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Research note

Several expressions for moments of sums of hyperbolic secant random variables

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Abstract: In this paper, we find several expressions for the moments of the hyperbolic secant distribution and the moments of the sum of two independent distributions. They are expressed in terms of Euler and Bernoulli numbers, zeta values, an integral representation, and certain infinite series.

Keywords: hyperbolic secant random variable; Euler numbers; Bernoulli numbers

1. Introduction

In recent years, probabilisitic extensions of many special numbers and polynomials have been intensively explored, following the introduction of probabilistic Stirling numbers of the second kind (see [1]). Specifically, some of those probabilisitc extensions and related results are as follows:

- [2] treated the intricate properties of degenerate Poisson random variables, including their moment generating function, the law of large numbers, and the central limit theorem;
- [3] derived a Spivey-type recurrence relation for the probabilistic r-Bell polynomials;
- [4] computed the expectations of random variables associated with Bernoulli and uniform random variables;
- [5] studied the probabilistic extensions of Bernoulli and Euler polynomials;
- [6] examined the expectations of random variables associated with fully degenerate Bernoulli and Euler polynomials;
- [7] explored the probabilistic degenerate Laguerre polynomials;
- [8] deduced the degenerate moment generating functions for several random variables; and
- [9] investigated the probabilistic degenerate poly-Bell polynomials.

In this paper, we deduce several expressions for the moments of the hyperbolic secant distribution and the moments of the sum of two independent distributions. They are expressed in terms of Euler and Bernoulli numbers, zeta values, an integral representation, and certain infinite series.

The structure of this paper is split into three sections: Section 1 provides a review of several key concepts. We remind the reader of Euler and Bernoulli numbers. Following this, we recall the Fourier series expansion for odd functions, the Riemann zeta function and its special values at even positive integers, and the uniform random variable on (a,b). Section 2 contains the main results. Theorems 2.1, 2.3, and 2.4 present three distinct expressions for the even moments of a hyperbolic secant random variable, whose the odd moments are zero; these expressions are given as an infinite series, in terms of Euler numbers, and as a difference of Hurwitz and Riemann zeta values at odd positive integers (up to a constant), respectively. Theorem 2.2 determines the moment generating function of the hyperbolic secant random variable using the Fourier expansion for the sine function. Theorems 2.5, 2.7, 2.9, and Corollary 2.6 focus on the even moments of the sum of two independent hyperbolic secant random variables, $X+Y$, whose odd moments also vanish; these moments are expressed in terms of Bernoulli numbers, certain integrals, infinite series, and special values of the Riemann zeta function at even positive integers, respectively. Theorem 2.8 derives the values of the Riemann zeta function at integers $k \geq 2$ as an infinite series that involves the unsigned Stirling numbers of the first kind. Section 3 concludes the paper.

In probability theory and statistics, the hyperbolic secant distribution is a continuous probability distribution whose probability density function and characteristic function are proportional to the hyperbolic secant function. A random variable follows a hyperbolic secant distribution if its probability density function is given by the following:

$$f(x) = \frac{1}{2}\operatorname{sech}\left(\frac{\pi}{2}x\right), \quad (-\infty < x < \infty), \quad (\text{see } [10,11]).$$

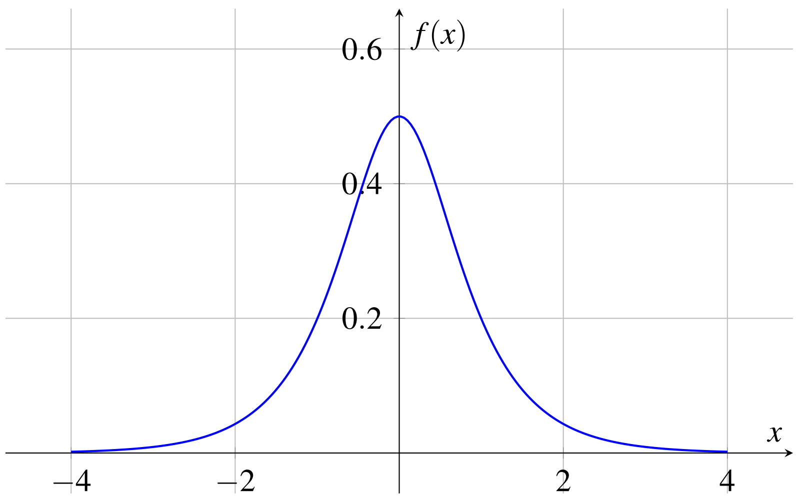

Figure 1. The shape of probability density function $\frac{1}{2}\operatorname{sech}(\frac{\pi}{2}x)$.

The hyperbolic secant distribution has several applications, particularly in finance, image processing, statistics, regression modeling, and plasma physics. Here is a brief account of those applications.

• Finance: Fischer [12] discussed the use of generalized hyperbolic secant distributions in finance. These distributions are a popular subclass of Perk's family and can model financial return distributions, such as those of stock indices and exchange rates. They are particularly useful because they can accommodate both thin and fat tails, which allow for a better fit to the "semi-heavy tails" often observed in financial assets compared to the normal distribution.

• Image processing: Castro-Macías et al. [13] applied a Hyperbolic Secant prior to Bayesian Blind Image Deconvolution. The hyperbolic secant distribution's heavy tails and peaked nature make it suitable for promoting sparsity, which is a desirable property to disentangle the stain mixture in histological images. This method improves the performance of blind color deconvolution compared to other methods that use different priors.

• Statistics: Ding [14] highlighted three natural occurrences of the hyperbolic-secant distribution. In Fisher's analysis of similarity between twins, the distribution arose from the Fisher's z-transformation of the intraclass correlation coefficient. It appears as the Jeffreys' prior for contingency tables, which is a specific type of prior probability distribution used in Bayesian inference. Additionally, it is naturally generated in the context of invalid instrumental variables.

• Regression modeling: Smyth [15] introduced a hyperbolic secant regression as a method to model cross-correlations. This approach provides a new way to estimate the correlation coefficient, particularly when dealing with bivariate normal samples. The regression model was constructed using a transformation related to the hyperbolic secant distribution.

• Plasma physics: Shagayda et al. [16] used the hyperbolic secant distribution to model the electron velocity distribution in a collisional, inhomogeneous plasma. They derived the moments (such as the mean and variance) of this distribution, which are crucial to understand the macroscopic properties of the plasma in the presence of crossed electric and magnetic fields. This application highlights the distribution's relevance beyond traditional statistical and financial modeling.

Moreover, Barndorff-Nielsen and Halgreen [17] proved that the hyperbolic secant distribution is infinitely divisible. Kaplya [10] introduced a new family of distributions that included both the hyperbolic secant distribution and the logistic distribution as special cases. Podgórski and Wallin [18] investigated the hyperbolic secant distribution in the context of generalized hyperbolic distributions. Yilmaz [11] introduced a new circular distribution called the inverse stereographic hyperbolic secant distribution.

As general references for this paper, one may refer to [19–21] for relevant books, [22] for study of some identities related to several special numbers, and [23–25] for investigations of some properties of random variables.

The Euler numbers $E_k,\ (k \geq 0)$, are defined by the following:

$$\frac{2}{e^x + e^{-x}} = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}, \quad (\text{see } [19,26]). \tag{1.1}$$

Note that $E_0 = 1,\ E_1 = 0,\ E_2 = -1,\ E_3 = 0,\ E_4 = 5,\ E_5 = 0,\ E_6 = -61,\ E_7 = 0,\ E_8 = 1385,\ E_9 = 0,\ E_{10} = -50521, \ldots$, and $E_{2k+1} = 0,\ (k \geq 0)$.
From (1.1), we note that

$$\sec x = \frac{2}{e^{ix} + e^{-ix}} = \sum_{n=0}^{\infty} \frac{i^n E_n}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n}, \quad (i = \sqrt{-1}), \quad (\text{see } [27]). \tag{1.2}$$

The Fourier series on $(-p, p)$ is given by

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{p}\right), \tag{1.3}$$

where

$$a_n = \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)dx, \quad \text{(see [21])}, \tag{1.4}$$

and f is an odd function.

The Bernoulli numbers are given by the following generating function:

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad \text{(see [5,28])}. \tag{1.5}$$

Note that the first few terms of B_n are given by the following:

$$\begin{aligned}
&B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66},\\
&B_{12} = -\frac{691}{2730}, B_{14} = \frac{7}{6}, B_{16} = -\frac{3617}{510}, B_{18} = \frac{43867}{798}, B_{20} = -\frac{174611}{330}, \ldots;\\
&B_{2k+1} = 0, \ (k \geq 1).
\end{aligned}$$

The Riemann zeta function is defined by the following:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad (\mathrm{Re}(s) > 1), \quad \text{(see [19,26])}.$$

Then, we have the following:

$$\zeta(2n) = \frac{(-1)^{n-1}(2\pi)^{2n}}{2(2n)!} B_{2n}, \quad (n \in \mathbb{N}). \tag{1.6}$$

Let X be the uniform random variable on (a,b). Then, the probability density function of X is given by the following(see [20]):

$$f(x) = \begin{cases} \frac{1}{b-a}, & \text{for } a < x < b, \\ 0, & \text{otherwise.} \end{cases} \tag{1.7}$$

2. Several expressions for moments of sums of hyperbolic secant random variables

Let X be the hyperbolic secant random variable whose probability density function is given by the following (see [12], Figure 1.):

$$f_X(x) = \frac{1}{2}\operatorname{sech}\frac{\pi}{2}x = \frac{1}{e^{\frac{\pi}{2}x} + e^{-\frac{\pi}{2}x}}, \quad (-\infty < x < \infty). \tag{2.1}$$

From (1.3) and (1.4), we note that

$$\sin xt = \sum_{n=1}^{\infty} a_n \sin nx, \tag{2.2}$$

where

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin(xt)\sin(nx)dx = \frac{2}{\pi}\int_0^{\pi} \frac{\cos(n-t)x - \cos(n+t)x}{2}dx \tag{2.3}$$

$$= \frac{1}{\pi}\left[\frac{\sin(n-t)x}{n-t} - \frac{\sin(n+t)x}{n+t}\right]_0^{\pi} = \frac{(-1)^{n-1}}{n^2-t^2}\frac{2n}{\pi}\sin\pi t.$$

Thus, by (2.2) and (2.3), we obtain the following:

$$\begin{aligned}
\sin\frac{\pi}{2}t &= \sum_{n=1}^{\infty} a_n \sin\frac{\pi}{2}n = \sum_{n=1}^{\infty} a_{2n-1}(-1)^{n-1} \\
&= \sum_{n=1}^{\infty}(-1)^{n-1}(-1)^{2n-2}\frac{2}{\pi}\sin(\pi t)\left(\frac{2n-1}{(2n-1)^2-t^2}\right) \\
&= \frac{2}{\pi}\sin(\pi t)\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{2n-1}{(2n-1)^2\left(1-\left(\frac{t}{2n-1}\right)^2\right)}\right) \\
&= \frac{2}{\pi}\sin(\pi t)\sum_{n=1}^{\infty}(-1)^{n-1}\sum_{k=0}^{\infty}\frac{t^{2k}}{(2n-1)^{2k+1}}.
\end{aligned} \tag{2.4}$$

From (2.4), we note that

$$\sum_{k=0}^{\infty}\frac{2}{\pi}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{2k+1}}\right)t^{2k} = \frac{\sin\frac{\pi}{2}t}{\sin\pi t} = \frac{1}{2}\frac{1}{\cos\frac{\pi}{2}t}. \tag{2.5}$$

Noting that

$$\int_{-\infty}^{\infty}\frac{x^n}{e^{\frac{\pi}{2}x}+e^{-\frac{\pi}{2}x}}dx = \begin{cases} 2\int_0^{\infty}\frac{x^n}{e^{\frac{\pi}{2}x}+e^{-\frac{\pi}{2}x}}dx, & \text{for } n \text{ even}, \\ 0, & \text{for } n \text{ odd}, \end{cases}$$

and using (2.1), we have the following:

$$\begin{aligned}
E\left[e^{Xt}\right] &= \int_{-\infty}^{\infty}\frac{e^{xt}}{e^{\frac{\pi}{2}x}+e^{-\frac{\pi}{2}x}}dx \\
&= \sum_{n=0}^{\infty}\frac{t^n}{n!}\int_{-\infty}^{\infty}\frac{x^n}{e^{\frac{\pi}{2}x}+e^{-\frac{\pi}{2}x}}dx \\
&= \sum_{n=0}^{\infty}\frac{2t^{2n}}{(2n)!}\int_0^{\infty}\frac{x^{2n}}{1+e^{-\pi x}}e^{-\frac{\pi}{2}x}dx \\
&= \sum_{n=0}^{\infty}\frac{2t^{2n}}{(2n)!}\sum_{k=0}^{\infty}(-1)^k\int_0^{\infty}e^{-\frac{\pi}{2}x(2k+1)}x^{2n}dx \\
&= \sum_{n=0}^{\infty}\frac{2t^{2n}}{(2n)!}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{2n+1}}\left(\frac{2}{\pi}\right)^{2n+1}\int_0^{\infty}e^{-y}y^{2n}dy \\
&= \sum_{n=0}^{\infty}\frac{2t^{2n}}{(2n)!}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{2n+1}}\left(\frac{2}{\pi}\right)^{2n+1}\Gamma(2n+1) \\
&= \sum_{n=0}^{\infty}\frac{t^{2n}}{(2n)!}\left(\frac{2^{2n+2}}{\pi^{2n+1}}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{2n+1}}\Gamma(2n+1)\right).
\end{aligned} \tag{2.6}$$

Therefore, by (2.6), we obtain the following theorem.

Theorem 2.1. *Let X be the hyperbolic secant random variable. For $n \geq 0$, we have the following:*

$$E\left[X^{2n}\right] = \frac{2^{2n+2}}{\pi^{2n+1}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} \Gamma(2n+1), \quad E\left[X^{2n+1}\right] = 0.$$

From (2.5) and (2.6), we have the following:

$$\begin{aligned} E\left[e^{Xt}\right] &= \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} \left(\frac{2^{2n+2}}{\pi^{2n+1}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} \Gamma(2n+1) \right) \\ &= \sum_{n=0}^{\infty} t^{2n} \left(\frac{2}{\pi}\right)^{2n} \left(\frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} \right) \\ &= 2 \sum_{n=0}^{\infty} \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} \left(\frac{2}{\pi} t\right)^{2n} \\ &= \frac{1}{\cos t} = \frac{2}{e^{it} + e^{-it}}, \quad (i = \sqrt{-1}). \end{aligned} \tag{2.7}$$

Therefore, by (2.7), we obtain the following theorem.

Theorem 2.2. *Let X be the hyperbolic secant random variable. Then, the moment generating function of X is given by the following:*

$$E\left[e^{Xt}\right] = \sec t = \frac{2}{e^{it} + e^{-it}}, \quad (i = \sqrt{-1}). \tag{2.8}$$

From (1.2) and (2.8), we note that

$$\sum_{n=0}^{\infty} E\left[X^n\right] \frac{t^n}{n!} = \frac{2}{e^{it} + e^{-it}} = \sum_{n=0}^{\infty} E_{2n} \frac{(-1)^n t^{2n}}{(2n)!}. \tag{2.9}$$

Therefore, by (2.9), we obtain the following theorem. Here, we recall that the variance of X is given by $\mathrm{Var}(X) = E[X^2] - E[X]^2$.

Theorem 2.3. *Let X be the hyperbolic secant random variable. For $n \geq 0$, we have the following:*

$$E\left[X^{2n}\right] = (-1)^n E_{2n}, \quad E\left[X^{2n+1}\right] = 0, \quad \text{and} \quad \mathrm{Var}(X) = 1.$$

By a simple calculation, we obtain the following:

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} &= 2 \sum_{k=1}^{\infty} \frac{1}{(4k-3)^{2n+1}} + \sum_{k=1}^{\infty} \frac{1}{(2k)^{2n+1}} - \sum_{k=1}^{\infty} \frac{1}{k^{2n+1}} - 1 \\ &= 2 \sum_{k=0}^{\infty} \frac{1}{(4k+1)^{2n+1}} + \frac{1}{2^{2n+1}} \sum_{k=1}^{\infty} \frac{1}{k^{2n+1}} - \sum_{k=1}^{\infty} \frac{1}{k^{2n+1}} - 1. \end{aligned} \tag{2.10}$$

Thus, by (2.10), we have the following:

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} = \frac{1}{2^{4n+1}} \zeta\left(2n+1, \frac{1}{4}\right) - \frac{2^{2n+1}-1}{2^{2n+1}} \zeta(2n+1), \tag{2.11}$$

where $\zeta(s,x)$ is the Hurwitz zeta function defined by (see [26])

$$\zeta(s,x)=\sum_{n=0}^{\infty}\frac{1}{(n+x)^{s}},\ (x>0).$$

Therefore, by (2.6) and (2.11), we obtain the following theorem.

Theorem 2.4. *Let X be the hyperbolic secant random variable. For $n\geq 0$, we have the following:*

$$E\left[X^{2n}\right]=(2n)!\left(\frac{2}{\pi}\right)^{2n}\frac{4}{\pi}\left(\frac{1}{2^{4n+1}}\zeta\left(2n+1,\frac{1}{4}\right)-\frac{2^{2n+1}-1}{2^{2n+1}}\zeta(2n+1)\right).$$

Let X and Y be independent hyperbolic secant random variables. Their probability density functions are given by the following:

$$f_X(x)=\frac{1}{e^{\frac{\pi}{2}x}+e^{-\frac{\pi}{2}x}},\quad f_Y(y)=\frac{1}{e^{\frac{\pi}{2}y}+e^{-\frac{\pi}{2}y}},\quad \text{where } -\infty<x,y<\infty.$$

From (2.7), the moment generating function of $X+Y$ is equal to the following:

$$\begin{aligned}E\left[e^{(X+Y)t}\right]&=E\left[e^{Xt}e^{Yt}\right]=E\left[e^{Xt}\right]E\left[e^{Yt}\right]\\&=\int_{-\infty}^{\infty}e^{xt}f_X(x)dx\int_{-\infty}^{\infty}e^{yt}f_Y(y)dy=\sec^2 t.\end{aligned}\tag{2.12}$$

Now, we observe that

$$\begin{aligned}i\tan x&=\frac{e^{ix}-e^{-ix}}{e^{ix}+e^{-ix}}=\frac{e^{2ix}-1}{e^{2ix}+1}\\&=1-\frac{2}{e^{2ix}+1}=1-\frac{2}{e^{2ix}-1}+\frac{4}{e^{4ix}-1}\end{aligned}\tag{2.13}$$

Thus, by (1.5) and (2.13), we obtain the following:

$$\begin{aligned}x\tan x&=-xi+\frac{2xi}{e^{2xi}-1}-\frac{4xi}{e^{4xi}-1}\\&=-xi+\sum_{n=0}^{\infty}\frac{2^n i^n B_n}{n!}x^n-\sum_{n=0}^{\infty}\frac{4^n i^n B_n}{n!}x^n\\&=\sum_{n=2}^{\infty}\frac{2^n i^n B_n}{n!}x^n-\sum_{n=2}^{\infty}\frac{4^n i^n B_n}{n!}x^n\\&=\sum_{n=1}^{\infty}\frac{(-1)^n 2^{2n}B_{2n}}{(2n)!}x^{2n}-\sum_{n=1}^{\infty}\frac{(-1)^n 4^{2n}B_{2n}}{(2n)!}x^{2n}\\&=\sum_{n=1}^{\infty}\frac{(-1)^n B_{2n}4^n(1-4^n)}{(2n)!}x^{2n}\\&=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}B_{2n+2}4^{n+1}(1-4^{n+1})}{(2n+2)!}x^{2n+2}.\end{aligned}\tag{2.14}$$

Thus, by (2.14), we have the following:

$$\tan x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} B_{2n+2} 4^{n+1} (1-4^{n+1})}{(2n+2)!} x^{2n+1}. \tag{2.15}$$

From (2.12) and (2.15), we note that

$$\begin{aligned} E\left[e^{(X+Y)t}\right] &= \sec^2 t = \frac{d}{dt} \tan t \\ &= \frac{d}{dt}\left(\sum_{n=0}^{\infty} \frac{(-1)^{n+1} B_{2n+2} 4^{n+1} (1-4^{n+1})}{(2n+2)!} t^{2n+1} \right) \\ &= \sum_{n=0}^{\infty} \frac{(2n+1)}{(2n+2)!} (-1)^{n+1} B_{2n+2} 4^{n+1} (1-4^{n+1}) t^{2n} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} B_{2n+2}}{2n+2} 4^{n+1} \left(1-4^{n+1}\right) \frac{t^{2n}}{(2n)!}. \end{aligned} \tag{2.16}$$

Therefore, by (2.16), we obtain the following theorem.

Theorem 2.5. *Let X and Y be independent hyperbolic secant random variables. Then, for $n \geq 0$, we have*

$$E\left[(X+Y)^{2n}\right] = (-1)^{n+1} \frac{B_{2n+2}}{2n+2} 2^{2n+2} \left(1-2^{2n+2}\right),$$

and

$$E\left[(X+Y)^{2n+1}\right] = 0.$$

From (1.6) and Theorem 2.5, we obtain the following corollary.

Corollary 2.6. *Let X and Y be independent hyperbolic secant random variables. For $n \geq 0$, we have the following:*

$$E\left[(X+Y)^{2n}\right] = 2\left(4^{n+1}-1\right) \frac{(2n+1)!}{\pi^{2n+2}} \zeta(2n+2).$$

Let X and Y be independent hyperbolic secant random variables, and let $Z = X+Y$. Then, we have the following:

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx \\ &= \int_{-\infty}^{\infty} \frac{1}{e^{\frac{\pi}{2}x} + e^{-\frac{\pi}{2}x}} \frac{1}{e^{\frac{\pi}{2}(z-x)} + e^{-\frac{\pi}{2}(z-x)}} dx \\ &= \int_{-\infty}^{\infty} \frac{e^{-\frac{\pi z}{2}} e^{\pi x}}{(1+e^{\pi x})(1+e^{\pi x} e^{-\pi z})} dx \\ &= \frac{2}{\pi} \int_0^{\infty} \frac{t e^{-\pi z/2}}{(1+t^2)(1+t^2 e^{-\pi z})} dt \\ &= \frac{2}{\pi} \frac{1}{e^{\frac{\pi z}{2}} - e^{-\frac{\pi z}{2}}} \int_0^{\infty} \left(\frac{t}{1+t^2} - \frac{t e^{-\pi z}}{1+t^2 e^{-\pi z}} \right) dt \end{aligned} \tag{2.17}$$

$$
\begin{aligned}
&= \frac{2}{\pi} \frac{1}{e^{\frac{\pi z}{2}} - e^{-\frac{\pi z}{2}}} \left[\frac{1}{2} \log \left(\frac{1+t^2}{1+t^2 e^{-\pi z}} \right) \right]_0^\infty \\
&= \frac{2}{\pi} \frac{1}{e^{\frac{\pi z}{2}} - e^{-\frac{\pi z}{2}}} \frac{1}{2} \log e^{\pi z} = \frac{z}{e^{\frac{\pi z}{2}} - e^{-\frac{\pi z}{2}}} = \frac{z}{2} \operatorname{csch} \frac{\pi z}{2}.
\end{aligned}
$$

Then, using (2.17), we get

$$
\begin{aligned}
E\left[Z^{2n}\right] &= E\left[(X+Y)^{2n}\right] = \int_{-\infty}^{\infty} f_{X+Y}(z) z^{2n} dz \\
&= \frac{1}{2} \int_{-\infty}^{\infty} z^{2n+1} \operatorname{csch} \frac{z\pi}{2} dz = \int_{-\infty}^{\infty} \frac{z^{2n+1}}{e^{\frac{\pi z}{2}} - e^{-\frac{\pi z}{2}}} dz.
\end{aligned} \tag{2.18}
$$

By (2.17) and (2.18), we obtain the following theorem.

Theorem 2.7. *Let X and Y be independent hyperbolic secant random variables. Then, the probability density function of $X+Y$ is given by*

$$
f_{X+Y}(z) = \frac{z}{2} \operatorname{csch} \frac{\pi z}{2},
$$

and

$$
E\left[(X+Y)^{2n}\right] = \frac{1}{2} \int_{-\infty}^{\infty} z^{2n+1} \operatorname{csch} \frac{\pi z}{2} dz = \int_{-\infty}^{\infty} \frac{z^{2n+1}}{e^{\frac{\pi z}{2}} - e^{-\frac{\pi z}{2}}} dz.
$$

Remark 2.8. *Here, we derived the probability density function of the sum of two independent hyperbolic secant random variables by directly computing the convolution $f_X * f_Y$ without using a complex analysis. It turns out that the probability density function of the sum of n independent hyperbolic secant random variables $X_1, X_2, \ldots, X_n$ was determined in [29]. To state Baten's result, we recall the following. Let $S_n = X_1 + X_2 + \cdots + X_n$, and let f_j be the probability density function of X_j, for $j = 1, 2, \ldots, n$. As $f_{S_n} = f_1 * f_2 * \cdots * f_n$, the characteristic function ϕ_{S_n} of S_n is given by the product $\phi_{S_n}(t) = \phi_1(t)\phi_2(t)\cdots\phi_n(t)$, where the characteristic function ϕ_j of X_j is given by the following:*

$$
\phi_j(t) = \int_{-\infty}^{\infty} e^{-itx} f_j(x) dx = \int_{-\infty}^{\infty} e^{-itx} \frac{1}{e^{\frac{\pi}{2}x} + e^{-\frac{\pi}{2}x}} dx = \operatorname{sech} t.
$$

Thus, the probability density function of S_n is equal to the following:

$$
f_{S_n}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx} (\operatorname{sech} t)^n dt = \frac{2^n}{\pi} \int_0^{\infty} \cos tx \frac{1}{(e^t + e^{-t})^n} dt. \tag{2.19}
$$

By applying the residue theorem for a contour integral, Baten determined (2.19) *in p.288 of [29], which is given by the following:*

$$
\begin{aligned}
f_{S_{2n}}(x) &= \frac{x \operatorname{csch} \frac{\pi x}{2}}{2(2n-1)!} \prod_{j=1}^{n-1} (x^2 + (2j)^2), \\
f_{S_{2n+1}}(x) &= \frac{\operatorname{sech} \frac{\pi x}{2}}{2(2n)!} \prod_{j=0}^{n-1} (x^2 + (2j+1)^2).
\end{aligned} \tag{2.20}
$$

From (2.20)*, it is immediate to see that*

$$\begin{aligned} E[(X_1+X_2+\cdots+X_{2n})^k] &= \frac{1}{(2n-1)!}\int_{-\infty}^{\infty}\frac{x^{k+1}}{e^{\frac{\pi x}{2}}-e^{-\frac{\pi x}{2}}}\prod_{j=1}^{n-1}(x^2+(2j)^2)dx, \\ E[(X_1+X_2+\cdots+X_{2n+1})^k] &= \frac{1}{(2n)!}\int_{-\infty}^{\infty}\frac{x^{k}}{e^{\frac{\pi x}{2}}+e^{-\frac{\pi x}{2}}}\prod_{j=0}^{n-1}(x^2+(2j+1)^2)dx. \end{aligned} \tag{2.21}$$

In particular, from (2.21)*, we see that all odd moments of $S_n = X_1+X_2+\cdots+X_n$ vanish for n.*

Recall that the unsigned Stirling numbers of the first kind are defined by the following:

$$\frac{1}{k!}\log^k\left(\frac{1}{1-t}\right) = \sum_{n=k}^{\infty}\begin{bmatrix} n \\ k \end{bmatrix}\frac{t^n}{n!}, \quad (k \geq 0). \tag{2.22}$$

Let $Y_{k+1} = X_1X_2\cdots X_{k+1}$ $(k \geq 1)$ be the product of $k+1$ independent uniform random variables X_j, $(j = 1,2,\ldots,k+1)$, on $(0,1)$ (see (1.7)). Then, from Theorem 1 of [30] with $a = 0$ and $b = 1$, the probability density function of Y_{k+1} is given by the following:

$$f_{Y_{k+1}}(x) = \begin{cases} \frac{1}{k!}\log^k(\frac{1}{x}), & \text{for } 0 < x < 1, \\ 0, & \text{otherwise}. \end{cases}$$

Then, using the transformation method, the probability density function of $1-Y_{k+1}$ is seen to be

$$f_{1-Y_{k+1}}(x) = \begin{cases} \frac{1}{k!}\log^k(\frac{1}{1-x}), & \text{for } 0 < x < 1, \\ 0, & \text{otherwise}. \end{cases} \tag{2.23}$$

Now, we compute $E[\frac{1}{1-Y_{k+1}}]$ in two different ways.
On the one hand, using (2.22) and (2.23), it is equal to

$$E\left[\frac{1}{1-Y_{k+1}}\right] = \int_0^1 \frac{1}{x}f_{1-Y_{k+1}}(x)dx = \sum_{n=k}^{\infty}\begin{bmatrix} n \\ k \end{bmatrix}\frac{1}{n!n}. \tag{2.24}$$

On the other hand, it is equal to

$$\begin{aligned} E\left[\frac{1}{1-Y_{k+1}}\right] &= \sum_{n=0}^{\infty}E[X_1^n]E[X_2^n]\cdots E[X_{k+1}^n] \\ &= \sum_{n=0}^{\infty}\frac{1}{(n+1)^{k+1}} = \zeta(k+1). \end{aligned} \tag{2.25}$$

By combining (2.24) and (2.25), we obtain the following theorem.

Theorem 2.9. *The Riemann zeta function $\zeta(s)$ has infinite series expressions at the integers $s = 2,3,\ldots,$ that involve the unsigned Stirling numbers of the first kind given by the following:*

$$\zeta(k+1) = \sum_{n=k}^{\infty}\frac{\begin{bmatrix} n \\ k \end{bmatrix}}{n!n}, \quad (k \geq 1).$$

Furthermore, by Corollary 2.6 and Theorem 2.9, we obtain the next theorem.

Theorem 2.10. *Let X and Y be independent hyperbolic secant random variables. Then, for $n \geq 0$, we have the following:*

$$E\left[(X+Y)^{2n}\right] = \frac{(2n+1)!2(4^{n+1}-1)}{\pi^{2n+2}} \sum_{k=2n+1}^{\infty} \frac{\left[{k \atop 2n+1}\right]}{k!k}.$$

Remark 2.11. *It is known that*

$$\cosh t = \prod_{k=0}^{\infty}\left(1+\left(\frac{2t}{(2k+1)\pi}\right)^2\right), \quad (\text{see [27, Corollary 4]}). \tag{2.26}$$

By (2.7) *and* (2.26)*, we obtain the following:*

$$E\left[e^{Xt}\right] = \prod_{k=0}^{\infty}\left(1-\left(\frac{2t}{(2k+1)\pi}\right)^2\right)^{-1},$$

where X is the hyperbolic secant random variable.

3. Conclusions

In recent years, many authors have investigated probabilisitic extensions of various special numbers and polynomials and their applications. Here, we studied several expressions for the moments of the hyperbolic secant random variable and for the sum of two independent random variables. Specifically, they are described below.

Let X be the hyperbolic secant random variable. Then, we showed that even moments of X are given by

$$\begin{aligned} E\left[X^{2n}\right] &= \frac{2^{2n+2}}{\pi^{2n+1}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} \Gamma(2n+1) \\ &= (-1)^n E_{2n} \\ &= (2n)!\left(\frac{2}{\pi}\right)^{2n} \frac{4}{\pi}\left(\frac{1}{2^{4n+1}}\zeta\left(2n+1,\frac{1}{4}\right) - \frac{2^{2n+1}-1}{2^{2n+1}}\zeta(2n+1)\right), \quad (n \geq 0), \end{aligned}$$

and all odd moments of X vanish.

Let X and Y be independent hyperbolic secant random variables. Then, we showed that even moments of $X+Y$ are given by

$$\begin{aligned} E\left[(X+Y)^{2n}\right] &= (-1)^{n+1}\frac{B_{2n+2}}{2n+2}2^{2n+2}\left(1-2^{2n+2}\right) \\ &= 2\left(4^{n+1}-1\right)\frac{(2n+1)!}{\pi^{2n+2}}\zeta(2n+2) \\ &= \frac{1}{2}\int_{-\infty}^{\infty} z^{2n+1}\operatorname{csch}\frac{z\pi}{2}dz \\ &= \frac{(2n+1)!2(4^{n+1}-1)}{\pi^{2n+2}} \sum_{k=2n+1}^{\infty} \frac{\left[{k \atop 2n+1}\right]}{k!k}, \quad (n \geq 0), \end{aligned}$$

and all odd moments of $X+Y$ vanish.

Our future work will continue to explore the probability theory, special functions, the analytic number theory, and their connections, alongside their applications to mathematics, physics, science, and engineering.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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