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Review

Observations on Abelian sandpile models for directed graphs

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In honor of Professor Roderick Melnik, in celebration of his contributions.

Abstract: We relate various characterizations of sandpile dynamics arising in the literature referred to as "Abelian sandpile models", and we rigorously establish the sense in which they are commutative and associative as algebraic structures. In particular, associativity is approached from different perspectives: directly for the addition of configurations followed by stabilization; as a property of homomorphic images of semigroups; via the composition of operators on configurations; and, more generally, from an asynchronous perspective. We show how the existing formulations all arise within a common framework as substructures or homomorphic images of a non-commutative semigroup of operators. The appendix gives results on the algebraic complexity of sandpile semigroups, with 1) two different proofs determining the Krohn–Rhodes complexity of the finite Abelian sandpile models and all finite Abelian semigroups, 2) exact values of the aperiodic complexity measure for directed non-Abelian sandpile semigroups, and 3) exhibits new kinds of non-Abelian sandpile semigroups of higher Krohn–Rhodes complexity.

Keywords: self-organized criticality; semigroups; sandpile model; recurrent group; complexity; algebraic structures; graph theory; asynchronous timing

1. Introduction

In 1987, the physicists Bak et al. [1, 2] suggested a notion of **self-organized criticality** – the ability of large, strongly dissipative spatial systems to drive themselves towards a critical state, causing a 'domino effect' of transition dynamics at that state. This behavior implies high unpredictability, is related to the formation of fractal spatial patterns, and obeys power-law energy dissipation. It is also fundamentally different from conventional criticality, such as in phase transitions, where changes are driven by an external control parameter like temperature. The canonical example of self-organized criticality phenomena, which accounts for the conventional name of the models studied here, is a continuous flow of sand grains onto the top of a sand pile. The pile becomes steeper and steeper approaching a critical slope, at which point an avalanche of grains partially flattens it. This moves the system away from instability, whereupon this process recurs indefinitely. The size of each avalanche is unpredictable, but the distribution of sizes follows a power law: tiny avalanches occur frequently, large ones are less common, and huge avalanches—known as global cascades or catastrophic events—are rare but non-negligible. Such robust self-driven behaviour of dynamical systems, leading to complex critical transitions, emerges from local interactions within the system that are driven by an internal forcing function pushing the system toward a limiting bound [3]. Another example is a line of blocks coupled by springs and pulled at constant velocity across a surface with friction: This setup leads to avalanches through stick-slip dynamics among groups of blocks. Numerous other examples of self-organized criticality in nature include earthquakes caused by shifts in tectonic plates (which can also be modeled with stick-slip dynamics), forest fires triggered by the accumulation of dry wood, snow avalanches resulting from the accumulation of snowfall on steep terrain, etc. These phenomena all exhibit self-organized criticality and are known to obey power-law distributions, with rare and unpredictable catastrophic events [3].

The sandpile model was first introduced as a two-dimensional cellular automaton by physicists [1, 2] as a simple illustration of the self-organized criticality phenomenon, which was motivated by understanding such complex phenomena as turbulence and $1/f$ noise. However, due to the abstract nature of the model, most subsequent studies address its algebraic and algorithmic aspects. The conventional **sandpile model** is defined on a fixed graph, where each vertex acts as a reservoir of sand grains, with capacity equal to its out-degree. Additionally, there is a special vertex—the **sink**—which is reachable from every vertex and absorbs all grains that fall into it. A vertex exceeding its capacity is called unstable, and a configuration containing such vertices can be stabilized by a sequence of topplings (avalanche). A toppling at an unstable vertex sends one sand grain to its neighbors along each of its outgoing edges. Each stable configuration is either **recurrent**, i.e., reachable from any other stable configuration, or **transient**, otherwise.*

From the physicists' perspective, the sandpile model has been treated as a Markov chain on the space of stable configurations [4–7], with attention given to probability distributions—e.g., the stationary distribution over recurrent configurations, or the distribution of avalanche duration and of the number of distinct vertices toppled in an avalanche-referred to as 'time' and 'space', respectively—both of which may exhibit striking power-law behaviour. Sandpile models have been studied on undirected graphs (such as rectangular grids) and directed (multi-)graphs. For directed graphs, the dynamics may be far richer and more intricate, enabling application to a broader class of systems with non-symmetric relationships—e.g., information flow, metabolic and reaction networks, or traffic systems. In such systems, directionality is crucial for modeling cause-and-effect relations and local-to-global propagation.

One variation of the model studied in the theory of computing is called the chip-firing game [8], in

*There are many equivalent definitions of the recurrence of a configuration.

which the process takes place on an arbitrary finite graph without a sink vertex, and each step of the game is the toppling of a single unstable vertex. In this view, the focus is on the stabilization process itself. For example, [9] shows the independence of the finiteness of the game and the terminating configuration from the steps made, characterizes the termination condition in terms of the total number of chips and the number of edges in the graph, and relates the game length to the eigenvalues of the graph's Laplace operator.

The conventional algebraic view on the sandpile model—on arbitrary directed multigraphs with a sink reachable from every vertex by some path—does not consider the intermediate steps of the stabilization process. Instead, it focuses on the set of stable configurations, equipped with a binary operation: pointwise addition of the grains at each site, followed by stabilization. This operation defines the structure known as the sandpile semigroup. This semigroup is known to be Abelian, whether or not the graph is directed. All the works we encountered in the literature either claim this fact or refer to the early work of Dhar [4]. A strong implication from the commutativity of the sandpile semigroup is the fact that its minimal ideal is a group, where the group elements correspond to the recurrent configurations of the graph.

The recurrent group of the Abelian sandpile is closely related to the Laplacian matrix of the graph [10], particularly, the group size is the matrix determinant, and the structure of the group can be obtained by finding the Smith normal form of the matrix. This last property relies on the fact that every finite Abelian group is a direct sum of cyclic groups. Thus, the structure of the recurrent group has been studied and/or completely established for many different types of graphs: for example, complete graphs [11]; for simple constructions, such as direct products of path and cyclic graphs [12]; or complete multipartite graphs and direct products of complete graphs [13]; for more unusual structures, such as 'flowers of polygon chains' [14]; or for some specific types of graphs, such as the n-cubes [15], wheel graphs [16], cone graphs [17], Paley graphs [18]; or for Cayley graphs of dihedral groups [19]; just to name a few.

A simple but inefficient iterative procedure for computing the identity configuration of the recurrent sandpile group [20] relies on the existence of the unique idempotent power in finite semigroups and the inevitability of reaching recurrent states by definition. In [21], a faster algorithm was proposed to find the recurrent identity element for the $n \times n$ grid that runs in $O(n^4 \log n)$ time. Notably, there is an efficient and general formula for the group identity of the recurrent group of a sandpile graph [22] based on the unique maximal stable configuration $\mathbf{u}_{\max}$ (i.e., with the number of sand grains at a vertex equalling its out-degree minus one); this formula includes only two stabilizations†, which, however, still makes it iterative in terms of topplings. A more recent work, for example, derives the exact sand-grain structure of the generator and identity configurations of the circle graph [23] by proving a non-iterative formula for the result of configuration addition and stabilization.

In [24], Babai and Toumpakari conduct a general study of the structure of the sandpile monoid, particularly establishing the connection between the lattice of idempotents and strongly connected components of the underlying directed graph. There are also several other variations of the model, such as stochastic sandpiles [25, 26], where a toppling sends a grain to a neighbor with probability p, changing the set of recurrent configurations; or non-Abelian sandpiles on directed rooted trees [27] which restricts grain addition to leaves only and adds the operation of pushing a grain from a vertex to

†In notation to be introduced below, the identity element of the recurrent group can be expressed as $\sigma(2\mathbf{u}_{\max} - \sigma(2\mathbf{u}_{\max}))$ where σ is the stabilization operator on sandpile configurations.

its parent along an edge toward the root; or the recently introduced extended sandpile [28], which is generated by allowing a non-negative real 'number of grains' at the border of the grid; or the divisible sandpile model [29] which also allows a real 'number of grains' at all vertices, including negative numbers representing holes. The above highlights the diverse approaches and perspectives in the study of the sandpile model, as well as the sustained interest in the subject to the present day.

In this paper, we present additional insights into fundamental aspects of the sandpile model, such as associativity and commutativity. While these properties are discussed in works such as [24] and [4], their justifications are often implicit, with some transitions in presentation treated as self-evident. We provide more rigorous proofs, clarifying the distinctions and connections between different interpretations of the dynamics, namely, how configurations are combined or how state transformations are performed on the graph. For many related facts or properties, we provide alternative proof techniques. Furthermore, in Appendix A2 we examine the Krohn–Rhodes complexity of finite Abelian semigroups, linking the sandpile model to a broader framework for analyzing discrete event dynamical systems, where the complexity measure describes the difficulty of emulating both the aperiodic and reversible aspects of the model while preserving its structure. These observations, along with our extended and detailed perspective on the fundamental properties of sandpile models, offer valuable insight into their theoretical foundations and support further exploration of their computational and dynamical aspects.

2. Abelian sandpile models

We assume the reader is familiar with standard mathematical definitions for graphs (e.g., digraph, multigraph, directed path) and algebraic structures (e.g., semigroups, monoids, homomorphisms, transformation semigroups, making an action faithful, etc.), but for convenience we give concise definitions in Appendix A1. In this paper, we generally adhere to the definitions and terminology introduced by Babai and Toumpakari [24], while clarifying connections to the operator view of Dhar [4] on sandpiles, and generalizing some of the concepts they presented. This yields new insights relating algebra and dynamics, and allows asynchronous sandpile processes in a unified framework.

An **Abelian sandpile model** is based on a digraph Γ, possibly a multigraph but loopless, with a unique sink—a vertex of out-degree 0—that is accessible from all vertices; that is, there is a directed path (possibly of length 0) from every vertex to the sink.[‡] The length of a shortest directed path from a site u to the sink is the *distance* of u to the sink. All the other vertices are called **sites**. The sink is denoted by s and the set of sites is denoted by $V_0 = V \setminus \{s\}$. Let $\deg^+(v)$ denote the **out-degree** of vertex v, i.e., the number of edges starting from vertex v, and let $\deg^-(v)$ denote its **in-degree**, the number of edges coming into v.

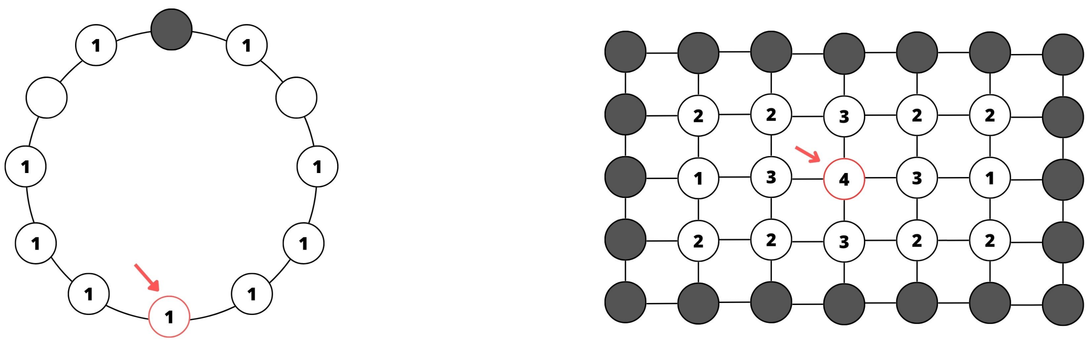
Definition 1. *A **configuration** $\mathbf{x}$ of the sandpile model based on digraph $\Gamma = (V, E)$ is a function $\mathbf{x} : V_0 \to \mathbb{N}$, where $\mathbb{N}$ is the set of nonnegative integers. The set of all configurations on Γ is denoted by $C = \mathbb{N}^{V_0}$ and is called the **configuration space** of Γ.*

The special configuration with one grain at site u but no grains at any other site is denoted by δ_u. The integer $\mathbf{x}_u$, denoting $\mathbf{x}(u)$, is thought of as the number of sand grains at site u (the **height** of the

[‡]Undirected graphs are included as a special case by replacing each undirected edge with a pair of directed edges, one in each direction, between the same two vertices. Thus, undirected sandpile models are fully covered by our general framework.

sandpile). A site $u \in V_0$ is **stable** if $\mathbf{x}_u < \deg^+(u)$. An unstable site, also called **critical**, can be **toppled**, sending one grain through each edge leaving u. Therefore, $\mathbf{x}_u$ is reduced by $\deg^+(u)$; and for each other site v, the height $\mathbf{x}_v$ is increased by a_{uv}, the number of edges from u to v.[§] Grains reaching the sink disappear, and the sink never topples. A configuration is **stable** if all sites are stable. A sequence of topplings that result in a stable configuration is called a **stabilizing sequence** or an **avalanche**. For C the set of all configurations, observe that the set of stable configurations $M = \sigma(C)$ is the image of the idempotent mapping $\sigma = \sigma^2$.

Example 1. The iterative stabilization process (avalanche) of the classical Abelian sandpile model is illustrated in the animated Figure 1 for two undirected graphs: a circle in Figure 1(a), and a two-dimensional grid in Figure 1(b), where all stable sites contain less than two and less than four sand grains respectively.[¶] The process starts by dropping a sand grain on the initially stable configuration, making the corresponding site unstable and causing the avalanche. At every step of the stabilizing sequence, all unstable sites topple synchronously, and grains that fall into sink vertices disappear. As discussed later, the toppling order does not matter, and the stable configuration that is reached eventually is unique.



(a) Circle stabilization. **(b)** Grid stabilization.

Figure 1. Stabilization process of the Abelian sandpile on a circle graph (left) and a grid (right) after adding a sand grain. The gray vertices denote the sink, the numbers on the vertices denote the number of grains they contain, and the white vertices are empty. To start the animation, click on the image (use Adobe Acrobat Reader).

Definition 2. *The **toppling digraph of** Γ is the directed graph on the configuration space in which there is an edge from* $\mathbf{x}$ *to* $\mathbf{y}$ *if* $\mathbf{y}$ *can be obtained from* $\mathbf{x}$ *by toppling at some site u. The subgraph of it consisting of* $\mathbf{x}$ *and all configurations that can be obtained from* $\mathbf{x}$ *by a sequence of topplings is called the **toppling graph of*** $\mathbf{x}$.

2.1. Key Facts

The Facts 1 and 3 below were proved in [24]; we provide alternative proofs using different methods.

[§]Classically, we have $a_{uv} \leq 1$ for digraphs, but we can make the same construction on multigraphs where the number a_{uv} of edges from u to v could be any non-negative number.

[¶]The multiple sink vertices in a grid can be equivalently replaced by a unique sink, and are only shown this way for illustration clarity, to preserve the grid structure.

Fact 1. *Starting with any configuration and toppling unstable sites in succession, we arrive at a stable configuration in a finite number of steps.*

Proof. Suppose we have an infinite sequence of topplings $\tau_1, \tau_2, \ldots$ with topplings τ_T indexed by positive integers $T \in \mathbb{N}^+$. As the initial configuration has a finite number of sand grains, this entails that after some time, no grains leave the graph. This means that for some T_1 the vertices at a distance $d = 1$ from the sink are not toppled at any $t > T_1$. However, this requires that after some $T_2 > T_1$ they do not receive any additional grains, which implies that no vertex at a distance $d = 2$ is toppled at $t > T_2$. Repeating this reasoning, we obtain that none of the vertices at a distance $d \leq D$ is toppled at $t > T_D$. As the graph is finite, letting D be the maximum distance from a graph vertex to the sink implies the finiteness of the toppling sequence, which is a contradiction.

Fact 2. *The toppling graph of* $\mathbf{x}$ *is a finite acyclic digraph.*

Proof. The only configurations reachable from $\mathbf{x}$ by topplings are those that have fewer or an equal number of grains. As both the graph Γ and the number of grains in $\mathbf{x}$ are finite, there is only a finite number of vertices in the toppling graph of $\mathbf{x}$. The existence of a cycle in the toppling graph of $\mathbf{x}$ would allow an infinite toppling sequence and contradict Fact 1.

Corollary 1. *There is an upper bound* $L(\mathbf{x})$ *on the length of paths to a stable configuration for any configuration* $\mathbf{x}$.

Fact 3. *Every possible avalanche from a given configuration* $\mathbf{x}$ *leads to the same stable configuration* $\sigma(\mathbf{x})$.

We prove Fact 3 in Lemma 2 below, which proceeds by induction on $L_{\mathbf{x}}$. This last fact enables us to define the following:

Definition 3. *For every configuration* $\mathbf{x} \in \mathbb{N}^{V_0}$, *the configuration* $\sigma(\mathbf{x})$ *is the unique stable configuration resulting from stabilizing* $\mathbf{x}$.

Regarding σ as a function on C, the configuration space, observe that $\sigma^2 = \sigma$, and the set $M = \sigma(C)$ of stable configurations is the image of σ.

An immediate consequence of Fact 3 is the following useful fact.

Fact 4 (Key Fact). *If configuration* $\mathbf{x}'$ *is in the toppling graph of* $\mathbf{x}$, *then* $\sigma(\mathbf{x}') = \sigma(\mathbf{x})$.

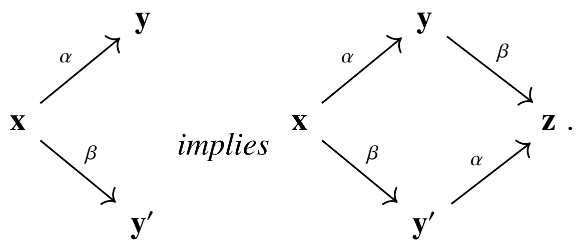
2.2. The sandpile dynamics graph of Γ

The following structure (abstracted from Dhar's work [4]), which models both adding sand grains and actual topplings, will be used to capture sandpile dynamics.

Definition 4. *Define* $\mathbf{D}(\Gamma)$ *the* **Dhar sandpile dynamics graph of** Γ *to be the directed graph with nodes* $\mathbf{x}$ *in* C *and the following arrows (i.e., edges): For each* u *in* V_0,
1) there exists an arrow $\mathbf{x} \xrightarrow{u} \mathbf{x}u$, *representing the addition of a sandgrain at site* u *to configuration* $\mathbf{x}$ *and the resulting configuration* $\mathbf{x}u = \mathbf{x} + \delta_u$.
2) If $\mathbf{x}_u \geq \deg^+(u)$, *then there is an arrow* $\mathbf{x} \xrightarrow{\tau_u} \mathbf{x}\tau_u$, *representing the unstable site* $\mathbf{x}$ *toppling and the resulting configuration* $\mathbf{x}\tau_u$.

A crucial property of $\mathbf{D} = \mathbf{D}(\Gamma)$ is that it includes just the 'proper' topplings, i.e., exactly those topplings that can actually occur. Thus, the toppling graph is a subgraph of $\mathbf{D}$. Observe that $\mathbf{D}$ has no loop edges and no multiple edges since an arrow of the form $\mathbf{x} \to \mathbf{y}$ is either the addition of a grain at a unique site u or a toppling at a unique site u with fewer grains in $\mathbf{y}$ than in $\mathbf{x}$.

Lemma 1 (Strong Local Confluence and Interchange Property)**.** *Whenever there are two distinct arrows of the form* $\mathbf{x} \xrightarrow{\alpha} \mathbf{y}$ *and* $\mathbf{x} \xrightarrow{\beta} \mathbf{y}'$ *in* $\mathbf{D}$*, then there are also two arrows* $\mathbf{y} \xrightarrow{\beta} \mathbf{z}$ *and* $\mathbf{y}' \xrightarrow{\alpha} \mathbf{z}'$ *in* $\mathbf{D}$*, and, moreover,* $\mathbf{z} = (\mathbf{x}\alpha)\beta = (\mathbf{x}\beta)\alpha = \mathbf{z}'$*. Diagrammatically,*



In other words, if α and β can both be applied to a configuration, then they can both be applied sequentially, in either order, yielding the same result.

Caveat: Note, since the arrows are distinct, $\alpha \neq \beta$. Therefore the Lemma does not wrongly entail that a proper toppling can be repeated if it can be executed at $\mathbf{x}$.

Proof. It is easy to see that 1) edges labelled u and v can always be 'interchanged': There are outgoing edges labelled u and v at every configuration corresponding to adding a one grain each at sites u and v. For $\mathbf{y} = \mathbf{x} + \delta_u$ and $\mathbf{y}' = \mathbf{x} + \delta_v$, we can traverse edges labelled u and v in either order to reach $\mathbf{z} = (\mathbf{x} + \delta_u) + \delta_v = (\mathbf{x} + \delta_v) + \delta_u$. 2) In case the outgoing labels are u and τ_v, then $\mathbf{x}$ is critical at v by the definition of arrows in the graph $\mathbf{D}$. Adding a grain at u leaves the configuration critical at v. Thus, adding a grain at u before or after toppling yields the same resulting configuration $\mathbf{z} = \mathbf{x}\tau_v + \delta_u$ having one more grain at u than the configuration obtained by toppling $\mathbf{x}$ at v. 3) If the outgoing labels are τ_u and τ_v then u and v are distinct sites since the arrows are distinct, and $\mathbf{x}$ is critical at both sites. Since toppling one site cannot decrease the number of grains at the other site, the second site can still be toppled. Toppling in either order gives the same configuration $\mathbf{z}$ with the following net change at every site w:

- If $w \neq u, v$, then $\mathbf{z}_w = \mathbf{x}_w + a_{uw} + a_{vw}$.
- If $w = u$, then $\mathbf{z}_w = \mathbf{x}_w + a_{vw} - \deg^+(w)$.
- If $w = v$, then $\mathbf{z}_w = \mathbf{x}_w + a_{uw} - \deg^+(w)$,
  recalling that a_{uv} denotes the number of edges from u to v.

Thus, $(\mathbf{x}\alpha)\beta = (\mathbf{x}\beta)\alpha$ holds whenever there are arrows with source $\mathbf{x}$ labelled α and β.

The next lemma constitutes a reformulation of [24, Facts 2.3, 2.4, 2.5] proven in a unified scheme in terms of our sandpile dynamics graph $\mathbf{D}$. In particular, a direct consequence of this lemma is that the following score function on sandpile configurations is well-defined.

Definition 5. *The* ***score of the configuration*** $\mathbf{x}$ *is the configuration* $s(\mathbf{x}) \in \mathbb{N}^{V_0}$ *where* $s(\mathbf{x})_v$ *is the number of topplings at the site* $v \in V_0$ *that occur on* some *path from* $\mathbf{x}$ *to a stable configuration.*

Lemma 2 (Extended Version of Fact 3 - Church-Rosser Property of Toppling and Well-definedness of Configuration Score)**.** *For every avalanche of a sandpile configuration* $\mathbf{x}$*, the number of occurrences of each toppling* τ_u*, the resulting stable configuration, and the number of topplings in the avalanche are independent of the toppling sequence. In particular, the score of* $\mathbf{x}$ *is well-defined.*

Proof. We prove this by induction on the maximum length of a toppling sequence from $\mathbf{x}$, denoted $L(\mathbf{x})$. That is, the length of a longest directed path starting at $\mathbf{x}$ in the toppling graph of $\mathbf{x}$.

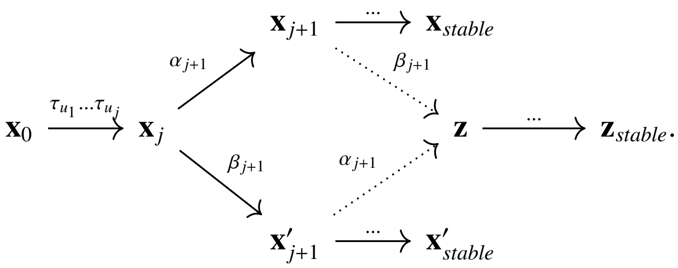
If $L(\mathbf{x}) = 0$, then $\mathbf{x} = \mathbf{x}'_m = \mathbf{x}_\ell = \sigma(\mathbf{x})$ is stable and the lemma holds for $\mathbf{x}$.

If $L(\mathbf{x}) = 1$, then the lemma holds for $\mathbf{x}$, as there is only one arrow out of $\mathbf{x}$ in the toppling graph of $\mathbf{x}$. If there are two arrows corresponding to say τ_u and τ_v, then by Lemma 1, there is a configuration $\mathbf{z}$ and arrows $\mathbf{x}\tau_v \stackrel{\tau_u}{\to} \mathbf{z}$ and $\mathbf{x}\tau_u \stackrel{\tau_v}{\to} \mathbf{z}$ such that $\mathbf{x}\tau_u\tau_v = \mathbf{x}\tau_v\tau_u = \mathbf{z}$. However, this means $L(\mathbf{x}) \geq 2$, a contradiction.

Now suppose that $L(\mathbf{x}) \geq 2$ and let $p = \mathbf{x}_0\alpha_1\mathbf{x}_1 \ldots \alpha_\ell \mathbf{x}_\ell$ and $q = \mathbf{x}_0\beta_1\mathbf{x}'_1 \ldots \beta_m\mathbf{x}'_m$ each be paths in the toppling graph of $\mathbf{x} = \mathbf{x}_0$ to a stable configuration, where each edge label α and β is a toppling τ_u for some $u \in V_0$. Since $L(\mathbf{x}) \geq 2$, then neither ℓ nor m can be zero, unless $\mathbf{x}$ is stable (and in this case $L(\mathbf{x}) = 0$).

It cannot be the case that p is the initial part of q (nor vice versa), unless $p = q$ since each ends in a stable configuration. If $p = q$ the lemma holds trivially. Therefore we may assume $p \neq q$ and let $j \geq 0$ be the first index of an $\mathbf{x}_j$ for which $\alpha_{j+1} \neq \beta_{j+1}$.

In the toppling graph of $\mathbf{x}_0$: The path p leads from $\mathbf{x}_0$ by zero or more topplings $\tau_{u_1}, \ldots, \tau_{u_{j-1}}$ to $\mathbf{x}_j$ and then along an edge labelled $\alpha_{j+1} = \tau_a$ and continues on to a stable configuration $\mathbf{x}_{stable}$, while path q also leads from $\mathbf{x}_0$ to $\mathbf{x}_j$ in the same way, but then along an edge labelled $\beta_{j+1} = \tau_b$ to $\mathbf{x}'_{j+1}$ and continues to some stable configuration $\mathbf{x}'_{stable}$ (See the diagram below). The dotted edges to a configuration $\mathbf{z}$ exist by the interchange property of $\mathbf{D}$ (Lemma 1). Meanwhile, there is also a toppling path from $\mathbf{z}$ to some stable $\mathbf{z}_{stable}$ by Fact 1. The induction hypothesis applies to $\mathbf{x}_{j+1}$, $\mathbf{z}$ and $\mathbf{x}'_{j+1}$ since the lengths of any paths from them to any stable site have length strictly less than $L(\mathbf{x}_0)$, as they can each be reached by non-trivial directed path from $\mathbf{x}_0$.



By induction applied to $\mathbf{x}_{j+1}$, the avalanche from $\mathbf{x}_{j+1}$ along p and the avalanche from $\mathbf{x}_{j+1}$ starting with β_{j+1} through $\mathbf{z}$ satisfy the lemma and stabilize to $\sigma(\mathbf{x}_j) = \sigma(\mathbf{z})$ and have the length and same number of topplings of each type.

Also, by induction applied to $\mathbf{x}'_{j+1}$, the avalanche from that node along q and the avalanche from that node starting with α_{j+1} through $\mathbf{z}$ satisfy the lemma and both stabilize to $\sigma(\mathbf{x}'_j) = \sigma(\mathbf{z})$ and have the length and same number of topplings of each type. It follows that $\mathbf{x}_\ell = \mathbf{x}'_m = \sigma(\mathbf{x})$, and $m = \ell$.

Moreover, by induction the score function s is well-defined for $\mathbf{x}_{j+1}$, $\mathbf{z}$ and $\mathbf{x}'_{j+1}$, $s(\mathbf{x}_{j+1}) = s(\mathbf{z}) + \delta_b$ and $s(\mathbf{x}'_{j+1}) = s(\mathbf{z}) + \delta_a$. Now, $s(\mathbf{x}_j)$ according to p is $s(\mathbf{x}_{j+1}) + \delta_a$ and according to q it is $s(\mathbf{x}'_{j+1}) + \delta_b$, and both give the same result $s(\mathbf{z}) + \delta_a + \delta_b$. Then $s(\mathbf{x}_0)$, both according to p and q, is this unique value (of $s(\mathbf{x}_j)$) plus $\delta_{u_1} + \cdots + \delta_{u_j}$ where $\tau_{u_1}, \ldots, \tau_{u_j}$ are the topplings in both paths from $\mathbf{x}_0$ to $\mathbf{x}_j$. This shows the uniqueness of the score as desired, independent of path.

Theorem 1. *Let $\mathbf{x}$ be a configuration in the Sandpile Graph of the directed graph Γ. Suppose there are paths $p : \mathbf{x} \to \mathbf{y}$ and $q : \mathbf{x} \to \mathbf{y}'$ that include as instances of the sandgrain additions $u_1, \ldots, u_n$ ($u_i \in V_0$, $n \geq 0$) in any order (possibly with repetitions) and no other edges other than topplings.*

Then p and q can be extended solely with toppling edges to paths p' and q' such that

(i). $p' = pw$ and $q' = qw'$.

(ii). The words over alphabet $A = \{u, \tau_u : u \in V_0\}$ labelling p' and q' have the same number of occurrences of each letter. That is,

$$|p'|_a = |q'|_a \text{ for all } a \in A$$

In particular, the letters of p' are a permutation of the letters of q'.

(iii). Both p' and q' are both from $\mathbf{x}$ to the stable configuration $\sigma(\mathbf{x}u_1 \ldots u_n)$.

(iv). Every path from $\mathbf{x}$ to a stable configuration containing exactly the additions $u_1, \ldots, u_n$ ends at the same stable configuration.

(v). Adding the sand grains to $\mathbf{x}$ from a configuration $\mathbf{x}'$ in any order and stabilizing results in a unique stable configuration $\sigma(\mathbf{x} + \mathbf{x}')$. Moreover, this is the case, even if the sand grains from $\mathbf{x}'$ are added during the course of iterated toppling.

Proof. Let $p = \mathbf{x}_0 a_1 \mathbf{x}_1 \ldots a_m \mathbf{x}_m$, showing vertices and edge labels, where we write $\mathbf{x}_0$ for $\mathbf{x}$. Find the first index j where $a_j = u_1$ occurs; if $j = 1$, do nothing. If $j > 1$, then a_{j-1} and u_1 are both edge labels at $\mathbf{x}_{j-2}$ (since a_{j-1} occurs after b_{j-2} and u_1 is always possible), then in the path we may replace

$$\mathbf{x}_{j-2} a_{j-1} \mathbf{x}_{j-1} u_1 \mathbf{x}_j$$

by

$$\mathbf{x}_{j-2} u_1 \mathbf{x}'_{j-1} a_{j-1} \mathbf{x}_j$$

by Lemma 1, for some $\mathbf{x}'_{j-1}$. This does not change the terminal vertex of the path nor its length.

Thus, we may place p by a path by another one with the same endpoint in which u_1 is the first edge label. We may do the same for the second addition of a grain u_2 to bring it into the second position, and so on, up to u_n, obtaining a path from $\mathbf{x}$ to $\mathbf{y}$ in which all the adding grain moves appear before any toppling moves.

(Intuitively, the grains can be added earlier without disabling any possible topplings since this change cannot decrease the number of grains at a site.)

Thus, we may assume that p and q both begin with edges labeled $u_1, \ldots, u_n$ in the same order, prior to any toppling edges, without affecting their lengths or where they terminate. That is, the two paths share their first n edges, whereupon both reach the same vertex $\mathbf{x}' = \mathbf{x}u_1 \ldots u_n$, and the remaining edges of both paths are all topplings.

By Fact 1, we know p and q can each be extended to a path p' resp. q' by appending toppling edges so each path ends in a stable configuration. Then each of p' and q' includes an avalanche from $\mathbf{x}'$

in the toppling graph of $\mathbf{x}'$. Hence, by Fact 1 and by Fact 3 and its extension above, the number of occurrences of any toppling τ_u in both avalanches is the same for all $u \in V_0$ and both end in the unique configuration $\sigma(\mathbf{x}_j)$.

3. Three ways of combining configurations

Now we introduce three different ways of combining configurations, the first of which was introduced by Babai and Toumpakari and is the standard way [24]. These endow the space of stable configurations M with an algebraic structure. We will show that in every case the defined structure is commutative in a precise sense. The detailed proofs of our generalizations are included in the next section.

To begin with, given two configurations $\mathbf{x}$ and $\mathbf{y}$, then construct their stable "sum" in two stages:

1) Pointwise addition of $\mathbf{x}$ and $\mathbf{y}$ to yield $\mathbf{x} + \mathbf{y}$ defined by

$$(\mathbf{x} + \mathbf{y})_v = \mathbf{x}_v + \mathbf{y}_v,$$

for each $v \in V_0$.

2) Stabilize the resulting configuration $\mathbf{x} + \mathbf{y}$ through repeated toppling as necessary, resulting in $\sigma(\mathbf{x} + \mathbf{y})$.

By the Fact 3 above, the result is a well-defined stable configuration $\sigma(\mathbf{x} + \mathbf{y})$.

Example 2. Classical addition on the set of stable configurations in an Abelian sandpile semigroup is illustrated for the circle graph in Figure 2. In the first stage, the pointwise addition makes two sites unstable; then in the second (animated) stage, a sequence of topplings leads to the unique stable configuration—the result of the addition.

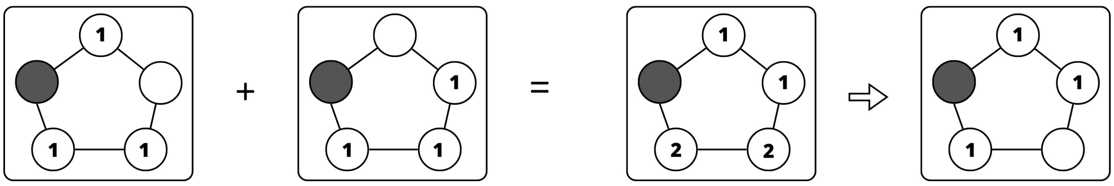


Figure 2. Addition of two stable configurations of the Abelian sandpile on a circle graph and iterative stabilization of the resulting configuration. The gray vertex is the sink, the numbers on the vertices denote the number of grains they contain, and the white vertices are empty. To start the animation, click on the third image (use Adobe Acrobat Reader).

Babai and Toumpakari denote this operation by $\mathbf{x} \oplus \mathbf{y}$ *when restricted to the set M of stable configurations (i.e.,* $\mathbf{x}$ *and* $\mathbf{y}$ *are fixed by* σ*).* However, one may study this operation more generally: for any configurations $\mathbf{x}, \mathbf{y} \in C$, define

$$\mathbf{x} \oplus \mathbf{y} := \sigma(\mathbf{x} + \mathbf{y}).$$

Since $\sigma(\mathbf{x} + \mathbf{y})$ is stable, M is closed under this operation.

Now, of course, operation + is commutative and associative on C with identity element $\mathbf{0}$, the *empty configuration* assigning zero to each site vertex (since $\mathbb{N}$, with addition, is an Abelian monoid with identity 0). Since addition + is commutative on C, $\oplus$ is commutative on C too. That is,

$$\mathbf{x} \oplus \mathbf{y} = \sigma(\mathbf{x} + \mathbf{y}) = \sigma(\mathbf{y} + \mathbf{x}) = \mathbf{y} \oplus \mathbf{x}.$$

The all-zero configuration $\mathbf{0}$ is a (two-sided) identity for the operation $\oplus$ on M as well: The equation

$$\mathbf{0} \oplus \mathbf{x} = \mathbf{x} \oplus \mathbf{0} = \sigma(\mathbf{x} + \mathbf{0}) = \sigma(\mathbf{x}) = \mathbf{x}$$

for all $\mathbf{x} \in M$, but $\mathbf{x} \oplus \mathbf{0} = \mathbf{x}$ *never* holds for (unstable) $\mathbf{x} \in C \setminus M$ since then $\sigma(\mathbf{x}) \neq \mathbf{x}$.
Indeed, C with operation $\oplus$ has no identity element, since any unstable configuration could not be fixed by this purported identity.

However, $\oplus$ is not obviously associative on C, nor on M. To show that M under $\oplus$ is a commutative monoid, it suffices to show that $\oplus$ is associative on C. In this paper, we shall prove this associativity in three very different ways: 1) By checking $(C, \oplus)$ satisfies the associative identity directly using the Key Fact (Fact 4). 2) Showing there is a surjective homomorphism from the monoid $(C, +)$ onto the set with binary operation $(M, \oplus)$, allowing the latter to inherit associativity from the former. 3) Constructing a transformation semigroup of mappings on C, the General Sandpile operator semigroup of Γ, having a subsemigroup isomorphic to $(M, \oplus)$.

Moreover, we shall see that $\oplus$ as defined above is a special case of *combining configurations that may be undergoing avalanches*.

1) **First way: The classical sandpile monoid**
   Let M be the set of all stable configurations. Babai and Toumpakari call M with the operation $\oplus$, the Sandpile monoid.

   The operation $\oplus$ is commutative on stable configurations since $\mathbf{x} + \mathbf{y} = \mathbf{x} + \mathbf{y}$ holds in $(C, +)$.

$$\sigma(\sigma(\mathbf{x}) + \sigma(\mathbf{y})) \quad = \quad \sigma(\mathbf{x} + \mathbf{y}), \qquad \text{since } \sigma(\mathbf{z}) = \mathbf{z} \text{ for } \mathbf{z} \in \{\mathbf{x}, \mathbf{y}\}. \tag{3.1}$$

   That is,

$$\sigma(\mathbf{x}) \oplus \sigma(\mathbf{y}) = \sigma(\mathbf{x} + \mathbf{y}),$$

   where $\mathbf{x}$ and $\mathbf{y}$ are in M. However, $\mathbf{x} + \mathbf{y}$ generally will not lie in M. For $\sigma : C \longrightarrow M$ to be a homomorphism requires that this hold arbitrary $\mathbf{x}, \mathbf{y}$ in C, not just in M.

   One way to prove associativity of $\oplus$ on stable configurations M by first showing that σ is a surjective homomorphism from $(C, +)$ onto $(M, \oplus)$, since then associativity (together with commutativity and an identity element) is inherited by $(M, \oplus)$ by writing the identities in $(C, +)$ and mapping them via σ onto elements of M combined using $\oplus$.

   We want to prove σ is a homomorphism with general configurations which are not necessarily stable, i.e., we want to show that $(C, \oplus)$ is an associative semigroup, with subsemigroup $(M, \oplus)$. This is proved in Theorem 2. Alternatively, we can prove $\sigma : (C, +) \rightarrow (M, \oplus)$ is a homomorphism, entailing that $(M, \oplus)$ is associative and commutative and has an identity since $(C, +)$ has these properties. This is done in Section 3.2.

2) **Second way:** Moreover, associativity of $\oplus$ holds even if $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$ are not necessarily stable. A corollary of the proof is that pointwise summing and then stabilizing is an Abelian operation on configurations (whether stable or not). This gives another Abelian sandpile semigroup $(C, \oplus)$ which is not a monoid, as shown below.

3) **Third way:** What is still open is whether a configuration plus another gives the same result if the addition of a grain or of a configuration happens *during* the process of stabilization. This would constitute a stronger form of commutativity. More simply, does adding a single grain to a configuration during, after, or before stabilization always yield the same configuration when stabilized? We show that the answer to these questions is yes below. This gives additional multiple alternative processes for the realization of $\oplus$ in $(C, \oplus)$.

3.1. Asynchronous multiplication and associativity

Now we will establish the associativity of the structures we have been discussing.

Theorem 2. *(Associativity of sandpile semigroups) Each of the following holds and each implies M is associative under binary operation $\oplus$:*

(i). The operation $\oplus$ is associative on the set of configurations C.

(ii). Stabilization $\sigma : (C, +) \twoheadrightarrow (M, \oplus)$ is a surjective homomorphism, hence $(M, \oplus)$ is a monoid since $(C, +)$ is.

(iii). $(M, \oplus)$ is isomorphic to a subsemigroup $\mathcal{SP}$ of the General Monoid of Sandpile Operators $\mathcal{GS}^+$ (defined in Section 4). This subsemigroup is itself also a homomorphic image of $\mathcal{GS}^+$.

Proof. There are respective consequences of the following results proved below: (i) Theorem 3 and Corollary 2, (ii) Lemma 3, and (iii) Theorem 5 parts (i) and (iii).

Theorem 3 (Asynchronous multiplication)**.** *The multiplication of sandpile configurations allowing a configuration to be added during an avalanche is well-defined. Precisely, let* $\mathbf{x}$ *and* $\mathbf{y}$ *be configurations (possibly consisting of single grains). Then if after any k topplings ($k \geq 0$) of* $\mathbf{x}$*,* $\mathbf{y}$ *is added to the current sandpile, the result after stabilization is always* $\mathbf{x} \oplus \mathbf{y}$.

Proof. The toppling graph X of $\mathbf{x}$ consists of all nodes and edges of all directed paths from $\mathbf{x}$ to $\sigma(x)$.

Consider the directed graph P obtained from X by adding $\mathbf{y}$ to every node in X. This is an injective graph morphism from X onto P, which is a subgraph of the toppling graph of $\mathbf{x} + \mathbf{y}$. That is, each toppling edge in X

$$\mathbf{x}' \to \mathbf{x}''$$

is mapped to the edge in P:

$$\mathbf{x}' + \mathbf{y} \to \mathbf{x}'' + \mathbf{y}.$$

This is a valid edge in the toppling graph of $\mathbf{x} + \mathbf{y}$, since toppling of a configuration $\mathbf{x}'$ in X yields a corresponding toppling of configuration $\mathbf{x}' + \mathbf{y}$ in P as there are sufficient grains in the configuration $\mathbf{x}' + \mathbf{y}$ to perform the same toppling, since all entries of $\mathbf{y}$ are non-negative.

Thus, it follows from the Key Fact (Fact 4) that there is a path from $\mathbf{x} + \mathbf{y}$ to $\sigma(\mathbf{x}) + \mathbf{y}$ in P. Now let $\mathbf{x}$ start to topple some number of steps k along the edges of X. Suppose configuration $\mathbf{x}'$ is reached, add $\mathbf{y}$

at that moment. This maps any path p from $\mathbf{x}'$ to $\sigma(\mathbf{x})$ to a path in the toppling graph of $\mathbf{x}+\mathbf{y}$ from $\mathbf{x}'+\mathbf{y}$ to $\sigma(\mathbf{x}) + \mathbf{y}$. All its edges are in the graph P, which is a subgraph of the toppling graph of $\mathbf{x} + \mathbf{y}$. Now by the Key Fact, there is a path from this node $\sigma(\mathbf{x}) + \mathbf{y}$ to the node $\sigma(\mathbf{x} + \mathbf{y})$ in the toppling graph of $\mathbf{x}+\mathbf{y}$. This proves that adding $\mathbf{y}$ at any time (after any number k of topplings from $\mathbf{x}$, including possibly no topplings or complete stabilization) results in a configuration $\mathbf{x}' + \mathbf{y}$ that stabilizes to $\sigma(\mathbf{x} + \mathbf{y})$.

Example 3. The illustration of asynchronous multiplication is shown in Figure 3 for two configurations $\mathbf{x}$ and $\mathbf{y}$ of the sandpile on a complete undirected graph on 5 vertices, where $\mathbf{x}$ is unstable. The top panel shows the pointwise sum of $\mathbf{x}$ and $\mathbf{y}$ followed by the stabilization sequence leading to $\mathbf{x}\oplus\mathbf{y}$. The middle panel shows the first three topplings of $\mathbf{x}$, denoting the resulting configuration by $\mathbf{x}'$, which is unstable, meaning we are still in the middle of the avalanche after 3 topplings. The third (bottom) panel shows the pointwise addition of $\mathbf{y}$ to $\mathbf{x}'$ followed by the stabilization process leading to $\mathbf{x}' \oplus \mathbf{y}$. As we can see, the resulting stable configurations are the same $\mathbf{x} \oplus \mathbf{y} = \mathbf{x}' \oplus \mathbf{y}$, as guaranteed by Theorem 3.

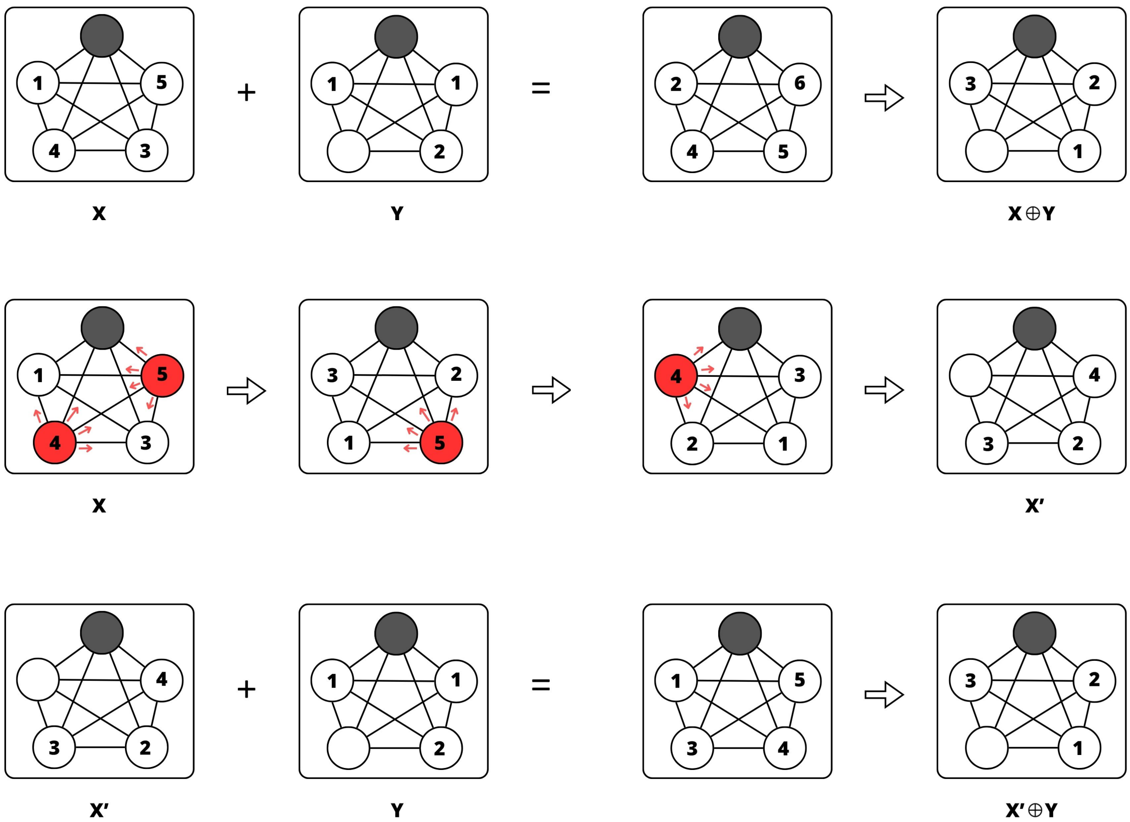


Figure 3. Addition of configurations during an avalanche compared to stabilization of their pointwise sum. The gray vertex is the sink, the numbers on the vertices denote the number of grains they contain, and the white vertices are empty. To start the animation in the top or bottom panel, click on the third image in the corresponding panel (use Adobe Acrobat Reader).

As a corollary, we obtain the associativity of $\oplus$ on C, proving Theorem 2(i).

Corollary 2 (Direct proof of associativity)**.** *For arbitrary sandpile configurations* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in C$*,*

$$(\mathbf{x} \oplus \mathbf{y}) \oplus \mathbf{z} = \mathbf{x} \oplus (\mathbf{y} \oplus \mathbf{z}).$$

The result is always $\sigma(\mathbf{x} + \mathbf{y} + \mathbf{z})$*. This holds even if the sandpiles are combined as avalanches are ongoing (in the sense of asynchronous multiplication in Theorem 2).*

Proof. Start with $(\mathbf{y}+\mathbf{z})$, and add $\mathbf{x}$ after k topplings of $\mathbf{y}+\mathbf{z}$. This gives an injective morphism from the toppling graph of $(\mathbf{y}+\mathbf{z})$ to the toppling graph of $(\mathbf{y}+\mathbf{z})+\mathbf{x}$, which stabilizes at $\sigma(\mathbf{x}+\mathbf{y}+\mathbf{z})$. Note that $\sigma(\mathbf{y}) + \sigma(\mathbf{z})$ is included in the toppling graph of $(\mathbf{y} + \mathbf{z})$, and therefore so is $\sigma(\sigma(\mathbf{y}) + \sigma(\mathbf{z}))$. Therefore, the configuration $\mathbf{x} + \sigma(\sigma(\mathbf{y}) + \sigma(\mathbf{z}))$ stabilizes to $\sigma(\mathbf{x} + \mathbf{y} + \mathbf{z})$.

Similarly, start with $(\mathbf{x} + \mathbf{y})$ and add $\mathbf{z}$ at some point in the stabilization process. This maps the toppling graph of $\mathbf{x}+\mathbf{y}$ injectively into the toppling graph of $(\mathbf{x}+\mathbf{y})+\mathbf{z}$, which stabilizes at $\sigma(\mathbf{x}+\mathbf{y}+\mathbf{z})$. In particular, $\sigma(\sigma(\mathbf{x}) + \sigma(\mathbf{y})) + \mathbf{z}$ also stabilizes to $\sigma(\mathbf{x} + \mathbf{y} + \mathbf{z})$.

This proves that $\oplus$, defined as pointwise addition followed by stabilization, is associative on both M and C, even under asynchronous combination of sandpile configurations.

Note that the monoid $(M, \oplus)$ is a subsemigroup of $(C, \oplus)$, which is not itself of monoid.

3.2. Homomorphism variant proof of associativity

The previous subsection shows $(C, \oplus)$ is a semigroup too by proving associativity directly, for the 2nd sandpile model. Moreover, it shows that one can add configurations in the course of stabilization (3rd sandpile model).

Babai and Toumapakari [24] proceed in a different way. The stabilization function σ maps C to M and seems like a homomorphism when restricted to M. Suppose $\mathbf{x}$ and $\mathbf{y}$ are in M (stable). Then

$$\begin{aligned} \sigma(\mathbf{x} + \mathbf{y}) &= \mathbf{x} \oplus \mathbf{y} && \text{by definition} \\ &= \sigma(\mathbf{x}) \oplus \sigma(\mathbf{y}) && \text{since } \mathbf{x} \text{ and } \mathbf{y} \text{ are stable.} \end{aligned}$$

That is, σ looks like it satisfies the homomorphism law.

Now the set of configurations $C = \mathbb{N}^{V_0}$ is a monoid (associative, commutative, and with identity) under pointwise addition $+$ in C. Therefore, all these properties are also true for any homomorphic images. Evidently, they see that the surjective map σ is a homomorphism from $(C, +)$ to $(M, \oplus)$, but do not give an explicit proof. Other authors in the literature studying sandpile monoids also omit justification of associativity when using the term "monoid" or "semigroup" as far as we could determine. Let us justify the assertion that $(M, \oplus)$ is the homomorphic image of the commutative monoid $(C, +)$.

To prove σ is actually is a homomorphism defined on C with $+$ (and thus establish associativity of $(M, \oplus)$ in another way), it suffices to consider general $\mathbf{x}$ and $\mathbf{y}$ in C and prove $\sigma(\mathbf{x} + \mathbf{y}) = \sigma(\mathbf{x}) \oplus \sigma(\mathbf{y})$.

Here we give an explicit proof:

Lemma 3. *The stabilization function* $\sigma : C \twoheadrightarrow M$ *is a homomorphism from the monoid* $(C, +)$ *onto* $(M, \oplus)$.

Proof. The function σ is trivially surjective since for $\mathbf{x} \in M$, $\sigma(x) = x$. To show, σ is a homomorphism, take any $\mathbf{x}, \mathbf{y} \in C$. Start with $\mathbf{x} + \mathbf{y}$, and make all the topplings in any avalanche that stabilizes $\mathbf{x}$. The result is $\sigma(\mathbf{x}) + \mathbf{y}$. Then carry out any avalanche (stabilization) for $\mathbf{y}$. The result is $\sigma(\mathbf{x}) + \sigma(\mathbf{y})$. This configuration is reachable from $\mathbf{x}+\mathbf{y}$. Therefore, by the Key Fact (Fact 4) above, $\sigma(\mathbf{x})+\sigma(\mathbf{y})$ stabilizes to $\sigma(\mathbf{x} + \mathbf{y})$. That is,

$$\sigma(\mathbf{x} + \mathbf{y}) = \sigma(\sigma(\mathbf{x}) + \sigma(\mathbf{y})).$$

The right-hand side equals $\sigma(\mathbf{x}) \oplus \sigma(\mathbf{y})$ by definition of $\oplus$. Thus,

$$\sigma(\mathbf{x} + \mathbf{y}) = \sigma(\mathbf{x}) \oplus \sigma(\mathbf{y}).$$

This proves σ is a homomorphism from $(C, +)$ onto $(M, \oplus)$.

In particular, $\oplus$ is associative on M.

Corollary 3. *$(M, \oplus)$ is an Abelian monoid.*

3.3. Two non-isomorphic semigroups mapping onto $(M, \oplus)$

We showed $(C, \oplus)$ is associative, hence an Abelian semigroup in Theorem 2. Note that $(C, \oplus)$ is *not* a monoid since $0 \oplus \mathbf{x} = \mathbf{x} \oplus 0 = \sigma(\mathbf{x}) \neq \mathbf{x}$, for any $\mathbf{x} \in C \setminus M$.

Theorem 4 (Retraction lemma)**.** *The function σ is a semigroup homomorphism from $(C, \oplus)$ onto the monoid $(M, \oplus)$. Moreover, inclusion $\iota : (M, \oplus) \hookrightarrow (C, \oplus)$ is a homomorphism such that $\sigma \circ \iota$ equals the identity on $(M, \oplus)$.*

Proof. We have already observed that σ is a surjective function from the set C of configurations onto the set of stable configurations M. Now, given any two configurations $\mathbf{x}$ and $\mathbf{y}$ in C, we have

$$\begin{aligned} \sigma(\mathbf{x} \oplus \mathbf{y}) &= \sigma(\sigma(\mathbf{x} + \mathbf{y})) \quad \text{by definition of } \oplus \\ &= \sigma(\mathbf{x} + \mathbf{y}) \quad \text{since } \sigma^2 = \sigma \text{ is idempotent} \\ &= \sigma(\mathbf{x}) \oplus \sigma(\mathbf{y}), \ \text{by Key Fact (Fact 4) since } \sigma(\mathbf{x}) + \sigma(\mathbf{y}) \text{ is in the toppling graph of } \mathbf{x} + \mathbf{y}. \end{aligned}$$

This proves σ is a homomorphism from $(C, \oplus)$ onto $(M, \oplus)$. Trivially, there is an inclusion homomorphism of semigroups $\iota : (M, \oplus) \to (C, \oplus)$, since for all $\mathbf{x}, \mathbf{y} \in M$,

$$\iota(\mathbf{x} \oplus \mathbf{y}) = \mathbf{x} \oplus \mathbf{y} = \iota(\mathbf{x}) \oplus \iota(\mathbf{y}).\text{[‖]}$$

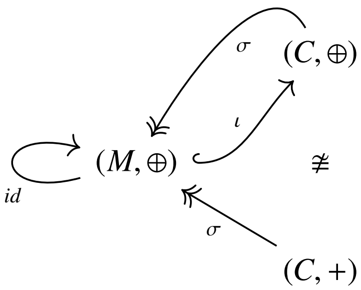
The composite homomorphism $\sigma \circ \iota$ is the identity homomorphism id on $(M, \oplus)$: since for $\mathbf{x} \in M$,

$$\sigma(\iota(\mathbf{x})) = \sigma(\mathbf{x}) = \mathbf{x} = id(\mathbf{x}),$$

using that $\mathbf{x}$ is stable.

Also, since the stabilization function σ is a surjective homomorphism from $(C, +)$ to $(M, \oplus)$ by Lemma 3, we have a commutative diagram of semigroups, where the two operations $\oplus$ and $+$ give non-isomorphic operations on C.

[‖]Caveat: In contrast, the inclusion function from $(M, \oplus)$ to $(C, +)$ is *not* a semigroup homomorphism, since the sum of stable configurations need not be stable, i.e., $\mathbf{x} \oplus \mathbf{y} = \mathbf{x} + \mathbf{y}$ does not hold in general for stable $\mathbf{x}$ and $\mathbf{y}$.



Remark. Perhaps surprisingly, $(M, \oplus)$ is the homomorphic image not only of $(C, \oplus)$ but also of $(C, +)$ under the *same* function σ. This holds despite the fact that $(C, +) \ncong (C, \oplus)$. In the case of $(C, +)$, σ is a monoid morphism mapping the empty configuration $\mathbf{0}$, the identity of $(C, +)$, to the identity of $(M, \oplus)$. However, σ is not a monoid morphism for $(C, \oplus)$, since that structure is not a monoid. Although $(C, \oplus)$ retracts onto $(M, \oplus)$, the monoid $(C, +)$ does not, as $(M, \oplus)$ is not a subsemigroup of $(C, +)$.

4. The general monoid of sandpile operators on configurations of digraph Γ

We now introduce a transformation monoid of operators on configurations, yielding a sandpile model in which the semigroups $(C, +)$ and $(M, \oplus)$ will appear as subsemigroups of a larger semigroup of operators acting on the set of configurations C. This monoid is closely related to the Dhar sandpile Dynamics Graph introduced above. Furthermore, $(C, \oplus)$ will appear as a homomorphic image within this framework. These constructions clarify, in a single framework, the relation of the sandpile semigroups to the discrete dynamical systems perspective introduced in a remarkable seminal paper of Dhar [4].

Following Dhar [4], consider the following operators on the state space $C = \mathbb{N}^{V_0}$ for a digraph $\Gamma = (V, E)$:

Add a grain at site u: Define $u : C \to C$ by adding a single grain at site u while leaving other sites unchanged. That is,

$$\mathbf{x} \cdot u := \mathbf{x} + \delta_u,$$

where δ_u is the element of C with one grain at u and no grains elsewhere and addition is pointwise in $(C, +)$.

Topple at site u: Define $\tau_u : C \to C$

$$((\mathbf{x})\tau_u)_v := \begin{cases} \mathbf{x}_v - \deg^+(u) & \text{if } \mathbf{x}_u \geq \deg^+(u) \text{ and } u = v \\ \mathbf{x}_v + a_{u,v} & \text{if } \mathbf{x}_u \geq \deg^+(u) \text{ and } u \neq v \\ \mathbf{x}_v & \text{otherwise,} \end{cases}$$

where $u, v \in V_0$ and $a_{u,v}$ is the number of edges from u to v.

Note that τ_u topples at site u if $\mathbf{x}$ has at least $\deg^+(u)$ sand grains, i.e., u sends one grain from u to each v along each edge from u to v, leaving other sites unchanged, and τ_u has *no effect* on $\mathbf{x}$ if there are fewer grains than the capacity $\deg^+(u)$ at site u.

Note that since $v \in V_0$, at least one grain will be lost from the system if u has an edge to the sink s.

Adding detail to the considerations of Dhar [4], we examine the commutation relations between these operators:

Lemma 4. *For $u, v \in V_0$ we have the following relations, where σ is the stabilization function on C,*

(i). $uv = vu$.

(ii). $u\tau_v\sigma = \tau_v u\sigma$.

(iii). $\tau_v\tau_u\sigma = \tau_u\tau_v\sigma$.

(iv). Neither $\tau_v\tau_u = \tau_u\tau_v$ nor $u\tau_v = \tau_v u$ hold in general.

(v). $v\sigma u\sigma = vu\sigma = uv\sigma = u\sigma v\sigma$.

(vi). $\sigma = \tau_v\sigma = \sigma\tau_v$.

Proof. (i) Trivial, since $(C, +)$ is commutative. We check (ii) at a configuration $\mathbf{x}$: if $u \neq v$, then adding a grain at u does not affect whether or not the $\mathbf{x}$ can topple by τ_v, so we can add the grain before or after τ_v. Thus, $\mathbf{x}u\tau_v = \mathbf{x}\tau_v u$; whence, $\mathbf{x}u\tau_v\sigma = \mathbf{x}\tau_v u\sigma$. Otherwise, $u = v$. If $\mathbf{x}_u \leq \deg^+(u) - 2$, then $\mathbf{x}\tau_u = \mathbf{x}$ and $\mathbf{x}\tau_u u = \mathbf{x}u = \mathbf{x}u\tau_u$. If $\mathbf{x}_u \geq \deg^+(u)$, then τ_u will topple site u and u will add one grain there, so $\mathbf{x}\tau_u u = \mathbf{x}u\tau_u$ so again $\mathbf{x}u\tau_v\sigma = \mathbf{x}\tau_v u\sigma$. If $\mathbf{x}_u = \deg^+(u) - 1$, then $\mathbf{x}\tau_u u = \mathbf{x}u$, whereas site u in $\mathbf{x}u$ is toppled by τ_u. It is *not* true that $\mathbf{x}\tau_u u = \mathbf{x}u\tau_u$ holds in this case, but $(\mathbf{x} + \delta_u)\tau_u$ is in the toppling graph of $\mathbf{x} + \delta_u$ so by the Key Fact (Fact 4),

$$((\mathbf{x} + \delta_u)\tau_u)\sigma = (\mathbf{x} + \delta_u)\sigma,$$

but $\mathbf{x} + \delta_u$ is $\mathbf{x}\tau_u u$. Thus, $\mathbf{x}\tau_u u\sigma = \mathbf{x}u\tau_u\sigma$ in every case.

If $u = v$, then the relation in (iii) holds trivially, so assume $u \neq v$. Consider any $\mathbf{x} \in C$. If u and v are not neighbors, then it is easy to see $\tau_v\tau_u = \tau_u\tau_v$. Also, if $\mathbf{x}_v$ and $\mathbf{x}_u$ have at least as many grains as the capacities of v and u, then toppling one leaves the other able to topple (since it can only increase the number of grains there). Therefore, toppling in either order results in the same net change at each vertex. Otherwise, if u and v are neighbors, then toppling one of them, say u, might then allow the other site v to topple, even though v did not have enough grains to topple first. Thus, $\tau_u\tau_v$ could decrease the number of grains at v, while $\tau_v\tau_u$ would, in this case, have the same effect at u on $\mathbf{x}$ but in particular could never decrease the number of grains at v. Therefore, $\mathbf{x}\tau_u\tau_v$ need not in general be equal to $\mathbf{x}\tau_v\tau_u$. This proves the first part of (iv). However, $\mathbf{x}\tau_v\tau_u$ and $\mathbf{x}\tau_v\tau_u$ are both in the toppling graph of $\mathbf{x}$, whence by the Key Fact (Fact 4), $\mathbf{x}\tau_v\tau_u\sigma = \mathbf{x}\tau_v\tau_u\sigma$, proving (iii). If a grain is added at u, and u originally had fewer grains than its capacity, then now it is possible to topple at u, potentially decreasing the number of grains at u by many, therefore in this case $u\tau_u \neq \tau_u u$. Thus, $u\tau_v$ is not in general equal to $\tau_v u$. This completes the proof of (iv). To prove (v), note that since $(\mathbf{x} + \delta_u)\sigma + \delta_v$ is in the toppling graph of $\mathbf{x} + \delta_u + \delta_v$; thus, by the Key Fact (Fact 4),

$$(\mathbf{x}u\sigma v)\sigma = (\mathbf{x}uv)\sigma,$$

whence the equation always holds. (vi) follows immediately from the Key Fact (Fact 4): Since $\mathbf{x}\tau_u$ is in the toppling graph of $\mathbf{x}$, it must equal $\mathbf{x}\sigma$, and since $\mathbf{x}\sigma$ is stable, it is fixed by τ_v. Note that (vi) implies the operators in (iii) equal σ.

Let us consider the question: what is the relation of stabilization $\sigma : C \to C$ to the toppling operations? Let $\tau = \Pi_{v\in V_0}\tau_v$, which topples once at every site in some fixed order.

Then by Facts 1 and 3 for all $\mathbf{x}$, there exists $n = n_{\mathbf{x}}$ such that $\mathbf{x}\tau^n$ is stable. That is,

$$\mathbf{x}\tau^n = \sigma(\mathbf{x}),$$

with n depending on $\mathbf{x}$. Here n is at most the length of any path from $\mathbf{x}$ to $\sigma(\mathbf{x})$ in the toppling graph of $\mathbf{x}$. For each $\mathbf{x} \in C$, the infinite sequence

$$\mathbf{x}, \mathbf{x}\tau, \ldots, \mathbf{x}\tau^k, \ldots$$

stabilizes at $\sigma(\mathbf{x}) = \mathbf{x}\tau^{n_x}$. Since $\mathbf{x}$ can have an arbitrary multiple number of grains at each node, there is no finite n for which this equation holds for all $\mathbf{x}$.

Therefore, while σ is not equal to any finite power of τ, the sequence τ^k may be viewed as converging to σ in a dynamical sense. Let us generate a *General Monoid of Sandpile Operators* for $\Gamma = (V, E)$ with non-sink nodes $V_0 \subsetneq V$,

$$\mathcal{GS}^+(\Gamma) = \langle u, \tau_u, \sigma : u \in V_0\rangle.$$

While the monoid $\mathcal{GS}^+$ fails to satisfy commutativity, it is closely related to the Dhar Sandpile Graph: it extends the (partial) transformations of C given by toppling in the graph to all configurations so that the result is a semigroup of transformations.

Let

$$\mathcal{SP} = \langle u\sigma : u \in V_0\rangle \cup \{\sigma\}.$$

By Lemma 4(iv), $\mathcal{GS}^+$ is not Abelian, but $\mathcal{SP}$ is Abelian and has identity σ by Lemma 4(v) and (vi). Both are, of course, associative since the multiplication is a function composition of operators on C in each case.

Theorem 5. *Both $\mathcal{SP}$ and $\mathcal{GS}^+$ are semigroups of operators acting on the set of configurations C. That is, $(C, \mathcal{GS}^+)$ and $(C, \mathcal{SP})$ are transformation semigroups. We have:*

(i). $(\mathcal{GS}^+)\sigma = \mathcal{SP}$ and, moreover, right multiplication by σ,

$$\nu \mapsto \nu\sigma, \text{ for } \nu \in \mathcal{GS}^+$$

is a homomorphism onto $\mathcal{SP}$. Moreover, together with the identity function on states C, this give a surjective morphism of transformation semigroups.
Note: $(C, \mathcal{SP})$ is not a transformation monoid since its identity σ is not the identity on C.

(ii). The commutative monoid generated by operators adding single sand grains $\langle u : u \in V_0\rangle \cong (C, +)$ is a submonoid of $\mathcal{GS}^+$.

(iii). The monoid $\mathcal{SP}$ is an Abelian subsemigroup of $\mathcal{GS}^+$ with two-sided identity σ and is isomorphic to the classical sandpile monoid $(M, \oplus)$.

(iv). $\mathcal{SP}$ is finite for finite Γ. Moreover, $\mathcal{SP}$ also acts faithfully on the set of stable configurations M.

Proof. (i) By Lemma 4(ii), (iii) and (v), if $\nu = \alpha_1 \dots \alpha_k$ where the α are generators of $\mathcal{GS}^+$, then in $\nu\sigma$, we can reorder the α_is arbitrarily and get the same resulting transformation of C. Moreover, $\nu\sigma = \prod_i(\alpha_i\sigma)$. We may remove any of the $\alpha_i = \sigma$ or of the form $\alpha_i = \tau_u$ by Lemma 4(vi). It follows that

$$\nu\sigma\nu'\sigma = \nu\nu'\sigma,$$

for all $\nu, \nu' \in \mathcal{GS}^+$. In particular, right multiplication by the operator σ is a homomorphism onto the subsemigroup $(GS^+)\sigma$, and its image is the monoid $\mathcal{SP}$ generated by the $u\sigma$ for $u \in V_0$ and σ. (ii) is clear. (iii) There is a monoid isomorphism from the semigroup of operators $\mathcal{SP}$ under function composition to $(M, \oplus)$ defined by $\eta(\nu) = \mathbf{0}\nu$. It is surjective since $\eta(\sigma) = \mathbf{0}$ and any other stable configuration $\mathbf{x} \in M$ can be obtained by adding the sand grains in $\mathbf{x}$ to the empty configuration: $\mathbf{0}u_1 \dots u_n$ with one u_i for each sandgrain at site u_i in $\mathbf{x}$. Then $\eta((u_1\sigma)\dots(u_n\sigma)) = \eta(u_1 \dots u_n\sigma) = \mathbf{0}u_1 \dots u_n\sigma = \mathbf{x}\sigma = \mathbf{x}$. It is injective since $\eta(\nu) = \eta(\nu')$ is $\mathbf{0}\nu = \mathbf{0}\nu'$; that is, ν and ν' each add the sand grains of a stable configuration to $\mathbf{0}$, whence they add the same number at each site, so $\nu = \nu'$.

The map η respects multiplication and the identity since

$$\eta(\nu\nu') = \mathbf{0}\nu\nu' = \mathbf{0}\nu\sigma\nu'\sigma = \mathbf{0}\nu\sigma\nu'\sigma^2 = (\mathbf{0}\nu\sigma + \mathbf{0}\nu'\sigma)\sigma = \eta(\nu) \oplus \eta(\nu') = \mathbf{0}\nu \oplus \mathbf{0}\nu' = \eta(\nu) \oplus \eta(\nu')$$

$$\text{and} \quad \eta(\sigma) = \mathbf{0}\sigma = \mathbf{0}.$$

(iv) If $\mathbf{x} \in M$, then by function application, σ acts as the identity on $\mathbf{x}$. For $\nu, \nu' \in \mathcal{SP}$, using function composition $\mathbf{x}(\nu\nu') = (\mathbf{x}\nu)\nu'$. This defines a monoid action of $\mathcal{SP}$ on M. If $\mathbf{x}\nu = \mathbf{x}\nu'$ for all $\mathbf{x} \in M$, then $\mathbf{0}\nu = \mathbf{0}\nu'$. This can be rewritten as $\eta(\nu) = \eta(\nu')$, so since by (iii), η is injective, we conclude $\nu = \nu'$. Therefore, the action is faithful. Indeed, ν and ν' have the effect of adding the same number of sand grains to each site and stabilizing.

Remark (Non-Abelian sandpile monoids): Define an operator **push** $p_e : C \to C$ for each edge e in Γ, that moves a grain along the edge e from its source vertex to its target, and has no effect if the source vertex is empty. Adjoining these operators to $\mathcal{GS}^+$, we obtain a semigroup of operators acting on sandpile configurations $\mathcal{GS}^+_p$, which is a more general construction. This more general construction gives rise to one of the non-Abelian sandpile model variations defined by Ayyer et al. [27]: For a directed tree Γ with sink s at the root, let $N(\Gamma)$ be the subsemigroup of $\mathcal{GS}^+_p$ generated by the grain addition operators u (restricted to the leaf vertices), toppling operators τ_u, and push operators p_e for all vertices and edges of Γ, respectively. The classical non-Abelian 'trickle-down' sandpile model of [27] appears by letting a subsemigroup of $\mathcal{GS}^+_p$ act on stable configurations: First let $N'(\Gamma)$ be the semigroup generated by all operators of the form $u\sigma$, $p_e\sigma$, or σ, where u is grain addition at some leaf vertex of Γ, p_e is a push operator along some edge e of Γ, and σ is stabilization. Then $(C, N'(\Gamma))$ is a transformation semigroup acting on the set of all configurations C, and clearly all members of $N'(\Gamma)$ map stable configurations M to stable configurations. Making this action of $N'(\Gamma)$ faithful** on M yields the classical non-Abelian sandpile monoid $(M, N''(\Gamma))$ when Γ is a directed tree. Allowing more general digraphs Γ than trees and the addition of sand grains at internal non-leaf vertices in the definition of $N'(\Gamma)$ gives rise to more general non-Abelian sandpile monoids $(M, N''(\Gamma))$. We examine

**See Appendix A1 for how to make an action faithful.

the complexity of the classical (Abelian and non-Abelian) as well as non-classical sandpile models in the Appendix A2.††

5. Discussion and conclusions

We considered the asynchronous combination of sandpiles as formalized in Theorem 3, and established that three distinct sandpile models satisfy associativity and commutativity by a variety of different methods. This yields three corresponding sandpile semigroups. Two of these semigroups (sandpile configurations with and without asynchronous multiplication) are infinite, and both retract onto the classical sandpile monoid, which is finite for finite directed graphs. In particular, the multiplication in the classical sandpile monoid can be carried out asynchronously without affecting the result. All these structures of interest arise by extending the partially defined transitions of the Dhar Sandpile Graph to total operators on the configuration space, and then taking substructures and homomorphic images of the resulting General Monoid of Sandpile Operators. Despite the rich behaviour associated with self-organized criticality, the Krohn–Rhodes complexity of finite Abelian sandpile monoids is at most 1. This implies that such monoids can be decomposed into a wreath product of simple building blocks—specifically, aperiodic flip-flop semigroups and at most one Abelian group of permutations—making them more amenable to structural analysis via Krohn–Rhodes theory. Likewise, aperiodic non-Abelian sandpile models can be incorporated into a variant of this framework, which in turn admits generalization to non-Abelian, non-aperiodic sandpile semigroups exhibiting higher algebraic complexity (see Remark at the end of Section 4 and Appendix A2).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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††In [27], this version of the sandpile is motivated by the interest of non-equilibrium statistical physics in systems with (infinite) reservoirs and a clear direction of the flow of particles.

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Appendix

A1. Basic graph-theoretic & algebraic notions

A **multigraph** $\Gamma = (V, E)$ is a set of vertices (or nodes) V and edges E. Each edge $e \in E$ has a unique **source** vertex $s(e) \in V$ and a unique **target** vertex $t(e)$. We shall assume our multigraphs are **loopless**, i.e., $s(e) \neq t(e)$ for all $e \in E$. Note that there may be many edges with the same source and target. Here we shall generally work in the finite case, i.e., $|V|, |E| < \infty$, although sandpile models on infinite grids also occur in the literature. A **directed graph** (or **digraph**) is a multigraph in which there is at most one edge $e \in E$ with source $s(e)$ and target $t(e)$. In that case, we can identify e with the ordered pair $(s(e), t(e))$. An **(undirected) graph** is a multigraph in which for every edge e there is a reverse edge $\overline{e}$ with source $s(\overline{e}) = t(e)$ and target $t(\overline{e}) = s(e)$, and moreover $\overline{\overline{e}} = e$. We shall work only with **loopless** multigraphs. In the case of graphs, we visualize Γ by drawing an undirected connection from $s(e)$ to $t(e)$.

The out-degree $\deg^+ v$ of a vertex v is the number of edges in E having source v:

$$\deg^+(v) = |\{e \in E \mid s(e) = v\}|.$$

Similarly, the in-degree of v denoted $\deg^-(v)$ is the number of edges coming into v, i.e., the number of vertices with target v:

$$\deg^-(v) = |\{e \in E \mid t(e) = v\}|.$$

A **semigroup** $(S, *)$ is a set S with an associative binary multiplication $* : S \times S \to S$. That is,

$$\alpha * (\beta * \gamma) = (\alpha * \beta) * \gamma,$$

for all $\alpha, \beta, \gamma \in S$. For example, the natural numbers $(\mathbb{N}, \cdot)$ under multiplication, or $(\mathbb{N}, +)$ under addition, however, not under subtraction, is associative, and the set of finite strings $(A^+, \cdot)$ under concatenation is a semigroup, as are vectors under vector addition. An important example is $(F_X, \circ)$ of functions from a set X to X under composition.

A semigroup is a **monoid** if it contains an *identity element* 1 satisfying $1 * s = s = s * 1$. One can show there can be at most one such identity element by multiplying any two such elements together: if $1' \in S$ were another one, then $1' = 1 \cdot 1' = 1$ (why?). A monoid such that for each $s \in S$ there is an $s' \in S$ with $ss' = 1 = s's$ is called a **group**. Important examples of groups include the real numbers under addition, or the non-zero real numbers under multiplication, polynomials under addition, and the symmetry group S_X consisting of all permutations of X under function composition. A **subsemigroup** T of a semigroup $(S, *)$ is a subset of S such that for all $t_1, t_2 \in T$, $t_1 * t_2 \in T$ always holds. In that case, $(T, *)$ is a semigroup using the restriction of the operation $*$ to elements of T. A **subgroup** G of a $(S, *)$ is a subsemigroup of S which is a group. Note that the identity element e of a subgroup G must be an **idempotent**, i.e., $e * e = e$, but that e need *not* be an identity element for S.

A function from a semigroup $(S, *)$ to another one $(T, \times)$ is called a **(homo)morphism of semigroups** if for all $s, s' \in S$, one has $\varphi(s * s') = \varphi(s) \times \varphi(s')$. If, in addition, S and T are both monoids with identity elements 1_S and 1_T, respectively, and $\varphi(1_S) = 1_T$, then φ is called a **monoid (homo)morphism**. It is easy to prove that a semigroup homomorphism between groups is automatically a monoid homomorphism. An injective homomorphism is called an *embedding*. Here the sandpile monoid will be a transformation semigroup (M, S) whose state set M consists of all stable configurations and S consists of operators acting faithfully on M, so S can be identified with a subsemigroup of F_X.

A **transformation semigroup** (X, S) consists of a set of states X and a semigroup $(S, *)$ such that each $s \in S$, gives a mapping (function) from X to X, denoted $x \mapsto x \cdot s$, we say s is an **operator** on X, and for all $s, s' \in S$ and $x \in X$, we have $(x \cdot s) \cdot s' = x \cdot (s * s')$. We then say S **acts** on X. A **permutation group** is a transformation semigroup for which $x \cdot 1 = x$ holds for all $x \in X$. In that case, it follows that each mapping $x \mapsto x \cdot s$ is a permutation of X. A transformation semigroup is **faithful** if for all $s \neq s' \in S$, there exists $x \in X$ such that $x \cdot s \neq x \cdot s'$. That is, no two elements of S give the same mapping from X to X. If S acts on X then we can always make the action faithful by replacing S by another semigroup $S/\equiv$ which identifies two elements s and s' in S if and only if $\forall x \in X, x \cdot s = x \cdot s'$. Then the induced multiplication of the equivalence classes given by $[s] * [s'] = [s * s']$ is well-defined, i.e., $s_1 \in [s]$ and $s_2 \in [s']$ implies $s_1 s_2 \in [s * s']$. Here $[s]$ denotes the equivalence class under $\equiv$ of s. If $(S, *)$ acts on X, i.e., each $s \in S$ gives mapping $x \mapsto x \cdot s \in S$ with $s * s'$ mapping x to $(x \cdot s) \cdot s'$, then one can make the action of S on X faithful by letting $S/\equiv$ act on S: then $x \cdot [s] = x \cdot s$ is well-defined for all $s \in [s]$. Suppose $x \cdot [s] = x \cdot [s']$ for all $x \in X$, $[s], [s'] \in S/\equiv$. Then the left-hand side is $x \cdot s$ and the right-hand side is $x \cdot s'$, for all $x \in X$. Thus, $s \equiv s'$, i.e., $[s] = [s']$, proving the action of $S/\equiv$ is faithful on X. Observe then that $\eta : S \twoheadrightarrow S/\equiv$ given by $\eta(s) = [s]$ is a surjective homomorphism of semigroups.

A set S with binary multiplication $*$ is called **Abelian** (or **commutative**) if $s * s' = s' * s$ for all $s, s' \in S$. Otherwise, $(S, *)$ is called non-Abelian. Often one omits the symbol $*$ or $\cdot$ for semigroup multiplication and action, respectively, if they are understood from context.

Fact A1. *Let S be a set with binary multiplication $*$ which is either associative, Abelian (commutative), a monoid, or a group, and $(T, \times)$ is a set with binary multiplication $\times$. Suppose $\varphi : S \twoheadrightarrow T$ is a surjective function with $\varphi(s * s') = \varphi(s) \times \varphi(t)$, then $(T, \times)$ is also associative, Abelian (commutative), or a monoid or group, respectively.*

Proof. These assertions are all easy to verify: Suppose $(S, *)$ is associative. Then since φ is surjective, for each $t_1, t_2, t_3 \in T$ choose $s_1, s_2, s_3 \in S$ with $\varphi(s_i) = t_i$ $(1 \le i \le 3)$. Then $t_1 \times (t_2 \times t_3) = \varphi(s_1) \times (\varphi(s_2) \times \varphi(s_3)) = \varphi(s_1) \times \varphi(s_2 * s_3) = \varphi(s_1 * (s_2 * s_3)) = \varphi((s_1 * s_2) * s_3) = \varphi(s_1 * s_2) \times \varphi(s_3) = (\varphi(s_1) \times \varphi(s_2)) \times \varphi(s_3) = (t_1 \times t_2) \times t_3$. This proves associativity of $(T, \times)$, since $t_1, t_2, t_3 \in T$ were arbitrary. The proof for commutativity is similar. If 1_S is an identity for $(S, *)$ then $\varphi(1_S)$ is an identity for $(T, \times)$: Since for any $t \in T$ we can choose an $s \in S$ mapping onto it, we have $\varphi(1_S) \times t = \varphi(1_S) \times \varphi(s) = \varphi(1_S * s) = \varphi(s) = t$. Similarly, $t \times \varphi(1_S) = t$. The other assertions are similarly straightforward.

If $A \subseteq S$, where S is a semigroup, we say that A *generates* S, if the smallest subset of S containing A and closed under $*$ is S. We then say A is a set of generators for S. If $A = \{a_1, \ldots, a_n\}$, we write $\langle a_1, \ldots, a_n \rangle$ for the smallest semigroup (under inclusion) which contains each $a_i \in A$ $(1 \le i \le n)$.

A *morphism of transformation semigroups* $\varphi : (X, S) \to (Y, T)$ is a homomorphism of semigroups $\varphi_1 : S \to T$ and a function $\varphi_2 : X \to Y$. such that

$$\varphi_2(x \cdot s) = \varphi_2(x) \cdot \varphi_1(s),$$

holds for all $x \in X$, $s \in S$. A morphism of transformation semigroups is called an *embedding* (or *injective*) if both φ_1 and φ_2 are injective. It is *surjective* if both φ_1 and φ_2 are.

A2. Algebraic complexity of sandpile semigroups

In this appendix, we collect results and observations on the algebraic complexity of various Abelian and non-Abelian sandpile models.

A2.1. Krohn–Rhodes complexity of Abelian sandpile semigroups

The Krohn–Rhodes complexity is the algebraic complexity measure for finite semigroups that represents the minimum number of simple building blocks needed to construct a system exhibiting such dynamics. As the sandpile monoid is a finite semigroup, we may apply Krohn–Rhodes theory to analyze its structure. We show that the classical Abelian sandpile semigroups have low complexity (at most 1). In contrast, the non-Abelian sandpile semigroups mentioned in Section 4 may have higher complexity (as shown in Appendix A2.3).

Theorem A1. *The Krohn–Rhodes complexity $cpx(S)$ of a finite Abelian semigroup S is at most one, $cpx(S) \leq 1$. It is 1 if and only if S contains a nontrivial group.*

It follows immediately that an Abelian sandpile monoid $(M, \oplus) \cong (\mathcal{SP}, \circ)$ has complexity at most one. Theorem A1 on the complexity of Abelian semigroups was briefly stated as a fact in [30] and [31] without giving formal proof. To address this, after first introducing all related definitions and facts from the underlying theory for completeness of the exposition, we present two different proofs for this theorem – the first using the powerful Fundamental Lemma of Complexity (stated as Lemma A1 below) and the second using only elementary methods. Although we came to this question while studying the complexity of Abelian sandpiles, the derived proofs only use general facts and do not rely on the sandpile graph or the model's properties.

Definition A1. *The transformation semigroup (X, S) **divides** the transformation semigroup (X', S'), denoted as $(X, S) \preceq (X', S')$, if there exist $Y \subseteq X'$, $T \subseteq S'$, a surjective mapping of states $\alpha_2 : Y \to X$ and a homomorphism of semigroups $\alpha_1 : T \to S$ such that*

$$\forall y \in Y,\ t \in T, \quad \alpha_2(y \cdot t) = \alpha_2(y) \cdot \alpha_1(t).$$

Definition A2. *Let S and T be the transformation semigroups acting on the sets of states X and Y respectively, denoted as (X, S) and (Y, T). The **wreath product** of two transformation semigroups (Y, T) and (X, S), denoted by $(Y, T) \wr (X, S)$, is a transformation semigroup $(Y \times X, W)$ that acts on the direct product $Y \times X$ of the state sets by the transformations $w \in W$ composed of two components satisfying the following:*

$$\forall w \in W,\ w = (f(x), s), \quad \text{such that } s \in S, \text{ and } f : X \to T,$$

where for every element $w \in W$ both the function $f : X \to T$ and the transformation $s \in S$ are uniquely determined and depend only on w. The semigroup W consists of all such possible transformations. Then, for every state, the wreath product transformation acts component-wise:

$$\forall (y, x) \in Y \times X, \quad (y, x) \cdot w = (y \cdot f(x), x \cdot s) \in Y \times X.$$

The above concepts of wreath product and division were introduced for the transformation semigroup (X, S), where semigroup elements $s \in S$ act on the states $x \in X$. Naturally, the above also applies to the semigroups acting on themselves with unit adjoined (S^1, S)—where the state set S^1 denotes S with an identity element adjoined if it does not already have one. We omit mention of the state set and write S below for simplicity.

Fact A2. (Krohn–Rhodes Theorem, e.g., [32]). *For any finite semigroup S, there is a set of groups G_i and aperiodic semigroups A_i (i.e., which do not contain any non-trivial subgroups), such that S divides the wreath product*

$$S \preceq A_n \wr G_n \wr \cdots \wr A_1 \wr G_1 \wr A_0.$$

This gives a wreath product decomposition of S for which A_i and G_i are called "building blocks". Here the G_i may be chosen to be divisors of S. Moreover, if G is a finite simple group and G divides S, then G divides one of the G_i. Furthermore, if the flip-flop FF (or either of its two 2-element subsemigroups) divides S, then it embeds in some A_i. Here FF is the 3-element aperiodic semigroup consisting of all mappings of the 2-element set to itself other than the transposition (i.e., the identity and two constant maps).

Definition A3. ([33]). *The **Krohn–Rhodes complexity** of semigroup S, denoted as $cpx(S)$, is the smallest number n of non-trivial groups G_i that must occur in any wreath product decomposition of the semigroup S. The complexity measure cpx is the unique point-wise maximal function that satisfies the following axioms:*

(i). $cpx(G) = 1$ *for any nontrivial group G,*

(ii). $cpx(FF) = 0$ *for the flip-flop FF,*

(iii). $cpx(S \wr T) \leq cpx(S) + cpx(T)$,

(iv). $cpx(S \times T) \leq \max(cpx(S), cpx(T))$,

(v). $S \preceq T \implies cpx(S) \leq cpx(T)$.

Definition A4. *The semigroup morphism $\phi : S \to T$ is called **aperiodic** if for all idempotents $e \in T$ the inverse image $\phi^{-1}(e)$ is an aperiodic semigroup.*

Fact A3. (Proposition from [32], p334). *For any homomorphism $\phi : S \to T$ between semigroups S and T the following properties are equivalent: (i) ϕ is aperiodic, (ii) if G is a group in S and $\phi(G)$ is a singleton, then G is a singleton, (iii) ϕ is injective on groups in S.*

Fact A4. (*e.g.*, [30]). *Let S be a finite semigroup. For an idempotent $e \in S$, $e^2 = e$, let G_e denote a maximum size subgroup of S containing e. The following holds:*

(a) Every element $s \in S$ has a unique idempotent power s^v ($v \geq 1$), i.e., $s^v s^v = s^v$, and there is an integer $\omega \geq 1$ such that for all $s \in S$, s^ω is the idempotent power of s.
(b) The unique idempotent $e = s^\omega$ of the subsemigroup $\langle s \rangle$ generated by any element $s \in S$ is the identity element of a finite cyclic subgroup $C = \{s^{\omega+k} : k \geq 0\}$.
(c) Every group $G \subseteq S$ whose identity element is e is contained in the group G_e (thus G_e is uniquely determined by e), and for two idempotents $e_1 \neq e_2$ the corresponding maximal groups are disjoint $G_{e_1} \cap G_{e_2} = \emptyset$.

Lemma A1 (Fundamental Lemma of Complexity [33]). *If $\phi : S \to T$ is a surjective aperiodic homomorphism of semigroups from S onto T, then $cpx(S) \leq cpx(T)$.*

Proof. [First proof for Theorem A1] Let $E \subseteq S, |E| = m$ be the set of all idempotents of S with a fixed numbering of elements $E = \{e_1, e_2, \ldots, e_m\}$ and G_k denote the maximal subgroup of S containing the idempotent e_k (some of the G_k may be trivial groups). Due to Fact A4(c) all G_k are disjoint. Denote the union $T = G_1 \sqcup \cdots \sqcup G_m$.

Due to Fact A4(a) there is an integer ω such that $\forall s \in S$, s^ω is an idempotent. Consider the semigroup morphism $\phi : S \to Im(\phi) \subseteq S$ defined as $\phi(s) = s^\omega s$, $\forall s \in S$. As multiplication in S

commutes, ϕ is a structure preserving mapping: $\forall s_1, s_2 \in S$, $\phi(s_1)\phi(s_2) = (s_1^\omega s_1)(s_2^\omega s_2) = s_1^\omega s_2^\omega s_1 s_2 = (s_1 s_2)^\omega s_1 s_2 = \phi(s_1 s_2)$. For any element $s \in S$, suppose $s^\omega = e_k$, then its image $\phi(s) = s^\omega s = s^{\omega+1} \in \langle s \rangle \subseteq G_k$ belongs to the corresponding group G_k, due to Fact A4(b),(c); therefore $Im(\phi) \subseteq T$. Moreover, every group element $s \in G_k$ is mapped to itself: $\phi(s) = s^\omega s = e_k s = s \in G_k$, so $T \subseteq Im(\phi)$ and ϕ is injective on groups. The mutual inclusion implies $Im(\phi) = T$, whence T is a subsemigroup of S. Thus we conclude that ϕ is a surjective semigroup homomorphism $\phi : S \to T$ that is injective on groups in S, therefore aperiodic due to Fact A3. The Fundamental Lemma of Complexity A1 implies $cpx(S) \leq cpx(T)$.

Consider a *new* element I acting as identity on S, that is $\forall s \in S,\ sI = Is = s$. Denote $S^I = S \sqcup \{I\}$; note that S^I is also Abelian. Let $Y = G_1^I \times \cdots \times G_m^I$ with component-wise multiplication denoted by $*$, where $G_i^I = G_i \sqcup \{I\}$. For each generator g of group $G_k \subseteq T$ there is the element $\vec{y_g} = (y_1, \ldots, y_m) \in Y$ such that $y_i = g$ for $i = k$ and $y_i = I$ for $i \neq k$. Let Y' be the semigroup $Y' \subseteq Y$ generated by elements $\vec{y_g}$ for all generators $g \in G_k$ and all groups $1 \leq k \leq m$. Consider the mapping ψ defined for all $\vec{y} = (y_1, \ldots, y_m) \in Y'$ as $\psi(\vec{y}) = \prod_{j=1}^m y_j$ with multiplication in S^I. As S^I is Abelian, for all $\vec{a} = (a_1, \ldots, a_m), \vec{b} = (b_1, \ldots, b_m) \in Y'$ we have $\psi(\vec{a})\psi(\vec{b}) = \prod_{j=1}^m a_j \prod_{j=1}^m b_j = \prod_{j=1}^m (a_j b_j) = \psi(\vec{a} * \vec{b})$. As $\forall \vec{y} \in Y'$, the image $\psi(\vec{y})$ is a product of at least one group element from T with or without the identity element I, the image $Im(\psi) = T$. Therefore, $\psi : Y' \to T$ is a surjective homomorphism with $Y' \subseteq Y$, hence we have a division $T \preceq Y$.

By the Definition A3 of Krohn–Rhodes complexity, $cpx(T) \leq cpx(Y)$ (axiom v) and $cpx(Y) \leq \max(cpx(G_i^I)) = \max(cpx(G_i)) \leq 1$ (axioms i and iv) since each $cpx(G_i^I) \leq 1$ as adjoining an identity does not change complexity. Thus, $cpx(S) \leq cpx(T) \leq cpx(Y) \leq 1$, completing the proof.

An alternative shorter proof is given below, it shows a 'rougher' emulation but does not use the Fundamental Lemma of Complexity (Lemma A1).

Proof. [Second proof for Theorem A1] Denote the distinct elements of the finite Abelian semigroup S by $s_1, \ldots, s_m$. The so-called monogenic semigroup $\langle s_j \rangle$, consisting of all positive powers of s_j, is the subsemigroup of S generated by the element $s_j \in S$. Consider the semigroup $Y = S_1^I \times \cdots \times S_m^I$ with component-wise multiplication denoted by $*$, where each $S_j^I = \langle s_j \rangle \sqcup \{I\}$ with a *new* element I acting as identity on S, i.e., $\forall s \in S, sI = Is = s$. Similarly define $S^I = S \sqcup \{I\}$; note that S^I is also Abelian.

Consider the mapping $\psi : Y \to S^I$ defined for all $\vec{y} = (y_1, \ldots, y_m) \in Y$ as $\psi(\vec{y}) = \prod_{j=1}^m y_j$ with multiplication in S^I. As S^I is Abelian, for all $\vec{a} = (a_1, \ldots, a_m), \vec{b} = (b_1, \ldots, b_m) \in Y'$ we have $\psi(\vec{a})\psi(\vec{b}) = \prod_{j=1}^m a_j \prod_{j=1}^m b_j = \prod_{j=1}^m (a_j b_j) = \psi(\vec{a} * \vec{b})$. Also, for every element $s_i \in S^I$ there is the element $\vec{y} = (I, \ldots, s_i, \ldots, I) \in Y$ that is mapped to it: $\psi(\vec{y}) = I^{i-1} s_i I^{m-i} = s_i$. Therefore, $\psi : Y \to S^I$ is a surjective homomorphism. This entails division $S^I \preceq Y$.

Therefore, $cpx(S^I) \leq cpx(Y) \leq \max_j \{cpx(S_j^I)\}$ by axioms v and iv of the Definition A3. Since adding an identity element does not increase complexity, $cpx(S) = cpx(S^I) \leq \max_j \{cpx(S_j^I)\} = \max_j \{cpx(S_j)\}$. As is easily verified, every monogenic semigroup $\langle s \rangle$ embeds in the direct product of a cyclic group and an aperiodic monogenic semigroup, therefore, it has complexity at most one $cpx(S_j) \leq 1$. Thus, we conclude $cpx(S) \leq 1$.

A2.2. Aperiodic complexity of classical non-Abelian sandpile semigroups

For finite aperiodic semigroups, it follows from the Krohn–Rhodes theorem that they can be constructed from flip-flops without using any group building blocks. Non-Abelian sandpile semigroups [27] arise when the digraph is a tree with a unique path from each node to the root r (sink node with $\deg^+(r) = 0$), together with the push operation p_e that moves a sand grain, if any, from node v to node u, defined for each edge $e = (v, u) \in E$, and addition operators a_v restricted to the leaf nodes only (i.e., nodes with $\deg^-(v) = 0$). In the aperiodic case, one can define the **aperiodic complexity** [34] of all finite aperiodic semigroups by dropping axiom i (since groups are not needed in the decomposition) and replacing axiom ii in Definition A3 with $cpx(FF) = 1$.

For these classical non-Abelian sandpile semigroups [27], we have the following

Theorem A2. *(Derets–Nehaniv [23]). Let S be the sandpile semigroup on a tree (directed acyclic digraph) $\Gamma = (V, E)$ where each edge is directed toward a unique sink at the root of the tree, and S is generated by two operators: adding a sand grain to the vertex a_v, defined for each node of the tree $v \in V$ such that $\deg^-(v) = 0$, i.e., leaf nodes, and pushing a sand grain (if any) p_e from node v to node u, defined for every edge $e = (v, u) \in E$. Then S embeds into the wreath product of direct products of flip-flop semigroups. In particular, S is aperiodic, and the aperiodic complexity of S equals the depth d of the tree (the number of edges in the longest path from a leaf to the root).*[‡‡]

A2.3. Non-Abelian sandpile semigroups of higher Krohn–Rhodes complexity

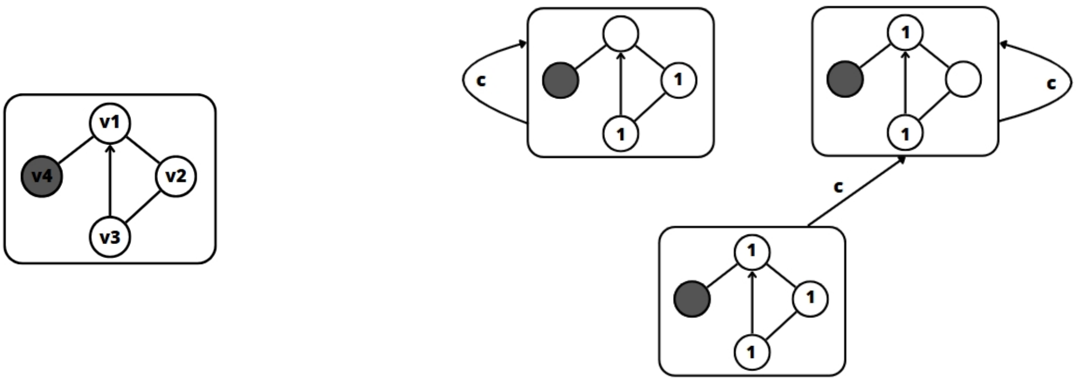
Using more general digraphs (beyond trees) and allowing addition of grains not only to the leaves but to all vertices, one finds finite non-Abelian sandpile semigroups containing non-Abelian groups with higher Krohn–Rhodes complexity than the classical Abelian and non-Abelian sandpile semigroups.

Example A1. Let us consider the digraph $\Gamma = (V, E)$ shown in Figure A1(a), where v_4 is the sink vertex and non-directed edges between vertices v and u denote the existence of both directed edges $v \to u$ and $u \to v$. We consider the generalization of the classical non-Abelian sandpile semigroup $(M, N''(\Gamma))$ acting on the set of stable configurations M (as in the Remark at the end of Section 4). Then $N''(\Gamma)$ is generated by 10 generators $N''(\Gamma) = \langle \sigma,\ a_k\sigma,\ p_{vu}\sigma \mid k \in V, e = (v, u) \in E \rangle$ where a_v is the operator of adding a grain to the vertex $v \in V$ and p_{vu} is the operator of pushing a grain, if any, from the vertex v to u along the edge $e = (v, u) \in E$. Then the sandpile semigroup $N''(\Gamma)$ has Krohn–Rhodes complexity strictly greater than one.

Proof. Consider the digraph in Figure A1(a) and the subset $X \subset M$ of stable configurations shown in Figure A1(b).[*] The transformations $c = a_1\sigma a_3\sigma p_{21}\sigma \in N''(\Gamma)$, $r = a_1\sigma \in N''(\Gamma)$ and $t = p_{31}\sigma p_{12}\sigma p_{14}\sigma p_{31}\sigma a_3\sigma \in N''(\Gamma)$ act on X as an elementary collapsing, rotation and transposition, as shown in Figure A1(b)–(d) respectively. These generate the semigroup $S = \langle c, r, t\rangle|_X$ isomorphic to the full transformation semigroup F_3, which consists of all 27 possible mappings on 3 points, i.e., $(\{1, 2, 3\}, F_3) \cong (X, S)$. As the Krohn–Rhodes complexity of the full transformation semigroup F_n on n points is $n - 1$ (see, e.g., [32]), the complexity of S is $cpx(S) = cpx(F_3) = 2$. Moreover, S is the homomorphic image of $N''(\Gamma)$ obtained by restricting its action to X and by making it faithful on X, i.e., $S = N''(\Gamma)|_X \twoheadleftarrow N''(\Gamma)$, therefore $S \prec N''(\Gamma) \implies cpx(S) \le cpx(N''(\Gamma))$; whence by axiom v for Krohn–Rhodes complexity, we have $cpx(N''(\Gamma)) \ge 2$.

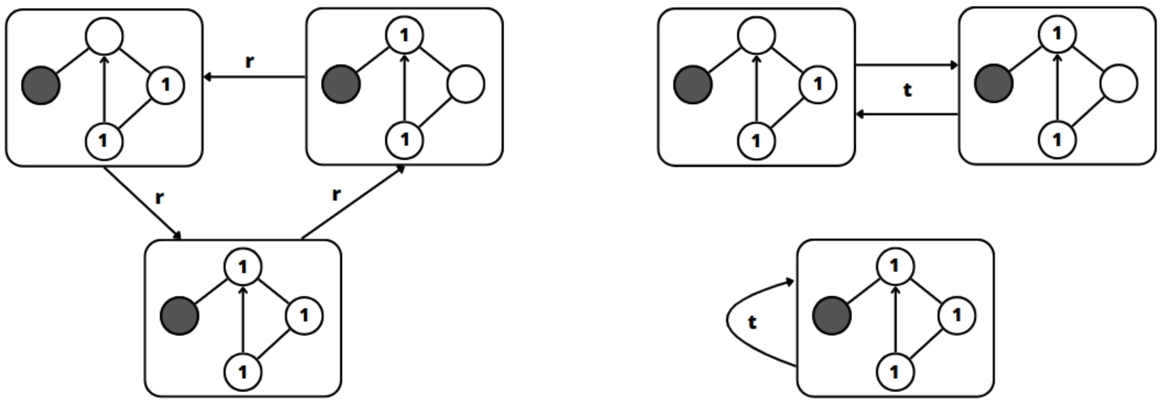
[‡‡]This result is also generalized to directed rooted acyclic multigraphs.

[*]The members of X also correspond to the recurrent configurations of the classical Abelian sandpile on the graph Γ where they form a minimal ideal isomorphic to the cyclic group C_3 of order 3 generated by a cyclic permutation on 3 points.



(a) Sandpile graph $\Gamma = (V, E)$.

(b) Elementary collapsing $c = a_1\sigma a_3\sigma p_{21}\sigma$.



(c) Rotation $r = a_1\sigma$

(d) Transposition $t = p_{31}\sigma p_{12}\sigma p_{14}\sigma p_{31}\sigma a_3\sigma$.

Figure A1. Non-Abelian sandpile semigroup of higher complexity. For the non-Abelian sandpile transformation semigroup $(M, N''(\Gamma))$ on the directed graph Γ depicted in (a), the action of transformations c, r, and t on the set X of recurrent configurations (a subset of the stable configurations M) is visualized in (b)–(d), respectively. In the configurations of (b)–(d), the sink vertex v_4 is gray, the empty vertices are white, and the rest of the vertices labeled by 1 contain a single sand grain.