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ERA, 34(4): 2462–2480.
DOI: 10.3934/era.2026113
Received: 29 December 2025
Revised: 11 February 2026
Accepted: 06 March 2026
Published: 23 March 2026

Research article

Nonlinear ξ-skew-Jordan triple higher derivations on prime $*$-algebras

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Abstract: Let $\mathfrak{B}$ denote a unital prime $*$-algebra over the complex field $\mathbb{C}$, and let $\xi \in \mathbb{C} \setminus \{0, \pm 1\}$. This paper characterized a family $\Psi = \{\varphi_m\}_{m\in\mathbb{N}}$ of maps (not necessarily additive) from $\mathfrak{B}$ into itself that satisfy the functional identity

$$\varphi_m(A \diamond_\xi B \diamond_\xi C) = \sum_{i+j+k=m} \varphi_i(A) \diamond_\xi \varphi_j(B) \diamond_\xi \varphi_k(C)$$

for all $A, B, C \in \mathfrak{B}$, where $A \diamond_\xi B = AB + \xi BA^*$. It was proved that Ψ satisfies the above identity if and only if Ψ is an additive higher $*$-derivation and $\varphi_m(\xi A) = \xi\varphi_m(A)$ holds for every $A \in \mathfrak{B}$. As an application, the result not only generalizes the structure of nonlinear ξ-skew-Jordan triple derivations on prime $*$-algebras, but also yields a description of nonlinear ξ-skew-Jordan triple higher derivations on several important operator algebras, including standard operator algebras and factor von Neumann algebras.

Keywords: nonlinear ξ-skew-Jordan triple higher derivation; additive higher $*$-derivation; unital $*$-algebras; standard operator algebras; von Neumann algebras

1. Introduction

Let $\mathcal{R}$ be a unital $*$-algebra over the complex field $\mathbb{C}$, whose involution $*$ satisfies $(xy)^* = y^*x^*$, $(x + y)^* = x^* + y^*$, and $((x)^*)^* = x$ for all $x, y \in \mathcal{R}$. We define the ξ-$*$-Jordan product by $A \diamond_\xi B = AB + \xi BA^*$ for all $A, B \in \mathcal{R}$. In the case where $\xi = 1$, this product reduces to $AB + BA^*$, which is known as the skew-Jordan product. Consequently, the general form $A \diamond_\xi B$ is also termed the ξ-skew-Jordan product. These products have recently garnered significant attention and are playing an increasingly important role in scientific research, as evidenced by [1–3].

An additive mapping $\delta_1 : \mathcal{R} \to \mathcal{R}$ is termed an additive $*$-derivation if it satisfies $\delta_1(y_1y_2) = \delta_1(y_1)y_2 + y_1\delta_1(y_2)$ and $\delta_1(x^*) = \delta_1(x)^*$. Without assuming additivity, a mapping $\delta_1 : \mathcal{R} \to \mathcal{R}$

is called a nonlinear ξ-skew-Jordan triple derivation provided that

$$\delta_1(y_1 \diamond_\xi y_2 \diamond_\xi y_3) = \delta_1(y_1) \diamond_\xi y_2 \diamond_\xi y_3 + y_1 \diamond_\xi \delta_1(y_2) \diamond_\xi y_3 + y_1 \diamond_\xi y_2 \diamond_\xi \delta_1(y_3) \tag{1.1}$$

for all $y_1, y_2, y_3 \in \mathcal{R}$. The case where $\xi = 1$ in (1.1) corresponds to a nonlinear skew Jordan triple derivation. The study of such maps defined via this product on $*$-algebras has attracted considerable scholarly interest, as seen in [4–6]. Research in this direction extends the Herstein Lie mapping research framework, which originated from [7].

In recent years, the study of mappings associated with the ξ-skew-Jordan product on $*$-algebras has attracted considerable attention, yielding numerous significant results (see, e.g., [8, 9]). A notable line of inquiry concerns maps preserving or deriving these products. For instance, Huo et al. [10] characterized such maps between von Neumann algebras, showing they are either linear or conjugate linear $*$-isomorphisms, depending on whether ξ is real. Subsequently, the authors of [11] investigated the structure of nonlinear ξ-skew-Jordan triple derivations δ_1, proving their additivity under the condition $|\xi| \notin \{0, -1\}$ and that δ_1 becomes an additive $*$-derivation if $\delta_1(I)$ is self-adjoint. However, their analysis did not cover the case where $|\xi| = 1$.

Building upon this, Zhang [9] extended the study to nonlinear ξ-skew-Jordan triple derivations on prime $*$-algebras. A key advancement in [9] was the establishment of additivity and the derivation property without the restrictive assumption $|\xi| \neq 1$, instead leveraging the existence of nontrivial idempotents. Importantly, the methodology developed in [9] is applicable to the context of [11], but not conversely. This observation directly inspires our current work, in which we employ mathematical induction, based on the framework of [9], to study the more general structure of nonlinear ξ-skew-Jordan triple higher derivations on prime $*$-algebras.

Higher derivations constitute an active research area in non-associative and non-commutative algebras (see [9, 12, 13] and the references therein). Let $\mathfrak{B}$ be a unital algebra over $\mathbb{C}$, and $\mathbb{N}$ the set of non-negative integers. A family $\Psi = \{\varphi_m\}_{m\in\mathbb{N}}$ of maps (without assuming additivity) on $\mathfrak{B}$, with $\varphi_0 = id_{\mathfrak{B}}$, is called:

(a) an additive higher $*$-derivation if for each $m \in \mathbb{N}$,

$$\varphi_m(y_1 y_2) = \sum_{i+j=m} \varphi_i(y_1)\varphi_j(y_2) \quad \text{and} \quad \varphi_m(y_1 + y_2) = \varphi_m(y_1) + \varphi_m(y_2)$$

for all $y_1, y_2 \in \mathfrak{B}$.

(b) a nonlinear ξ-skew-Jordan triple higher derivation if for each $m \in \mathbb{N}$,

$$\varphi_m(y_1 \diamond_\xi y_2 \diamond_\xi y_3) = \sum_{u+v+w=m} \varphi_u(y_1) \diamond_\xi \varphi_v(y_2) \diamond_\xi \varphi_w(y_3) \tag{1.2}$$

for all $y_1, y_2, y_3 \in \mathfrak{B}$ and for each $m \in \mathbb{N}$.

Note that if Ψ is such a higher derivation, then φ_1 is a nonlinear ξ-skew-Jordan triple derivation. While every higher ξ-derivation is a ξ-skew-Jordan triple higher derivation, the converse is not generally true. Given the result from [9] that every nonlinear ξ-skew-Jordan triple derivation on prime $*$-algebras is an additive $*$-derivation, a natural question arises: Under what conditions does a nonlinear ξ-skew-Jordan triple higher derivation become an additive higher $*$-derivation? This question is the primary motivation for our work.

The main objective of this paper is to characterize the structure of families $\Psi = \{\varphi_m\}_{m\in\mathbb{N}}$ satisfying (1.2) on a prime $*$-algebra $\mathfrak{B}$, thereby proving that they are additive higher $*$-derivations (see Theorem 2.1). As an immediate corollary, we obtain the structure of a single nonlinear ξ-skew-Jordan triple derivation on $\mathfrak{B}$.

Furthermore, we note that the proof techniques developed herein are also applicable to $*$-algebras $\mathcal{A}$ containing a nontrivial projection P_1 and a unit I satisfying the conditions: $X\mathcal{A}P_1 = 0$ implies $X = 0$, and $X\mathcal{A}(I - P_1) = 0$ implies $X = 0$. Consequently, our results extend to important classes of $*$-algebras, including standard operator algebras, factor von Neumann algebras, and von Neumann algebras of type I_1.

2. Main results

In this section, we investigate the structure of nonlinear ξ-skew-Jordan triple higher derivations on unital prime $*$-algebras. To facilitate our discussion, we first recall the notion of unital prime $*$-algebras and introduce some key notations.

Let $\mathfrak{B}$ be a unital prime $*$-algebra with identity I and containing a nontrivial projection e_1 (i.e., $e_1 \neq 0$, $e_1^2 = e_1 = e_1^*$). Define $e_2 = I - e_1$. The algebra $\mathfrak{B}$ is said to be prime if, for any $x, y \in \mathfrak{B}$, the condition $x\mathfrak{B}y = 0$ implies that $x = 0$ or $y = 0$.

Using the Peirce decomposition relative to the projection e_1, we may express $\mathfrak{B}$ as the direct sum

$$\mathfrak{B} = e_1\mathfrak{B}e_1 + e_1\mathfrak{B}e_2 + e_2\mathfrak{B}e_1 + e_2\mathfrak{B}e_2,$$

where we denote

$$\mathfrak{B}_{ij} = e_i\mathfrak{B}e_j \text{ for } i, j \in \{1, 2\}.$$

These components satisfy the following multiplicative rules:

(a) $\mathfrak{B}_{ij}\mathfrak{B}_{lk} = 0$ whenever $j \neq l$,
(b) $\mathfrak{B}_{ij}\mathfrak{B}_{jk} \subseteq \mathfrak{B}_{ik}$ for all $i, j, k \in \{1, 2\}$.

Here, we present the main conclusion of the paper as the following theorem.

Theorem 2.1. *Let $\mathfrak{B}$ be a unital prime $*$-algebra with identity element I and let $\xi \in \mathbb{C} \setminus \{0, -1\}$. Then $\Psi = \{\varphi_m\}_{m\in\mathbb{N}}$ is a nonlinear ξ-skew-Jordan triple higher derivation satisfying the equation*

$$\varphi_m(y_1 \diamond_\xi y_2 \diamond_\xi y_3) = \sum_{u+v+w=m} \varphi_u(y_1) \diamond_\xi \varphi_v(y_2) \diamond_\xi \varphi_w(y_3)$$

for all $y_1, y_2, y_3 \in \mathfrak{B}$ if and only if $\Psi = \{\varphi_m\}_{m\in\mathbb{N}}$ is an additive higher $*$-derivation and $\varphi_m(\xi y) = \xi\varphi_m(y)$ for all $y \in \mathfrak{B}$.

Note that the nonlinear ξ-skew-Jordan triple higher derivations degenerate into nonlinear ξ-skew-Jordan triple derivations for $m = 1$ in $\Psi = \{\varphi_m\}_{m\in\mathbb{N}}$, so with the aid of Theorem 2.1, we immediately obtain the following corollary.

Corollary 2.2. *[9] Let $\mathfrak{B}$ be a unital prime $*$-algebra with identity element I and let $\xi \in \mathbb{C} \setminus \{0, -1\}$. Then $\varphi_1 : \mathfrak{B} \to \mathfrak{B}$ is a nonlinear ξ-skew-Jordan triple derivation satisfying the equation*

$$\varphi_1(y_1 \diamond_\xi y_2 \diamond_\xi y_3) = \sum_{u+v+w=1} \varphi_u(y_1) \diamond_\xi \varphi_v(y_2) \diamond_\xi \varphi_w(y_3)$$

for all $y_1, y_2, y_3 \in \mathfrak{B}$ *if and only if* $\varphi_1 : \mathfrak{B} \to \mathfrak{B}$ *is an additive* $*$*-derivation and* $\varphi_1(\xi y) = \xi\varphi_1(y)$ *for all* $y \in \mathfrak{B}$.

In order to facilitate readers' understanding, we split it into two propositions.

Proposition 2.3. *Let* $\mathfrak{B}$ *be a unital prime* $*$*-algebra with identity element I, and let* $\xi \in \mathbb{C} \setminus \{ 0, -1 \}$. *Then* $\Psi = \{\varphi_m\}_{m\in\mathbb{N}}$ *is a nonlinear* ξ*-skew-Jordan triple higher derivation satisfying equation* (1.2) *if and only if* $\Psi = \{\varphi_m\}_{m\in\mathbb{N}}$ *is an additive mapping.*

The proof of Proposition 2.3 is based on the induction on m.

Proof. Clearly, we only need to prove the necessity. The proof will be realized via a series of claims.

Claim 1. $\varphi_m(0) = 0$.

Proceed by induction on $m \geq 1$. The case where $m = 1$ follows from [9, Claim 1]. For $m > 1$, it is clear that

$$\begin{aligned}
\varphi_m(0) &= \varphi_m(0 \diamond_\xi 0 \diamond_\xi 0)\\
&= \sum_{i+j+k=m} \varphi_i(0) \diamond_\xi \varphi_j(0) \diamond_\xi \varphi_k(0)\\
&= \varphi_m(0) \diamond_\xi 0 \diamond_\xi 0 + 0 \diamond_\xi \varphi_m(0) \diamond_\xi 0 + 0 \diamond_\xi 0 \diamond_\xi \varphi_m(0) + \sum_{\substack{i+j+k=m,\\ i,j,k<m}} \varphi_i(0) \diamond_\xi \varphi_j(0) \diamond_\xi \varphi_k(0)\\
&= 0.
\end{aligned}$$

Claim 2. $\varphi_m(a_{12} + a_{21}) = \varphi_m(a_{12}) + \varphi_m(a_{21})$ for every $a_{ij} \in \mathfrak{B}_{ij}$ with $i \neq j \in \{1, 2\}$.

Proceed by induction on $m \geq 1$. Define $J = \varphi_m(a_{12}+a_{21})-\varphi_m(a_{12})-\varphi_m(a_{21})$. For case where $m = 1$, the result holds by [9, Claim 2], i.e.,

$$\varphi_1(a_{12} + a_{21}) = \varphi_1(a_{12}) + \varphi_1(a_{21})$$

for every $a_{ij} \in \mathfrak{B}_{ij}$ with $i \neq j \in \{1, 2\}$.

Suppose that

$$\varphi_r(a_{12} + a_{21}) = \varphi_r(a_{12}) + \varphi_r(a_{21})$$

for all $1 \leq r < m$ and every $a_{ij} \in \mathfrak{B}_{ij}$ with $i \neq j \in \{1, 2\}$. Consider the expression

$$\varphi_m(I \diamond_\xi (\frac{e_1 - \frac{1}{\xi}e_2}{1 + \xi}) \diamond_\xi (a_{12} + a_{21})).$$

On one hand,

$$\begin{aligned}
&\varphi_m(I \diamond_\xi (\frac{e_1 - \frac{1}{\xi}e_2}{1 + \xi}) \diamond_\xi (a_{12} + a_{21}))\\
= \ &\varphi_m(I) \diamond_\xi (\frac{e_1 - \frac{1}{\xi}e_2}{1 + \xi}) \diamond_\xi (a_{12} + a_{21}) + I \diamond_\xi \varphi_m(\frac{e_1 - \frac{1}{\xi}e_2}{1 + \xi}) \diamond_\xi (a_{12} + a_{21}) + I \diamond_\xi (\frac{e_1 - \frac{1}{\xi}e_2}{1 + \xi}) \diamond_\xi \varphi_m(a_{12} + a_{21})\\
&+ \sum_{\substack{i+j+k=m,\\ i,j,k<m}} \varphi_i(I) \diamond_\xi \varphi_j(\frac{e_1 - \frac{1}{\xi}e_2}{1 + \xi}) \diamond_\xi \varphi_k(a_{12} + a_{21})\\
= \ &\sum_{i+j+k=m} \varphi_i(I) \diamond_\xi \varphi_j(\frac{e_1 - \frac{1}{\xi}e_2}{1 + \xi}) \diamond_\xi \varphi_k(a_{12} + a_{21}).
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
&\varphi_m(I \diamond_\xi (\frac{e_1 - \frac{1}{\xi}e_2}{1+\xi}) \diamond_\xi (a_{12} + a_{21}))\\
=\ &\varphi_m(I \diamond_\xi (\frac{e_1 - \frac{1}{\xi}e_2}{1+\xi}) \diamond_\xi a_{12}) + \varphi_m(I \diamond_\xi (\frac{e_1 - \frac{1}{\xi}e_2}{1+\xi}) \diamond_\xi a_{21})\\
=\ &\sum_{i+j+k=m} \varphi_i(I) \diamond_\xi \varphi_j(\frac{e_1 - \frac{1}{\xi}e_2}{1+\xi}) \diamond_\xi \varphi_k(a_{12}) + \sum_{i+j+k=m} \varphi_i(I) \diamond_\xi \varphi_j(\frac{e_1 - \frac{1}{\xi}e_2}{1+\xi}) \diamond_\xi \varphi_k(a_{21})\\
=\ &\varphi_m(I) \diamond_\xi (\frac{e_1 - \frac{1}{\xi}e_2}{1+\xi}) \diamond_\xi (a_{12} + a_{21}) + I \diamond_\xi \varphi_m(\frac{e_1 - \frac{1}{\xi}e_2}{1+\xi}) \diamond_\xi (a_{12} + a_{21}) + I \diamond_\xi (\frac{e_1 - \frac{1}{\xi}e_2}{1+\xi}) \diamond_\xi (\varphi_m(a_{12}) + \varphi_m(a_{21}))\\
&+ \sum_{\substack{i+j+k=m,\\ i,j,k<m}} \varphi_i(I) \diamond_\xi \varphi_j(\frac{e_1 - \frac{1}{\xi}e_2}{1+\xi}) \diamond_\xi \varphi_k(a_{12} + a_{21}).
\end{aligned}
$$

Comparing both expressions yields

$$
I \diamond_\xi (\frac{e_1 - \frac{1}{\xi}e_2}{1+\xi}) \diamond_\xi J = 0.
$$

This implies

$$
(1+\xi)J_{11} - (1+\overline{\xi})J_{22} + (\xi - \frac{1}{\overline{\xi}})J_{21} = 0,
$$

where $\overline{\xi}$ is the complex conjugate of the complex number ξ. Since $\xi \neq 0, -1$, we conclude that $J_{11} = J_{22} = 0$.

Using $a_{12} \diamond_\xi e_1 \diamond_\xi I = 0$, we can write

$$
\begin{aligned}
&\varphi_m((a_{12} + a_{21}) \diamond_\xi e_1 \diamond_\xi I)\\
=\ &\varphi_m(a_{12} + a_{21}) \diamond_\xi e_1 \diamond_\xi I + (a_{12} + a_{21}) \diamond_\xi \varphi_m(e_1) \diamond_\xi I + (a_{12} + a_{21}) \diamond_\xi e_1 \diamond_\xi \varphi_m(I)\\
&+ \sum_{\substack{i+j+k=m,\\ i,j,k<m}} \varphi_i(a_{12} + a_{21}) \diamond_\xi \varphi_j(e_1) \diamond_\xi \varphi_k(I)\\
=\ &\sum_{i+j+k=m} \varphi_i(a_{12} + a_{21}) \diamond_\xi \varphi_j(e_1) \diamond_\xi \varphi_k(I)
\end{aligned}
$$

and

$$
\begin{aligned}
&\varphi_m((a_{12} + a_{21}) \diamond_\xi e_1 \diamond_\xi I)\\
=\ &\varphi_m(a_{12} \diamond_\xi e_1 \diamond_\xi I) + \varphi_m(a_{21} \diamond_\xi e_1 \diamond_\xi I)\\
=\ &\sum_{i+j+k=m} \varphi_i(a_{12}) \diamond_\xi \varphi_j(e_1) \diamond_\xi \varphi_k(I) + \sum_{i+j+k=m} \varphi_i(a_{21}) \diamond_\xi \varphi_j(e_1) \diamond_\xi \varphi_k(I)\\
=\ &(\varphi_m(a_{12}) + \varphi_m(a_{21})) \diamond_\xi e_1 \diamond_\xi I + (a_{12} + a_{21}) \diamond_\xi \varphi_m(e_1) \diamond_\xi I + (a_{12} + a_{21}) \diamond_\xi e_1 \diamond_\xi \varphi_m(I)\\
&+ \sum_{\substack{i+j+k=m,\\ i,j,k<m}} (\varphi_i(a_{12}) + \varphi_i(a_{21})) \diamond_\xi \varphi_j(e_1) \diamond_\xi \varphi_k(I).
\end{aligned}
$$

Then we get

$$J \diamond_\xi e_1 \diamond_\xi I = 0.$$

We can arrive at $(1 + |\xi|^2)J_{21} + 2\xi J_{21}^* = 0$. So $J_{21} = 0$. Using the same calculation technique, $J_{12} = 0$ holds. To sum up, $J = 0$.

Claim 3. Let $p, q, l \in \{1, 2\}$ with $p \neq q$, and then $\varphi_m(a_{ll} + a_{pq}) = \varphi_m(a_{ll}) + \varphi_m(a_{pq})$ for all $a_{ll} \in \mathfrak{B}_{ll}$ and $a_{pq} \in \mathfrak{B}_{pq}$.

Indeed, we will use induction on m. Let us first consider the case of $m = 1$. By the proof of [9, Claim 3], we know that

$$\varphi_1(a_{ll} + a_{pq}) = \varphi_1(a_{ll}) + \varphi_1(a_{pq}),$$

for all $a_{ll} \in \mathfrak{B}_{ll}$ and $a_{pq} \in \mathfrak{B}_{pq}$, where $p, q, l \in \{1, 2\}$ with $p \neq q$.

For $1 \leq r < m$, suppose that

$$\varphi_r(a_{ll} + a_{pq}) = \varphi_r(a_{ll}) + \varphi_r(a_{pq}),$$

for all $a_{ll} \in \mathfrak{B}_{ll}$ and $a_{pq} \in \mathfrak{B}_{pq}$, where $p, q, l \in \{1, 2\}$ with $p \neq q$. Now, we only prove $p = l = 1$ and $q = 2$. The other cases can be obtained by similar calculation techniques. We introduce the notation $J = \varphi_m(a_{11} + a_{12}) - \varphi_m(a_{11}) - \varphi_m(a_{12})$ for all $a_{11} \in \mathfrak{B}_{11}$ and $a_{12} \in \mathfrak{B}_{12}$.

Since $I \diamond_\xi \frac{e_2}{1+\xi} \diamond_\xi a_{11} = 0$, in combination with Claim 1, we have

$$\begin{aligned}
&\varphi_m(I) \diamond_\xi \frac{e_2}{1+\xi} \diamond_\xi (a_{11} + a_{12}) + I \diamond_\xi \varphi_m(\frac{e_2}{1+\xi}) \diamond_\xi (a_{11} + a_{12}) + I \diamond_\xi \frac{e_2}{1+\xi} \diamond_\xi \varphi_m(a_{11} + a_{12}) \\
&\quad + \sum_{\substack{i+j+k=m,\\ i,j,k<m}} \varphi_i(I) \diamond_\xi \varphi_j(\frac{e_2}{1+\xi}) \diamond_\xi \varphi_k(a_{11} + a_{12}) \\
= &\sum_{i+j+k=m} \varphi_i(I) \diamond_\xi \varphi_j(\frac{e_2}{1+\xi}) \diamond_\xi \varphi_k(a_{11} + a_{12}) \\
= &\ \varphi_m(I \diamond_\xi \frac{e_2}{1+\xi} \diamond_\xi (a_{11} + a_{12})) \\
= &\ \varphi_m(I \diamond_\xi \frac{e_2}{1+\xi} \diamond_\xi a_{11}) + \varphi_m(I \diamond_\xi \frac{e_2}{1+\xi} \diamond_\xi a_{12}) \\
= &\sum_{i+j+k=m} \varphi_i(I) \diamond_\xi \varphi_j(\frac{e_2}{1+\xi}) \diamond_\xi \varphi_k(a_{11}) + \sum_{i+j+k=m} \varphi_i(I) \diamond_\xi \varphi_j(\frac{e_2}{1+\xi}) \diamond_\xi \varphi_k(a_{12}) \\
= &\ \varphi_m(I) \diamond_\xi \frac{e_2}{1+\xi} \diamond_\xi (a_{11} + a_{12}) + I \diamond_\xi \varphi_m(\frac{e_2}{1+\xi}) \diamond_\xi (a_{11} + a_{12}) + I \diamond_\xi \frac{e_2}{1+\xi} \diamond_\xi (\varphi_m(a_{11}) + \varphi_m(a_{12})) \\
&\quad + \sum_{\substack{i+j+k=m,\\ i,j,k<m}} \varphi_i(I) \diamond_\xi \varphi_j(\frac{e_2}{1+\xi}) \diamond_\xi \varphi_k(a_{11} + a_{12}).
\end{aligned}$$

Then we get

$$I \diamond_\xi \frac{e_2}{1+\xi} \diamond_\xi J = 0.$$

We can obtain $J_{12} = J_{21} = J_{22} = 0$.

On the other hand, as maintained by $I \diamond_\xi \frac{e_1-\frac{1}{\xi}e_2}{1+\xi} \diamond_\xi a_{12} = 0$, we have

$$\begin{aligned}
&\varphi_m(I) \diamond_\xi \frac{e_1-\frac{1}{\xi}e_2}{1+\xi} \diamond_\xi (a_{11}+a_{12}) + I \diamond_\xi \varphi_m(\frac{e_1-\frac{1}{\xi}e_2}{1+\xi}) \diamond_\xi (a_{11}+a_{12}) + I \diamond_\xi \frac{e_1-\frac{1}{\xi}e_2}{1+\xi} \diamond_\xi \varphi_m(a_{11}+a_{12}) \\
&+ \sum_{\substack{i+j+k=m,\\ i,j,k<m}} \varphi_i(I) \diamond_\xi \varphi_j(\frac{e_1-\frac{1}{\xi}e_2}{1+\xi}) \diamond_\xi \varphi_k(a_{11}+a_{12}) \\
= &\sum_{i+j+k=m} \varphi_i(I) \diamond_\xi \varphi_j(\frac{e_1-\frac{1}{\xi}e_2}{1+\xi}) \diamond_\xi \varphi_k(a_{11}+a_{12}) \\
= &\varphi_m(I \diamond_\xi \frac{e_1-\frac{1}{\xi}e_2}{1+\xi} \diamond_\xi (a_{11}+a_{12})) \\
= &\varphi_m(I \diamond_\xi \frac{e_1-\frac{1}{\xi}e_2}{1+\xi} \diamond_\xi a_{11}) + \varphi_m(I \diamond_\xi \frac{e_1-\frac{1}{\xi}e_2}{1+\xi} \diamond_\xi a_{12}) \\
= &\sum_{i+j+k=m} \varphi_i(I) \diamond_\xi \varphi_j(\frac{e_1-\frac{1}{\xi}e_2}{1+\xi}) \diamond_\xi \varphi_k(a_{11}) + \sum_{i+j+k=m} \varphi_i(I) \diamond_\xi \varphi_j(\frac{e_1-\frac{1}{\xi}e_2}{1+\xi}) \diamond_\xi \varphi_k(a_{12}) \\
= &\varphi_m(I) \diamond_\xi (\frac{(e_1-\frac{1}{\xi}e_2)}{1+\xi}) \diamond_\xi (a_{11}+a_{12}) + I \diamond_\xi \varphi_m(\frac{e_1-\frac{1}{\xi}e_2}{1+\xi}) \diamond_\xi (a_{11}+a_{12}) + I \diamond_\xi (\frac{e_1-\frac{1}{\xi}e_2}{1+\xi}) \diamond_\xi (\varphi_m(a_{11})+\varphi_m(a_{12})) \\
&+ \sum_{\substack{i+j+k=m,\\ i,j,k<m}} \varphi_i(I) \diamond_\xi \varphi_j(\frac{e_1-\frac{1}{\xi}e_2}{1+\xi}) \diamond_\xi \varphi_k(a_{11}+a_{12}),
\end{aligned}$$

which implies that

$$I \diamond_\xi \frac{e_1-\frac{1}{\xi}e_2}{1+\xi} \diamond_\xi J = 0.$$

Then we obtain $J_{11} = 0$. To sum up, $J = 0$.

Claim 4. For all $a_{ij} \in \mathfrak{B}_{ij}$ with $i, j \in \{1, 2\}$, we have

$$\begin{aligned}
\varphi_m(a_{11}+a_{12}+a_{21}) &= \varphi_m(a_{11}) + \varphi_m(a_{12}) + \varphi_m(a_{21}), \\
\varphi_m(a_{12}+a_{21}+a_{22}) &= \varphi_m(a_{12}) + \varphi_m(a_{21}) + \varphi_m(a_{22}).
\end{aligned}$$

We prove this claim by induction on m as well. In fact, in the case of $m = 1$, due to [9, Claim 4], we have $\varphi_1(a_{11}+a_{12}+a_{21}) = \varphi_1(a_{11}) + \varphi_1(a_{12}) + \varphi_1(a_{21})$.

Suppose that

$$\varphi_r(a_{11}+a_{12}+a_{21}) = \varphi_r(a_{11}) + \varphi_r(a_{12}) + \varphi_r(a_{21})$$

for all $1 \le r < m$. Let us simplify the proof by introducing the symbol $J = \varphi_m(a_{11}+a_{12}+a_{21}) -$

$\varphi_m(a_{11}) - \varphi_m(a_{12}) - \varphi_m(a_{21})$. In line with Claim 2, we have

$$\begin{aligned}
&\varphi_m(I) \diamond_\xi \frac{e_2}{1+\xi} \diamond_\xi (a_{11}+a_{12}+a_{21}) + I \diamond_\xi \varphi_m(\frac{e_2}{1+\xi}) \diamond_\xi (a_{11}+a_{12}+a_{21}) + I \diamond_\xi \frac{e_2}{1+\xi} \diamond_\xi \varphi_m(a_{11}+a_{12}+a_{21}) \\
&+ \sum_{\substack{i+j+k=m,\\ i,j,k<m}} \varphi_i(I) \diamond_\xi \varphi_j(\frac{e_2}{1+\xi}) \diamond_\xi \varphi_k(a_{11}+a_{12}+a_{21}) \\
=& \sum_{i+j+k=m} \varphi_i(I) \diamond_\xi \varphi_j(\frac{e_2}{1+\xi}) \diamond_\xi \varphi_k(a_{11}+a_{12}+a_{21}) \\
=& \varphi_m(I \diamond_\xi \frac{e_2}{1+\xi} \diamond_\xi (a_{11}+a_{12}+a_{21})) \\
=& \varphi_m(I \diamond_\xi \frac{e_2}{1+\xi} \diamond_\xi (a_{11}+a_{12})) + \varphi_m(I \diamond_\xi \frac{e_2}{1+\xi} \diamond_\xi a_{21}) \\
=& \sum_{i+j+k=m} \varphi_i(I) \diamond_\xi \varphi_j(\frac{e_2}{1+\xi}) \diamond_\xi \varphi_k(a_{11}+a_{12}) + \sum_{i+j+k=m} \varphi_i(I) \diamond_\xi \varphi_j(\frac{e_2}{1+\xi}) \diamond_\xi \varphi_k(a_{21}) \\
=& \varphi_m(I) \diamond_\xi (\frac{e_2}{1+\xi}) \diamond_\xi (a_{11}+a_{12}+a_{21}) + I \diamond_\xi \varphi_m(\frac{e_2}{1+\xi}) \diamond_\xi (a_{11}+a_{12}+a_{21}) + I \diamond_\xi (\frac{e_2}{1+\xi}) \diamond_\xi (\varphi_m(a_{11}) + \varphi_m(a_{12}) + \varphi_m(a_{21})) \\
&+ \sum_{\substack{i+j+k=m,\\ i,j,k<m}} \varphi_i(I) \diamond_\xi \varphi_j(\frac{e_2}{1+\xi}) \diamond_\xi (\varphi_k(a_{11}+a_{12}) + \varphi_k(a_{21})),
\end{aligned}$$

i.e.,

$$I \diamond_\xi \frac{e_2}{1+\xi} \diamond_\xi J = 0.$$

It follows from the last equation that $J_{12} = J_{21} = J_{22} = 0$.

On the other hand, as stated by $I \diamond_\xi \frac{-\frac{1}{\xi}e_1+e_2}{1+\xi} \diamond_\xi a_{21} = 0$, we have

$$\begin{aligned}
&\varphi_m(I) \diamond_\xi \frac{-\frac{1}{\xi}e_1+e_2}{1+\xi} \diamond_\xi (a_{11}+a_{12}+a_{21}) + I \diamond_\xi \varphi_m(\frac{-\frac{1}{\xi}e_1+e_2}{1+\xi}) \diamond_\xi (a_{11}+a_{12}+a_{21}) + I \diamond_\xi \frac{-\frac{1}{\xi}e_1+e_2}{1+\xi} \diamond_\xi \varphi_m(a_{11}+a_{12}+a_{21}) \\
&+ \sum_{\substack{i+j+k=m,\\ i,j,k<m}} \varphi_i(I) \diamond_\xi \varphi_j(\frac{-\frac{1}{\xi}e_1+e_2}{1+\xi}) \diamond_\xi \varphi_k(a_{11}+a_{12}+a_{21}) \\
=& \sum_{i+j+k=m} \varphi_i(I) \diamond_\xi \varphi_j(\frac{-\frac{1}{\xi}e_1+e_2}{1+\xi}) \diamond_\xi \varphi_k(a_{11}+a_{12}+a_{21}) \\
=& \varphi_m(I \diamond_\xi \frac{-\frac{1}{\xi}e_1+e_2}{1+\xi} \diamond_\xi (a_{11}+a_{12}+a_{21})) \\
=& \varphi_m(I \diamond_\xi \frac{-\frac{1}{\xi}e_1+e_2}{1+\xi} \diamond_\xi (a_{11}+a_{12})) + \varphi_m(I \diamond_\xi \frac{-\frac{1}{\xi}e_1+e_2}{1+\xi} \diamond_\xi a_{21}) \\
=& \sum_{i+j+k=m} \varphi_i(I) \diamond_\xi \varphi_j(\frac{-\frac{1}{\xi}e_1+e_2}{1+\xi}) \diamond_\xi \varphi_k(a_{11}+a_{12}) + \sum_{i+j+k=m} \varphi_i(I) \diamond_\xi \varphi_j(\frac{-\frac{1}{\xi}e_1+e_2}{1+\xi}) \diamond_\xi \varphi_k(a_{21}) \\
=& \varphi_m(I) \diamond_\xi (\frac{-\frac{1}{\xi}e_1+e_2}{1+\xi}) \diamond_\xi (a_{11}+a_{12}+a_{21}) + I \diamond_\xi \varphi_m(\frac{-\frac{1}{\xi}e_1+e_2}{1+\xi}) \diamond_\xi (a_{11}+a_{12}+a_{21}) \\
&+ I \diamond_\xi (\frac{-\frac{1}{\xi}e_1+e_2}{1+\xi}) \diamond_\xi (\varphi_m(a_{11}) + \varphi_m(a_{12}) + \varphi_m(a_{21})) + \sum_{\substack{i+j+k=m,\\ i,j,k<m}} \varphi_i(I) \diamond_\xi \varphi_j(\frac{-\frac{1}{\xi}e_1+e_2}{1+\xi}) \diamond_\xi (\varphi_k(a_{11}+a_{12}) + \varphi_k(a_{21})),
\end{aligned}$$

i.e.,

$$I \diamond_\xi \frac{-\frac{1}{\xi}e_1 + e_2}{1+\xi} \diamond_\xi J = 0.$$

It follows from the last equation that $J_{11} = 0$. Therefore, $J = 0$, i.e., $\varphi_m(a_{11} + a_{12} + a_{21}) = \varphi_m(a_{11}) + \varphi_m(a_{12}) + \varphi_m(a_{21})$ for all $a_{11} \in \mathfrak{B}_{11}, a_{12} \in \mathfrak{B}_{12}, a_{21} \in \mathfrak{B}_{21}$.

As for the second conclusion, we can prove that it is also true by using similar computational techniques and technical means.

Claim 5. For all $a_{ij} \in \mathfrak{B}_{ij}$ such that $i, j \in \{1, 2\}$, we have

$$\varphi_m(a_{11} + a_{12} + a_{21} + a_{22}) = \varphi_m(a_{11}) + \varphi_m(a_{12}) + \varphi_m(a_{21}) + \varphi_m(a_{22}).$$

In fact, we also need to check its correctness by complete induction on m. In the case of $m = 1$, due to [9, Claim 5], we get

$$\varphi_1(a_{11} + a_{12} + a_{21} + a_{22}) = \varphi_1(a_{11}) + \varphi_1(a_{12}) + \varphi_1(a_{21}) + \varphi_1(a_{22}).$$

Suppose that

$$\varphi_r(a_{11} + a_{12} + a_{21} + a_{22}) = \varphi_r(a_{11}) + \varphi_r(a_{12}) + \varphi_r(a_{21}) + \varphi_r(a_{22})$$

for all $1 \leq r < m$.

Define $J = \varphi_m(a_{11} + a_{12} + a_{21} + a_{22}) - \varphi_m(a_{11}) - \varphi_m(a_{12}) - \varphi_m(a_{21}) - \varphi_m(a_{22})$ for all $a_{ij} \in \mathfrak{B}_{ij}$ such that $i, j \in \{1, 2\}$.

As believed by Claim 4, we pick up

$$\begin{aligned}
&\varphi_m(I) \diamond_\xi \frac{e_1}{1+\xi} \diamond_\xi (a_{11} + a_{12} + a_{21} + a_{22}) + I \diamond_\xi \varphi_m(\frac{e_1}{1+\xi}) \diamond_\xi (a_{11} + a_{12} + a_{21} + a_{22}) \\
&\quad + I \diamond_\xi \frac{e_1}{1+\xi} \diamond_\xi \varphi_m(a_{11} + a_{12} + a_{21} + a_{22}) + \sum_{\substack{i+j+k=m,\\ i,j,k<m}} \varphi_i(I) \diamond_\xi \varphi_j(\frac{e_1}{1+\xi}) \diamond_\xi \varphi_k(a_{11} + a_{12} + a_{21} + a_{22}) \\
= &\sum_{i+j+k=m} \varphi_i(I) \diamond_\xi \varphi_j(\frac{e_1}{1+\xi}) \diamond_\xi \varphi_k(a_{11} + a_{12} + a_{21} + a_{22}) \\
= &\varphi_m(I \diamond_\xi \frac{e_1}{1+\xi} \diamond_\xi (a_{11} + a_{12} + a_{21} + a_{22})) \\
= &\varphi_m(I \diamond_\xi \frac{e_1}{1+\xi} \diamond_\xi (a_{11} + a_{12} + a_{21})) + \varphi_m(I \diamond_\xi \frac{e_1}{1+\xi} \diamond_\xi a_{22}) \\
= &\sum_{i+j+k=m} \varphi_i(I) \diamond_\xi \varphi_j(\frac{e_1}{1+\xi}) \diamond_\xi \varphi_k(a_{11} + a_{12} + a_{21}) + \sum_{i+j+k=m} \varphi_i(I) \diamond_\xi \varphi_j(\frac{e_1}{1+\xi}) \diamond_\xi \varphi_k(a_{22}) \\
= &\varphi_m(I) \diamond_\xi \frac{e_1}{1+\xi} \diamond_\xi (a_{11} + a_{12} + a_{21} + a_{22}) + I \diamond_\xi \varphi_m(\frac{e_1}{1+\xi}) \diamond_\xi (a_{11} + a_{12} + a_{21} + a_{22}) \\
&\quad + I \diamond_\xi \frac{e_1}{1+\xi} \diamond_\xi (\varphi_m(a_{11}) + \varphi_m(a_{12}) + \varphi_m(a_{21}) + \varphi_m(a_{22})) \\
&\quad + \sum_{\substack{i+j+k=m,\\ i,j,k<m}} \varphi_i(I) \diamond_\xi \varphi_j(\frac{e_1}{1+\xi}) \diamond_\xi (\varphi_k(a_{11}) + \varphi_k(a_{12}) + \varphi_k(a_{21}) + \varphi_k(a_{22})),
\end{aligned}$$

i.e.,

$$I \diamond_\xi \frac{e_1}{1+\xi} \diamond_\xi J = 0.$$

It follows from the above last equation that $J_{11} = J_{12} = J_{21} = 0$.

Using $\frac{e_2}{1+\xi}$ instead of $\frac{e_1}{1+\xi}$ in the above calculation, combined with the prime property of the algebra $\mathfrak{B}$, combined with a similar calculation technique, it is found that $J_{22} = 0$ is true.

From the above calculation, it can be concluded that $J = 0$, i.e., $\varphi_m(a_{11} + a_{12} + a_{21} + a_{22}) = \varphi_m(a_{11}) + \varphi_m(a_{12}) + \varphi_m(a_{21}) + \varphi_m(a_{22})$.

Claim 6. For all $a_{ij}, b_{ij} \in \mathfrak{B}_{ij}$ with $i \neq j \in \{1, 2\}$, we have

$$\varphi_m(a_{ij} + b_{ij}) = \varphi_m(a_{ij}) + \varphi_m(b_{ij}).$$

In fact, set $J = \varphi_m(a_{ij} + b_{ij}) - \varphi_m(a_{ij}) - \varphi_m(b_{ij})$ for all $a_{ij}, b_{ij} \in \mathfrak{B}_{ij}$ with $i \neq j \in \{1, 2\}$. We will use mathematical induction for m to prove that $J = 0$.

According to [9, Claim 6], it follows that the mapping φ_1 satisfies the relation

$$\varphi_1(a_{ij} + b_{ij}) = \varphi_1(a_{ij}) + \varphi_1(b_{ij})$$

for all $a_{ij}, b_{ij} \in \mathfrak{B}_{ij}$ with $i \neq j \in \{1, 2\}$, which is the basis of mathematical induction. Furthermore we assume that the mapping φ_r satisfying condition $1 \leq r < m$ coincides with equality

$$\varphi_r(a_{ij} + b_{ij}) = \varphi_r(a_{ij}) + \varphi_r(b_{ij})$$

for all $a_{ij}, b_{ij} \in \mathfrak{B}_{ij}$ with $i \neq j \in \{1, 2\}$.

According to $I \diamond_\xi \frac{e_i+a_{ij}}{1+\xi} \diamond_\xi (e_j + b_{ij}) = a_{ij} + b_{ij} + \xi a_{ij}^* + \xi b_{ij} a_{ij}^*$, and in combination with Claim 5, we can obtain

$$\begin{aligned}
& \varphi_m(a_{ij} + b_{ij}) + \varphi_m(\xi a_{ij}^*) + \varphi_m(\xi b_{ij} a_{ij}^*) \\
= \; & \varphi_m(I \diamond_\xi \frac{e_i + a_{ij}}{1 + \xi} \diamond_\xi (e_j + b_{ij})) \\
= \; & \sum_{i+j+k=m} \varphi_i(I) \diamond_\xi \varphi_j(\frac{e_i + a_{ij}}{1 + \xi}) \diamond_\xi \varphi_k(e_j + b_{ij}) \\
= \; & \varphi_m(I) \diamond_\xi \frac{e_i + a_{ij}}{1 + \xi} \diamond_\xi (e_j + b_{ij}) + I \diamond_\xi \varphi_m(\frac{e_i + a_{ij}}{1 + \xi}) \diamond_\xi (e_j + b_{ij}) + I \diamond_\xi \frac{e_i + a_{ij}}{1 + \xi} \diamond_\xi \varphi_m(e_j + b_{ij}) \\
& + \sum_{\substack{i+j+k=m, \\ i,j,k<m}} \varphi_i(I) \diamond_\xi \varphi_j(\frac{e_i + a_{ij}}{1 + \xi}) \diamond_\xi \varphi_k(e_j + b_{ij}) \\
= \; & \varphi_m(I) \diamond_\xi \frac{e_i + a_{ij}}{1 + \xi} \diamond_\xi (e_j + b_{ij}) + I \diamond_\xi (\varphi_m(\frac{e_i}{1 + \xi}) + \varphi_m(\frac{a_{ij}}{1 + \xi})) \diamond_\xi (e_j + b_{ij}) + I \diamond_\xi \frac{e_i + a_{ij}}{1 + \xi} \diamond_\xi (\varphi_m(e_j) + \varphi_m(b_{ij})) \\
& + \sum_{\substack{i+j+k=m, \\ i,j,k<m}} \varphi_i(I) \diamond_\xi (\varphi_j(\frac{e_i}{1 + \xi}) + \varphi_j(\frac{a_{ij}}{1 + \xi})) \diamond_\xi (\varphi_k(e_j) + \varphi_k(b_{ij})) \\
= \; & \varphi_m(I \diamond_\xi \frac{e_i}{1 + \xi} \diamond_\xi e_j) + \varphi_m(I \diamond_\xi \frac{e_i}{1 + \xi} \diamond_\xi b_{ij}) + \varphi_m(I \diamond_\xi \frac{a_{ij}}{1 + \xi} \diamond_\xi b_{ij}) + \varphi_m(I \diamond_\xi \frac{a_{ij}}{1 + \xi} \diamond_\xi e_j) \\
= \; & \varphi_m(a_{ij} + \xi a_{ij}^*) + \varphi_m(b_{ij}) + \varphi_m(\xi b_{ij} a_{ij}^*) \\
= \; & \varphi_m(a_{ij}) + \varphi_m(\xi a_{ij}^*) + \varphi_m(b_{ij}) + \varphi_m(\xi b_{ij} a_{ij}^*).
\end{aligned}$$

Therefore, it can be obtained that $\varphi_m(a_{ij}+b_{ij}) = \varphi_m(a_{ij})+\varphi_m(b_{ij})$ for all $a_{ij}, b_{ij} \in \mathfrak{B}_{ij}$ with $i \neq j \in \{1, 2\}$.

Claim 7. For all $a_{ii}, b_{ii} \in \mathfrak{B}_{ii}$ with $i \in \{1, 2\}$, we have

$$\varphi_m(a_{ii} + b_{ii}) = \varphi_m(a_{ii}) + \varphi_m(b_{ii}).$$

Indeed, we have set $J = \varphi_m(a_{ii} + b_{ii}) - \varphi_m(a_{ii}) - \varphi_m(b_{ii})$ for all $a_{ii}, b_{ii} \in \mathfrak{B}_{ii}$ with $i \in \{1, 2\}$.

We still use mathematical induction on index m to prove that $J = 0$. According to [9, Claim 7], the mapping φ_1 satisfies the equation

$$\varphi_1(a_{ii} + b_{ii}) = \varphi_1(a_{ii}) + \varphi_1(b_{ii})$$

for all $a_{ii}, b_{ii} \in \mathfrak{B}_{ii}$ with $i \in \{1, 2\}$, which provides the basis for induction.

Suppose that

$$\varphi_r(a_{ii} + b_{ii}) = \varphi_r(a_{ii}) + \varphi_r(b_{ii})$$

for all $1 \leq r < m$. Now let us prove that $J = 0$.

For $i \neq j$, we obtain

$$\begin{aligned}
&\varphi_m(e_j) \diamond_\xi \frac{e_j}{1+\xi} \diamond_\xi (a_{ii} + b_{ii}) + e_j \diamond_\xi \varphi_m(\frac{e_j}{1+\xi}) \diamond_\xi (a_{ii} + b_{ii}) + e_j \diamond_\xi \frac{e_j}{1+\xi} \diamond_\xi \varphi_m(a_{ii} + b_{ii}) \\
&\quad + \sum_{\substack{u+v+w=m,\\ u,v,w<m}} \varphi_u(e_j) \diamond_\xi \varphi_v(\frac{e_j}{1+\xi}) \diamond_\xi \varphi_w(a_{ii} + b_{ii}) \\
= &\sum_{u+v+w=m} \varphi_u(e_j) \diamond_\xi \varphi_v(\frac{e_j}{1+\xi}) \diamond_\xi \varphi_w(a_{ii} + b_{ii}) \\
= &\varphi_m(e_j \diamond_\xi \frac{e_j}{1+\xi} \diamond_\xi (a_{ii} + b_{ii})) \\
= &\varphi_m(e_j \diamond_\xi \frac{e_j}{1+\xi} \diamond_\xi a_{ii}) + \varphi_m(e_j \diamond_\xi \frac{e_j}{1+\xi} \diamond_\xi b_{ii}) \\
= &\sum_{u+v+w=m} \varphi_u(e_j) \diamond_\xi \varphi_v(\frac{e_j}{1+\xi}) \diamond_\xi \varphi_w(a_{ii}) + \sum_{u+v+w=m} \varphi_u(e_j) \diamond_\xi \varphi_v(\frac{e_j}{1+\xi}) \diamond_\xi \varphi_w(b_{ii}) \\
= &\varphi_m(e_j) \diamond_\xi \frac{e_j}{1+\xi} \diamond_\xi (a_{ii} + b_{ii}) + e_j \diamond_\xi \varphi_m(\frac{e_j}{1+\xi}) \diamond_\xi (a_{ii} + b_{ii}) + e_j \diamond_\xi (\frac{e_j}{1+\xi}) \diamond_\xi (\varphi_m(a_{ii}) + \varphi_m(b_{ii})) \\
&\quad + \sum_{\substack{u+v+w=m,\\ u,v,w<m}} \varphi_u(e_j) \diamond_\xi \varphi_v(\frac{e_j}{1+\xi}) \diamond_\xi (\varphi_w(a_{ii}) + \varphi_w(b_{ii})),
\end{aligned}$$

i.e.,

$$e_j \diamond_\xi \frac{e_j}{1+\xi} \diamond_\xi J = 0.$$

It follows from the above last equation that $J_{ij} = J_{ji} = J_{jj} = 0$.

Finally, we will prove that $J_{ii} = 0$. According to Claim 6, we have

$$\begin{aligned}
&\varphi_m(e_i) \diamond_\xi (a_{ii} + b_{ii}) \diamond_\xi c_{ij} + e_i \diamond_\xi \varphi_m(a_{ii} + b_{ii}) \diamond_\xi c_{ij} + e_i \diamond_\xi (a_{ii} + b_{ii}) \diamond_\xi \varphi_m(c_{ij}) + \sum_{\substack{u+v+w=m,\\ u,v,w<m}} \varphi_u(e_i) \diamond_\xi \varphi_v(a_{ii} + b_{ii}) \diamond_\xi \varphi_w(c_{ij}) \\
= &\sum_{u+v+w=m} \varphi_u(e_i) \diamond_\xi \varphi_v(a_{ii} + b_{ii}) \diamond_\xi \varphi_w(c_{ij}) \\
= &\varphi_m(e_i \diamond_\xi (a_{ii} + b_{ii}) \diamond_\xi c_{ij}) \\
= &\varphi_m(e_i \diamond_\xi a_{ii} \diamond_\xi c_{ij}) + \varphi_m(e_i \diamond_\xi b_{ii} \diamond_\xi c_{ij}) \\
= &\sum_{u+v+w=m} \varphi_u(e_i) \diamond_\xi \varphi_v(a_{ii}) \diamond_\xi \varphi_w(c_{ij}) + \sum_{u+v+w=m} \varphi_u(e_i) \diamond_\xi \varphi_v(b_{ii}) \diamond_\xi \varphi_w(c_{ij}) \\
= &\varphi_m(e_i) \diamond_\xi (a_{ii} + b_{ii}) \diamond_\xi c_{ij} + e_i \diamond_\xi (\varphi_m(a_{ii}) + \varphi_m(b_{ii})) \diamond_\xi c_{ij} + e_i \diamond_\xi (a_{ii} + b_{ii}) \diamond_\xi \varphi_m(c_{ij}) \\
&\quad + \sum_{\substack{u+v+w=m,\\ u,v,w<m}} \varphi_u(e_i) \diamond_\xi \varphi_v(a_{ii} + b_{ii}) \diamond_\xi \varphi_w(c_{ij}),
\end{aligned}$$

i.e.,

$$e_i \diamond_\xi J \diamond_\xi c_{ij} = 0.$$

It follows from the above last equation and prime property of algebra $\mathfrak{B}$ that $J_{ii} = 0$. To sum up, $J = 0$, i.e., $\varphi_m(a_{ii} + b_{ii}) = \varphi_m(a_{ii}) + \varphi_m(b_{ii})$ for all $a_{ii}, b_{ii} \in \mathfrak{B}_{ii}$ with $i \in \{1, 2\}$.

According to Claims 5–7, we can obtain that Claim 8 is established.

Claim 8. φ_m is additive, i.e., $\varphi_m(x + y) = \varphi_m(x) + \varphi_m(y)$ for all $x, y \in \mathfrak{B}$.

In fact, for any $x, y \in \mathfrak{B}$ that satisfy relations $x = a_{11} + a_{12} + a_{21} + a_{22}$ and $y = b_{11} + b_{12} + b_{21} + b_{22}$, respectively, we can obtain

$$\begin{aligned}
\varphi_m(x + y) &= \varphi_m(a_{11} + a_{12} + a_{21} + a_{22} + b_{11} + b_{12} + b_{21} + b_{22}) \\
&= \varphi_m(a_{11} + b_{11}) + \varphi_m(a_{12} + b_{12}) + \varphi_m(a_{21} + b_{21}) + \varphi_m(a_{22} + b_{22}) \\
&= \varphi_m(a_{11}) + \varphi_m(a_{12}) + \varphi_m(a_{21}) + \varphi_m(a_{22}) + \varphi_m(b_{12}) + \varphi_m(b_{11}) + \varphi_m(b_{21}) + \varphi_m(b_{22}) \\
&= \varphi_m(x) + \varphi_m(y).
\end{aligned}$$

To sum up, this proposition holds. □

Proposition 2.4. *Let $\mathfrak{B}$ be a unital prime $*$-algebra with identity element I. Let $\xi \in \mathbb{C} \setminus \{0, \pm 1\}$. Then $\Psi = \{\varphi_m\}_{m \in \mathbb{N}}$ is a nonlinear ξ-skew-Jordan triple higher derivation satisfying the equation*

$$\varphi_m(y_1 \diamond_\xi y_2 \diamond_\xi y_3) = \sum_{u+v+w=m} \varphi_u(y_1) \diamond_\xi \varphi_v(y_2) \diamond_\xi \varphi_w(y_3)$$

for all $y_1, y_2, y_3 \in \mathfrak{B}$ if and only if $\Psi = \{\varphi_m\}_{m \in \mathbb{N}}$ is an additive higher $*$-derivation and $\varphi_m(\xi y) = \xi \varphi_m(y)$ for all $y \in \mathfrak{B}$.

Proof. We only prove necessity, as sufficiency is straightforward. The additivity of φ_m has been established. We proceed by considering two cases.

Case 1. $|\xi| = 1$ and $\xi \neq 1, -1$.

We first show that $\varphi_m(I) = 0$ and $\varphi_m(\mathrm{i}I) = 0$, where i is the imaginary unit. The proof proceeds by complete induction on m. For $m = 1$, the result follows from [9, Case 1, Theorem 2.2], i.e., $\varphi_1(I) = 0$ and $\varphi_1(\mathrm{i}I) = 0$, where i is the imaginary unit.

Suppose that $\varphi_r(I) = 0$ and $\varphi_r(\mathrm{i}I) = 0$ for all $1 \leq r < m$.

In the following we show that $\varphi_m(I) = 0$ and $\varphi_m(\mathrm{i}I) = 0$.

Consider the relation $I \diamond_\xi \mathrm{i}I \diamond_\xi \mathrm{i}I = 0$, and we obtain

$$\begin{aligned}
0 &= \varphi_m(I \diamond_\xi \mathrm{i}I \diamond_\xi \mathrm{i}I) \\
&= \sum_{i+j+k=m} \varphi_i(I) \diamond_\xi \varphi_j(\mathrm{i}I) \diamond_\xi \varphi_k(\mathrm{i}I) \\
&= \varphi_m(I) \diamond_\xi \mathrm{i}I \diamond_\xi \mathrm{i}I + I \diamond_\xi \varphi_m(\mathrm{i}I) \diamond_\xi \mathrm{i}I + I \diamond_\xi \mathrm{i}I \diamond_\xi \varphi_m(\mathrm{i}I) \\
&= \mathrm{i}\varphi_m(\mathrm{i}I) + \xi \mathrm{i}\varphi_m(\mathrm{i}I) + \mathrm{i}\varphi_m(\mathrm{i}I)^* + \xi \mathrm{i}\varphi_m(\mathrm{i}I)^* \\
&= \mathrm{i}(1 + \xi)(\varphi_m(\mathrm{i}I) + \varphi_m(\mathrm{i}I)^*).
\end{aligned}$$

This implies $\varphi_m(\mathrm{i}I) = -\varphi_m(\mathrm{i}I)^*$.

On the other hand, note that $\mathrm{i}(1-\xi)I = \mathrm{i}I \diamond_\xi I \diamond_\xi I$, and we can get

$$\begin{aligned}
2\varphi_m(\mathrm{i}(1-\xi)I) &= \varphi_m(\mathrm{i}I \diamond_\xi I \diamond_\xi I)\\
&= \sum_{i+j+k=m} \varphi_i(\mathrm{i}I) \diamond_\xi \varphi_j(I) \diamond_\xi \varphi_k(I)\\
&= \varphi_m(\mathrm{i}I) \diamond_\xi I \diamond_\xi I + \mathrm{i}I \diamond_\xi \varphi_m(I) \diamond_\xi I + \mathrm{i}I \diamond_\xi I \diamond_\xi \varphi_m(I)\\
&= 2(1-\xi)\varphi_m(\mathrm{i}I) + 3\mathrm{i}(1-\xi)\varphi_m(I) + \mathrm{i}(1-\xi)\varphi_m(I)^*.
\end{aligned}$$

With the help of $2\mathrm{i}(1+\xi)I = I \diamond_\xi I \diamond_\xi \mathrm{i}I$, we have

$$\begin{aligned}
2\varphi_m(\mathrm{i}(1+\xi)I) &= \varphi_m(I \diamond_\xi I \diamond_\xi \mathrm{i}I)\\
&= \sum_{i+j+k=m} \varphi_i(I) \diamond_\xi \varphi_j(I) \diamond_\xi \varphi_k(\mathrm{i}I)\\
&= \varphi_m(I) \diamond_\xi I \diamond_\xi \mathrm{i}I + I \diamond_\xi \varphi_m(I) \diamond_\xi \mathrm{i}I + I \diamond_\xi I \diamond_\xi \varphi_m(\mathrm{i}I)\\
&= \mathrm{i}(3+\xi)\varphi_m(I) + \mathrm{i}(1+3\xi)\varphi_m(I)^* + 2\mathrm{i}(1+\xi)\varphi_m(\mathrm{i}I).
\end{aligned}$$

It follows from the above last two equations that

$$\varphi_m(I)^* = \frac{\xi-3}{\xi+1}\varphi_m(I).$$

By taking the conjugate of the above last equation, we obtain

$$\varphi_m(I) = \frac{\overline{\xi}-3}{\overline{\xi}+1}\varphi_m(I)^* = \frac{\overline{\xi}-3}{\overline{\xi}+1} \times \frac{\xi-3}{\xi+1}\varphi_m(I).$$

If $\varphi_m(I) \neq 0$, then $\xi = 1$, which is a contradiction. Thus

$$\varphi_m(I) = 0.$$

The next step is to prove that $\varphi_m(\mathrm{i}I) = 0$.

For all $a \in \mathfrak{B}$, with the help of $2(1+\xi)a = I \diamond_\xi I \diamond_\xi a$ and $\varphi_r(I) = 0$ for $1 \leq r \leq m$, we have

$$\begin{aligned}
2\varphi_m((1+\xi)a) &= \varphi_m(I \diamond_\xi I \diamond_\xi a)\\
&= \sum_{i+j+k=m} \varphi_i(I) \diamond_\xi \varphi_j(I) \diamond_\xi \varphi_k(a)\\
&= I \diamond_\xi I \diamond_\xi \varphi_m(a) = 2(1+\xi)\varphi_m(a),
\end{aligned}$$

i.e.,

$$\varphi_m((1+\xi)a) = (1+\xi)\varphi_m(a).$$

For all $a, b \in \mathfrak{B}$, with the help of $(1+\xi)(ab+ba^*) = I \diamond_\xi a \diamond_\xi b$, we have

$$\begin{aligned}
\varphi_m((1+\xi)(ab+ba^*)) &= \varphi_m(I \diamond_\xi a \diamond_\xi b)\\
&= \sum_{i+j+k=m} \varphi_i(I) \diamond_\xi \varphi_j(a) \diamond_\xi \varphi_k(b)\\
&= \sum_{j+k=m} I \diamond_\xi \varphi_j(a) \diamond_\xi \varphi_k(b)\\
&= (1+\xi) \sum_{j+k=m} (\varphi_j(a)\varphi_k(b) + \varphi_k(b)\varphi_j(a)^*).
\end{aligned}$$

This together with the equation $\varphi_m((1+\xi)a) = (1+\xi)\varphi_m(a)$ indicates that

$$\varphi_m(ab + ba^*) = \sum_{j+k=m} (\varphi_j(a)\varphi_k(b) + \varphi_k(b)\varphi_j(a)^*) \tag{2.1}$$

for all $a, b \in \mathfrak{B}$.

It follows from $\varphi_m(I) = 0$ and $2(a + \xi a^*) = a \diamond_\xi I \diamond_\xi I$ that

$$2\varphi_m(a) + 2\varphi_m(\xi a^*) = \varphi_m(a \diamond_\xi I \diamond_\xi I) = \varphi_m(a) \diamond_\xi I \diamond_\xi I = 2\varphi_m(a) + 2\xi\varphi_m(a)^*.$$

Then, we have

$$\varphi_m(\xi a^*) = \xi\varphi_m(a)^*.$$

With the help of the equation $\varphi_m((1+\xi)a) = (1+\xi)\varphi_m(a)$, we have $\varphi_m(\xi a^*) = \xi\varphi_m(a^*)$. Then

$$\varphi_m(a^*) = \varphi_m(a)^*. \tag{2.2}$$

According to $\varphi_m(I) = 0$ and $\varphi_m((1+\xi)a) = (1+\xi)\varphi_m(a)$, combining with the equation $2\varphi_m((\xi-1)I) = \varphi_m(\mathrm{i}I \diamond_\xi I \diamond_\xi \mathrm{i}I) = 4i(1-\xi)\varphi_m(\mathrm{i}I)$, we can get

$$\varphi_m(\mathrm{i}I) = 0.$$

For all $A = A^*$, with the help of (2.1) and (2.2), we have

$$2\varphi_m(\mathrm{i}A) = \varphi_m(A\mathrm{i} + \mathrm{i}A^*) = \sum_{j+k=m} (\varphi_j(A)\varphi_k(\mathrm{i}I) + \varphi_k(\mathrm{i}I)\varphi_j(A)^*) = 2\mathrm{i}\varphi_m(A),$$

i.e., $\varphi_m(\mathrm{i}A) = \mathrm{i}\varphi_m(A)$. Thus for all $B \in \mathfrak{B}$, since $B = B_1 + \mathrm{i}B_2$, where $B_1 = \frac{1}{2}(B+B^*)$ and $B_2 = \frac{1}{2\mathrm{i}}(B-B^*)$, we obtain

$$\begin{aligned} \varphi_m(\mathrm{i}B) &= \varphi_m(\mathrm{i}(B_1 + \mathrm{i}B_2)) = \varphi_m(\mathrm{i}B_1 - B_2)) \\ &= \mathrm{i}\varphi_m(B_1) - \varphi_m(B_2) = \mathrm{i}(\varphi_m(B_1) + \mathrm{i}\varphi_m(B_2)) \\ &= \mathrm{i}(\varphi_m(B_1) + \varphi_m(\mathrm{i}B_2)) = \mathrm{i}\varphi_m(B), \end{aligned}$$

i.e., $\varphi_m(\mathrm{i}B) = \mathrm{i}\varphi_m(B)$.

For all $a, b \in \mathfrak{B}$, from (2.1), we have

$$\begin{aligned} \varphi_m(-ab + ba^*) &= \varphi_m((\mathrm{i}a)(\mathrm{i}b) + (\mathrm{i}b)(\mathrm{i}a)^*) \\ &= \sum_{j+k=m} (\varphi_j(\mathrm{i}a)\varphi_k(\mathrm{i}b) + \varphi_k(\mathrm{i}b)\varphi_j(\mathrm{i}a)^*) \\ &= \sum_{j+k=m} (-\varphi_j(a)\varphi_k(b) + \varphi_k(b)\varphi_j(a)^*). \end{aligned}$$

It follows from the above last equation and (2.1) that

$$\varphi_m(ab) = \sum_{j+k=m} \varphi_j(a)\varphi_k(b).$$

Then $\varphi = \{\varphi_m\}_{m\in\mathbb{N}}$ is a higher $*$-derivation and $\varphi_m(\xi a) = \xi\varphi_m(a)$ for all $a \in \mathcal{B}$.

Case 2. $|\xi| \neq 1$.

We show that $\varphi_m(I) = 0$ and $\varphi_m(\mathrm{i}I) = 0$ by complete induction on m.

The base case $m = 1$ is given by [9, Case 2, Theorem 2.2], i.e.,

$$\varphi_1(I) = 0 \text{ and } \varphi_1(\mathrm{i}I) = 0.$$

Assume that

$$\varphi_r(I) = 0 \text{ and } \varphi_r(\mathrm{i}I) = 0$$

for all $1 \le r < m$.

It follows from the equation $I \diamond_\xi \mathrm{i}I \diamond_\xi \mathrm{i}I = \mathrm{i}I \diamond_\xi \mathrm{i}I \diamond_\xi I$ and the inductive hypothesis that $\varphi_m(I \diamond_\xi \mathrm{i}I \diamond_\xi \mathrm{i}I) = \varphi_m(\mathrm{i}I \diamond_\xi \mathrm{i}I \diamond_\xi I)$. Thus

$$\begin{aligned} & \varphi_m(I) \diamond_\xi \mathrm{i}I \diamond_\xi \mathrm{i}I + I \diamond_\xi \varphi_m(\mathrm{i}I) \diamond_\xi \mathrm{i}I + I \diamond_\xi \mathrm{i}I \diamond_\xi \varphi_m(\mathrm{i}I) \\ = \ & \varphi_m(\mathrm{i}I) \diamond_\xi \mathrm{i}I \diamond_\xi I + \mathrm{i}I \diamond_\xi \varphi_m(\mathrm{i}I) \diamond_\xi I + \mathrm{i}I \diamond_\xi \mathrm{i}I \diamond_\xi \varphi_m(I). \end{aligned}$$

Furthermore, we have

$$\begin{aligned} & 2\mathrm{i}\varphi_m(\mathrm{i}I) - |\xi|^2\mathrm{i}\varphi_m(\mathrm{i}I) + |\xi|^2\mathrm{i}\varphi_m(\mathrm{i}I)^* + \xi\mathrm{i}\varphi_m(\mathrm{i}I) + \xi\mathrm{i}\varphi_m(\mathrm{i}I)^* - \varphi_m(I) + |\xi|^2\varphi_m(I) \\ = \ & -\varphi_m(I) + |\xi|^2\varphi_m(I) + 2\mathrm{i}\varphi_m(\mathrm{i}I) - \xi\mathrm{i}\varphi_m(\mathrm{i}I) - \xi\mathrm{i}\varphi_m(\mathrm{i}I)^* - |\xi|^2\mathrm{i}\varphi_m(\mathrm{i}I) + |\xi|^2\mathrm{i}\varphi_m(\mathrm{i}I)^*. \end{aligned}$$

Then $2\xi\mathrm{i}(\varphi_m(\mathrm{i}I) + \varphi_m(\mathrm{i}I)^*) = 0$, which implies

$$\varphi_m(\mathrm{i}I) = -\varphi_m(\mathrm{i}I)^*. \tag{2.3}$$

On the other hand, it follows from $\mathrm{i}I \diamond_\xi \mathrm{i} \diamond_\xi \mathrm{i}I = \mathrm{i}I \diamond_\xi \mathrm{i} \diamond_\xi \mathrm{i}I$ that $\varphi_m(\mathrm{i}I \diamond_\xi \mathrm{i} \diamond_\xi \mathrm{i}I) = \varphi_m(\mathrm{i}I \diamond_\xi \mathrm{i} \diamond_\xi \mathrm{i}I)$. Thus

$$\begin{aligned} & \varphi_m(\mathrm{i}I) \diamond_\xi \mathrm{i} \diamond_\xi \mathrm{i}I + \mathrm{i}I \diamond_\xi \varphi_m(\mathrm{i}) \diamond_\xi \mathrm{i}I + \mathrm{i}I \diamond_\xi \mathrm{i} \diamond_\xi \varphi_m(\mathrm{i}I) \\ = \ & \varphi_m(\mathrm{i}I) \diamond_\xi \mathrm{i} \diamond_\xi \mathrm{i}I + \mathrm{i}I \diamond_\xi \varphi_m(\mathrm{i}) \diamond_\xi \mathrm{i}I + \mathrm{i}I \diamond_\xi \mathrm{i} \diamond_\xi \varphi_m(\mathrm{i}I). \end{aligned}$$

In line with (2.3), we obtain

$$(1 - |\xi|^2)\varphi_m(\mathrm{i}I) = (1 - |\xi|^2)\mathrm{i}\varphi_m(I),$$

which implies

$$\varphi_m(\mathrm{i}I) = \mathrm{i}\varphi_m(I).$$

Hence,

$$\varphi_m(I)^* = \varphi_m(I).$$

According to the above last equation and $\varphi_m(\mathrm{i}I) = \mathrm{i}\varphi_m(I)$, we have

$$\begin{aligned} & \varphi_m(\mathrm{i}(1 - 2\xi + |\xi|^2)I) \\ = \ & \varphi_m(\mathrm{i}I \diamond_\xi I \diamond_\xi I) \\ = \ & \sum_{i+j+k=m} \varphi_i(\mathrm{i}I) \diamond_\xi \varphi_j(I) \diamond_\xi \varphi_k(I) \\ = \ & \varphi_m(\mathrm{i}I) \diamond_\xi I \diamond_\xi I + \mathrm{i}I \diamond_\xi \varphi_m(I) \diamond_\xi I + \mathrm{i}I \diamond_\xi I \diamond_\xi \varphi_m(I) \\ = \ & \varphi_m(\mathrm{i}I) - 2\xi\varphi_m(\mathrm{i}I) + |\xi|^2\varphi_m(\mathrm{i}I) + \mathrm{i}\varphi_m(I) - 4\mathrm{i}\varphi_m(I) + 2\mathrm{i}|\xi|^2\varphi_m(I). \end{aligned}$$

On the other hand, in keeping with $\mathrm{i}(1+2\xi+|\xi|^2)I = I \diamond_\xi I \diamond_\xi \mathrm{i}I$, we have

$$\begin{aligned}
&\varphi_m(\mathrm{i}(1+2\xi+|\xi|^2)I)\\
=\ &\varphi_m(I \diamond_\xi I \diamond_\xi \mathrm{i}I)\\
=\ &\sum_{i+j+k=m} \varphi_i(I) \diamond_\xi \varphi_j(I) \diamond_\xi \varphi_k(\mathrm{i}I)\\
=\ &\varphi_m(I) \diamond_\xi I \diamond_\xi \mathrm{i}I + I \diamond_\xi \varphi_m(I) \diamond_\xi \mathrm{i}I + I \diamond_\xi I \diamond_\xi \varphi_m(\mathrm{i}I)\\
=\ &2\mathrm{i}\varphi_m(I) + 4\mathrm{i}\varphi_m(I) + 2\mathrm{i}|\xi|^2\varphi_m(I) + \mathrm{i}\varphi_m(I) + 2\xi\varphi_m(\mathrm{i}I) + |\xi|^2\varphi_m(\mathrm{i}I).
\end{aligned}$$

It follows from the above last equations that

$$\varphi_m(\mathrm{i}(1+|\xi|^2)I) = \varphi_m(\mathrm{i}I) + 2\mathrm{i}\varphi_m(I) + |\xi|^2\varphi_m(\mathrm{i}I) + 2\mathrm{i}|\xi|^2\varphi_m(I).$$

In compliance with $\varphi_m(\mathrm{i}I) = -\varphi_m(\mathrm{i}I)^*$ and $\varphi_m(I)^* = \varphi_m(I)$, we obtain

$$\begin{aligned}
&\varphi_m(\mathrm{i}(1-|\xi|^2)I)\\
=\ &\varphi_m(I \diamond_\xi \mathrm{i}I \diamond_\xi I)\\
=\ &\sum_{i+j+k=m} \varphi_i(I) \diamond_\xi \varphi_j(\mathrm{i}I) \diamond_\xi \varphi_k(I)\\
=\ &\varphi_m(I) \diamond_\xi \mathrm{i}I \diamond_\xi I + I \diamond_\xi \varphi_m(\mathrm{i}I) \diamond_\xi I + I \diamond_\xi \mathrm{i}I \diamond_\xi \varphi_m(I)\\
=\ &\mathrm{i}\varphi_m(I) - \mathrm{i}|\xi|^2\varphi_m(I) + \varphi_m(\mathrm{i}I) - |\xi|^2\varphi_m(\mathrm{i}I) + \mathrm{i}\varphi_m(I) - \mathrm{i}|\xi|^2\varphi_m(I).
\end{aligned}$$

In accordance with the above last equations, we have $4\mathrm{i}\varphi_m(I) = 0$, thus

$$\mathrm{i}\varphi_m(I) = 0.$$

It follows from $\varphi_m(\mathrm{i}I) = \mathrm{i}\varphi_m(I)$ that

$$\varphi_m(\mathrm{i}I) = 0.$$

For all $B \in \mathfrak{B}$, in relation to $\varphi_m(\mathrm{i}I) = 0$, we obtain

$$\varphi_m(B \diamond_\xi \mathrm{i}I \diamond_\xi \mathrm{i}I) = \varphi_m(B) \diamond_\xi \mathrm{i}I \diamond_\xi \mathrm{i}I,$$

which implies

$$\varphi_m(|\xi|^2 B) = |\xi|^2\varphi_m(B).$$

From $\varphi_m(I) = 0$, we have

$$\varphi_m(B \diamond_\xi I \diamond_\xi I) = \varphi_m(B) \diamond_\xi I \diamond_\xi I.$$

Then we have

$$\varphi_m(\xi B^*) = \xi\varphi_m(B)^*$$

via the equation $\varphi_m(|\xi|^2 B) = |\xi|^2\varphi_m(B)$.

On the other hand, from $\varphi_m(I) = 0$, we have

$$\varphi_m(I \diamond_\xi I \diamond_\xi B^*) = I \diamond_\xi I \diamond_\xi \varphi_m(B^*).$$

Then, we have

$$\varphi_m(\xi B^*) = \xi\varphi_m(B^*).$$

It follows from $\varphi_m(\xi B^*) = \xi\varphi_m(B)^*$ and the above last equation that

$$\varphi_m(B^*) = \varphi_m(B)^*$$

for all $B \in \mathfrak{B}$.

As believed by $\varphi_m(|\xi|^2 B) = |\xi|^2\varphi_m(B)$, for all $B \in \mathfrak{B}$, we obtain

$$\begin{aligned}(1-|\xi|^2)\varphi_m(\mathrm{i}B) = {} & \varphi_m(\mathrm{i}(1-|\xi|^2)B) = \varphi_m(I \diamond_\xi \mathrm{i}I \diamond_\xi B) \\ = {} & I \diamond_\xi \mathrm{i}I \diamond_\xi \varphi_m(B) = (1-|\xi|^2)\mathrm{i}\varphi_m(B).\end{aligned}$$

Hence

$$\varphi_m(\mathrm{i}B) = \mathrm{i}\varphi_m(B).$$

For all $A, B \in \mathfrak{B}$, we have

$$\begin{aligned}& \varphi_m(AB + \xi AB + \xi BA^* + |\xi|^2 BA^*) \\ = {} & \varphi_m(I \diamond_\xi A \diamond_\xi B) \\ = {} & \sum_{i+j+k=m} \varphi_i(I) \diamond_\xi \varphi_j(A) \diamond_\xi \varphi_k(B) \\ = {} & \sum_{j+k=m} I \diamond_\xi \varphi_j(A) \diamond_\xi \varphi_k(B).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}& \varphi_m(AB + \xi AB - \xi BA^* - |\xi|^2 BA^*) \\ = {} & \varphi_m(I \diamond_\xi \mathrm{i}A \diamond_\xi (-\mathrm{i}B)) \\ = {} & \sum_{i+j+k=m} \varphi_i(I) \diamond_\xi \varphi_j(\mathrm{i}A) \diamond_\xi \varphi_k(-\mathrm{i}B) \\ = {} & \sum_{j+k=m} I \diamond_\xi \varphi_j(\mathrm{i}A) \diamond_\xi \varphi_k(-\mathrm{i}B).\end{aligned}$$

From the above last equations, we have

$$\varphi_m((1+\xi)AB) = \sum_{j+k=m} (1+\xi)\varphi_j(A)\varphi_k(B)$$

for all $A, B \in \mathfrak{B}$. From the equation $\varphi_m(\xi B^*) = \xi\varphi_m(B^*)$, we have

$$(1+\xi)\varphi_m(AB) = \sum_{j+k=m} (1+\xi)\varphi_j(A)\varphi_k(B)$$

for all $A, B \in \mathfrak{B}$. Hence $\varphi_m(AB) = \sum_{j+k=m} \varphi_j(A)\varphi_k(B)$ for all $A, B \in \mathfrak{B}$.

Combining Cases 1 and 2, the proposition is finished.

In summary, it can be concluded that mapping φ_m is an additive higher $*$-derivation on $\mathfrak{B}$.

□

Proof of Theorem 2.1. By means of Propositions 2.3 and 2.4, we show that $\Psi = \{\varphi_m\}_{m\in\mathbb{N}}$ is a nonlinear ξ-skew-Jordan triple higher derivation satisfying the equation

$$\varphi_m(y_1 \diamond_\xi y_2 \diamond_\xi y_3) = \sum_{u+v+w=m} \varphi_u(y_1) \diamond_\xi \varphi_v(y_2) \diamond_\xi \varphi_w(y_3)$$

for all $y_1, y_2, y_3 \in \mathfrak{B}$ if and only if $\Psi = \{\varphi_m\}_{m\in\mathbb{N}}$ is an additive higher $*$-derivation and $\varphi_m(\xi y) = \xi\varphi_m(y)$ for all $y \in \mathfrak{B}$, which shows that the necessity of Theorem 2.1 holds.

3. Conclusions

In this paper, we have investigated the structure of nonlinear ξ-skew-Jordan triple higher derivations on a unital prime $*$-algebra over the complex field $\mathbb{C}$, where $\xi \in \mathbb{C} \setminus \{0, \pm 1\}$. Under certain conditions, we have proved that every family $\Psi = \{\varphi_m\}_{m\in\mathbb{N}}$ of maps (not necessarily additive) from $\mathfrak{B}$ into itself is a nonlinear ξ-skew-Jordan triple higher derivation if and only if it is an additive higher $*$-derivation and satisfies $\varphi_m(\xi A) = \xi\varphi_m(A)$ for all $A \in \mathfrak{B}$ and $m \in \mathbb{N}$. As an application, our result not only generalizes the structure of nonlinear ξ-skew-Jordan triple derivations on prime $*$-algebras, but also yields a description of nonlinear ξ-skew-Jordan triple higher derivations on several important operator algebras, including standard operator algebras and factor von Neumann algebras.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by the Youth Fund of Anhui Natural Science Foundation (Grant No. 2008085QA01), Key Projects of University Natural Science Research Project of Anhui Province (Grant No. KJ2019A0107), and Open Research Fund of Hubei Key Laboratory of Mathematical Sciences (Central China Normal University, Wuhan 430079, China).

Conflict of interest

The authors declare there is no conflict of interest.

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