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*Research article*

# Open-loop equilibrium strategy of time-inconsistent deterministic linear-quadratic problems

**Wei Ji[1,*], Yuanjiao Lei[1,2], Yinju Wei[1] and Yandan Zhang[1]**

[1] College of Science, Guiyang University, Guiyang 550005, China

[2] School of Mathematics and Big Data, Guizhou Education University, Guiyang 550018, China

* **Correspondence:** Email: weiji2021@126.com.

**Abstract:** This paper investigates time-inconsistent linear-quadratic control problems within an open-loop framework, where the objective functional incorporates a non-exponential discounting structure. By introducing a two-point boundary value problem and a Riccati-type equation with an error function, it is established, under appropriate conditions, that the existence of an open-loop equilibrium strategy for the time-inconsistent control problem is equivalent to the existence of solutions to both the two-point boundary value problem and the Riccati-type equation. Furthermore, through an illustrative example, it is demonstrated that there exists an essential distinction between open-loop and closed-loop equilibria in time-inconsistent control problems.

**Keywords:** time-inconsistent; open-loop equilibrium; linear-quadratic; error function; two-point boundary value problem; Riccati-type equation

## 1. Introduction

The issue of time inconsistency has a long historical lineage, tracing back to the mid-18th century. Due to its wide-ranging applications in finance, economics, psychology, political science, and management science, it has been extensively studied by scholars, yielding a rich body of research. Following the mathematical formalization by Strotz [1] in the 1950s, the mathematical theory of time inconsistency and related issues has garnered significant attention, with key contributions documented in [2–6] and the references therein.

Because time-inconsistent problems do not satisfy the dynamic programming principle, the optimality framework from classical control theory (i.e., time-consistent problems) cannot be directly applied, necessitating alternative approaches. Based on the existing literature, research on time inconsistency largely revolves around two central questions: (1) How should a solution to a time-inconsistent problem be appropriately defined? (2) Can the theoretical framework developed for time-consistent problems be extended to the time-inconsistent setting, and if so, what fundamental distinctions arise?

Concerning the first question, substantial scholarly attention has led to significant advances. For instance, Yong [7] introduced the concept of closed-loop equilibrium by discretizing the time horizon and treating the time-inconsistent problem as a limit of its classical counterpart, achieving notable results [8,9]. Further studies on closed-loop formulations can be found in [10–12] and related references. Hu et al. [13] proposed the notion of open-loop equilibrium and discussed its uniqueness in [14]. He et al. [15] introduced and systematically analyzed the concepts of strong, weak, and regular equilibria, while Bayraktar et al. [16] defined strong, mild, and weak equilibria and provided corresponding discussions. Zhou [17] further proposed the concept of almost strong equilibrium and examined its relationship with strong equilibrium.

Regarding the second question, considerable efforts have been made by researchers, leading to fruitful outcomes. Under solvability conditions for forward-backward differential equations and Riccati-type equations, Yong [8] constructed open-loop equilibria and, under solvability of Riccati-type systems, derived corresponding closed-loop equilibrium. In a one-dimensional setting, Hu et al. [13, 14] established the equivalence between the existence of open-loop equilibrium solutions and the solvability of forward-backward stochastic differential equations. Ni et al. [18] examined, in discrete time, the equivalence between the existence of open-loop equilibria and the solvability of forward-backward stochastic equations. Within a closed-loop framework, Cai et al. [12] extended the classical equivalence among solution existence for linear-quadratic optimal control problems, solvability of two-point boundary value problems, and solvability of Riccati-type equations to time-inconsistent control problems. They further derived solvability conditions for the Riccati equation, enabling the construction of closed-loop equilibrium controls for time-inconsistent problems. Building on [12], Peng et al. [19] investigated the existence and uniqueness of closed-loop solutions.

To the best of our knowledge, when the objective functional incorporates quadratic and linear terms in both state and control variables, includes cross-terms between state and control, and exhibits dependence on the initial time $t$ in a non-exponential discounting form, the following questions warrant further investigation: (1) In the open-loop framework, does an equivalence exist among the existence of equilibrium solutions for time-inconsistent linear-quadratic control problems, the solvability of two-point boundary value problems, and the solvability of Riccati-type equations? (2) Is there a connection between open-loop and closed-loop equilibria in time-inconsistent problems, and if so, what are their fundamental differences?

Motivated by the above questions, this paper examines time-inconsistent linear-quadratic control problems. By introducing a two-point boundary value problem and a Riccati-type equation with an error function, we establish, under appropriate conditions, the equivalence among the existence of open-loop equilibrium solutions, the solvability of the two-point boundary value problem, and the solvability of the Riccati-type equation. Furthermore, through illustrative examples, we elucidate the essential distinctions between open-loop and closed-loop equilibria in time-inconsistent control problems, thereby highlighting the fundamental differences between time-inconsistent and classical control problems.

The structure of this paper is organized as follows: Section 2 introduces the model setup, key definitions, necessary notation, and fundamental assumptions required for subsequent analysis. Section 3 explores the equivalence properties of open-loop equilibria by incorporating a two-point boundary value problem and Riccati-type equations with an error function under suitable hypotheses. Section 4 analyzes the intrinsic relationship between open-loop and closed-loop equilibria in time-inconsistent problems by an example. Finally, Section 5 provides concluding remarks on this paper.

## 2. Problem setting

For notational simplicity, we introduce the following notation, which will be utilized in subsequent sections.

$$D([0,T]) = \{(t,s) \in [0,T] \times [0,T] \mid 0 \le t \le s \le T\},$$

$$C([0,T];\mathbb{R}) = \{f : [0,T] \to \mathbb{R} \mid f \text{ is continuous}\},$$

$$C^1([0,T];\mathbb{R}) = \left\{f : [0,T] \to \mathbb{R} \mid \frac{df}{dt}(t) \text{ are continuous}\right\},$$

$$C^{1,0}(D[0,T];\mathbb{R}^m) = \left\{f : D[0,T] \to \mathbb{R}^m \mid f \text{ and } \frac{\partial f}{\partial t}(t,s) \text{ are continuous}\right\},$$

$$\Phi_C(s,t) = exp\left\{\int_t^s C(\tau)d\tau\right\}, \quad \text{for any } s,t \in [0,T], \tag{2.1}$$

$$\Psi(s,t) = exp\left\{\int_t^s \Big(C(\tau) - D(\tau)G^{-1}(\tau,\tau)\left(D^\top(\tau)P(\tau) + F(\tau,\tau)\right)\Big)d\tau\right\}, \quad \text{for any } s,t \in [0,T]. \tag{2.2}$$

Let $T > 0$ be the terminal time and $u \subset \mathbb{R}^m$. For any $(t,z) \in [0,T] \times \mathbb{R}^n$, consider the following linear control system

$$\begin{cases} \dot{Z}(s) = C(s)Z(s) + D(s)u(s), & s \in [t,T], \\ Z(t) = z \end{cases} \tag{2.3}$$

and the cost functional

$$\begin{aligned} L(t,z;u(\cdot)) = \int_t^T & (\langle E(t,s)Z(s),Z(s)\rangle + \langle 2e(t,s),Z(s)\rangle + \langle 2F(t,s)Z(s),u(s)\rangle \\ & + \langle G(t,s)u(s),u(s)\rangle + \langle 2g(t,s),u(s)\rangle)\,ds + \langle H(t)Z(T),Z(T)\rangle + \langle 2h(t),Z(T)\rangle. \end{aligned} \tag{2.4}$$

Here, $C, D, E, F, G, H, e, g,$ and $h$ are matrices (or vectors) of appropriate dimensions. Consider the following optimization problem:

**Problem (I)**. For any $(t,z) \in [0,T] \times \mathbb{R}^n$, find a control $\bar{u}(\cdot) \in L^2([0,T];\mathbb{R}^m)$ such that cost functional $L(t,z;u(\cdot))$ is minimized.

**Remark 2.1**. Problem (I) is evidently distinct from classical optimal control problems, as the cost functional in Problem (I) exhibits explicit dependence on $t$. This implies that the cost functional varies with time $t$, meaning we are not optimizing a single problem but rather a family of problems. Consequently, classical methods for traditional control problems—such as the Bellman dynamic programming principle—are no longer applicable. Even the optimality definition from classical optimal control theory cannot be directly employed here, necessitating alternative approaches.

**Definition 2.1**. [14] Let $\bar{u}(\cdot) \in L^2([0,T];\mathbb{R}^m)$ be given and $\bar{Z}(\cdot)$ be the corresponding state trajectory. Define the perturbed control

$$u^\varepsilon(s) = \begin{cases} v, & s \in (t, t+\varepsilon], \\ \bar{u}(s), & s \in [0,t] \bigcup (t+\varepsilon, T]. \end{cases} \tag{2.5}$$

The control $\bar{u}$ is called an open-loop equilibrium strategy if it satisfies the following condition:

$$\lim_{\varepsilon \searrow 0} \frac{L\Big(t, \bar{Z}(t); u^{\varepsilon}(\cdot)\Big) - L\Big(t, \bar{Z}(t); \bar{u}(\cdot)\Big)}{\varepsilon} \geq 0, \quad \text{for any } t \in [0, T], v \in \mathbb{R}^m. \tag{2.6}$$

**Remark 2.2**. Since classical methods (such as the Bellman dynamic programming principle) cannot be applied to Problem (I), we introduce perturbations locally and propose the concept of equilibrium to replace the optimality in classical control problems for our investigation.

For the purpose of this study, we impose the following assumptions.

$(S1)$ $C \in L^1([0,T]; \mathbb{R}^{n\times n})$ and $D \in L^2([0,T]; \mathbb{R}^{n\times m})$.

$(S2)$ $G \in C(D[0,T]; \mathbb{S}^m)$ such that for all $(t, s) \in D[0,T]$, $G(t,s)$ is a symmetric positive definite matrix.

$(S3)$ $E \in C^{1,0}(D[0,T]; \mathbb{S}^n)$ and $H \in C^{1,0}(D[0,T]; \mathbb{S}^n)$ such that for all $(t, s) \in D[0,T]$, $E(t,s)$ and $H(t,s)$ are symmetric positive semi-definite matrices.

$(S4)$ $e \in C^{1,0}(D[0,T]; \mathbb{R}^n)$, $F \in C^{1,0}(D[0,T]; \mathbb{R}^{m\times n})$, $g \in C(D[0,T]; \mathbb{R}^m)$, and $h \in C^1([0,T]; \mathbb{R}^n)$.

$(S5)$ For any $t, s \in [0,T]$, $E_t(t,s)$, $e_t(t,s)$, $F_t(t,s)$, $\dot{H}(t)$, and $\dot{h}(t)$ are continuous, where

$$E_t(t,s) = \frac{\partial E}{\partial t}(t,s),\ e_t(t,s) = \frac{\partial e}{\partial t}(t,s),\ F_t(t,s) = \frac{\partial F}{\partial t}(t,s),\ \dot{H}(t) = \frac{dH}{dt}(t),\ \text{and } \dot{h}(t) = \frac{dh}{dt}(t).$$

## 3. Main results

We first examine the properties of the control system under the perturbation (2.5). Consider a given control $\bar{u}(\cdot) \in L^2([0,T]; \mathbb{R}^m)$ and let $u^{\varepsilon}(\cdot)$ be defined by (2.5). The corresponding state trajectories, $\bar{Z}(\cdot)$ and $Z^{\varepsilon}(\cdot)$, are governed by the following systems, respectively:

$$\begin{cases} \dot{\bar{Z}}(s) = C(s)\bar{Z}(s) + D(s)\bar{u}(s), \quad s \in [0,T], \\ \bar{Z}(0) = z_0, \end{cases} \tag{3.1}$$

and

$$\begin{cases} \dot{Z}^{\varepsilon}(s) = C(s)Z^{\varepsilon}(s) + D(s)u^{\varepsilon}(s), \quad s \in [0,T], \\ Z^{\varepsilon}(0) = z_0. \end{cases} \tag{3.2}$$

**Proposition 3.1**. Let assumption (S1) hold. Then the following convergence holds in $C([0,T]; \mathbb{R}^n)$:

$$Z^{\varepsilon}(\cdot) \to \bar{Z}(\cdot) \quad \text{as} \quad \varepsilon \to 0.$$

*Proof.* Let $\bar{Z}$ and $Z^{\varepsilon}$ satisfy (3.1) and (3.2), respectively. Then we easily obtain

$$\bar{Z}(s) = \begin{cases} \Phi_C^{\top}(s,0)z_0 + \int_0^s \Phi_C^{\top}(s,\tau)D(\tau)\bar{u}(\tau)d\tau, & s \in [0,t], \\ \Phi_C^{\top}(s,0)z_0 + \int_0^t \Phi_C^{\top}(t,\tau)D(\tau)\bar{u}(\tau)d\tau + \int_t^s \Phi_C^{\top}(s,\tau)D(\tau)\bar{u}(\tau)d\tau, & s \in [t, t+\varepsilon], \\ \Phi_C^{\top}(s,0)z_0 + \int_0^t \Phi_C^{\top}(t,\tau)D(\tau)\bar{u}(\tau)d\tau + \int_t^{t+\varepsilon} \Phi_C^{\top}(t+\varepsilon,\tau)D(\tau)\bar{u}(\tau)d\tau & \\ \qquad + \int_{t+\varepsilon}^s \Phi_C^{\top}(s,\tau)D(\tau)\bar{u}(\tau)d\tau, & s \in [t+\varepsilon, T] \end{cases}$$

and

$$Z^{\varepsilon}(s) = \begin{cases} \Phi_C^{\top}(s,0)z_0 + \int_0^s \Phi_C^{\top}(s,\tau)D(\tau)\bar{u}(\tau)d\tau, & s \in [0,t], \\ \Phi_C^{\top}(s,0)z_0 + \int_0^t \Phi_C^{\top}(t,\tau)D(\tau)\bar{u}(\tau)d\tau + \int_t^s \Phi_C^{\top}(s,\tau)D(\tau)v d\tau, & s \in [t, t+\varepsilon], \\ \Phi_C^{\top}(s,0)z_0 + \int_0^t \Phi_C^{\top}(t,\tau)D(\tau)\bar{u}(\tau)d\tau + \int_t^{t+\varepsilon} \Phi_C^{\top}(t+\varepsilon,\tau)D(\tau)v d\tau & \\ \qquad + \int_{t+\varepsilon}^s \Phi_C^{\top}(s,\tau)D(\tau)\bar{u}(\tau)d\tau, & s \in [t+\varepsilon, T]. \end{cases}$$

Thus,

$$Z^{\varepsilon}(s) - \bar{Z}(s) = \begin{cases} 0, & s \in [0,t], \\ \int_t^s \Phi_C^{\top}(s,\tau)D(\tau)\left(v - \bar{u}(\tau)\right)d\tau, & s \in [t, t+\varepsilon], \\ \int_t^{t+\varepsilon} \Phi_C^{\top}(t+\varepsilon,\tau)D(\tau)\left(v - \bar{u}(\tau)\right)d\tau & s \in [t+\varepsilon, T]. \end{cases}$$

This implies that

$$Z^{\varepsilon}(\cdot) \to \bar{Z}(\cdot) \ \text{ as } \ \varepsilon \to 0.$$

□

**Theorem 3.2**. Suppose that assumptions (S1)–(S5) hold. Then Problem (I) admits an open-loop equilibrium strategy if and only if the two-point boundary value problem

$$\begin{cases} \bar{Z}(t) = \Phi_C(t,0)z_0 + \int_0^t \Phi_C(t,\tau)D(\tau)\bar{u}(\tau)d\tau, & \\ \lambda(t) = \Phi_C^{\top}(T,t)\left(H(t)\bar{Z}(T) + h(t)\right) & \text{a.e. } t \in [0,T] \\ \qquad + \int_t^T \Phi_C^{\top}(s,t)\left(E(t,s)\bar{Z}(s) + F^{\top}(t,s)\bar{u}(t) + e(t,s)\right)ds, & \end{cases} \tag{3.3}$$

has a solution in $C([0,T];\mathbb{R}^n) \times C([0,T];\mathbb{R}^n)$, where the equilibrium strategy $\bar{u}(\cdot)$ is given by

$$\bar{u}(t) = -G^{-1}(t,t)\left(D^{\top}(t)\lambda(t) + F(t,t)\bar{Z}(t) + g(t,t)\right), \quad \text{a.e. } t \in [0,T]. \tag{3.4}$$

*Proof.* Let $\bar{u}(\cdot) \in L^2([0,T];\mathbb{R}^m)$ be a given control, $u^{\varepsilon}(\cdot)$ be defined by (2.5), and $\bar{Z}(\cdot)$, $Z^{\varepsilon}(\cdot)$ satisfy Eqs (3.1) and (3.2), respectively. We first examine the variation of the control system with respect to the trajectory. Define

$$W^{\varepsilon}(s) = \frac{1}{\varepsilon}\left(Z^{\varepsilon}(s) - \bar{Z}(s)\right), \quad s \in [0,T]. \tag{3.5}$$

Then $W^{\varepsilon}(0) = 0$. This leads to

$$\dot{W}^{\varepsilon}(s) = \frac{1}{\varepsilon}\left(\dot{Z}^{\varepsilon}(s) - \dot{\bar{Z}}(s)\right) = C(s)W^{\varepsilon}(s) + \frac{1}{\varepsilon}D(s)\left(u^{\varepsilon}(s) - \bar{u}(s)\right), \quad s \in [0,T].$$

It follows readily from the proof of Proposition 3.1 that

$$W^{\varepsilon}(s) = \begin{cases} 0, & s \in [0,t], \\ \frac{1}{\varepsilon}\int_t^s \Phi_C(s,\tau)D(\tau)\left(v - \bar{u}(\tau)\right)d\tau, & s \in [t, t+\varepsilon], \\ \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \Phi_C(s,\tau)D(\tau)\left(v - \bar{u}(\tau)\right)d\tau, & s \in [t+\varepsilon, T]. \end{cases} \tag{3.6}$$

Next, we compute the variation of the cost functional with respect to the control, and we obtain

$$\frac{L(t,\bar{Z}(t);u^{\varepsilon}(\cdot)) - L(t,\bar{Z}(t);\bar{u}(\cdot))}{\varepsilon}$$

$$
\begin{aligned}
=&\frac{1}{\varepsilon}\int_t^T (\langle E(t,s)Z^\varepsilon(s), Z^\varepsilon(s)\rangle + 2\langle e(t,s), Z^\varepsilon(s)\rangle + \langle 2F(t,s)Z^\varepsilon(s), u^\varepsilon(s)\rangle)\,ds \\
&+ \frac{1}{\varepsilon}\int_t^T (\langle G(t,s)u^\varepsilon(s), u^\varepsilon(s)\rangle + 2\langle g(t,s), u^\varepsilon(s)\rangle)\,ds + \frac{1}{\varepsilon}(\langle H(t)Z^\varepsilon(T), Z^\varepsilon(T)\rangle + \langle 2h(t), Z^\varepsilon(T)\rangle) \\
&- \frac{1}{\varepsilon}\int_t^T \Big(\langle E(t,s)\bar{Z}(s), \bar{Z}(s)\rangle + 2\langle e(t,s), \bar{Z}(s)\rangle + 2\langle F(t,s)\bar{Z}(s), \bar{u}(s)\rangle\Big)\,ds \\
&- \frac{1}{\varepsilon}\int_t^T (\langle G(t,s)\bar{u}(s), \bar{u}(s)\rangle + 2\langle g(t,s), \bar{u}(s)\rangle)\,ds - \frac{1}{\varepsilon}\Big(\langle H(t)\bar{Z}(T), \bar{Z}(T)\rangle + \langle 2h(t), \bar{Z}(T)\rangle\Big).
\end{aligned}
$$

For notational simplicity, we proceed with the computations one by one.

$$
\begin{aligned}
&\frac{1}{\varepsilon}\int_t^T \Big(\langle E(t,s)Z^\varepsilon(s), Z^\varepsilon(s)\rangle - \langle E(t,s)\bar{Z}(s), \bar{Z}(s)\rangle\Big)\,ds \\
=&\int_t^{t+\varepsilon} \langle E(t,s)\Big(Z^\varepsilon(s) + \bar{Z}(s)\Big), \frac{Z^\varepsilon(s) - \bar{Z}(s)}{\varepsilon}\rangle ds + \int_{t+\varepsilon}^T \langle E(t,s)\Big(Z^\varepsilon(s) + \bar{Z}(s)\Big), \frac{Z^\varepsilon(s) - \bar{Z}(s)}{\varepsilon}\rangle ds \\
=&\int_t^{t+\varepsilon} \langle E(t,s)\Big(Z^\varepsilon(s) + \bar{Z}(s)\Big), \frac{1}{\varepsilon}\int_t^s \Phi_C(s,\tau)D(\tau)(v - \bar{u}(\tau))\,d\tau\rangle ds \\
&+ \int_{t+\varepsilon}^T \langle E(t,s)\Big(Z^\varepsilon(s) + \bar{Z}(s)\Big), \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \Phi_C(s,\tau)D(\tau)(v - \bar{u}(\tau))\,d\tau\rangle ds \\
=&\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \langle D^\top(\tau)\int_\tau^T \Phi_C^\top(s,\tau)E(t,s)\Big(Z^\varepsilon(s) + \bar{Z}(s)\Big)\,ds, v - \bar{u}(\tau)\rangle d\tau.
\end{aligned}
$$

By Proposition 3.1, we obtain

$$
\begin{aligned}
&\lim_{\varepsilon\searrow 0}\frac{1}{\varepsilon}\int_t^T \Big(\langle E(t,s)Z^\varepsilon(s), Z^\varepsilon(s)\rangle - \langle E(t,s)\bar{Z}(s), \bar{Z}(s)\rangle\Big)\,ds \\
=&\langle 2D^\top(t)\int_t^T \Phi_C^\top(s,t)E(t,s)\bar{Z}(s)ds, v - \bar{u}(t)\rangle, \qquad \text{a.e. } t \in [0,T].
\end{aligned}
\tag{3.7}
$$

Let

$$
\lambda_1 = \int_t^T \Phi_C^\top(s,t)E(t,s)\bar{Z}(s)ds.
$$

Following an analogous procedure to the above computation, we rigorously derive the following results

$$
\lim_{\varepsilon\searrow 0}\frac{2}{\varepsilon}\int_t^T \Big(\langle e(t,s), Z^\varepsilon(s)\rangle - \langle e(t,s), \bar{Z}(s)\rangle\Big)\,ds = \langle 2D^\top(t)\int_t^T \Phi_C^\top(s,t)e(t,s)ds, v - \bar{u}(t)\rangle, \quad \text{a.e. } t \in [0,T].
\tag{3.8}
$$

Define

$$
\lambda_2 = \int_t^T \Phi_C^\top(s,t)e(t,s)ds.
$$

Furthermore, we obtain

$$
\begin{aligned}
&\lim_{\varepsilon\searrow 0}\frac{2}{\varepsilon}\int_t^T \Big(\langle F(t,s)Z^\varepsilon(s), u^\varepsilon(s)\rangle - \langle F(t,s)\bar{Z}(s), \bar{u}(s)\rangle\Big)\,ds \\
=&\langle 2F(t,t)\bar{Z}(t), v - \bar{u}(t)\rangle + \langle 2D^\top(t)\int_t^T \Phi_C^\top(s,t)F^\top(t,s)\bar{u}(s)ds, v - \bar{u}(t)\rangle, \qquad \text{a.e. } t \in [0,T].
\end{aligned}
\tag{3.9}
$$

Let

$$\lambda_3 = \int_t^T \Phi_C^\top(s,t)F^\top(t,s)\bar{u}(s)ds.$$

Subsequent calculations yield

$$\lim_{\varepsilon\searrow 0}\frac{1}{\varepsilon}\left(\langle H(t)Z^\varepsilon(T), Z^\varepsilon(T)\rangle - \langle H(t)\bar{Z}(T), \bar{Z}(T)\rangle\right) = \langle 2D^\top(t)\Phi_C^\top(T,t)H(t)\bar{Z}(T), v-\bar{u}(t)\rangle, \quad \text{a.e. } t\in[0,T]. \tag{3.10}$$

Define

$$\lambda_4 = \Phi_C^\top(T,t)H(t)\bar{Z}(T).$$

Using the same calculation method, we can also obtain

$$\lim_{\varepsilon\searrow 0}\frac{2}{\varepsilon}\left(\langle h(t), Z^\varepsilon(T)\rangle - \langle h(t), \bar{Z}(T)\rangle\right) = \langle 2D^\top(t)\Phi_C^\top(T,t)h(t), v-\bar{u}(t)\rangle, \qquad \text{a.e. } t\in[0,T]. \tag{3.11}$$

Set

$$\lambda_5 = \Phi_C^\top(T,t)h(t).$$

Finally, we establish

$$\lim_{\varepsilon\searrow 0}\frac{1}{\varepsilon}\int_t^T \left(\langle G(t,s)u^\varepsilon(s), u^\varepsilon(s)\rangle - \langle G(t,s)\bar{u}(s), \bar{u}(s)\rangle\right)ds = \langle G(t,t)\left(v+\bar{u}(t)\right), v-\bar{u}(t)\rangle, \qquad \text{a.e. } t\in[0,T], \tag{3.12}$$

and

$$\lim_{\varepsilon\searrow 0}\frac{2}{\varepsilon}\int_t^T \left(\langle g(t,s), u^\varepsilon(s)\rangle - \langle g(t,s), \bar{u}(s)\rangle\right)ds = \langle 2g(t,t), v-\bar{u}(t)\rangle, \qquad \text{a.e. } t\in[0,T]. \tag{3.13}$$

Let $\lambda = \lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5$. By consolidating the results from Eqs (3.7)–(3.13), we derive the following expression for the limit of the difference quotient:

$$\begin{aligned}&\lim_{\varepsilon\searrow 0}\frac{L\left(t,\bar{Z}(t);u^\varepsilon(\cdot)\right) - L\left(t,\bar{Z}(t);\bar{u}(\cdot)\right)}{\varepsilon}\\ &=\left\langle G(t,t)\left(v+\bar{u}(t)+2G^{-1}(t,t)\left(D^\top(t)\lambda(t)+F(t,t)\bar{Z}(t)+g(t,t)\right)\right), v-\bar{u}(t)\right\rangle, \qquad \text{a.e. } t\in[0,T].\end{aligned} \tag{3.14}$$

By the positive definiteness of the matrix $G(t,t)$, the function

$$\lim_{\varepsilon\searrow 0}\frac{L\left(t,\bar{Z}(t);u^\varepsilon(\cdot)\right) - L\left(t,\bar{Z}(t);\bar{u}(\cdot)\right)}{\varepsilon} \geq 0, \qquad \text{a.e. } t\in[0,T].$$

when

$$\bar{u}(t) = -G^{-1}(t,t)\left(D^\top(t)\lambda(t)+F(t,t)\bar{Z}(t)+g(t,t)\right), \qquad \text{a.e. } t\in[0,T].$$

This implies that $\bar{u}$ is an equilibrium strategy.

Conversely, suppose that Problem (I) admits an equilibrium control $\bar{u}(\cdot)$, with $u^{\varepsilon}(\cdot)$ defined by (2.5). Let $\bar{Z}(\cdot)$ and $Z^{\varepsilon}(\cdot)$ satisfy Eqs (3.1) and (3.2), respectively. Define the adjoint process

$$\begin{aligned}\tilde{\lambda}(t) &= \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \\ &= \int_t^T \Phi_C^{\top}(s,t)\Big(E(t,s)\bar{Z}(s) + F^{\top}(t,s)\bar{u}(s) + e(t,s)\Big)ds + \Phi_C^{\top}(T,t)\Big(H(t)\bar{Z}(T) + h(t)\Big).\end{aligned} \tag{3.15}$$

Analogous to the previous calculation, we obtain the following expression:

$$\begin{aligned}&\lim_{\varepsilon\searrow 0}\frac{L\Big(t,\bar{Z}(t);u^{\varepsilon}(\cdot)\Big) - L\Big(t,\bar{Z}(t);\bar{u}(\cdot)\Big)}{\varepsilon} \\ &= \Big\langle G(t,t)\Big(v + \bar{u}(t) + 2G^{-1}(t,t)\Big(D^{\top}(t)\tilde{\lambda}(t) + F(t,t)\bar{Z}(t) + g(t,t)\Big)\Big), v - \bar{u}(t)\Big\rangle, \qquad \text{a.e. } t\in[0,T].\end{aligned} \tag{3.16}$$

Define the first variation of the cost functional as

$$\widetilde{L}\Big(t,\bar{Z}(t);v\Big) = \lim_{\varepsilon\searrow 0}\frac{L\Big(t,\bar{Z}(t);u^{\varepsilon}(\cdot)\Big) - L\Big(t,\bar{Z}(t);\bar{u}(\cdot)\Big)}{\varepsilon}, \qquad \text{a.e. } t\in[0,T]. \tag{3.17}$$

From (3.17), it is evident that $\widetilde{L}(t,\bar{Z}(t);v)$ is strictly convex in $v$. Since $\bar{u}(\cdot)$ is an equilibrium control, it follows that $\widetilde{L}(t,\bar{Z}(t);v) \geq 0$ for all admissible $v$. Consequently, $\widetilde{L}(t,\bar{Z}(t);v)$ admits a unique minimizer point, and it is given by

$$\tilde{v} = -G^{-1}(t,t)\Big(D^{\top}(t)\tilde{\lambda}(t) + F(t,t)\bar{Z}(t) + g(t,t)\Big). \tag{3.18}$$

By the uniqueness of this minimizer $\tilde{v}$, we deduce that the equilibrium strategy must satisfy

$$\tilde{u}(t) = G^{-1}(t,t)\Big(D^{\top}(t)\tilde{\lambda}(t) + F(t,t)\bar{Z}(t) + g(t,t)\Big), \qquad \text{a.e. } t\in[0,T]. \tag{3.19}$$

Substituting (3.19) into the state Eq (2.3) yields the following control system:

$$\begin{cases}\dot{\tilde{Z}}(t) = C(t)\tilde{Z}(t) - D(t)G^{-1}(t,t)\Big(D^{\top}(t)\tilde{\lambda}(t) + F(t,t)\tilde{Z}(t) + g(t,t)\Big), \qquad \text{a.e. } t\in[0,T], \\ \tilde{Z}(0) = z_0.\end{cases} \tag{3.20}$$

This differential equation admits a unique solution, which can be represented as

$$\tilde{Z}(t) = \Phi_C(t,0)z_0 - \frac{1}{2}\int_0^t \Phi_C(t,\tau)D(\tau)G^{-1}(\tau,\tau)\Big(D^{\top}(\tau)\tilde{\lambda}(\tau) + F(\tau,\tau)\tilde{Z}(\tau) + g(\tau,\tau)\Big)d\tau, \ \ \text{a.e. } t\in[0,T]. \tag{3.21}$$

Furthermore, substituting (3.19) into (3.15) gives

$$\begin{aligned}\tilde{\lambda}(t) =& \Phi_C^{\top}(T,t)\Big(H(t)\tilde{Z}(T) + h(t)\Big) + \int_t^T \Phi_C^{\top}(\tau,t)\Big(E(t,\tau)\tilde{Z}(\tau) \\ &- F(t,\tau)G^{-1}(\tau,\tau)\Big(D^{\top}(\tau)\tilde{\lambda}(\tau) + F(\tau,\tau)\tilde{Z}(\tau) + g(\tau,\tau)\Big) + e(t,\tau)\Big)d\tau, \qquad \text{a.e. } t\in[0,T].\end{aligned} \tag{3.22}$$

Combining (3.21) and (3.22), we have that $(\tilde{Z},\tilde{\lambda})$ solves the two-point boundary value problem (3.3) for a.e. $t\in[0,T]$.

□

For the ease of exposition, we introduce the following notation before presenting the subsequent results. We first introduce the following two-point boundary value problems:

$$\begin{cases}\bar{Z}(t) = \Phi_C(t,0)z_0 - \int_0^t \Phi_C(t,\tau)D(\tau)G^{-1}(\tau,\tau)\left(D^\top(\tau)\lambda(\tau) + F(\tau,\tau)\bar{Z}(\tau) + g(\tau,\tau)\right)d\tau, \\ \lambda(t) = \Phi_C^\top(T,t)\left(H(t)\bar{Z}(T) + h(t)\right) + \int_t^T \Phi_C^\top(\tau,t)\left(E(t,\tau)\bar{Z}(\tau)\right. & t\in[0,T], \\ \qquad \left. -F^\top(t,\tau)G^{-1}(\tau,\tau)\left(D^\top(\tau)\lambda(\tau) + F(\tau,\tau)\bar{Z}(\tau) + g(\tau,\tau)\right) + e(t,\tau)\right)d\tau, \end{cases} \tag{3.23}$$

and Riccati-type equations

$$\begin{cases} \dot{P}(t) + P(t)C(t) + C^\top(t)P(t) + E(t,t) - \hat{E}(t,t) \\ \qquad - \left(D^\top(t)P(t) + F(t,t)\right)^\top G^{-1}(t,t)\left(D^\top(t)P(t) + F(t,t)\right) = 0, \quad t\in[0,T], \\ P(T) = H(T), \end{cases} \tag{3.24}$$

where

$$\begin{aligned}\hat{E}(t,t) =& \Phi_C^\top(T,t)\dot{H}(t)\Psi(T,t) \\ &+ \int_t^T \Phi_C^\top(\tau,t)\left(E_t(t,\tau) - F_t^\top(t,\tau)G^{-1}(\tau,\tau)\left(D^\top(\tau)P(\tau) + F(\tau,\tau)\right)\right)\Psi(\tau,t)d\tau\end{aligned} \tag{3.25}$$

and $\Phi_C$ and $\Psi$ are given by (2.1) and (2.2), respectively. $\hat{E}$ is referred to as the error function in this paper.

For the purpose of this research, we introduce an auxiliary function $\varphi$, which satisfies the following ordinary differential equations:

$$\begin{cases} \dot{\varphi}(t) + \left(C(t) - D(t)G^{-1}(t,t)\left(D^\top(t)P(t) + F(t,t)\right)\right)^\top \varphi(t) + e(t,t) \\ \qquad - \hat{e}(t,t) - \left(D^\top(t)P(t) + F(t,t)\right)^\top G^{-1}(t,t)g(t,t) = 0, \quad t\in[0,T], \\ \varphi(T) = h(T), \end{cases} \tag{3.26}$$

where

$$\hat{e}(t,t) = \Phi_C^\top(T,t)\dot{h}(t) + \int_t^T \Phi_C^\top(\tau,t)\left(e_t(t,\tau) - F_t^\top(t,\tau)G^{-1}(\tau,\tau)\left(D^\top(\tau)\varphi(\tau) + g(\tau,\tau)\right)\right)d\tau \tag{3.27}$$

and $\Phi_C$ and $\Psi$ are given by (2.1) and (2.2), respectively.

It is clear that (3.26) is linear ordinary differential equations (LODEs), so then it is easy to obtain the following result.

**Proposition 3.3**. [19] Let assumptions (S1)–(S5) hold. For any $P \in C\left([0,T];\mathbb{R}^{n\times n}\right)$, LODEs (3.26) admit unique solution in $C([0,T];\mathbb{R}^n)$.

**Theorem 3.4**. Let assumptions (S1)–(S5) hold. Then the two-point boundary value problems (3.23) admit a solution in $C\left([0,T];\mathbb{R}^n\right)\times C\left([0,T];\mathbb{R}^n\right)$ if and only if the Riccati-type equations (3.24) admit solutions in $C\left([0,T];\mathbb{R}^{n\times n}\right)$.

*Proof.* Let $P(\cdot) \in C\left([0,T];\mathbb{R}^{n\times n}\right)$ be a solution of Riccati-type equations (3.24), then there exists $\tilde{\varphi} \in C\left([0,T];\mathbb{R}^n\right)$ that solves Eqs (3.26) by Proposition 3.3. Define

$$\tilde{\lambda}(t) = P(t)\bar{Z}(t) + \tilde{\varphi}(t) \tag{3.28}$$

and

$$\bar{Z}(t) = \Psi(t,0)z_0 - \int_0^t \Psi(t,\tau)D(\tau)G^{-1}(\tau,\tau)\left(D^\top(\tau)\tilde{\varphi}(\tau) + g(\tau,\tau)\right)d\tau, \tag{3.29}$$

where $\Psi$ is defined by (2.2). We now verify that $\left(\bar{Z}(\cdot), \tilde{\lambda}(\cdot)\right) \in C([0,T];\mathbb{R}^n) \times C([0,T];\mathbb{R}^n)$ is a solution to the two-point boundary value problems (3.23).

On the one hand, note that $\bar{Z}(\cdot)$ satisfies the following ordinary differential equations:

$$\begin{cases} \dot{\bar{Z}}(t) = \left(C^\top(t) - F^\top(t,t)G^{-1}(t,t)\left(D^\top(t)P(t) + F(t,t)\right)\right)\bar{Z}(t) \\ \qquad -D(t)G^{-1}(t,t)\left(D^\top(t)\tilde{\varphi}(t) + g(t,t)\right), \qquad t \in [0,T], \\ \bar{Z}(0) = z_0. \end{cases} \tag{3.30}$$

On the other hand, differentiating both sides of Eq (3.28) with respect to the time variable $t$ yields

$$\dot{\tilde{\lambda}}(t) = \dot{P}(t)\bar{Z}(t) + P(t)\dot{\bar{Z}}(t) + \dot{\tilde{\varphi}}(t). \tag{3.31}$$

Substituting Eqs (3.24) and (3.30) into Eq (3.31), we can have

$$\begin{aligned} \dot{\tilde{\lambda}}(t) = &- C^\top(t)\tilde{\lambda}(t) - \left(E(t,t) - \hat{E}(t,t) - F(t,t)G^{-1}(t,t)\left(D^\top(t)P(t) + F(t,t)\right)\right)\bar{Z}(t) \\ &+ F^\top(t,t)G^{-1}(t,t)\left(D^\top(t)\tilde{\varphi}(t) + g(t,t)\right) - \left(e(t,t) - \hat{e}(t,t)\right). \end{aligned} \tag{3.32}$$

Combining the expression for $\hat{E}$ in (3.25) and $\hat{e}$ in (3.27), we obtain

$$\begin{aligned} \tilde{\lambda}(t) =& \Phi_C^\top(T,t)\left(H(t)\bar{Z}(T) + h(t)\right) + \int_t^T \Phi_C^\top(\tau,t)\Big(E(t,\tau)\bar{Z}(\tau) \\ &-F^\top(t,\tau)G^{-1}(\tau,\tau)\left(D^\top(\tau)\tilde{\lambda}(\tau) + F(\tau,\tau)\bar{Z}(\tau) + g(\tau,\tau)\right) + e(t,\tau)\Big)d\tau. \end{aligned} \tag{3.33}$$

From (3.30) and (3.33), it follows that the two-point boundary value problems (3.23) admit a solution $\left(\bar{Z}(\cdot), \tilde{\lambda}(\cdot)\right)$.

Now, let $\left(\hat{Z}(\cdot), \hat{\lambda}(\cdot)\right)$ be a solution to the two-point boundary value problems (3.23). We then have

$$\begin{cases} \dot{\hat{Z}}(s) = C(s)\hat{Z}(s) - D(s)G^{-1}(s,s)\left(D^\top(s)\hat{\lambda}(s) + F(s,s)\hat{Z}(s) + g(s,s)\right), \quad s \in [0,T], \\ \hat{Z}(0) = z_0. \end{cases} \tag{3.34}$$

and

$$\begin{cases} \dot{\hat{\lambda}}(s) = -C^\top(s)\hat{\lambda}(s) - \Big(E(s,s)\hat{Z}(s) + F^\top(s,s)G^{-1}(s,s)\left(D^\top(s)\hat{\lambda}(s) + F(s,s)\hat{Z}(s) + g(s,s)\right) \\ \qquad +e(s,s)\Big) + \int_s^T \Phi_C^\top(\tau,s)\Big(E_s(s,\tau)\hat{Z}(\tau) + F_s^\top(s,\tau)G^{-1}(\tau,\tau)\Big(D^\top(\tau)\hat{\lambda}(\tau) + F(\tau,\tau)\hat{Z}(\tau) \\ \qquad +g(\tau,\tau)\Big) + e_s(s,\tau)\Big)d\tau + \Phi_C^\top(T,s)\left(\dot{H}(s)\hat{Z}(T) + h(s)\right), \qquad s \in [0,T], \\ \hat{\lambda}(T) = H(T)\hat{Z}(T) + h(T). \end{cases} \tag{3.35}$$

We now define the following expression for any $\hat{P}(\cdot) \in C([0,T];\mathbb{R}^{n\times n})$ and $\hat{\varphi} \in C([0,T];\mathbb{R}^n)$:

$$\hat{\lambda}(s) = \hat{P}(s)\hat{Z}(s) + \hat{\varphi}(s), \quad s \in [0,T], \tag{3.36}$$

where $\hat{\varphi}$ is a solution of (3.26). It is evident that $\hat{P}$ is continuously differentiable. We thus have

$$\dot{\hat{\lambda}}(s) = \dot{\hat{P}}(s)\hat{Z}(s) + \hat{P}(s)\dot{\hat{Z}}(s) + \dot{\hat{\varphi}}(s), \quad s \in [0,T]. \tag{3.37}$$

From Eqs (3.26), (3.34), (3.35), and (3.37), we obtain

$$\begin{aligned}0 =&\dot{\hat{P}}(s) + \hat{P}(s)C(s) + C^\top(s)\hat{P}(s) + E(s,s) - \Phi_C^\top(T,s)\dot{H}(s)\Psi(T,s)\\&- \int_s^T \Phi_C^\top(\tau,s)\left(E_s(s,\tau) - F_s^\top(s,\tau)G^{-1}(\tau,\tau)\left(D^\top(t)\hat{P}(\tau) + F(\tau,\tau)\right)\right)\Psi(\tau,s)d\tau\\&- \left(D^\top(s)\hat{P}(s) + F(s,s)\right)^\top G^{-1}(s,s)\left(D^\top(s)\hat{P}(s) + F(s,s)\right), \quad s \in [0,T].\end{aligned}$$

That is

$$\begin{aligned}0 =&\dot{\hat{P}}(s) + \hat{P}(s)C(s) + C^\top(s)\hat{P}(s) + E(s,s) - \hat{E}(s,s)\\&- \left(D^\top(s)\hat{P}(s) + F(s,s)\right)^\top G^{-1}(s,s)\left(D^\top(s)\hat{P}(s) + F(s,s)\right), \quad s \in [0,T],\end{aligned}$$

where $\hat{E}$ is given by (3.25). From Eqs (3.23), (3.26), and (3.36), we obtain $\hat{P}(T) = H(T)$. This demonstrates that the Riccati-type equations (3.24) admit a solution in $C\left([0,T];\mathbb{R}^{n\times n}\right)$.

□

## 4. Example

In reference [12], the authors constructed the following Riccati-type equations for the linear closed-loop equilibrium of the time-inconsistent linear-quadratic control problem. They also discussed the solvability of the Riccati-type equations and subsequently derived the closed-loop equilibrium.

$$\begin{cases}\dot{P}(t) + P(t)C(t) + C^\top(t)P(t) + E(t,t) - \tilde{E}(t,t)\\ \qquad - \left(D^\top(t)P(t) + F(t,t)\right)^\top G^{-1}(t,t)\left(D^\top(t)P(t) + F(t,t)\right) = 0, \quad t \in [0,T],\\ P(T) = H(T),\end{cases} \tag{4.1}$$

where

$$\begin{aligned}\tilde{E}(t,t) =&\Psi^\top(T,t)\dot{H}(t)\Psi(T,t)\\&- \int_t^T \Psi^\top(\tau,t)\frac{\partial}{\partial t}\left(E(t,\tau) + \Gamma^\top(\tau)G(t,\tau)\Gamma(\tau) - \Gamma^\top(\tau)F(t,\tau) - F(t,\tau)\Gamma(\tau)\right)\Psi(\tau,t)d\tau,\end{aligned} \tag{4.2}$$

$$\Gamma(\tau) = G^{-1}(\tau,\tau)\left(D^\top(\tau)P(\tau) + F(\tau,\tau)\right), \tag{4.3}$$

and $\Psi$ is given by (2.2).

As we previously noted, the Riccati-type equations (3.24) for the time-inconsistent linear-quadratic problem lack symmetry, making them generally unsolvable. However, in special cases, we can establish the solvability of the corresponding Riccati-type equations by constructing specific examples, thereby obtaining the open-loop equilibrium strategy. Meanwhile, based on the solvability of the Riccati-type equations (4.1) in [12], we construct the closed-loop equilibrium strategy to elucidate the fundamental differences between open-loop and closed-loop equilibrium strategies in time-inconsistent problems.

**Example 4.1**. Let $C, E, e, F, g, h = 0$ and $D, G = 1$. Let $H = t$. For any $(t,z) \in [0,T] \times \mathbb{R}^n$, we consider the following control systems:

$$\begin{cases}\dot{Z}(s) = u(s), \quad s \in [t,T],\\ Z(t) = z,\end{cases} \tag{4.4}$$

and cost functionals:

$$L(t, z; u) = \int_t^T u^2(s)ds + tZ^2(T), \tag{4.5}$$

where $u(\cdot) \in L^2[0, T]$.

**Problem (II)**. For any $(t, z) \in [0, T] \times \mathbb{R}^n$, we want to find a control function $\bar{u}(\cdot) \in L^2[0, T]$ such that the cost functional $L(t, z; u)$ is minimized.

On the one hand, we can derive the equation under the open-loop equilibrium framework according to (3.24).

$$\begin{cases} \dot{P}(s) = P^2(s) + 1, \quad s \in [t, T], \\ P(T) = T. \end{cases} \tag{4.6}$$

A straightforward calculation leads to

$$P(s) = tan[s - T + arctan(T)], \quad s \in [t, T]. \tag{4.7}$$

The corresponding control trajectory is given by

$$Z(s) = \frac{cos[s - T + arctan(T)]}{cos[t - T + arctan(T)]} z, \quad s \in [t, T],$$

and the open-loop equilibrium strategy is

$$\bar{u}(s) = \frac{cos[t - T + arctan(T)] sin[s - T + arctan(T)]}{cos^2[s - T + arctan(T)]}, \quad s \in [0, T]. \tag{4.8}$$

On the other hand, the Riccati-type equations can be derived from (4.1) in the sense of closed-loop equilibrium strategy.

$$\begin{cases} \dot{P}(s) = P^2(s) + exp\{-2 \int_s^T P(\tau)d\tau\}, \quad s \in [t, T], \\ P(T) = T. \end{cases} \tag{4.9}$$

It is evident that the Riccati-type equations (3.24) under the open-loop equilibrium framework and the Riccati-type equations (4.1) within the closed-loop equilibrium framework exhibit significant differences for the time-inconsistent problems. Moreover, the solution to Eqs (3.24) does not satisfy Eqs (4.1). More importantly, within our current analytical capabilities, we are unable to obtain a closed-form analytical solution for Eqs (4.1).

This example not only demonstrates the significant differences between open-loop and closed-loop equilibrium strategies in time-inconsistent problems but also reveals the fundamental distinction between time-inconsistent and time-consistent problems.

## 5. Concluding remarks

We discuss time-inconsistent control problems within the open-loop framework and establish the equivalence among the existence of time-consistent open-loop equilibria, the solvability of two-point boundary value problems, and the solvability of Riccati-type equations with an error function. Furthermore, we construct an example to illustrate the fundamental distinction between open-loop and closed-loop equilibria in time-inconsistent problems, unlike classical control problems (time-consistent problems). Compared with [2], our cost functional is more general, and an additional ordinary differential equation needs to be constructed when establishing the solvability of two-point boundary value problems and Riccati-type equations. Once a closed-loop control is obtained, it is straightforward to construct an open-loop control. These results generalize the existing literature. Compared with [3], we derive necessary and sufficient conditions for the equivalence among the existence of time-consistent open-loop equilibria, the solvability of two-point boundary value problems, and the solvability of Riccati-type equations with an error function. Compared with [8], our Riccati-type equations (3.24) contain only one parameter. Compared with [12], we extend their closed-loop results to the open-loop case, and our cost functional is more general.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare there is no conflicts of interest.

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