Electronic Research Archive

https://www.aimspress.com/journal/era

ERA, 33(9): 5616–5637.
DOI: 10.3934/era.2025250
Received: 06 June 2025
Revised: 15 July 2025
Accepted: 01 August 2025
Published: 17 September 2025

Research article

Bi-continuous semigroups on sequentially complete random Saks spaces

Leilei Wei, Xia Zhang* **and Ming Liu**

School of Mathematical Sciences, TianGong University, Tianjin 300387, China

* **Correspondence:** Email: zhangxia@tiangong.edu.cn.

Abstract: In this paper, we first introduced the Riemann integral for an abstract-valued function from a finite closed real interval to a sequentially complete random Saks space and gave the fundamental theorem of calculus for an L^0-Lipschitz function. Then we investigated some important properties peculiar to bi-continuous semigroups on a sequentially complete random Saks space. Finally, based on the above work, we established the Hille-Yosida generation theorem for such bi-continuous semigroups, which extends and improves several known results.

Keywords: random Saks spaces; bi-continuous semigroups; Hille-Yosida generation theorem; (ε, λ)-topology

1. Introduction

It is well known that the classical theory of strongly continuous semigroups on Banach spaces has been widely applied to partial differential equations, ergodic theory, Markov processes, and so on [1,2]. However, in many situations, such as the case of the Ornstein-Uhlenbeck semigroup on $C_b(H)$ of bounded continuous functions on a separable Hilbert space H, the semigroup fails to be strongly continuous. To overcome this continuity deficiency, in 2003, Kühnemund [3] introduced the concept of bi-continuous semigroups and established the Hille-Yosida generation theorem for such semigroups. The idea is to equip the underlying Banach space with an additional locally convex topology, which is coarser than the norm topology. It is noteworthy that such spaces are exactly sequentially complete Saks spaces. In fact, a Saks space is a triple $(X, \|\cdot\|, \tau)$ consisting of a normed space $(X, \|\cdot\|)$ and a coarser locally convex Hausdorff topology τ on X such that the norm $\|\cdot\|$ is the supremum taken over some directed system of continuous seminorms that generates the topology τ, see [4,5]. In 2004, Albanese and Mangino studied the Trotter-Kato theorem for bi-continuous semigroups and further applied it to Feller semigroups [6]. Besides, the Lumer-Phillips theorem and the perturbation theory for such semigroups were also given in [7–9].

The notion of a random normed module (briefly, an RN module) is a random generalization of that of a normed space and plays an important role in the study of random functional analysis. Thanks to Gigli [10] and Guo's [11] independent pioneering contributions, the theory of RN modules has obtained a systematic and deep development, and has been successfully applied to several important fields such as random equations [12–14] and nonsmooth differential geometry on metric measure spaces [10, 15]. With the development of RN modules, the theory of operator semigroups is no longer limited to classical spaces. In recent years, some progress has been made in the study of strongly continuous semigroups on complete RN modules [16–18]. In this paper, a complete RN module $(S, \|\cdot\|)$ is further endowed with another (ε, λ)-topology induced by a family of L^0-seminorms $\mathcal{P}$, which is coarser than the (ε, λ)-topology induced by the L^0-norm $\|\cdot\|$. Such a space, which is called a random Saks space in this paper, is a random generalization of a classical Saks space. The purpose of this paper is to study bi-continuous semigroups on a sequentially complete random Saks space. Specifically, we will first introduce the Riemann integral for an abstract-valued function in the random setting and further establish the Hille-Yosida generation theorem for such bi-continuous semigroups.

This paper is arranged as follows: In Section 2, we will present some preliminaries; in Section 3, we will establish the Riemann integral for abstract-valued functions from a finite closed real interval to a sequentially complete random Saks space; in Section 4, we will investigate several important properties peculiar to bi-continuous semigroups in the random setting, and then in Theorem 4.3, we will establish the Hille-Yosida generation theorem for such bi-continuous semigroups.

2. Preliminaries

In this paper, we start with some notations. $\mathbb{K}$ denotes the real scalar field $\mathbb{R}$ or the complex scalar field $\mathbb{C}$, $\mathbb{R}^+$ the set of nonnegative real numbers, $\mathbb{N}$ the set of positive integers, and $(\Omega, \mathcal{F}, P)$ a given probability space. Besides, $L^0(\mathcal{F}, \mathbb{K})$ denotes the algebra of equivalence classes of $\mathbb{K}$-valued $\mathcal{F}$-measurable random variables on Ω under the usual algebraic operations and $\bar{L}^0(\mathcal{F}, \mathbb{R})$ the set of equivalence classes of extended real-valued $\mathcal{F}$-measurable random variables on Ω.

Proposition 2.1 ([19]). *$\bar{L}^0(\mathcal{F}, \mathbb{R})$ is a complete lattice under the partial order $\leq$: $f \leq g$ if and only if $f^0(\omega) \leq g^0(\omega)$ for almost all ω in Ω, where f^0 and g^0 are arbitrarily chosen representatives of f and g in $\bar{L}^0(\mathcal{F}, \mathbb{R})$, and the following statements hold.*

1) For any $G \subset \bar{L}^0(\mathcal{F}, \mathbb{R})$ and $G \neq \emptyset$, there are two sequences $\{\xi_n, n \in \mathbb{N}\}$ and $\{\eta_n, n \in \mathbb{N}\}$ in G such that $\bigvee_{n\geq 1} \xi_n = \bigvee G$ and $\bigwedge_{n\geq 1} \eta_n = \bigwedge G$, where $\bigvee G$ and $\bigwedge G$ denote the supremum and the infimum of G, respectively;

2) If G is directed upward (resp., downward), i.e., for any g_1 and g_2 in G, there exists some $g_3 \in G$ such that $g_1 \bigvee g_2 \leq g_3$ (resp., $g_1 \wedge g_2 \geq g_3$), and then the present $\{\xi_n,\ n \in \mathbb{N}\}$ (resp., $\{\eta_n, n \in \mathbb{N}\}$) can be chosen as nondecreasing (resp., nonincreasing).

In this paper, I_D denotes the characteristic function of D for any $D \in \mathcal{F}$ and $\tilde{I}_D$ denotes the equivalence class of I_D. As usual, for any $f, h \in \bar{L}^0(\mathcal{F}, \mathbb{R})$, the relation $f > h$ is defined by $f \geq h$ and $f \neq h$, and for any $A \in \mathcal{F}$, $f > h$ on A means $f^0(\omega) > h^0(\omega)$ for almost all $\omega \in A$, where f^0 and h^0 are arbitrarily chosen representatives of f and h, respectively. Besides, let $D = \left\{\omega \in \Omega \mid f^0(\omega) > h^0(\omega)\right\}$, and then we always use $[f > h]$ for the equivalence class of D and often write $I_{[f>h]}$ for $\tilde{I}_D$. One can also understand such notations as $I_{[f\leq h]}$, $I_{[f\neq h]}$, and $I_{[f=h]}$.

Specifically, $L^0_+(\mathcal{F}) = \{\eta \in L^0(\mathcal{F},\mathbb{R}) \mid \eta \geq 0\}$ and $L^0_{++}(\mathcal{F}) = \{\eta \in L^0(\mathcal{F},\mathbb{R}) \mid \eta > 0 \text{ on } \Omega\}$.

Definition 2.1 ([11]). *An ordered pair $(S,\|\cdot\|)$ is called a random normed module (briefly, an RN module) over $\mathbb{K}$ with base $(\Omega,\mathcal{F},P)$ if S is a left module over the algebra $L^0(\mathcal{F},\mathbb{K})$ and $\|\cdot\|$ is a mapping from S to $L^0_+(\mathcal{F})$ satisfying the following three axioms.*
(RN-1) $\|\eta x\| = |\eta|\cdot\|x\|$ for any $\eta \in L^0(\mathcal{F},\mathbb{K})$ and $x \in S$;
(RN-2) $\|x+y\| \leq \|x\| + \|y\|$ for any x and $y \in S$;
(RN-3) $\|x\| = 0 \Rightarrow x = \theta$ (the null element of S).

As usual, $\|\cdot\|$ is called the L^0-norm on S and $\|x\|$ is called the L^0-norm of x in S. If $\|\cdot\| : S \to L^0_+(\mathcal{F})$ only satisfies (RN-1) and (RN-2), then it is called an L^0-seminorm on S.

Definition 2.2 ([11]). *An ordered pair $(S,\mathcal{P})$ is called a random locally convex module (briefly, an RLC module) over $\mathbb{K}$ with base $(\Omega,\mathcal{F},P)$ if S is a left module over the algebra $L^0(\mathcal{F},\mathbb{K})$ and $\mathcal{P}$ is a family of mappings from S to $L^0_+(\mathcal{F})$ such that the following three axioms hold.*
(RLC-1) $\bigvee\{p(x) \mid p \in \mathcal{P}\} = 0$ if and only if $x = \theta$ (the null element of S);
(RLC-2) $p(\xi x) = |\xi| p(x)$ for any $p \in \mathcal{P}$, $\xi \in L^0(\mathcal{F},\mathbb{K})$, and $x \in S$;
(RLC-3) $p(x+y) \leq p(x) + p(y)$ for any $p \in \mathcal{P}$ and $x, y \in S$.
Besides, if $\mathcal{P}$ consists of a single L^0-norm $\|\cdot\|$, then the RLC module $(S,\mathcal{P})$ reduces to an RN module $(S,\|\cdot\|)$.

Let $(S,\mathcal{P})$ be an RLC module and $\mathcal{P}_f$ the family of all nonempty finite subsets of $\mathcal{P}$. For any given $Q \in \mathcal{P}_f$, define a mapping $\|\cdot\|_Q$: $S \to L^0_+(\mathcal{F})$ by $\|x\|_Q = \bigvee\{p(x) \mid p \in Q\}$ for any $x \in S$, and then $\|\cdot\|_Q$ is an L^0-seminorm on S. For any countable partition $\{D_n, n \in \mathbb{N}\}$ of Ω to $\mathcal{F}$ and any sequence $\{Q_n, n \in \mathbb{N}\}$ in $\mathcal{P}_f$, define a mapping $\sum_{n=1}^{\infty} I_{D_n}\|\cdot\|_{Q_n}$: $S \to L^0_+(\mathcal{F})$ by $\left(\sum_{n=1}^{\infty} I_{D_n}\|\cdot\|_{Q_n}\right)(x) = \sum_{n=1}^{\infty} I_{D_n}\|x\|_{Q_n}$ for any $x \in S$. Then $\sum_{n=1}^{\infty} I_{D_n}\|\cdot\|_{Q_n}$ is an L^0-seminorm on S. Moreover, if $\sum_{n=1}^{\infty} I_{D_n}\|\cdot\|_{Q_n} \in \mathcal{P}$ for any countable partition $\{D_n, n \in \mathbb{N}\}$ of Ω to $\mathcal{F}$ and any sequence $\{Q_n, n \in \mathbb{N}\}$ in $\mathcal{P}_f$, then $\mathcal{P}$ is said to have the countable concatenation property.

Let $(S,\mathcal{P})$ be an *RLC* module over $\mathbb{K}$ with base $(\Omega,\mathcal{F},P)$. For any given $\varepsilon > 0$ and $0 < \lambda < 1$, let

$$V_\theta(Q,\varepsilon,\lambda) = \{x \in S \mid P\{\omega \in \Omega \mid \|x\|_Q(\omega) < \varepsilon\} > 1-\lambda\}$$

for any $Q \in \mathcal{P}_f$. Then the family $\{V_\theta(Q,\varepsilon,\lambda) \mid Q \in \mathcal{P}_f, \varepsilon > 0, 0 < \lambda < 1\}$ forms a local basis of some Hausdorff linear topology on S, which is called the (ε,λ)-topology induced by $\mathcal{P}$.

Remark 2.1. *Throughout this paper, an RLC module $(S,\mathcal{P})$ is always endowed with the (ε,λ)-topology induced by $\mathcal{P}$. Besides, one should note that a net $\{z_\alpha, \alpha \in \Gamma\}$ in S converges to some $z_0 \in S$ in the (ε,λ)-topology induced by $\mathcal{P}$ if and only if the net $\{p(z_\alpha - z_0), \alpha \in \Gamma\}$ converges to 0 in probability P for any $p \in \mathcal{P}$. Further, if $(S,\mathcal{P})$ reduces to an RN module $(S,\|\cdot\|)$, then a sequence $\{z_n, n \in \mathbb{N}\}$ in S converges to some $z_0 \in S$ in the (ε,λ)-topology induced by $\|\cdot\|$ if and only if the sequence $\{\|z_n - z_0\|, n \in \mathbb{N}\}$ converges to 0 in probability P.*

Proposition 2.2 ([17]). *Let $[s,t]$ be a finite closed real interval and $(S,\|\cdot\|)$ a complete RN module. If g: $[s,t] \to S$ is continuously differentiable and $\bigvee\left\{\left\|\frac{g(u_1)-g(u_2)}{u_1-u_2}\right\| \mid u_1,\ u_2 \in [s,t] \text{ and } u_1 \neq u_2\right\} \in L^0_+(\mathcal{F})$, then g' is Riemann integrable on $[s,t]$ and $\int_s^t g'(u)du = g(t) - g(s)$.*

Given an RLC module $(S,\mathcal{P})$, let $(S,\mathcal{P})^* = \{f\colon S \to L^0(\mathcal{F},\mathbb{K}) \mid f$ is a continuous module homomorphism from $(S,\mathcal{P})$ to $(L^0(\mathcal{F},\mathbb{K}),|\cdot|)\}$. Then $(S,\mathcal{P})^*$ is called the random conjugate space of $(S,\mathcal{P})$.

Proposition 2.3 ([11]). *Suppose $(S,\mathcal{P})$ is an RLC module and $\mathcal{P}$ has the countable concatenation property. Then $f \in (S,\mathcal{P})^*$ if and only if there are $\eta \in L^0_+(\mathcal{F})$ and $Q \in \mathcal{P}_f$ such that $|f(x)| \leq \eta\|x\|_Q$ for any $x \in S$.*

Proposition 2.4 ([20]). *Suppose that g is a continuous function from $[s,t]$ to $L^0(\mathcal{F},\mathbb{R})$ satisfying $\bigvee_{u\in[s,t]}|g(u)| \in L^1(\mathcal{F},\mathbb{R})$, where $L^1(\mathcal{F},\mathbb{R}) = \left\{\xi \in L^0(\mathcal{F},\mathbb{R}) \mid \int_\Omega |\xi| dP < \infty\right\}$, and then*

$$\int_\Omega \left[\int_s^t g(u)du\right] dP = \int_s^t \left[\int_\Omega g(u)dP\right] du.$$

Given an RLC module $(S,\mathcal{P})$, a continuous module homomorphism from $(S,\mathcal{P})$ to $(L^0(\mathcal{F},\mathbb{K}),|\cdot|)$ is called a canonical module homomorphism on $(S,\mathcal{P})$; $(S,\mathcal{P})$ is said to admit enough canonical module homomorphisms if for each nonzero element z in $(S,\mathcal{P})$ there exists at least one canonical module homomorphism f on $(S,\mathcal{P})$ such that $f(z)$ is a nonzero element in $(L^0(\mathcal{F},\mathbb{K}),|\cdot|)$.

Proposition 2.5 ([21]). *An RLC module $(S,\mathcal{P})$ admits enough canonical module homomorphisms.*

3. The Riemann integral for an abstract-valued function from a finite closed real interval to a sequentially complete random Saks space

Definition 3.1. *Let $(S,\|\cdot\|)$ be a complete RN module over $\mathbb{K}$ with base $(\Omega,\mathcal{F},P)$ and $\mathcal{T}$ a Hausdorff linear topology on S such that $\mathcal{T}$ is coarser than the (ε,λ)-topology induced by $\|\cdot\|$. Then the triple $(S,\|\cdot\|,\mathcal{T})$ is called a random Saks space over $\mathbb{K}$ with base $(\Omega,\mathcal{F},P)$ if there exists a directed system $\mathcal{P}$ of L^0-seminorms such that $\mathcal{T}$ coincides with the (ε,λ)-topology induced by $\mathcal{P}$, and*

$$\|x\| = \bigvee_{p\in\mathcal{P}} p(x)$$

for any $x \in S$.

A sequence $\{z_n, n \in \mathbb{N}\} \subseteq S$ is said to be $\|\cdot\|$-bounded if $\bigvee_{n\in\mathbb{N}}\|z_n\| \in L^0_+(\mathcal{F})$. Further, if any $\|\cdot\|$-bounded $\mathcal{T}-$Cauchy sequence converges in $(S,\mathcal{T})$, then the triple $(S,\|\cdot\|,\mathcal{T})$ is called a sequentially complete random Saks space.

Remark 3.1. *In the following, for any given random Saks space $(S,\|\cdot\|,\mathcal{T})$, we always denote by $\mathcal{P}$ the family of L^0-seminorms satisfying Definition 3.1.*

Let $(S,\|\cdot\|,\mathcal{T})$ be a random Saks space and define a mapping $\|\cdot\|_1\colon S \to [0,\infty]$ by

$$\|x\|_1 = \int_\Omega \|x\| dP.$$

Let

$$L^1(S) = \{x \in S \mid \|x\|_1 < \infty\},$$

and then $\left(L^1(S), \|\cdot\|_1\right)$ is a Banach space. Define a mapping ρ_p: $L^1(S) \to \mathbb{R}^+$ by

$$\rho_p(x) = \int_\Omega p(x)dP$$

for any $p \in \mathcal{P}$, and then $\left\{\rho_p\right\}_{p\in\mathcal{P}}$ is a family of seminorms in $L^1(S)$. Denote by τ the locally convex topology induced by the family of seminorms $\left\{\rho_p\right\}_{p\in\mathcal{P}}$. Clearly, $\left(L^1(S), \tau\right)$ is a locally convex space.

Let $\mathcal{N} = \{u_0, u_1, \ldots, u_n\}$ be a finite partition of $[s,t]$, i.e., $s = u_0 < u_1 < \cdots < u_n = t$ and $|\mathcal{N}| = \max_{1\le i\le n}(\Delta u_i)$, where $\Delta u_i = u_i - u_{i-1}$. Besides, let $\bar{\mathcal{N}}([s,t])$ denote the set of all partitions of $[s,t]$, $[\mathcal{N}] = \prod_{k=1}^n [u_{k-1}, u_k] \subseteq \mathbb{R}^n$, and $\mathcal{D} = \{(\mathcal{N}, \xi)\colon\ \mathcal{N} \in \bar{\mathcal{N}}([s,t]),\ \xi \in [\mathcal{N}]\}$. As usual, for any $\mathcal{N}, \mathcal{M} \in \bar{\mathcal{N}}([s,t])$ and $\xi \in [\mathcal{N}], \eta \in [\mathcal{M}]$, $(\mathcal{N}, \xi) \ge (\mathcal{M}, \eta)$ means $|\mathcal{N}| \le |\mathcal{M}|$, and then $(\mathcal{D}, \ge)$ is a directed set.

Suppose $(S, \|\cdot\|, \mathcal{T})$ is a random Saks space and g is a function from $[s,t]$ to S. For any $\mathcal{N} = \{u_0, u_1, \ldots, u_n\} \in \bar{\mathcal{N}}([s,t])$ and $\xi = (\xi_1, \ldots, \xi_n) \in [\mathcal{N}]$, let $R(g, \mathcal{N}, \xi) = \sum_{i=1}^n g(\xi_i)\Delta u_i$, and then $\{R(g, \mathcal{N}, \xi)\colon (\mathcal{N}, \xi) \in \mathcal{D}\}$ is a net in S.

Definition 3.2. *Let $(S, \|\cdot\|, \mathcal{T})$ be a random Saks space and g a function from $[s,t]$ to S. Then g is said to be $\mathcal{T}-$Riemann integrable on $[s,t]$ if there is some I in S satisfying that for any $\varepsilon > 0$ and $0 < \lambda < 1$, there is $\delta(\varepsilon, \lambda) > 0$ such that for any $p \in \mathcal{P}$,*

$$P\{\omega \in \Omega \mid p(R(g, \mathcal{N}, \xi) - I)(\omega) < \varepsilon\} > 1 - \lambda$$

for any $\mathcal{N} \in \bar{\mathcal{N}}([s,t])$ and $\xi \in [\mathcal{N}]$ whenever $|\mathcal{N}| < \delta(\varepsilon, \lambda)$. In addition, I is called the $\mathcal{T}-$Riemann integral of g over $[s,t]$, denoted by $\int_s^t g(u)du$.

Definition 3.3. *Let $(S, \|\cdot\|, \mathcal{T})$ be a random Saks space and g a function from $[s,t]$ to S. For any $u \in (s,t)$, if $\mathcal{T} - \lim\limits_{h\to 0} \frac{g(u+h)-g(u)}{h}$ exists, then we say g is $\mathcal{T}-$differentiable at u, denoting this limit by $g'(u)$ or $\frac{dg(u)}{du}$. Analogously, one can define the right $\mathcal{T}-$derivative of g at s and the left $\mathcal{T}-$derivative of g at t. Then, as usual, $g'(u)$ is called the $\mathcal{T}-$derivative of g at $u \in [s,t]$. If g is $\mathcal{T}-$differentiable for any $u \in [s,t]$, then g is said to be $\mathcal{T}-$differentiable on $[s,t]$.*

Theorem 3.1. *Let $(S, \|\cdot\|, \mathcal{T})$ be a sequentially complete random Saks space and g: $[s,t] \to S$ a continuous function with respect to $\mathcal{T}$. If $\bigvee_{u\in[s,t]} \|g(u)\| \in L^0_+(\mathcal{F})$, then g is $\mathcal{T}-$Riemann integrable on $[s,t]$.*

Proof. Let $\eta = \bigvee_{u\in[s,t]} \|g(u)\|$, and then $\eta \in L^0_+(\mathcal{F})$. Set

$$\mathcal{E}_k = \{k - 1 \le \eta < k\}$$

for any $k \in \mathbb{N}$, and then $\mathcal{E}_i \bigcap \mathcal{E}_j = \emptyset$ for any $i, j \in \mathbb{N}$ with $i \ne j$ and $\bigcup_{k=1}^\infty \mathcal{E}_k = \Omega$. Define a mapping g_k: $[s,t] \to S$ by

$$g_k(u) = I_{\mathcal{E}_k} g(u)$$

for any $k \in \mathbb{N}$, and then $g_k(u) \in (L^1(S), \tau)$ for any $u \in [s,t]$. Since for any $p \in \mathcal{P}$ and $k \in \mathbb{N}$, $p(g_k(u) - g_k(u_0)) \le 2k$ for any $u_0, u \in [s,t]$ and $p(g_k(u) - g_k(u_0)) \to 0$ in probability P as $u \to u_0$, due to Lebesgue's dominated convergence theorem, we have

$$\lim_{u\to u_0} \rho_p(g_k(u) - g_k(u_0)) = 0,$$

i.e., g_k: $[s,t] \to (L^1(S),\tau)$ is continuous. It is easy to check that $\{R(g_k,\mathcal{N},\xi)\colon (\mathcal{N},\xi)\in\mathcal{D}\}$ is a $\tau-$Cauchy net in $L^1(S)$ for any $k\in\mathbb{N}$. Since for any $\mathcal{N},\mathcal{M}\in\bar{\mathcal{N}}([s,t])$ and $\xi\in[\mathcal{N}],\eta\in[\mathcal{M}]$,

$$\begin{aligned}&P\{\omega\in\Omega \mid p(R(g_k,\mathcal{N},\xi)-R(g_k,\mathcal{M},\eta))(\omega)\geq\varepsilon\}\\ &\leq\frac{1}{\varepsilon}\int_\Omega p(R(g_k,\mathcal{N},\xi)-R(g_k,\mathcal{M},\eta))dP\end{aligned}$$

for any $\varepsilon>0$, $k\in\mathbb{N}$ and $p\in\mathcal{P}$, it follows that $\{R(g_k,\mathcal{N},\xi)\colon (\mathcal{N},\xi)\in\mathcal{D}\}$ is a $\mathcal{T}-$Cauchy net in S. Further, by $\sum_{k=1}^{\infty}P(\mathcal{E}_k)=P(\bigcup_{k=1}^{\infty}\mathcal{E}_k)=P(\Omega)=1$, we have that $\{R(g,\mathcal{N},\xi)\colon (\mathcal{N},\xi)\in\mathcal{D}\}$ is a $\mathcal{T}-$Cauchy net in S. Consequently, for any $\varepsilon_1>0$ and $0<\lambda_1<1$, there is an $O(\varepsilon_1,\lambda_1)\in\bar{\mathcal{N}}([s,t])$ such that for any $p\in\mathcal{P}$

$$P\{\omega\in\Omega \mid p(R(g,\mathcal{N},\xi)-R(g,\mathcal{M},\eta))(\omega)<\varepsilon_1\}>1-\lambda_1$$

for any $\xi\in[\mathcal{N}],\eta\in[\mathcal{M}]$ whenever $\mathcal{N},\mathcal{M}\in\bar{\mathcal{N}}([s,t])$ satisfy $\mathcal{N},\mathcal{M}\geq O(\varepsilon_1,\lambda_1)$.

For any $n\in\mathbb{N}$, let $\mathcal{N}_n=\left\{s,s+\frac{(t-s)}{n},s+\frac{2(t-s)}{n},\ldots,t\right\}$ and $\xi_{[n]}=(\xi_1,\ldots,\xi_n)\in[\mathcal{N}_n]$, where $\xi_i=s+\frac{i(t-s)}{n}$ $(i=1\ldots,n)$, and then $\{R(g,\mathcal{N}_n,\xi_{[n]}),n\in\mathbb{N}\}$ is a $\mathcal{T}-$Cauchy sequence in S. Since $\bigvee_{u\in[s,t]}\|g(u)\|\in L^0_+(\mathcal{F})$, it follows that $\bigvee_{n\in\mathbb{N}}\|R(g,\mathcal{N}_n,\xi_{[n]})\|\in L^0_+(\mathcal{F})$. By the sequential completeness of S, there is some I in S satisfying that

$$\mathcal{T}-\lim_{n\to\infty}R(g,\mathcal{N}_n,\xi_{[n]})=I,$$

i.e., for any $\varepsilon_2>0$ and $0<\lambda_2<1$, there is an $M\in\mathbb{N}$ satisfying that for any $p\in\mathcal{P}$

$$P\{\omega\in\Omega \mid p(R(g,\mathcal{N}_n,\xi_{[n]})-I)(\omega)<\varepsilon_2\}>1-\lambda_2$$

for any $n>M$.

According to

$$p(R(g,\mathcal{N},\xi)-I)\leq p(R(g,\mathcal{N},\xi)-R(g,\mathcal{N}_n,\xi_n))+p(R(g,\mathcal{N}_n,\xi_n)-I)$$

for any $p\in\mathcal{P}$ and $n\in\mathbb{N}$, it follows that for any $\varepsilon_3>0$, $0<\lambda_3<1$, there is an $O(\varepsilon_3,\lambda_3)\in\bar{\mathcal{N}}([s,t])$ such that

$$P\{\omega\in\Omega \mid p(R(g,\mathcal{N},\xi)-I)(\omega)<\varepsilon_3\}>1-\lambda_3$$

for any $\xi\in[\mathcal{N}]$ whenever $\mathcal{N}\in\bar{\mathcal{N}}([s,t])$ satisfies $\mathcal{N}\geq O(\varepsilon_3,\lambda_3)$, i.e., there is $\delta(\varepsilon_3,\lambda_3)>0$ (i.e., $\delta(\varepsilon_3,\lambda_3):=|O(\varepsilon_3,\lambda_3)|$) such that for any $p\in\mathcal{P}$,

$$P\{\omega\in\Omega \mid p(R(g,\mathcal{N},\xi)-I)(\omega)<\varepsilon_3\}>1-\lambda_3$$

for any $\mathcal{N}\in\bar{\mathcal{N}}([s,t])$ and $\xi\in[\mathcal{N}]$ whenever $|\mathcal{N}|<\delta(\varepsilon_3,\lambda_3)$. Thus we have that g is $\mathcal{T}-$Riemann integrable on $[s,t]$. □

Suppose that $(S,\|\cdot\|,\mathcal{T})$ is a random Saks space over $\mathbb{K}$ with base $(\Omega,\mathcal{F},P)$. A function g: $[s,t]\to S$ is said to be L^0-Lipschitz on $[s,t]$ if there is an $\eta\in L^0_+(\mathcal{F})$ satisfying $\|g(u_1)-g(u_2)\|\leq\eta|u_1-u_2|$ for any $u_1,u_2\in[s,t]$.

In the remainder of this paper, we always assume that $\mathcal{P}$ has the countable concatenation property.

Theorem 3.2. *Let* $(S, \|\cdot\|, \mathcal{T})$ *be a sequentially complete random Saks space and* g: $[s, t] \to S$ *a continuously differentiable function with respect to* $\mathcal{T}$. *If* g *is* L^0*–Lipschitz on* $[s, t]$, *then*

$$\int_s^t g'(u)du = g(t) - g(s).$$

Proof. Let $\zeta = \bigvee \left\{ \left\| \frac{g(u_1)-g(u_2)}{u_1-u_2} \right\| \mid u_1,\ u_2 \in [s,t] \text{ and } u_1 \neq u_2 \right\}$, and then $\zeta \in L^0_+(\mathcal{F})$ since g is L^0–Lipschitz on $[s, t]$. Since $\|g'(u)\| \leq \bigvee \left\{ \left\| \frac{g(u_1)-g(u_2)}{u_1-u_2} \right\| \mid u_1,\ u_2 \in [s,t] \text{ and } u_1 \neq u_2 \right\}$ for any $u \in [s, t]$, we have $\bigvee_{u \in [s,t]} \|g'(u)\| \in L^0_+(\mathcal{F})$. By Theorem 3.1, one can obtain that g' is $\mathcal{T}$–Riemann integrable on $[s, t]$.

For any $f \in (S, \mathcal{P})^*$, let $G(u) = f[g(u)]$ for any $u \in [s, t]$, and according to Proposition 2.3, there are $\eta_1 \in L^0_+(\mathcal{F})$ and $Q_1 \in \mathcal{P}_f$ such that

$$\begin{aligned} \left| \frac{G(u_1) - G(u_2)}{u_1 - u_2} \right| &= \left| \frac{f[g(u_1)] - f[g(u_2)]}{u_1 - u_2} \right| \\ &\leq \eta_1 \left\| \frac{g(u_1) - g(u_2)}{u_1 - u_2} \right\|_{Q_1} \\ &\leq \eta_1 \left\| \frac{g(u_1) - g(u_2)}{u_1 - u_2} \right\| \end{aligned}$$

for any $u_1,\ u_2 \in [s, t]$ and $u_1 \neq u_2$. Thus we have that $\bigvee \left\{ \left| \frac{G(u_1)-G(u_2)}{u_1-u_2} \right| \mid u_1,\ u_2 \in [s,t] \text{ and } u_1 \neq u_2 \right\} \in L^0_+(\mathcal{F})$ since g is L^0–Lipschitz on $[s, t]$. For any $u \in [s, t]$ and $f \in (S, \mathcal{P})^*$, there are $\eta_2 \in L^0_+(\mathcal{F})$ and $Q_2 \in \mathcal{P}_f$ such that

$$\begin{aligned} \left| \frac{G(u_1) - G(u_2)}{u_1 - u_2} - f[g'(u)] \right| &= \left| \frac{f[g(u_1)] - f[g(u_2)]}{u_1 - u_2} - f[g'(u)] \right| \\ &\leq \eta_2 \left\| \frac{g(u_1) - g(u_2)}{u_1 - u_2} - g'(u) \right\|_{Q_2} \end{aligned}$$

for any $u_1,\ u_2 \in [s, t]$ and $u_1 \neq u_2$. Further, by the $\mathcal{T}$–differentiability of g, we have that G is differentiable and $G'(u) = f[g'(u)]$ for any $u \in [s, t]$. Similarly, one can prove that G': $[s, t] \to L^0(\mathcal{F}, \mathbb{K})$ is continuous. Due to Proposition 2.2, one has

$$G(t) - G(s) = \int_s^t G'(u)du, \tag{3.1}$$

i.e.,

$$\int_s^t f[g'(u)]du = f[g(t)] - f[g(s)].$$

Let $\mathcal{N} = \{u_0, u_1, \ldots, u_n\} \in \bar{\mathcal{N}}([s, t])$ and $\xi = (\xi_1, \ldots, \xi_n) \in [\mathcal{N}]$ be fixed, since g' is $\mathcal{T}$–Riemann integrable on $[s, t]$ and $f[g']$ is Riemann integrable on $[s, t]$, and it follows that for any $\varepsilon > 0$ and $p \in \mathcal{P}$,

$$\lim_{n \to \infty} P \left\{ \omega \in \Omega \mid p \left(\sum_{i=1}^n g'(\xi_i)\Delta u_i - \int_s^t g'(u)du \right)(\omega) > \varepsilon \right\} = 0,$$

and

$$\lim_{n\to\infty} P\left\{\omega \in \Omega \mid \left|\sum_{i=1}^{n} f[g'(\xi_i)]\Delta u_i - \int_s^t f[g'(u)]du\right|(\omega) > \varepsilon\right\} = 0.$$

Moreover, for any $f \in (S,\mathcal{P})^*$, there are $\eta_3 \in L^0_+(\mathcal{F})$ and $Q_3 \in \mathcal{P}_f$ such that

$$\begin{aligned}\left|f\left(\sum_{i=1}^{n} g'(\xi_i)\Delta u_i\right) - f\left(\int_s^t g'(u)du\right)\right| &= \left|f\left(\sum_{i=1}^{n} g'(\xi_i)\Delta u_i - \int_s^t g'(u)du\right)\right| \\ &\leq \eta_3 \left\|\sum_{i=1}^{n} g'(\xi_i)\Delta u_i - \int_s^t g'(u)du\right\|_{Q_3}.\end{aligned}$$

For any $f \in (S,\mathcal{P})^*$, since

$$f\left(\sum_{i=1}^{n} g'(\xi_i)\Delta u_i\right) = \sum_{i=1}^{n} f[g'(\xi_i)]\Delta u_i,$$

it follows that

$$\begin{aligned}\left|\int_s^t f[g'(u)]du - f\left(\int_s^t g'(u)du\right)\right| &\leq \left|\int_s^t f[g'(u)]du - \sum_{i=1}^{n} f[g'(\xi_i)]\Delta u_i\right| \\ &\quad + \left|f\left(\sum_{i=1}^{n} g'(\xi_i)\Delta u_i\right) - \sum_{i=1}^{n} f[g'(\xi_i)]\Delta u_i\right| \\ &\quad + \left|f\left(\sum_{i=1}^{n} g'(\xi_i)\Delta u_i\right) - f\left(\int_s^t g'(u)du\right)\right| \\ &\leq \left|\int_s^t f[g'(u)]du - \sum_{i=1}^{n} f[g'(\xi_i)]\Delta u_i\right| \\ &\quad + \eta_3 \left\|\sum_{i=1}^{n} g'(\xi_i)\Delta u_i - \int_s^t g'(u)du\right\|_{Q_3}.\end{aligned}$$

Letting $n \to \infty$ in the above inequality, one has

$$\int_s^t f[g'(u)]du = f\left(\int_s^t g'(u)du\right)$$

for any $f \in (S,\mathcal{P})^*$. Further, according to (3.1), we get

$$f\left(\int_s^t g'(u)du - g(t) + g(s)\right) = 0 \tag{3.2}$$

for any $f \in (S,\mathcal{P})^*$. By Proposition 2.5, if $\left(\int_s^t g'(u)du - g(t) + g(s)\right)$ is a nonzero element in $(S,\mathcal{T})$, then there exists at least one canonical module homomorphism f_0 on $(S,\mathcal{T})$ such that $f_0\left(\int_s^t g'(u)du - g(t) + g(s)\right)$ is a nonzero element in $L^0(\mathcal{F},\mathbb{K})$, which contradicts (3.2). Thus we have

$$g(t) - g(s) = \int_s^t g'(u)du.$$

□

Let $(S, \|\cdot\|, \mathcal{T})$ be a random Saks space over $\mathbb{K}$ with base $(\Omega, \mathcal{F}, P)$. A linear operator T from $(S, \|\cdot\|)$ to $(S, \|\cdot\|)$ is said to be almost surely bounded if there is an $\eta \in L^0_+(\mathcal{F})$ satisfying $\|Tz\| \leq \eta \cdot \|z\|$ for any $z \in S$. Denote by $\mathcal{B}(S)$ the $L^0(\mathcal{F}, \mathbb{K})$-module of almost surely bounded linear operators from $(S, \|\cdot\|)$ to $(S, \|\cdot\|)$.

Lemma 3.1. *Let $(S, \|\cdot\|, \mathcal{T})$ be a sequentially complete random Saks space and $T \in \mathcal{B}(S)$. Then, for any $z \in S$ and $t \geq 0$, the series $\sum_{k=0}^{\infty} \frac{(tT)^k}{k!} z$ converges in $(S, \mathcal{T})$.*

Proof. Since $T \in \mathcal{B}(S)$, it follows that there is an $\eta \in L^0_+(\mathcal{F})$ satisfying $\|Tz\| \leq \eta\|z\|$ for any $z \in S$. For any $z \in S$ and $n \in \mathbb{N}$, let

$$e^{tT}_n z := \sum_{k=0}^{n} \frac{(tT)^k}{k!} z$$

for any $t \geq 0$, and then

$$\begin{aligned}
\|e^{tT}_n z\| &= \|\sum_{k=0}^{n} \frac{(tT)^k}{k!} z\| \\
&\leq \sum_{k=0}^{n} \frac{t^k}{k!} \eta^k \|z\| \\
&\leq \sum_{k=0}^{\infty} \frac{t^k}{k!} \eta^k \|z\| \\
&= e^{t\eta}\|z\|,
\end{aligned}$$

i.e., $\bigvee_{n \in \mathbb{N}} \|e^{tT}_n z\| \in L^0_+(\mathcal{F})$.

Since

$$\begin{aligned}
p\left(e^{tT}_n z - e^{tT}_m z\right) &\leq p\left(\sum_{k=0}^{n} \frac{(tT)^k}{k!} z - \sum_{k=0}^{m} \frac{(tT)^k}{k!} z\right) \\
&= p\left(\sum_{k=n+1}^{m} \frac{(tT)^k}{k!} z\right) \\
&\leq \left\|\sum_{k=n+1}^{m} \frac{(tT)^k}{k!} z\right\| \\
&\leq \sum_{k=n+1}^{m} \frac{t^k}{k!} \eta^k \|z\|
\end{aligned}$$

for any $n, m \in \mathbb{N}$, $z \in S$ with $n < m$, it follows that $\{e^{tT}_n z, n \in \mathbb{N}\}$ is a $\mathcal{T}-$ Cauchy sequence in S. By the sequential completeness of S, we have

$$\mathcal{T} - \lim_{n \to \infty} e^{tT}_n z = \sum_{k=0}^{\infty} \frac{(tT)^k}{k!} z$$

for any $z \in S$. □

4. Bi-continuous semigroups on sequentially complete random Saks spaces

In the remainder of this section, we always assume that $(S, \|\cdot\|, \mathcal{T})$ is a sequentially complete random Saks space over $\mathbb{K}$ with base $(\Omega, \mathcal{F}, P)$.

Definition 4.1. *A family $\{V(t)\colon t \geq 0\} \subseteq \mathcal{B}(S)$ is called a bi-continuous semigroup on S if*

(i) $V(0) = I$.

(ii) $V(t + s) = V(t)V(s)$ for any $s, t \geq 0$.

(iii) The family $\{V(t)\colon t \geq 0\}$ is exponentially bounded, i.e., $\|V(t)z\| \leq Me^{at}\|z\|$ for any $t \geq 0$, $z \in S$, and some $a \in L^0(\mathcal{F}, \mathbb{R})$, $M \in L^0_{++}(\mathcal{F})$ with $M \geq 1$.

(iv) $\{V(t)\colon t \geq 0\}$ is strongly $\mathcal{T}-$continuous, i.e., for any $z \in S$, the mapping $t \longmapsto V(t)z$ from $\mathbb{R}^+$ into S is continuous with respect to $\mathcal{T}$.

(v) $\{V(t)\colon t \geq 0\}$ is locally bi-equicontinuous, i.e., for any $l \geq 0$ and $\|\cdot\|$-bounded sequence $\{z_n, n \in \mathbb{N}\} \subseteq S$ with $\mathcal{T} - \lim_{n\to\infty} z_n = 0$, we have

$$\mathcal{T} - \lim_{n\to\infty} V(t)z_n = 0$$

uniformly for any $t \in [0, l]$.

Remark 4.1. *When $(\Omega, \mathcal{F}, P)$ is trivial, i.e., $\mathcal{F} = \{\Omega, \emptyset\}$, then the sequentially complete random Saks space $(S, \|\cdot\|, \mathcal{T})$ reduces to an ordinary sequentially complete Saks space and Definition 4.1 reduces to Definition 3 in [3]. Thus Definition 4.1 is a non-trivial generalization of Definition 3 in [3].*

We denote by $H(M, a)$ the set of all bi-continuous semigroups $\{V(t)\colon t \geq 0\}$ on S such that $\|V(t)z\| \leq Me^{at}\|z\|$ for any $t \geq 0, z \in S$, and some $a \in L^0(\mathcal{F}, \mathbb{R}), M \in L^0_{++}(\mathcal{F})$ with $M \geq 1$.

Definition 4.2. *Suppose $\{V(t)\colon t \geq 0\}$ is a bi-continuous semigroup on S. Define*

$$D(A) = \left\{ z \in S : \mathcal{T} - \lim_{t\to 0} \frac{V(t)z - z}{t} \text{ exists and } \bigvee_{t\in(0,1]} \frac{\|V(t)z - z\|}{t} \in L^0_+(\mathcal{F}) \right\}$$

and

$$Az = \mathcal{T} - \lim_{t\to 0} \frac{V(t)z - z}{t}$$

for any $z \in D(A)$. Then the mapping $A\colon D(A) \to S$ is called the infinitesimal generator of $\{V(t)\colon t \geq 0\}$.

Definition 4.3. *A family $\{V(t)\colon t \geq 0\} \subseteq \mathcal{B}(S)$ is said to be globally bi-equicontinuous if for any $\|\cdot\|$-bounded sequence $\{z_n, n \in \mathbb{N}\} \subseteq S$ with $\mathcal{T} - \lim_{n\to\infty} z_n = 0$, we have*

$$\mathcal{T} - \lim_{n\to\infty} V(t)z_n = 0$$

uniformly for any $t \geq 0$.

Definition 4.4. *A subset $E \subseteq S$ is said to be bi-dense in S if for any $z \in S$ there exists a $\|\cdot\|$-bounded sequence $\{z_n, n \in \mathbb{N}\} \subseteq E$ such that $\mathcal{T} - \lim_{n\to\infty} z_n = z$.*

A function g from $\mathbb{R}^+$to S is said to be locally L^0-Lipschitz if for any $L > 0$, there is a $\xi_L \in L^0_+(\mathcal{F})$ such that $\|g(s_1) - g(s_2)\| \leq \xi_L |s_1 - s_2|$ for any $s_1, s_2 \in [0, L]$.

Theorem 4.1. *Let $\{V(t)\colon t \geq 0\} \in H(M,a)$ be a bi-continuous semigroup on S with the infinitesimal generator A. Define a mapping $g\colon \mathbb{R}^+ \to S$ by $g(t) = V(t)z$ for any $z \in D(A)$, and then g is locally L^0−Lipschitz.*

Proof. For any $L \in (0,1]$ and $t \in (0,L]$, we have

$$\bigvee_{t\in(0,L]} \frac{\|V(t)z - z\|}{t} \leq \bigvee_{t\in(0,1]} \frac{\|V(t)z - z\|}{t} \in L^0_+(\mathcal{F}) \tag{4.1}$$

for any $z \in D(A)$.

For any $L > 1$ and $t \in (0,L]$, one has

$$\begin{aligned} \bigvee_{t\in(0,L]} \frac{\|V(t)z - z\|}{t} &\leq \bigvee_{t\in(0,1]} \frac{\|V(t)z - z\|}{t} + \bigvee_{t\in(1,L]} \frac{\|V(t)z - z\|}{t} \\ &\leq \bigvee_{t\in(0,1]} \frac{\|V(t)z - z\|}{t} + Me^{aL}\|z\| + \|z\| \\ &\in L^0_+(\mathcal{F}) \end{aligned} \tag{4.2}$$

for any $z \in D(A)$.

For any $L > 0$ and $t \in [0,L]$, let

$$\xi_L = \bigvee_{t\in(0,1]} \frac{\|V(t)z - z\|}{t} + Me^{aL}\|z\| + \|z\|$$

for any $z \in D(A)$, and then $\xi_L \in L^0_+(\mathcal{F})$. Combining (4.1) and (4.2), for any $L > 0$ and $t \in [0,L]$, one can obtain

$$\|V(t)z - z\| \leq \xi_L t$$

for any $z \in D(A)$. Further, for any $L > 0$, we have

$$\begin{aligned} \|V(t)z - V(s)z\| &= \|V(s)(V(t-s)z - z)\| \\ &\leq Me^{aL}\xi_L(t-s) \end{aligned}$$

for any $z \in D(A)$, $s,t \in [0,L]$ and $t \geq s$.

Consequently, for any $L > 0$, we get

$$\|V(t)z - V(s)z\| \leq Me^{aL}\xi_L|t-s|$$

for any $z \in D(A)$ and $s,t \in [0,L]$, which shows that g is locally L^0−Lipschitz. □

Lemma 4.1. *Let $\{V(t)\colon t \geq 0\} \in H(M,a)$ be a bi-continuous semigroup on S, and then $\{e^{-\alpha t}V(t)\colon t \geq 0\}$ is globally bi-equicontinuous for any $\alpha \in L^0(\mathcal{F},\mathbb{R})$ with $\alpha > a$ on Ω.*

Proof. Let $\{z_n, n \in \mathbb{N}\} \subseteq S$ be a $\|\cdot\|$-bounded sequence such that $\mathcal{T} - \lim_{n\to\infty} z_n = 0$. Since for any $n \in \mathbb{N}$,

$$\begin{aligned} p\left(e^{-\alpha t}V(t)z_n\right) &\leq e^{-\alpha t}\|V(t)z_n\| \\ &\leq e^{-(\alpha-a)t}M\|z_n\| \end{aligned}$$

for any $p \in \mathcal{P}$, $\alpha \in L^0(\mathcal{F}, \mathbb{R})$ with $\alpha > a$ on Ω, it follows that for any $\varepsilon, \lambda > 0$, there is $t_0 \geq 0$ such that

$$P\left\{\omega \in \Omega \mid e^{-(\alpha-a)t} M\|z_n\|(\omega) \geq \frac{\varepsilon}{2}\right\} \leq \frac{\lambda}{2}$$

for any $t > t_0$. Further, by the local bi-equicontinuity of $\{V(t)\colon t \geq 0\}$, for any $p \in \mathcal{P}$, there exists $n_0 \in \mathbb{N}$ such that

$$P\left\{\omega \in \Omega \mid \bigvee_{0 \leq t \leq t_0} p\left(e^{-\alpha t} V(t) z_n\right)(\omega) \geq \frac{\varepsilon}{2}\right\} \leq \frac{\lambda}{2}$$

for any $n \geq n_0$. For any $n \in \mathbb{N}$ and $p \in \mathcal{P}$, due to

$$\bigvee_{t \geq 0} p\left(e^{-\alpha t} V(t) z_n\right) \leq \bigvee_{0 \leq t \leq t_0} p\left(e^{-\alpha t} V(t) z_n\right) + \bigvee_{t > t_0} p\left(e^{-\alpha t} V(t) z_n\right)$$

for any $\alpha \in L^0(\mathcal{F}, \mathbb{R})$ with $\alpha > a$ on Ω, we have

$$\begin{aligned}
&P\left\{\omega \in \Omega \mid \bigvee_{t \geq 0} p\left(e^{-\alpha t} V(t) z_n\right)(\omega) \geq \varepsilon\right\} \\
&\leq P\left\{\omega \in \Omega \mid e^{-(\alpha-a)t} M\|z_n\|(\omega) \geq \frac{\varepsilon}{2}\right\} \\
&\quad + P\left\{\omega \in \Omega \mid \bigvee_{0 \leq t \leq t_0} p\left(e^{-\alpha t} V(t) z_n\right)(\omega) \geq \frac{\varepsilon}{2}\right\} \\
&\leq \lambda
\end{aligned}$$

for any $n \geq n_0$. □

Theorem 4.2. *Let $\{V(t)\colon t \geq 0\}$ be a bi-continuous semigroup on S with the infinitesimal generator A. Then*

(a) for any $z \in D(A)$ and $t \geq 0$, we have $V(t)z \in D(A)$ and

$$\frac{d}{dt} V(t) z = A V(t) z = V(t) A z;$$

(b) for any $z \in D(A)$ and $t \geq 0$,

$$V(t)z - z = \int_0^t V(s) A z \, ds = \int_0^t A V(s) z \, ds;$$

(c) A is bi-closed, i.e., for any sequences $\{z_n, n \in \mathbb{N}\} \subseteq D(A)$ with $\bigvee_{n \in \mathbb{N}} \|z_n\|, \bigvee_{n \in \mathbb{N}} \|A z_n\| \in L^0_+(\mathcal{F})$ such that $\mathcal{T} - \lim_{n \to \infty} z_n = z$ and $\mathcal{T} - \lim_{n \to \infty} A z_n = y$, $z \in D(A)$ and $Az = y$;

(d) for any $z \in S$ and $t \geq 0$, $\int_0^t V(s) z \, ds \in D(A)$ and

$$A \int_0^t V(s) z \, ds = V(t) z - z.$$

Proof. (*a*) Let $t \geq 0$ be fixed and we have

$$V(t)Az = V(t)\left[\mathcal{T} - \lim_{h \to 0^+} \frac{V(h)z - z}{h}\right] = \mathcal{T} - \lim_{h \to 0^+} \frac{V(h+t)z - V(t)z}{h} \tag{4.3}$$

for any $z \in D(A)$. Besides, since

$$AV(t)z = \mathcal{T} - \lim_{h \to 0^+} \frac{V(h)V(t)z - V(t)z}{h} = \mathcal{T} - \lim_{h \to 0^+} \frac{V(h+t)z - V(t)z}{h} \tag{4.4}$$

for any $z \in D(A)$, it follows that $V(\cdot)z$ is right $\mathcal{T}$−differentiable at t and $AV(t)z = V(t)Az$.

For any $h \in (0, t)$, one can obtain that

$$\begin{aligned} & p\left(\frac{V(t)z - V(t-h)z}{h} - V(t)Az\right) \\ & \leq p\left(V(t-h)\left[\frac{V(h)z - z}{h} - Az\right]\right) + p\left(V(t-h)[Az - V(h)Az]\right) \end{aligned}$$

for any $p \in \mathcal{P}$ and $z \in D(A)$, and we get

$$\mathcal{T} - \lim_{h \to 0^+} \frac{V(t)z - V(t-h)z}{h} = V(t)Az = AV(t)z. \tag{4.5}$$

Combining (4.3)–(4.5), one has

$$\frac{dV(t)z}{dt} = AV(t)z = V(t)Az$$

for any $z \in D(A)$ and $t \geq 0$.

(*b*) Due to (a), one has $\frac{dV(t)z}{dt} = AV(t)z = V(t)Az$ for any $z \in D(A)$ and $t \geq 0$. Further, according to Theorem 4.1, the mapping $t \mapsto V(t)z$ is locally L^0-Lipschitz for any $z \in D(A)$. Thus, due to Theorem 3.2, we have

$$V(t)z - z = \int_0^t V(s)Az\, ds = \int_0^t AV(s)z\, ds$$

for any $z \in D(A)$ and $t \geq 0$.

(*c*) Suppose that there is a sequence $\{z_n, n \in \mathbb{N}\} \subseteq D(A)$ with $\bigvee_{n\in\mathbb{N}} \|z_n\|, \bigvee_{n\in\mathbb{N}} \|Az_n\| \in L^0_+(\mathcal{F})$ such that $\mathcal{T} - \lim_{n\to\infty} z_n = z$ and $\mathcal{T} - \lim_{n\to\infty} Az_n = y$, and we have that $p(Az_n - y)$ converges to 0 in probability P as $n \to \infty$ for any $p \in \mathcal{P}$. Set

$$\eta_t = \bigvee_{\substack{s\in[0,t] \\ n\in\mathbb{N}}} \|V(s)Az_n - V(s)y\|$$

for any $t \geq 0$, and we have $\eta_t \in L^0_+(\mathcal{F})$ since $\{V(s)\colon s \geq 0\}$ is exponentially bounded. Let

$$\mathcal{E}_{k,t} = \{k - 1 \leq \eta_t < k\}$$

for any $k \in \mathbb{N}$ and $t \geq 0$, and then $\mathcal{E}_{i,t} \cap \mathcal{E}_{j,t} = \emptyset$ for any $i, j \in \mathbb{N}$ with $i \neq j$, and further $\bigcup_{k=1}^{\infty} \mathcal{E}_{k,t} = \Omega$. Clearly, for any $k \in \mathbb{N}$ and $t \geq 0$, $p\left(I_{\mathcal{E}_{k,t}} V(s)Az_n - I_{\mathcal{E}_{k,t}} V(s)y\right)$ converges to 0 in probability P as $n \to \infty$

for any $s \in [0,t]$. For any $s \in [0,t]$ and $k \in \mathbb{N}$, we have $p\left(I_{\mathcal{E}_{k,t}} V(s)Az_n - I_{\mathcal{E}_{k,t}} V(s)y\right) \leq k$ for any $p \in \mathcal{P}$ and $n \in \mathbb{N}$. Due to Lebesgue's dominated convergence theorem, one can obtain

$$\lim_{n\to\infty} \int_\Omega p\left(I_{\mathcal{E}_{k,t}} V(s)Az_n - I_{\mathcal{E}_{k,t}} V(s)y\right) dP = 0.$$

For any $k \in \mathbb{N}$ and $t \geq 0$, by Proposition 2.4, we have

$$\begin{aligned}
&\int_\Omega p\left(\int_0^t I_{\mathcal{E}_{k,t}} V(s)Az_n - I_{\mathcal{E}_{k,t}} V(s)y\, ds\right) dP \\
&\leq \int_\Omega \int_0^t p\left(I_{\mathcal{E}_{k,t}} V(s)Az_n - I_{\mathcal{E}_{k,t}} V(s)y\right) ds\, dP \\
&= \int_0^t \int_\Omega p\left(I_{\mathcal{E}_{k,t}} V(s)Az_n - I_{\mathcal{E}_{k,t}} V(s)y\right) dP\, ds
\end{aligned}$$

for any $p \in \mathcal{P}$ and $n \in \mathbb{N}$, which implies that

$$p\left(\int_0^t I_{\mathcal{E}_{k,t}} V(s)Az_n - I_{\mathcal{E}_{k,t}} V(s)y\, ds\right)$$

converges to 0 in probability P as $n \to \infty$. According to $\sum_{k=1}^\infty P\left(\mathcal{E}_{k,t}\right) = P\left(\bigcup_{k=1}^\infty \mathcal{E}_{k,t}\right) = P(\Omega) = 1$, one can obtain

$$\mathcal{T} - \lim_{n\to\infty} \int_0^t V(s)Az_n\, ds = \int_0^t V(s)y\, ds$$

for any $t \geq 0$. Further, due to (b), we have

$$V(s)z - z = \mathcal{T} - \lim_{n\to\infty} (V(s)z_n - z_n) = \mathcal{T} - \lim_{n\to\infty} \int_0^t V(s)Az_n\, ds = \int_0^t V(s)y\, ds$$

for any $t \geq 0$, and thus $z \in D(A)$ and $Az = y$.

(d) This is obvious from (b) and (c). □

Remark 4.2. *If $\mathcal{F} = \{\Omega, \emptyset\}$, then Theorem 4.2 reduces to Proposition 11 in [3]. Moreover, in the context of non-trivial random Saks spaces, the (ε, λ)-topology induced by $\mathcal{P}$ is too weak, and the method used in Proposition 11 from [3] cannot be directly generalized. Therefore, we employ the technique of measurable partitions on S to prove Theorem 4.2.*

Suppose A: $D(A) \to S$ is a module homomorphism and $\rho(A) = \{\eta \in L^0(\mathcal{F}, \mathbb{K})$: $\eta I - A$ is bijective and $(\eta I - A)^{-1} \in \mathcal{B}(S)\}$, then $\rho(A)$ is called the resolvent set of A. Further, if $\eta \in \rho(A)$, then $R(\eta, A) = (\eta I - A)^{-1}$ is called the resolvent of A.

Lemma 4.2. *Let $\{V(t)$: $t \geq 0\} \in H(M, a)$ be a bi-continuous semigroup on S with the infinitesimal generator A. If $\eta \in L^0(\mathcal{F}, \mathbb{R})$ with $\eta > a$ on Ω, then*

(a)

$$R(\eta, A)z = \int_0^\infty e^{-\eta t} V(t)z dt$$

for any $z \in S$;

(b)

$$\mathcal{T} - \lim_{\eta\to\infty} \eta R(\eta, A)z = z$$

for any $z \in S$.

Proof. (*a*) Step 1. For any $\eta \in L^0(\mathcal{F}, \mathbb{R})$ with $\eta > a$ on Ω, define a mapping $R(\eta) : S \to S$ by

$$R(\eta)z = \int_0^\infty e^{-\eta t} V(t) z dt$$

for any $z \in S$, and then

$$\begin{aligned} p(R(\eta)z) &= p\left(\int_0^\infty e^{-\eta t} V(t) z dt\right) \\ &\le \int_0^\infty e^{-\eta t} p(V(t)z) dt \\ &\le \int_0^\infty e^{-\eta t} \|V(t)z\| dt \\ &\le \frac{M}{\eta - a}\|z\|. \end{aligned}$$

Thus the module homomorphism $R(\eta)$ is well-defined for any $\eta \in L^0(\mathcal{F}, \mathbb{R})$ with $\eta > a$ on Ω.

Step 2. For any $z \in S$, let

$$\xi = \bigvee_{s \ge 0} \left\| e^{-\eta s} V(s) z - z \right\|$$

for any $\eta \in L^0(\mathcal{F}, \mathbb{R})$ with $\eta > a$ on Ω. Since

$$\begin{aligned} \left\| e^{-\eta s} V(s) z - z \right\| &\le \left\| e^{-\eta s} V(s) z \right\| + \|z\| \\ &\le M e^{-(\eta - a)s} \|z\| + \|z\| \end{aligned}$$

for any $s \ge 0$, we have $\xi \in L^0_+(\mathcal{F})$. Set

$$\mathcal{E}_k = \{k - 1 \le \xi < k\}$$

for any $k \in \mathbb{N}$, and then $\mathcal{E}_i \cap \mathcal{E}_j = \emptyset$ for any $i, j \in \mathbb{N}$ with $i \neq j$, and further $\bigcup_{k=1}^\infty \mathcal{E}_k = \Omega$. Since for any $z \in S$, $k \in \mathbb{N}$, and $p \in \mathcal{P}$, $p\left(I_{\mathcal{E}_k} e^{-\eta s} V(s) z - I_{\mathcal{E}_k} z\right)$ converges to 0 in probability P as $s \to 0$ and $p\left(I_{\mathcal{E}_k} e^{-\eta s} V(s) z - I_{\mathcal{E}_k} z\right) \le k$, it follows from Lebesgue's dominated convergence theorem that $\int_\Omega p\left(I_{\mathcal{E}_k} e^{-\eta s} V(s) z - I_{\mathcal{E}_k} z\right) dP$ converges to 0 as $s \to 0$. Thus, by Proposition 2.4, one has

$$\begin{aligned} &\frac{1}{h} \int_\Omega p\left(\int_0^h I_{\mathcal{E}_k} e^{-\eta s} V(s) z - I_{\mathcal{E}_k} z ds\right) dP \\ &\le \frac{1}{h} \int_\Omega \int_0^h p\left(I_{\mathcal{E}_k} e^{-\eta s} V(s) z - I_{\mathcal{E}_k} z\right) ds dP \\ &= \frac{1}{h} \int_0^h \int_\Omega p\left(I_{\mathcal{E}_k} e^{-\eta s} V(s) z - I_{\mathcal{E}_k} z\right) dP ds \\ &\le \max_{s \in [0,h]} \int_\Omega p\left(I_{\mathcal{E}_k} e^{-\eta s} V(s) z - I_{\mathcal{E}_k} z\right) dP \\ &\to 0 \; as \; h \to 0 \end{aligned}$$

for any $z \in S$, $k \in \mathbb{N}$, and $p \in \mathcal{P}$, which implies that $\frac{1}{h}p\left(\int_0^h I_{\mathcal{E}_k}e^{-\eta s}V(s)z - I_{\mathcal{E}_k}zds\right)$ converges to 0 in probability P as $h \to 0$. Since $\sum_{k=1}^{\infty} P(\mathcal{E}_k) = P\left(\bigcup_{k=1}^{\infty}\mathcal{E}_k\right) = P(\Omega) = 1$, it follows that

$$\mathcal{T} - \lim_{h\to 0}\frac{1}{h}\int_0^h e^{-\eta s}V(s)zds = z$$

for any $z \in S$, $\eta \in L^0(\mathcal{F}, \mathbb{R})$ with $\eta > a$ on Ω.

Step 3. For any $z \in S$ and $h > 0$, we have

$$\begin{aligned}
&\frac{V(h)\int_0^\infty e^{-\eta t}V(t)zdt - \int_0^\infty e^{-\eta t}V(t)zdt}{h}\\
&= \frac{1}{h}\int_0^\infty e^{-\eta t}V(t+h)zdt - \frac{1}{h}\int_0^\infty e^{-\eta t}V(t)zdt\\
&= \frac{1}{h}\int_h^\infty e^{-\eta(t-h)}V(t)zdt - \frac{1}{h}\int_0^\infty e^{-\eta t}V(t)zdt\\
&= \frac{1}{h}\left(e^{\eta h} - 1\right)\int_0^\infty e^{-\eta t}V(t)zdt - \frac{1}{h}e^{\eta h}\int_0^h e^{-\eta t}V(t)zdt
\end{aligned}$$

for any $\eta \in L^0(\mathcal{F}, \mathbb{R})$ with $\eta > a$ on Ω. Letting $h \to 0$ in the above equality, we get

$$A\int_0^\infty e^{-\eta t}V(t)zdt = \eta\int_0^\infty e^{-\eta t}V(t)zdt - z,$$

i.e., $(\eta - A)\int_0^\infty e^{-\eta t}V(t)zdt = z$. Further, according to Theorem 4.2, one can obtain $A\int_0^\infty e^{-\eta t}V(t)zdt = \int_0^\infty e^{-\eta t}V(t)Azdt$ for any $z \in D(A)$, $\eta \in L^0(\mathcal{F}, \mathbb{R})$ with $\eta > a$ on Ω, i.e., $\int_0^\infty e^{-\eta t}V(t)(\eta - A)zdt = z$. Thus we have

$$R(\eta, A)z = \int_0^\infty e^{-\eta t}V(t)zdt$$

for any $z \in S$, $\eta \in L^0(\mathcal{F}, \mathbb{R})$ with $\eta > a$ on Ω.

(b) Since $\{V(t)\colon t \geq 0\}$ is strongly $\mathcal{T}-$ continuous, it follows that for any $\varepsilon, \lambda > 0$, there is $t_0 > 0$ such that

$$P\{\omega \in \Omega \mid p(V(t)z - z)(\omega) \geq \varepsilon\} \leq \lambda$$

for any $z \in S$, $p \in \mathcal{P}$, and $t \in [0, t_0]$. For any $\eta \in L^0(\mathcal{F}, \mathbb{R})$ with $\eta > a$ on Ω, since

$$\begin{aligned}
p(\eta R(\eta, A)z - z) &= p\left(\int_0^\infty \eta e^{-\eta t}V(t)zdt - \int_0^\infty \eta e^{-\eta t}zdt\right)\\
&\leq p\left(\int_0^{t_0} \eta e^{-\eta t}(V(t)z - z)dt\right) + p\left(\int_{t_0}^\infty \eta e^{-\eta t}(V(t)z - z)dt\right)\\
&\leq \int_0^{t_0} \eta e^{-\eta t}p(V(t)z - z)dt + \int_{t_0}^\infty \eta e^{-\eta t}p(V(t)z - z)dt\\
&\leq \bigvee_{t\in[0,t_0]} p(V(t)z - z)\int_0^\infty \eta e^{-\eta t}dt + \int_{t_0}^\infty \eta e^{-\eta t}\|V(t)z - z\|dt\\
&\leq \bigvee_{t\in[0,t_0]} p(V(t)z - z) + \|z\|\left[M\frac{\eta}{\eta - a}e^{-(\eta-a)t_0} + e^{-\eta t_0}\right]
\end{aligned}$$

for any $z \in S$ and $p \in \mathcal{P}$, it follows that $p(\eta R(\eta, A)z - z)$ converges to 0 in probability P as $\eta \to \infty$, which completes the proof. □

Lemma 4.3. *Let $\{V(t)\colon t \geq 0\} \in H(M, a)$ be a bi-continuous semigroup on S with the infinitesimal generator A. Then $D(A)$ is bi-dense in S, i.e., for any $z \in S$, there exists a $\|\cdot\|$-bounded sequence $\{z_n, n \in \mathbb{N}\} \subseteq D(A)$ such that $\mathcal{T} - \lim_{n\to\infty} z_n = z$.*

Proof. For any $n \in \mathbb{N}$, let $\eta_n \in L^0(\mathcal{F}, \mathbb{R})$ with $\eta_n > a$ on Ω such that $\eta_n \to \infty$ as $n \to \infty$. Since

$$\eta_n R(\eta_n, A)z = \eta_n \int_0^\infty e^{-\eta_n t} V(t)z dt$$

for any $z \in S$ and $n \in \mathbb{N}$, it follows that

$$\begin{aligned}\|\eta_n R(\eta_n, A)z\| &= \eta_n \left\| \int_0^\infty e^{-\eta_n t} V(t)z dt \right\| \\ &\leq \eta_n \int_0^\infty \left| e^{-\eta_n t} \right| \|V(t)z\| dt \\ &\leq \frac{M\eta_n}{\eta_n - a} \|z\|,\end{aligned}$$

i.e., $\bigvee_{n\in\mathbb{N}} \|\eta_n R(\eta_n, A)z\| \in L^0_+(\mathcal{F})$. For any given $z \in S$, define

$$z_{\eta_n} := \begin{cases} \eta_n R(\eta_n, A)z, & \text{if } \eta_n > a \text{ on } \Omega, \\ 0, & \text{else,} \end{cases}$$

and then $\{z_{\eta_n}, n \in \mathbb{N}\} \subseteq D(A)$ and $\bigvee_{n\in\mathbb{N}} \left\| z_{\eta_n} \right\| \in L^0_+(\mathcal{F})$. According to Lemma 4.2, we get $\mathcal{T} - \lim_{n\to\infty} z_{\eta_n} = z$. □

Theorem 4.3 (The Hille-Yosida-type result)**.** *Suppose $A\colon D(A) \to S$ is a module homomorphism and $a \in L^0(\mathcal{F}, \mathbb{R})$, $M \in L^0_{++}(\mathcal{F})$ with $M \geq 1$. Then the following statements are equivalent.*

(*a*) *A is the infinitesimal generator of a bi-continuous semigroup $\{V(t)\colon t \geq 0\} \in H(M, a)$ on S.*

(*b*) (b_1)*. $D(A)$ is bi-dense in S.*

(b_2)*. $(\eta - a)^k \|R(\eta, A)^k\| \leq M$ for any $\eta \in L^0(\mathcal{F}, \mathbb{R})$ with $\eta > a$ on Ω and $k \in \mathbb{N}$.*

(b_3)*. For any $\eta, \alpha \in L^0(\mathcal{F}, \mathbb{R})$, the family $\{(\eta - \alpha)^k R(\eta, A)^k\colon k \in \mathbb{N}, \eta > \alpha \text{ on } \Omega\}$ is globally bi-equicontinuous for any $\alpha > a$ on Ω.*

Proof. (a) ⇒ (b). According to Lemma 4.3, we have that (b_1) holds.

For (b_2), without loss of generality, we can assume that k = 1. For any $z \in S$, by Lemma 4.2, we get

$$\begin{aligned}\|R(\eta, A)z\| &= \left\| \int_0^\infty e^{-\eta t} V(t)z dt \right\| \\ &\leq \int_0^\infty e^{-\eta t} \|V(t)z\| dt \\ &\leq \frac{M}{\eta - a} \|z\|\end{aligned}$$

for any $\eta \in L^0(\mathcal{F}, \mathbb{R})$ with $\eta > a$ on Ω, i.e., $(\eta - a)\|R(\eta, A)\| \leq M$.

For (b_3), suppose that there is a sequence $\{z_n, n \in \mathbb{N}\} \subseteq S$ such that $\bigvee_{n\in\mathbb{N}} \|z_n\| \in L^0_+(\mathcal{F})$ and $\mathcal{T} - \lim_{n\to\infty} z_n = 0$. Then, for any $p \in \mathcal{P}$ and $\alpha \in L^0(\mathcal{F}, \mathbb{R})$ with $\alpha > a$ on Ω, we have

$$\begin{aligned} p\left((\eta-\alpha)^k R(\eta, A)^k z_n\right) &= p\left((\eta-\alpha)^k \int_0^\infty \ldots \int_0^\infty e^{-\eta(t_1+\ldots+t_k)} V(t_1+\ldots+t_k) z_n dt_1 \ldots dt_k\right) \\ &= p\left((\eta-\alpha)^k \int_0^\infty \ldots \int_0^\infty e^{-(\eta-\alpha)(t_1+\ldots+t_k)} e^{-\alpha(t_1+\ldots+t_k)} V(t_1+\ldots+t_k) z_n dt_1 \ldots dt_k\right) \\ &\leq \bigvee_{t\geq 0} p\left(e^{-\alpha t} V(t) z_n\right) \end{aligned}$$

for any $k \in \mathbb{N}$, $\eta \in L^0(\mathcal{F}, \mathbb{R})$ with $\eta > \alpha$ on Ω. Due to Lemma 4.1, one has the family $\{(\eta-\alpha)^k R(\eta, A)^k \colon k \in \mathbb{N}, \eta > \alpha \textit{ on } \Omega\}$ is globally bi-equicontinuous.

$(b) \Rightarrow (a)$. Step 1. Let $T \in \mathcal{B}(S)$, and define a mapping $e^{tT}\colon S \to S$ by

$$e^{tT} z = \sum_{k=0}^{\infty} \frac{(tT)^k}{k!} z$$

for any $z \in S$ and $t \geq 0$. According to Lemma 3.1, the module homomorphism e^{tT} is well-defined.

For any $n \in \mathbb{N}$, let $\eta_n \in L^0(\mathcal{F}, \mathbb{R})$ with $\eta_n > a$ on Ω such that $\eta_n \to \infty$ as $n \to \infty$. Set $\Phi_{\eta_n} = \eta_n(\eta_n - A)^{-1}$ for any $n \in \mathbb{N}$, and then $\eta_n(\Phi_{\eta_n} - I) = \eta_n AR(\eta_n, A)$. Let

$$V_n(t)z = e^{\eta_n(\Phi_{\eta_n} - I)t} z$$

for any $n \in \mathbb{N}$, $t \geq 0$, and $z \in S$, and it is easy to check that $V_n(0)z = z$, $\mathcal{T} - \lim_{t\to t_0} V_n(t)z = V_n(t_0)z$ for any $t_0 \geq 0$, and $V_n(s+t)z = V_n(s)V_n(t)z$ for any $t, s \geq 0$. For any $t \geq 0$, $z \in S$, and $\alpha \in L^0(\mathcal{F}, \mathbb{R})$ with $\alpha > a$ on Ω, we have

$$\begin{aligned} p\left(V_n(t)z\right) &= e^{-\eta_n t} p\left(e^{\eta_n \Phi_{\eta_n} t} z\right) \\ &\leq e^{-\eta_n t} \sum_{k=0}^{\infty} \frac{t^k \eta_n^{2k}}{k!} p\left((\eta_n - A)^{-k} z\right) \\ &= e^{-\eta_n t} \sum_{k=0}^{\infty} \frac{t^k \eta_n^{2k}}{k!} \frac{1}{(\eta_n - \alpha)^k} p\left((\eta_n - \alpha)^k (\eta_n - A)^{-k} z\right) \end{aligned}$$

for any $\eta_n > 2\alpha$ on Ω, which shows that $\{V_n(t)\colon t \geq 0\}$ is locally bi-equicontinuous. Moreover, for any $t \geq 0$,

$$\begin{aligned} \|V_n(t)z\| &= e^{-\eta_n t} \left\|e^{\eta_n \Phi_{\eta_n} t} z\right\| \\ &\leq e^{-\eta_n t} \sum_{k=0}^{\infty} \frac{t^k \eta_n^{2k}}{k!} \left\|(\eta_n - A)^{-k} z\right\| \\ &\leq e^{-\eta_n t} \sum_{k=0}^{\infty} \frac{t^k \eta_n^{2k}}{k!} \frac{M}{(\eta_n - a)^k} \|z\| \\ &= M e^{\frac{a\eta_n t}{\eta_n - a}} \|z\| \\ &\leq M e^{2at} \|z\| \end{aligned}$$

for any $\eta_n > 2a$ on Ω, $z \in S$, and $n \in \mathbb{N}$, which shows that $\{V_n(t)\colon t \geq 0\}$ is exponentially bounded. Consequently, for any $n \in \mathbb{N}$, $\{V_n(t)\colon t \geq 0\}$ is a bi-continuous semigroup on S.

Step 2. Since $R(\eta, A) - R(\mu, A) = (\mu - \eta)R(\mu, A)R(\eta, A)$ for any $\eta, \mu \in L^0(\mathcal{F}, \mathbb{R})$ with $\eta, \mu > a$ on Ω, it follows that the family $\{\Phi_{\eta_n}, n \in \mathbb{N}\}$ is a commutative set. Thus for any $t, s \geq 0$, $V_m(t)\Phi_{\eta_n} = \Phi_{\eta_n}V_m(t)$ and $V_m(t)V_n(s) = V_n(s)V_m(t)$ for any $m, n \in \mathbb{N}$. For any $t \geq 0$, let $y(s) = V_n(t - s)V_m(s)z$ for any $z \in D(A)$ and $s \in [0, t]$, and we have $y'(s) = V_n(t - s)V_m(s)(\Phi_{\eta_m}Az - \Phi_{\eta_n}Az)$. By Theorem 4.1, y is locally L^0-Lipschitz. According to Theorem 3.2, one can obtain

$$\begin{aligned} V_m(t)z - V_n(t)z &= \int_0^t y'(s)ds \\ &= \int_0^t V_n(t-s)V_m(s)\left(\Phi_{\eta_m}Az - \Phi_{\eta_n}Az\right)ds \end{aligned}$$

for any $z \in D(A)$, $t \geq 0$, and $m, n \in \mathbb{N}$. Thus for any $p \in \mathcal{P}$, we have

$$p\left(V_m(t)z - V_n(t)z\right) \leq t \bigvee_{s\in[0,t]} p(V_n(t-s)V_m(s)(\Phi_{\eta_m}Az - \Phi_{\eta_n}Az) \tag{4.6}$$

for any $z \in D(A)$, $t \geq 0$, and $m, n \in \mathbb{N}$. Due to Lemma 4.2 and the locally bi-equicontinuity of $\{V_n(t)\colon t \geq 0\}$ for any $n \in \mathbb{N}$, we have $\{V_n(t)z, n \in \mathbb{N}\}$ is a $\mathcal{T}-$Cauchy sequence in S for any $z \in D(A)$. Since $D(A)$ is bi-dense in S, it follows that for any $z \in S$, there exists a $\|\cdot\|$-bounded sequence $\{y_k, k \in \mathbb{N}\} \subseteq D(A)$ such that

$$\mathcal{T} - \lim_{k\to\infty} y_k = z.$$

By the locally bi-equicontinuity of $\{V_n(t)\colon t \geq 0\}$ for any $n \in \mathbb{N}$, for any $l \geq 0$, one has

$$\mathcal{T} - \lim_{k\to\infty} V_n(t)y_k = V_n(t)z$$

uniformly for any $t \in [0, l]$. Since for any $z \in S$ and $t \geq 0$,

$$\begin{aligned} p(V_m(t)z - V_n(t)z) \leq\ & p(V_m(t)z - V_m(t)y_k) + p(V_n(t)z - V_n(t)y_k) \\ & + p(V_m(t)y_k - V_n(t)y_k) \end{aligned}$$

for any $p \in \mathcal{P}$ and $m, n \in \mathbb{N}$, it follows that $\{V_n(t)z, n \in \mathbb{N}\}$ is a $\mathcal{T}-$Cauchy sequence in S. By the sequential completeness of S, one has that $\mathcal{T} - \lim\limits_{n\to\infty} V_n(t)z$ exists for any $z \in S$ and $t \geq 0$, and we put

$$\mathcal{T} - \lim_{n\to\infty} V_n(t)z = V(t)z.$$

Step 3. We show that $\{V(t)\colon t \geq 0\}$ is a bi-continuous semigroup on S. Clearly, $\{V(t)\colon t \geq 0\}$ is exponentially bounded and $V(0)z = \mathcal{T} - \lim\limits_{n\to\infty} V_n(0)z = z$ for any $z \in S$. For any $z \in S$ and $t, s \geq 0$, we have

$$\mathcal{T} - \lim_{n\to\infty} V_n(t+s)z = V(t+s)z$$

and

$$\mathcal{T} - \lim_{n\to\infty} V_n(t)V_n(s)z = V(t)V(s)z.$$

Thus

$$V(t+s) = V(t)V(s)$$

for any $t, s \geq 0$. Next, we prove that

$$\mathcal{T} - \lim_{t \to t_0} V(t)\, z = V(t_0)\, z$$

for any $z \in S$ and $t_0 \geq 0$. For any $p \in \mathcal{P}$,

$$\begin{aligned} p\,(V(t)z - V(t_0)\, z) \leq\ & p(V(t)z - V_n(t)z) + p(V_n(t)z - V_n(t_0)\, z) \\ & + p\,(V_n(t_0)\, z - V(t_0)\, z) \end{aligned}$$

for any $z \in S$ and $t, t_0 \geq 0$, which implies that

$$\mathcal{T} - \lim_{t \to t_0} V(t)z = V(t_0)\, z.$$

Finally, it remains to show that $\{V(t)\colon t \geq 0\}$ is locally bi-equicontinuous. Let $\{z_k, k \in \mathbb{N}\} \subseteq S$ with $\bigvee_{k \in \mathbb{N}} \|z_k\| \in L^0_+(\mathcal{F})$ and $\mathcal{T} - \lim\limits_{k \to \infty} z_k = 0$, and we have

$$p(V(t)z_k) \leq p\,(V(t)z_k - V_n(t)z_k) + p\,(V_n(t)z_k)$$

for any $p \in \mathcal{P}$. Consequently, for any $l_1 \geq 0$, we get

$$\mathcal{T} - \lim_{k \to \infty} V(t)z_k = 0$$

uniformly for any $t \in [0, l_1]$.

Step 4. Next, we show that A is the infinitesimal generator of $\{V(t)\colon t \geq 0\}$. Suppose that $\widetilde{A}$ is the infinitesimal generator of $\{V(t)\colon t \geq 0\}$. For any $z \in D(A)$ and $p \in \mathcal{P}$, we have

$$\begin{aligned} & p\left(\frac{V(t)z - z}{t} - Az\right) \\ & \quad \leq p\left(\frac{V(t)z - z}{t} - \frac{V_m(t)z - z}{t}\right) \\ & \qquad + p\left(\frac{V_m(t)z - z}{t} - \Phi_{\eta_m} Az\right) + p\left(\Phi_{\eta_m} Az - Az\right) \end{aligned}$$

for any $m \in \mathbb{N}$ and $t > 0$. By Lemma 4.2 and (4.6), one can obtain

$$\mathcal{T} - \lim_{t \to 0} \frac{V(t)z - z}{t} = Az$$

for any $z \in D(A)$, which shows that $D(A) \subset D(\widetilde{A})$ and $Az = \widetilde{A}z$ for any $z \in D(A)$. Conversely, let $E = \eta z - \widetilde{A}z$ for any $z \in D(\widetilde{A})$ and $\eta, a \in L^0(\mathcal{F}, \mathbb{R})$ with $\eta > a$ on Ω, and we have

$$\begin{aligned} E &= (\eta - A)(\eta - A)^{-1}E \\ &= \eta(\eta - A)^{-1}E - \widetilde{A}(\eta - A)^{-1}E \\ &= (\eta - \widetilde{A})(\eta - A)^{-1}E. \end{aligned}$$

Thus $z = (\eta - \widetilde{A})^{-1}E = (\eta - A)^{-1}E \in D(A)$. □

Remark 4.3. *If $\mathcal{P}$ consists of a single L^0-norm $\|\cdot\|$, then the sequentially complete random Saks space $(S, \|\cdot\|, \mathcal{T})$ reduces to a complete RN module $(S, \|\cdot\|)$ and the bi-continuous semigroup $\{V(t)\colon t \geq 0\}$ reduces to an exponentially bounded C_0-semigroup on $(S, \|\cdot\|)$. Besides, if $\mathcal{F} = \{\Omega, \emptyset\}$, then the sequentially complete random Saks space $(S, \|\cdot\|, \mathcal{T})$ reduces to an ordinary sequentially complete Saks space and the bi-continuous semigroup $\{V(t)\colon t \geq 0\}$ reduces to an ordinary bi-continuous semigroup, which shows that Theorem 4.3 is an extension of Theorem 11, Corollary 1 in [16] and Theorem 16 in [3].*

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant No. 12171361) and the Humanities and Social Science Foundation of the Ministry of Education (Grant No. 20YJC790174).

Conflict of interest

The authors declare there are no conflicts of interest.

References

1. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, 1983. https://doi.org/10.1007/978-1-4612-5561-1
2. K. Engel, R. Nagel, One-parameter semigroups for linear evolution equations, *Semigroup Forum*, **63** (2001), 278–280. https://doi.org/10.1007/s002330010042
3. F. Kühnemund, A Hille-Yosida theorem for bi-continuous semigroups, *Semigroup Forum*, **67** (2003), 205–225. https://doi.org/10.1007/s00233-002-5000-3
4. J. B. Cooper, *Saks Spaces and Applications to Functional Analysis*, Elsevier, 1978.
5. K. Kruse, F. L. Schwenninger, On equicontinuity and tightness of bi-continuous semigroups, *J. Math. Anal. Appl.*, **509** (2022), 125985. https://doi.org/10.1016/j.jmaa.2021.125985
6. A. A. Albanese, E. Mangino, Trotter-Kato theorems for bi-continuous semigroups and applications to Feller semigroups, *J. Math. Anal. Appl.*, **289** (2004), 477–492. https://doi.org/10.1016/j.jmaa.2003.08.032
7. C. Budde, S. Wegner, A Lumer-Phillips type generation theorem for bi-continuous semigroups, *Z. Anal. Anwend.*, **41** (2022), 65–80. https://doi.org/10.4171/ZAA/1695
8. B. Farkas, Perturbations of bi-continuous semigroups, *Studia Math.*, **161** (2004), 147–161.
9. C. Budde, B. Farkas, A Desch-Schappacher perturbation theorem for bi-continuous semigroups, *Math. Nachr.*, **293** (2020), 1053–1073. https://doi.org/10.1002/mana.201800534

10. N. Gigli, *Nonsmooth Differential Geometry-An Approach Tailored for Spaces with Ricci Curvature Bounded from Below*, American Mathematical Society, 2018. http://doi.org/10.1090/memo/1196
11. T. Guo, Relations between some basic results derived from two kinds of topologies for a random locally convex module, *J. Funct. Anal.*, **258** (2010), 3024–3047. https://doi.org/10.1016/j.jfa.2010.02.002
12. T. Guo, E. Zhang, Y. Wang, Z. Guo, Two fixed point theorems in complete random normed modules and their applications to backward stochastic equations, *J. Math. Anal. Appl.*, **483** (2020), 123644. https://doi.org/10.1016/j.jmaa.2019.123644
13. M. Liu, X. Zhang, L. Dai, Trotter-Kato approximations of impulsive neutral SPDEs in Hilbert spaces, *Acta Math. Sin. (Engl. Ser.)*, **40** (2024), 1229–1243. https://doi.org/10.1007/s10114-023-1553-8
14. M. Liu, X. Zhang, L. Dai, Trotter-Kato approximations of semilinear stochastic evolution equations in Hilbert spaces, *J. Math. Phys.*, **64** (2023), 043506. https://doi.org/10.1063/5.0137515
15. M. Lučić, E. Pasqualetto, I. Vojnović, On the reflexivity properties of Banach bundles and Banach modules, *Banach J. Math. Anal.*, **18** (2024), 7. https://doi.org/10.1007/s43037-023-00315-9
16. D. H. Thang, T. C. Son, N. Thinh, Semigroups of continuous module homomorphisms on complex complete random normed modules, *Lith. Math. J.*, **59** (2019), 229–250. https://doi.org/10.1007/s10986-019-09442-z
17. T. Guo, X. Zhang, Stone's representation theorem of a group of random unitary operators on complete complex random inner product modules, *Sci. Sin. Math.*, **42** (2012), 181–202.
18. X. Zhang, H. Zhang, B. Wang, M. Liu, The Cauchy initial value problem in complete random normed modules, *Math. Methods Appl. Sci.*, **43** (2020), 4007–4015. https://doi.org/10.1002/mma.6169
19. N. Dunford, J. T. Schwartz, *Linear Operators*, Interscience, 1958.
20. X. Zhang, M. Liu, On almost surely bounded semigroups of random linear operators, *J. Math. Phys.*, **54** (2013), 053517. https://doi.org/10.1063/1.4805049
21. T. Guo, S. Peng, A characterization for an $L(\mu, \mathbb{K})$-topological module to admit enough canonical module homomorphisms, *J. Math. Anal. Appl.*, **263** (2001), 580–599. https://doi.org/10.1006/jmaa.2001.7637

AIMS Press