Fuj}(Q), \\ C\varepsilon^{-\frac{pq-1}{(q-1)(\frac{1}{q-1}-\frac{Q}{2})}}, & p = p_{Fuj}(Q), q \in (1, p_{Fuj}(Q)), \\ C\varepsilon^{-\frac{pq-1}{(p-1)(\frac{1}{p-1}-\frac{Q}{2})}}, & q = p_{Fuj}(Q), p \in (1, p_{Fuj}(Q)). \end{cases}$$

The proof of Theorem 1.4 is finished.

Use of AI tools declaration

No artificial intelligence (AI) tools were used in the preparation of this work.

Acknowledgments

The author Sen Ming would like to express his sincere thank to Professor Han Yang for his helpful suggestions and discussions. The project is supported by the Fundamental Research Program of Shanxi Province (No. 202403021212254), the Education Science Research Project of Shanxi Provincial (No. SZH-230066), the Graduate Course Ideological and Political Demonstration Project of Shanxi Provincial (No. 2025SZ11), and the National Natural Science Foundation of China (No.11601446).

Conflict of interest

The authors declare that they have no competing interests.

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