



Research article

Applications of nonlinear integrable equations in artificial intelligence

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Abstract: This study begins with the nonlinear coupled Korteweg-de Vries equation and successfully solves its Lax pair using the prolongation structure method. Based on the obtained Lax pair, the Bäcklund transformation and superposition formula are further derived. These results provide a theoretical foundation for understanding the dynamic behavior of nonlinear systems and offer new approaches for constructing exact solutions. The study highlights the potential of the Bäcklund transformation and superposition formula in improving computational efficiency, enhancing model generalization capabilities, and increasing prediction accuracy. Furthermore, this paper explores the application prospects of these mathematical tools in the field of AI demonstrating significant advantages in simulating complex physical phenomena, such as waves and turbulence. Experimental results show that the AI model incorporating Bäcklund transformation improves computational efficiency by 40 percent and reduces prediction error by 25 percent in turbulence forecasting. Integrating traditional mathematical methods with modern AI technologies not only significantly enhances the performance of existing algorithms but also opens up new possibilities for addressing complex real-world problems. This research lays the groundwork for future interdisciplinary studies, encouraging more scholars to engage in this cutting-edge field. Through cross-disciplinary integration, it is expected to promote the joint advancement of nonlinear science and AI technology.

Keywords: prolongation structure; Lax pair; Bäcklund transformation; artificial intelligence

1. Introduction

Nonlinear phenomena are ubiquitous in nature, and their dynamical behaviors can typically be described by nonlinear partial differential equations [1]. The Korteweg-de Vries (KdV) equation and its coupled forms serve as paradigmatic examples, demonstrating significant application value in the fields of fluid mechanics and plasma physics [2, 3].

Analytical solutions to such nonlinear equations present considerable challenges. The prolongation structure method provides a systematic approach to investigating the integrability and exact solutions

of these equations through the construction of Lax pairs and Bäcklund transformations [4, 5]. These mathematical tools not only reveal the intrinsic structure of nonlinear systems but also establish the theoretical foundation for this study [6, 7].

Distinct from existing research, the innovations of this work are manifested in the following:

- 1) The first systematic application of the prolongation structure method to derive Bäcklund transformations for coupled KdV equations
- 2) A pioneering proposal to apply Bäcklund transformations in the field of artificial intelligence (AI), achieving efficient compression and transformation of large models

This interdisciplinary approach not only expands the application boundaries of nonlinear mathematical theory but also provides novel mathematical tools for AI model optimization. By establishing fundamental connections between mathematical models and AI algorithms, this research opens new avenues for addressing complex computational problems.

2. Prolongation structure of nonlinear integrable equations

Given the coupled KdV equations as

$$\begin{cases} u_t = -u_{xxx} + 6uu_x, \\ v_t = -4v_{xxx} + 6uv_x + 3u_xv. \end{cases} \quad (2.1)$$

If $v = 0$, the above coupled system reverts to the standard KdV equation.

To transform this system into an equivalent set of first-order partial differential equations, new variables are introduced as follows: $p, q, r,$ and s .

Let $u_x = p, u_{xx} = p_x = q, v_x = r,$ and $v_{xx} = r_x = s$. Based on these substitutions, we can rewrite Eq (2.1) accordingly. Thus, if we proceed with these substitutions, we get

$$\begin{cases} u_x = p, \\ v_x = r, \\ u_t = -q_x + 6up, \\ v_t = -4s_x + 6ur + 3pv. \end{cases} \quad (2.2)$$

Define a set of exterior differential 2-forms on the manifold $U = \{u, v, p, q, r, s, x, t\}$.

$$\begin{cases} \alpha_1 = dt \wedge du + dx \wedge dt p, \\ \alpha_2 = dt \wedge dp + dx \wedge dt q, \\ \alpha_3 = dt \wedge dv + dx \wedge dt r, \\ \alpha_4 = dt \wedge dr + dx \wedge dt s, \\ \alpha_5 = -dx \wedge du + dt \wedge dq + dx \wedge dt(6up), \\ \alpha_6 = -dx \wedge dv + 4dt \wedge ds + dx \wedge dt(6ur) + 3dx \wedge dt(pv). \end{cases} \quad (2.3)$$

where d denotes the exterior derivative, and \wedge represents the wedge product. The first four terms of Eq (2.3) correspond to the terms introducing new variables, and the last two terms correspond to the terms from the original equations.

By taking the exterior derivative of Eq (2.3), we obtain

$$\begin{cases} d\alpha_1 = dx \wedge \alpha_2, \\ d\alpha_2 = dx \wedge \alpha_5, \\ d\alpha_3 = dx \wedge \alpha_4, \\ d\alpha_4 = dx \wedge 4\alpha_6, \\ d\alpha_5 = dx \wedge 6(\alpha_1 p + \alpha_2 u), \\ d\alpha_6 = dx \wedge 6(\alpha_1 r + \alpha_4 u + \frac{1}{2}\alpha_2 v + \frac{1}{2}\alpha_3 p). \end{cases} \quad (2.4)$$

Therefore, $I = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ forms a closed ideal on the manifold U .

If the 2-forms α_i are restricted to the solution manifold U and set to zero, then Eq (2.1) can be obtained.

Now, introduce n one-forms ω^k

$$w^k = dy^k + F^k(x, t, u, v, p, q, r, s, y^i)dx + G^k(x, t, u, v, p, q, r, s, y^i)dt, \quad (k = 1, \dots, n). \quad (2.5)$$

Here, F^k and G^k represent the Lax pair at the k -th iteration, satisfying the isospectral condition $\partial_t F - \partial_x G + [F, G] = 0$. y^i are prolongation variables, and it is required that they form a new closed ideal with α_i , meaning that ω^k must satisfy the condition to ensure this closure [8]

$$d\omega^k = \sum_{j=1}^6 f_j^k \alpha^j + \sum_{j=1}^n \eta_j^k \wedge \omega^j \quad (k = 1, 2, 3, \dots, n), \quad (2.6)$$

where f_j^i are 0-forms (scalar functions), and η_j^i are 1-forms. From Eq (2.6) and by eliminating f_j^i , we can derive the following system of partial differential equations that $F^k(u, v, p, q, r, s, y^i)$ and $G^k(u, v, p, q, r, s, y^i)$ must satisfy:

$$\begin{cases} F_p = F_q = F_r = F_s = 0, \\ G_q = -F_u, \quad G_s = -4F_v, \\ 6upF_u + 6urF_v + 3pvF_v - pG_u - rG_v - qG_p - sG_r - [G, F] = 0. \end{cases} \quad (2.7)$$

Integrating the above expressions with respect to q and s , we obtain

$$G = -qF_u - 4sF_v + G_1(u, v, p, r, y^i). \quad (2.8)$$

Substituting Eq (2.8) into Eq (2.7), we get

$$G_1 = \frac{1}{2}p^2 F_{uu} + 2r^2 F_{vv} + rpF_{uv} + 4rpF_{vu} + 4r[F_v, F] + p[F_u, F] + G_2(u, v, y^i). \quad (2.9)$$

Substituting Eq (2.9) back into Eq (2.8) and comparing coefficients, we arrive at the following system of equations:

$$\begin{cases} F_{uuu} = F_{vvv} = [F_{uu}, F] = [F_{vv}, F] = F_{vuu} = F_{uvv} = 0, \\ -6uF_u - 3vF_v + [[F_u, F], F] + G_{2u} = 0, \\ -6uF_v + 4[[F_v, F], F] + G_{2v} = 0, \\ [G_2, F] = 0. \end{cases} \quad (2.10)$$

From Eqs (2.7)–(2.10), let

$$F = X_1 + uX_2 + vX_3 + uvX_4 + u^2X_5 + v^2X_6. \quad (2.11)$$

Here, $X_i (i = 1, 2, 3, 4, 5, 6)$ depend only on the prolongation variables y^j .

Let $[X_2, X_1] = X_5$, $[X_4, X_1] = X_6$, and $[X_3, X_1] = X_7$. Substituting F into Eq (2.10), and by utilizing the Jacobi identity, we can derive the following relations:

$$\begin{cases} [X_4, X_2] = [X_4, X_3] = [X_5, X_3] = 0, [X_6, X_i] = 0, (i = 1, 2, 3, 4), \\ G_2 = 2u^2X_2 + 4uvX_3 + 4u^2vX_4 + uvX_4 - u[X_5, X_1] + u^2[X_5, X_2] \\ -4v[X_7, X_1] - 2v^2[X_7, X_3] - 8uv[X_6, X_1] + X_0. \end{cases} \quad (2.12)$$

Substitute G_2 into the equation $[G_2, F] = 0$, and we further obtain the following relations:

$$\begin{cases} [[X_5, X_1], X_1] + [X_0, X_2] = 0, \\ 3X_7 - [[X_5, X_1], X_3] + [[X_7, X_1], X_2] = 0, \\ 2X_5 - \frac{1}{2}[[X_5, X_2], X_1] - [[X_5, X_1], X_2] = 0, \\ [[X_7, X_1], X_1] + [X_0, X_3] = [X_0, X_1] = 0, \\ [[X_5, X_2], X_i] = 0, (i = 2, 3, 4) \quad [[X_7, X_3], X_i] = 0, (i = 1, 2, 3, 4). \end{cases} \quad (2.13)$$

Since $[X_2, X_1] = X_5$, from Eq (2.13), we have $[X_5, X_2] = X_2$ and $[X_2, X_1] = X_5$, indicating that X_2 is a nilpotent element and X_5 is a central element.

Note that Eq (2.1) has the following scale symmetry:

$$x \rightarrow \lambda^{-1}x, \quad t \rightarrow \lambda^{-3}t, \quad \phi \rightarrow \lambda^2\phi, \quad \varphi \rightarrow \lambda^2\varphi. \quad (2.14)$$

By requiring ω^i to be invariant under this transformation, we have

$$F \rightarrow \lambda F, \quad G \rightarrow \lambda^3 G. \quad (2.15)$$

Under the scale symmetry (2.14), the generators $\{X_i, i = 0, 1, 2, \dots, 7\}$ transform as

$$\begin{aligned} X_0 &\rightarrow \lambda X_0, & X_1 &\rightarrow \lambda^{-1}X_1, & X_2 &\rightarrow \lambda^{-1}X_2, \\ X_3 &\rightarrow \lambda^{-3}X_3, & X_4 &\rightarrow X_4, & X_5 &\rightarrow X_5, & X_6 &\rightarrow X_6, & X_7 &\rightarrow X_7. \end{aligned} \quad (2.16)$$

Here λ is a constant. By using the basis elements $\{X_i, i = 1, 2, \dots, 7\}$, we try to embed the prolongation algebra into $\mathfrak{sl}(n+1, \mathbb{C})$. Starting from the case $n = 1$, we find that $\mathfrak{sl}(1, \mathbb{C})$, $\mathfrak{sl}(2, \mathbb{C})$, and $\mathfrak{sl}(3, \mathbb{C})$ can not be the whole algebra. The simplest nontrivial closure is in terms of $\mathfrak{sl}(4, \mathbb{C})$. Take from $[X_5, X_2] = X_2$ and $[X_2, X_1] = X_5$, we can embed the prolongation algebra \mathfrak{y} .

Let

$$\begin{cases} [X_5, X_2] = X_8, & [X_5, X_1] = X_9, & [X_7, X_1] = X_{10}, \\ X_2 = e_-, & X_5 = -h. \end{cases} \quad (2.17)$$

Thus, we obtain the matrix representation of the generators as follows $\{X_i, i = 1, 2, \dots, 7\}$.

$$X_0 = \begin{pmatrix} 0 & -4\lambda & 0 & 0 \\ 4\lambda^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4\lambda \\ 0 & 0 & 4\lambda^4 & 0 \end{pmatrix}, \quad X_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & \lambda & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

$$\begin{aligned}
X_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}, & X_4 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & X_5 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & -3 & 0 & -1 \end{pmatrix}. \\
X_6 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & X_7 &= \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix}, & X_8 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \\
X_9 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\lambda & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \lambda & 0 \end{pmatrix}, & X_{10} &= \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2}\lambda & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2}\lambda & 0 & -\frac{1}{2}\lambda & 0 \end{pmatrix}.
\end{aligned} \tag{2.18}$$

By substituting the above generators into Eqs (2.8)–(2.13), (2.17), and (2.18), we obtain the explicit expressions for F and G .

$$\begin{aligned}
F &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ u - v + \lambda & 0 & v & 0 \\ 0 & 0 & 0 & -1 \\ -v & 0 & u + v + \lambda & 0 \end{pmatrix}, \\
G &= \begin{pmatrix} u_x - 4v_x & 2u - 2v - 4\lambda & 4v_x & 2v \\ A & -u_x + 4v_x & -4v_{xx} + 4uv - 2v\lambda & -4v_x \\ -4v_x + 3u_x & -2v & u_x + 4v_x & 2u + 2v - 4\lambda \\ 4v_{xx} + 4uv - 2v\lambda & 4v_x - 3u_x & B & -u_x - 4v_x \end{pmatrix},
\end{aligned} \tag{2.19}$$

where

$$\begin{aligned}
A &= 4v_{xx} - u_{xx} - 4uv - 2u\lambda - 2v\lambda + 2u^2 + 4\lambda^2, \\
B &= 4v_{xx} + 4uv - 2v\lambda.
\end{aligned}$$

If we require $\omega^i|_U = 0$, then we can obtain

$$\begin{pmatrix} y^1 \\ y^2 \\ y^3 \\ y^4 \end{pmatrix}_x = -F \begin{pmatrix} y^1 \\ y^2 \\ y^3 \\ y^4 \end{pmatrix}, \quad \begin{pmatrix} y^1 \\ y^2 \\ y^3 \\ y^4 \end{pmatrix}_t = -G \begin{pmatrix} y^1 \\ y^2 \\ y^3 \\ y^4 \end{pmatrix}. \tag{2.20}$$

From the compatibility condition $y_{xt} = y_{tx}$, it is straightforward to derive the original Eq (2.1).

3. The Bäcklund transformation of nonlinear integrable equations

Now, we construct the Bäcklund transformation. The Bäcklund transformation typically involves introducing a Bäcklund parameter θ and two auxiliary functions $\phi(x, t)$ and $\psi(x, t)$, which satisfy

certain auxiliary equations. For the obtained Lax pair F and G , we can construct a Bäcklund transformation of the following form:

$$\phi' = (\lambda I - S)\phi, \quad (3.1)$$

where S is a matrix satisfying

$$\begin{aligned} S_x &= F'S - SF, \\ S_t &= G'S - SG. \end{aligned} \quad (3.2)$$

To maintain the covariance of the Lax pair, the new potential functions u' and v' corresponding to the Lax pair F' and G' must satisfy

$$\begin{aligned} \phi'_x &= F'\phi', \\ \phi'_t &= G'\phi'. \end{aligned}$$

Let ϕ_0 be the eigenfunction for $\lambda = \lambda_0$,

$$F\phi_0 = \lambda_0\phi_0. \quad (3.3)$$

The matrix S can be expressed as

$$S = H\Lambda H^{-1}, \quad (3.4)$$

where Λ is a diagonal matrix, and H is a matrix composed of the eigenfunctions ϕ_0 . We can set

$$S = \lambda I - \begin{pmatrix} S_{11} & S_{12} & 0 & 0 \\ S_{21} & S_{22} & 0 & 0 \\ 0 & 0 & S_{33} & S_{34} \\ 0 & 0 & S_{43} & S_{44} \end{pmatrix}, \quad (3.5)$$

From $S_x = F'S - SF$, where S_x is

$$S_x = \begin{pmatrix} S_{11x} & S_{12x} & 0 & 0 \\ S_{21x} & S_{22x} & 0 & 0 \\ 0 & 0 & S_{33x} & S_{34x} \\ 0 & 0 & S_{43x} & S_{44x} \end{pmatrix}$$

$F'S$ is

$$F'S = \begin{pmatrix} -S_{11} & -S_{12} & 0 & 0 \\ (u' - v' + \lambda)S_{11} & (u' - v' + \lambda)S_{12} & S_{33}v' & S_{34}v' \\ 0 & 0 & -S_{43} & -S_{44} \\ v'S_{11} & v'S_{12} & (u' + v' + \lambda)S_{33} & (u' + v' + \lambda)S_{34} \end{pmatrix}$$

and SF is

$$SF = \begin{pmatrix} (u - v + \lambda)S_{12} & -S_{11} & S_{12}v & -S_{34} \\ (u - v + \lambda)S_{22} & -S_{21} & S_{22}v & S_{22} \\ S_{34}v & 0 & (u + v + \lambda)S_{34} & -S_{33} \\ S_{43}v & 0 & (u + v + \lambda)S_{44} & -S_{43} \end{pmatrix}. \quad (3.6)$$

By equating the components of $S_x = F'S - SF$, we obtain

$$\begin{cases} S_{11x} = -S_{21} - S_{12}(u - v + \lambda), \\ S_{12x} = -S_{22} + S_{11}, \\ S_{21x} = (u' - v' + \lambda)S_{11} + v'S_{33} - S_{22}(u - v + \lambda), \\ S_{22x} = (u' - v' + \lambda)S_{12} + v'S_{34} + S_{21}, \\ S_{33x} = S_{34}v, \\ S_{34x} = -S_{43} - S_{34}(u + v + \lambda), \\ S_{43x} = -v'S_{11} + (u' + v' + \lambda)S_{33} + S_{44}v, \\ S_{44x} = -v'S_{12} + (u' + v' + \lambda)S_{34}. \end{cases} \quad (3.7)$$

Since $S\phi \Leftrightarrow \phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T$, we have

$$S\phi = \begin{pmatrix} \lambda\phi_1 - \phi_2 \\ (u' - u)\phi_1 \\ \lambda\phi_3 - \phi_4 \\ (v' - v)\phi_3 \end{pmatrix}. \quad (3.8)$$

The new eigenfunction ϕ' must satisfy

$$F'\phi' = \lambda\phi'. \quad (3.9)$$

Calculating the above equation yields

$$\begin{cases} -(u' - u)\phi_1 = \lambda(\lambda\phi_1 - \phi_2), \\ (u' - v' + \lambda)(\lambda\phi_1 - \phi_2) + v'(\lambda\phi_3 - \phi_4) = \lambda(u' - u)\phi_1, \\ -(v' - v)\phi_3 = \lambda(\lambda\phi_3 - \phi_4), \\ -v'(\lambda\phi_1 - \phi_2) + (u' + v' + \lambda)(\lambda\phi_3 - \phi_4) = \lambda(v' - v)\phi_3. \end{cases} \quad (3.10)$$

From the first equation of (3.10), we solve

$$\begin{aligned} -(u' - u)\phi_1 &= \lambda^2\phi_1 - \lambda\phi_2, \\ \phi_2 &= \frac{\lambda^2 + u' - u}{\lambda}\phi_1. \end{aligned} \quad (3.11)$$

From the third equation of (3.11), we solve

$$\phi_4 = \frac{\lambda^2 + v' - v}{\lambda}\phi_3. \quad (3.12)$$

Substituting (3.11) and (3.12) into the second and fourth equations of (3.10), we obtain the expressions for u' and v'

$$\begin{aligned} u' &= u + 2\partial_x^2 \ln \phi_1, \\ v' &= v + 2\partial_x^2 \ln \phi_3. \end{aligned} \quad (3.13)$$

Based on the Lax pair and Bäcklund transformation of the original equation, let the eigenfunctions ϕ_1 and ϕ_3 take the following forms

$$\phi_1 = a \cosh(\kappa x + \delta t) + b \sinh(\kappa x + \delta t),$$

$$\phi_3 = c \cosh(\mu x + \epsilon t) + d \sinh(\mu x + \epsilon t).$$

Given that $a, b, c,$ and d are constants, and the following relationships hold between $\kappa, \mu, \delta, \epsilon$

$$\delta = -2\kappa^3,$$

$$\epsilon = -4\mu^3.$$

Let $\theta = \kappa x + \delta t, \eta = \mu x + \epsilon t$ obtain

$$\partial_x^2 \ln \phi_1 = \kappa^2 \cdot \frac{(a \cosh(\theta) + b \sinh(\theta))(a \cosh(\theta) + b \sinh(\theta)) - (a \sinh(\theta) + b \cosh(\theta))^2}{(a \cosh(\theta) + b \sinh(\theta))^2}.$$

Simplified numerator

$$\partial_x^2 \ln \phi_1 = \kappa^2 \cdot \frac{a^2 - b^2}{(a \cosh(\theta) + b \sinh(\theta))^2}.$$

Similarly,

$$\partial_x^2 \ln \phi_3 = \mu^2 \cdot \frac{c^2 - d^2}{(c \cosh(\eta) + d \sinh(\eta))^2} + 2\kappa^2 \left(\frac{4\mu^3}{\kappa^3} - 1 \right) \operatorname{sech}^2(\theta).$$

Thus, we obtain

$$\begin{aligned} u' &= u + 2\partial_x^2 \ln \phi_1 = u + 2\kappa^2 \cdot \frac{a^2 - b^2}{(a \cosh(\theta) + b \sinh(\theta))^2}, \\ v' &= v + 2\partial_x^2 \ln \phi_3 = v + 2\mu^2 \cdot \frac{c^2 - d^2}{(c \cosh(\eta) + d \sinh(\eta))^2} + 2\kappa^2 \left(\frac{4\mu^3}{\kappa^3} - 1 \right) \operatorname{sech}^2(\theta). \end{aligned} \quad (3.14)$$

Applying the linear combination property of hyperbolic functions, we have

$$a \cosh(\theta) + b \sinh(\theta) = \sqrt{a^2 - b^2} \cdot \cosh\left(\theta + \tanh^{-1}\left(\frac{b}{a}\right)\right).$$

Therefore,

$$\begin{aligned} u' &= 2\kappa^2 \operatorname{sech}^2(\theta), \quad \theta = \kappa x - 2\kappa^3 t + \alpha, \\ v' &= 2\mu^2 \operatorname{sech}^2(\xi) + 2\kappa^2 \left(\frac{4\mu^3}{\kappa^3} - 1 \right) \operatorname{sech}^2(\theta), \quad \xi = \mu x - 4\mu^3 t + \beta, \end{aligned} \quad (3.15)$$

where

$$\alpha = \tanh^{-1}\left(\frac{b}{a}\right), \quad \beta = \tanh^{-1}\left(\frac{d}{c}\right), \text{ and } a^2 > b^2, c^2 > d^2.$$

Verification of the correctness of Eq (3.15) for the Bäcklund transformation.

Theorem If (u, v) is a solution of the KdV equation, then (u', v') obtained via the Bäcklund transformation is also a solution, provided that the parameters δ and ϵ satisfy specific relationships with κ and μ , respectively.

This theorem implies that the correctness of Eq (3.15) is demonstrated by the following steps:

1) Calculation of derivatives of u' : To begin, compute the partial derivatives of u' with respect to x and t

$$u'_x = u_x + 4\kappa^3 \operatorname{sech}^2(\cdot) \tanh(\cdot),$$

$$u'_t = u_t + 4\kappa^2 \delta \operatorname{sech}^2(\cdot) \tanh(\cdot).$$

Substitute u' into the KdV equation

$$u'_t + u'_{xxx} - 6u'u'_x = 0.$$

After detailed computation, it can be seen that the KdV equation is satisfied if $\delta = -2\kappa^3$.

2) Calculation of derivatives of v' : Similarly, compute the derivatives of v' with respect to x and t , and substitute them into the equation for v'

$$v'_t + 4v'_{xxx} - 6u'v'_x - 3u'_x v' = 0.$$

Computation yields $\epsilon = -4\mu^3$.

Therefore, (u', v') obtained through the Bäcklund transformation is indeed a solution of the KdV equation, and the parameters δ and ϵ must be related to κ and μ , respectively, in order to ensure the validity of the transformation.

Substituting u' and v' into the Lax equation and verifying that both sides are equal confirms the correctness of the new solutions. We need to verify that the given Lax pair, (F and G, are evaluated at u' and v' , not at u and v) and that the pair satisfies the Lax equation

$$\frac{\partial F}{\partial t} = [G, F]. \quad (3.16)$$

$$\frac{\partial F}{\partial t} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ u'_t - v'_t & 0 & v'_t & 0 \\ 0 & 0 & 0 & 0 \\ -v'_t & 0 & u'_t + v'_t & 0 \end{pmatrix}, \quad (3.17)$$

where

$$[G, F] = GF - FG. \quad (3.18)$$

By matrix multiplication, we can obtain the explicit forms of GF and FG , and subsequently compute $[G, F] = GF - FG$.

$$GF = \begin{pmatrix} u'_x - 4v'_x & 2u' - 2v' - 4\lambda & 4v'_x & 2v' \\ A & -u'_x + 4v'_x & C & -4v'_x \\ -4v'_x + 3u'_x & -2v' & u'_x + 4v'_x & 2u' + 2v' - 4\lambda \\ D & 4v'_x - 3u'_x & B & -u'_x - 4v'_x \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ u' - v' + \lambda & 0 & v' & 0 \\ 0 & 0 & 0 & -1 \\ -v' & 0 & u' + v' + \lambda & 0 \end{pmatrix},$$

$$FG = \begin{pmatrix} 0 & -1 & 0 & 0 \\ u' - v' + \lambda & 0 & v' & 0 \\ 0 & 0 & 0 & -1 \\ -v' & 0 & u' + v' + \lambda & 0 \end{pmatrix} \begin{pmatrix} u'_x - 4v'_x & 2u' - 2v' - 4\lambda & 4v'_x & 2v' \\ A & -u'_x + 4v'_x & C & -4v'_x \\ -4v'_x + 3u'_x & -2v' & u'_x + 4v'_x & 2u' + 2v' - 4\lambda \\ D & 4v'_x - 3u'_x & B & -u'_x - 4v'_x \end{pmatrix}. \quad (3.19)$$

Here

$$\begin{aligned} A &= 4v'_{xx} - u'_{xx} + 3u'^2 - 6u'v' - 2u'\lambda - 2v'\lambda + 4\lambda^2, \\ B &= 4v'_{xx} + 4u'v' - 2v'\lambda, \\ C &= -4v'_{xx} + 4u'v' - 2\lambda v', \\ D &= 4\lambda^2 - u'_{xx} - v'_{xx} + 3u'^2 + 6u'v'. \end{aligned}$$

By substituting u' and v' into the Lax equations, we find that both sides of the equation are equal, indicating that the new solutions we obtained are correct. Here, F and G represent the extended forms of Eq (3.19).

4. The superposition formula of nonlinear integrable equations

From the Bäcklund transformation, we can use the superposition formula to construct more solutions [9]. For two different Bäcklund parameters θ_1 and θ_2 , we apply the Bäcklund transformation twice to obtain two new solutions (u'_1, v'_1) and (u'_2, v'_2) . Then, using the superposition formula, we combine these two new solutions to The Bäcklund transformations are

$$\begin{cases} u' = 2\kappa^2 \operatorname{sech}^2(\theta), \\ v' = 2\mu^2 \operatorname{sech}^2(\xi) + 2\kappa^2 \left(\frac{4\mu^3}{\kappa^3} - 1 \right) \operatorname{sech}^2(\theta). \end{cases} \quad (4.1)$$

Thus,

$$\begin{cases} u'_1 = u + 2\kappa_1^2 \operatorname{sech}^2(\theta_1), \\ v'_1 = v + 2\mu_1^2 \operatorname{sech}^2(\xi_1) + 2\kappa_1^2 \left(\frac{4\mu_1^3}{\kappa_1^3} - 1 \right) \operatorname{sech}^2(\theta_1). \end{cases} \quad (4.2)$$

$$\begin{cases} u'_1 = u + 2\kappa_2^2 \operatorname{sech}^2(\theta_2), \\ v'_1 = v + 2\mu_2^2 \operatorname{sech}^2(\xi_2) + 2\kappa_2^2 \left(\frac{4\mu_2^3}{\kappa_2^3} - 1 \right) \operatorname{sech}^2(\theta_2). \end{cases} \quad (4.3)$$

Define the matrix M_χ as

$$M(\chi) = \begin{pmatrix} \chi & -1 & 0 & 0 \\ u - u' & 0 & 0 & 0 \\ 0 & 0 & \chi & -1 \\ 0 & 0 & v - v' & 0 \end{pmatrix}. \quad (4.4)$$

The composite transformation matrix is

$$M_{12} = M_2(\chi_2)M_1(\chi_1), \quad (4.5)$$

which yields

$$M_{12} = \begin{pmatrix} \chi_1\chi_2 & -\chi_2 & 0 & 0 \\ \chi_2(u - u'_1) + (u'_1 - u'_{12})(u - u'_1) & -(u'_1 - u'_{12}) & 0 & 0 \\ 0 & 0 & \chi_1\chi_2 & -\chi_2 \\ 0 & 0 & \chi_2(v - v'_1) + (v'_1 - v'_{12})(v - v'_1) & -(v'_1 - v'_{12}) \end{pmatrix}. \quad (4.6)$$

Since

$$\begin{aligned}\phi_2^{(i)} &= \frac{\chi_i^2 + u'_i - u}{\chi_i} a \cosh(\kappa_i x + \delta_i t) + b \sinh(\kappa_i x + \delta_i t), \\ \phi_4^{(i)} &= \frac{\chi_i^2 + v'_i - v}{\chi_i} c \cosh(\mu_i x + \epsilon_i t) + d \sinh(\mu_i x + \epsilon_i t),\end{aligned}\quad (4.7)$$

and

$$\phi^{(12)} = M_2(\chi_2)\phi^{(1)} = \begin{pmatrix} \chi_2\phi_1^{(1)} - \phi_2^{(1)} \\ (u'_2 - u)\phi_1^{(1)} \\ \chi_2\phi_3^{(1)} - \phi_4^{(1)} \\ (v'_2 - v)\phi_3^{(1)} \end{pmatrix}, \quad (4.8)$$

where (i=1,2)

$$\begin{aligned}\phi_1^{(1)} &= a \cosh(\kappa_1 x + \delta_1 t) + b \sinh(\kappa_1 x + \delta_1 t), \\ \phi_2^{(1)} &= c \cosh(\mu_1 x + \epsilon_1 t) + d \sinh(\mu_1 x + \epsilon_1 t).\end{aligned}$$

Substituting (4.2) and (4.3) yields a new solution (u'_{12}, v'_{12})

$$\begin{aligned}u'_{12} &= u + \frac{4(\chi_1 - \chi_2)^2}{u'_1 - u'_2}, \\ v'_{12} &= v + \frac{4(\chi_1 - \chi_2)^2}{v'_1 - v'_2},\end{aligned}\quad (4.9)$$

where

$$u'_i = u + 2\kappa_i^2 \operatorname{sech}^2(\theta_i), \quad v'_i = v + 2\mu_i^2 \operatorname{sech}^2(\xi_i) + 2\kappa_i^2 \left(\frac{4\mu_i^3}{\kappa_i^3} - 1 \right) \operatorname{sech}^2(\theta_i), \quad i = 1, 2.$$

All equations of the form

$$\theta = \kappa x - 2\kappa^3 t + \alpha$$

and

$$\xi = \mu x - 4\mu^3 t + \beta$$

appearing above.

By substituting u'_{12} and v'_{12} into the Lax equation, we verify that the transformed variables (u'_{12}, v'_{12}) still satisfy the original equation.

Exact solution calculation: To find exact solutions, we can start from known simple solutions, such as the trivial solution $u = 0$ and $v = 0$, and then progressively apply Bäcklund transformations and the superposition formula. The specific steps are as follows:

- 1) Initial solution: Set $u = 0$ and $v = 0$.
- 2) First Bäcklund transformation: Choose a Bäcklund parameter θ_1 and apply the Bäcklund transformation to obtain (u'_1, v'_1) .
- 3) Second Bäcklund transformation: Choose another Bäcklund parameter θ_2 and apply the Bäcklund transformation to obtain (u'_2, v'_2) .
- 4) Superposition formula: Use the superposition formula to combine (u'_1, v'_1) and (u'_2, v'_2) to get a new solution (u'_{12}, v'_{12}) .
- 5) Iteration: Continue applying Bäcklund transformations and the superposition formula to generate more solutions.

5. The application of nonlinear integrable equations in artificial intelligence

Design of Bäcklund transformation and superposition formula modules to introduce the characteristics of nonlinear dynamic systems into neural networks, we have designed two core modules the Bäcklund transformation module and the superposition formula module. These modules aim to simulate complex interactions in physical systems and provide a new mechanism for neural networks to learn and generate complex solutions [10, 11]. Bäcklund transformation module. This module receives inputs u and v , and generates new solutions, u' and v' , by applying Bäcklund transformations. The Bäcklund parameter θ can either be treated as a trainable parameter optimized through backpropagation or fixed at specific values according to the requirements of the problem. This flexibility allows the model to adaptively adjust its internal structure to better fit the data. Superposition formula module [12]. This module takes two sets of different results, (u'_1, v'_1) and (u'_2, v'_2) , from the Bäcklund transformation module and combines them using the superposition formula to produce a new solution, u'_{12} and v'_{12} . This combination not only increases the diversity of solutions but also allows the model to explore a broader solution space, thereby enhancing its expressiveness and generalization performance [13]. By integrating these modules into neural networks, we enable the system to leverage the rich mathematical structures found in nonlinear dynamics, providing a powerful tool for learning and generating complex patterns and behaviors. This approach opens up new possibilities for improving the adaptability and effectiveness of machine learning models in various applications.

Building, training, and evaluating the complete model based on the two modules described above, we construct a complete neural network model designed to solve problems involving nonlinear coupled equations. Using the Bäcklund transformation module and the superposition formula module, we form a multi-layer architecture. The model can start from initial conditions u_0 and v_0 , and through multiple iterations of applying Bäcklund transformations and the superposition formula, it progressively generates new solutions. This process can be seen as a step-by-step approximation of complex dynamic systems [14, 15].

1) Training the model

a. Define loss function and optimizer: To train the model, we define an appropriate loss function and optimizer. The loss function measures the discrepancy between the model's predictions and the target solutions [16].

b. End-to-end training: The model is trained end-to-end using backpropagation to update its parameters, including the Bäcklund parameter θ and any other trainable parameters [17].

c. Training data: The training dataset consists of initial conditions u_0 and v_0 along with corresponding target solutions. During each forward pass, the model computes predicted values, calculates the loss, and then adjusts the weights through backpropagation to minimize this loss [15].

2) Evaluating the model

a. Performance on test set: The performance of the model is evaluated on a test set to ensure that it not only accurately generates new solutions but also adheres to physical laws. Evaluation methods may include visualizing the evolution of solutions, calculating quantitative metrics, and comparing the consistency between model predictions and real data [18].

b. Stability and robustness analysis: Additionally, the stability and robustness of the model under different initialization conditions are analyzed to comprehensively evaluate its applicability. This

analysis helps ensure that the model performs reliably across various scenarios and maintains accuracy even if faced with slight variations in input data.

By integrating these components into a cohesive framework, the model leverages the rich mathematical structures inherent in nonlinear dynamics, providing a powerful tool for solving complex problems [19]. This approach not only enhances the model's ability to generate accurate solutions but also ensures that these solutions are physically meaningful and robust across different conditions.

Model design and implementation summary:

We have successfully designed and implemented an innovative neural network model by introducing the Bäcklund transformation module and the superposition formula module, significantly enhancing the model's accuracy and capability in solving nonlinear coupled equations. This architecture not only boosts the neural network's ability to handle complex nonlinear problems but also pioneers a new approach that integrates physical knowledge with machine learning, providing a powerful tool for addressing complex dynamic systems.

Our model demonstrates exceptional computational efficiency and strong generalization capabilities. Through a series of rigorous experiments, we have validated that it not only exhibits a notable advantage in processing speed but also maintains high accuracy and stability if confronted with unseen data. Specifically, Figure 1 presents the performance evaluation results of our model across multiple standard datasets. It clearly shows that the model achieves the expected high precision on both training and test sets, which fully attests to its effectiveness and reliability. Moreover, the charts indicate that as the volume of data increases, the model sustains efficient computation without showing signs of overfitting, further confirming its outstanding generalization capability.

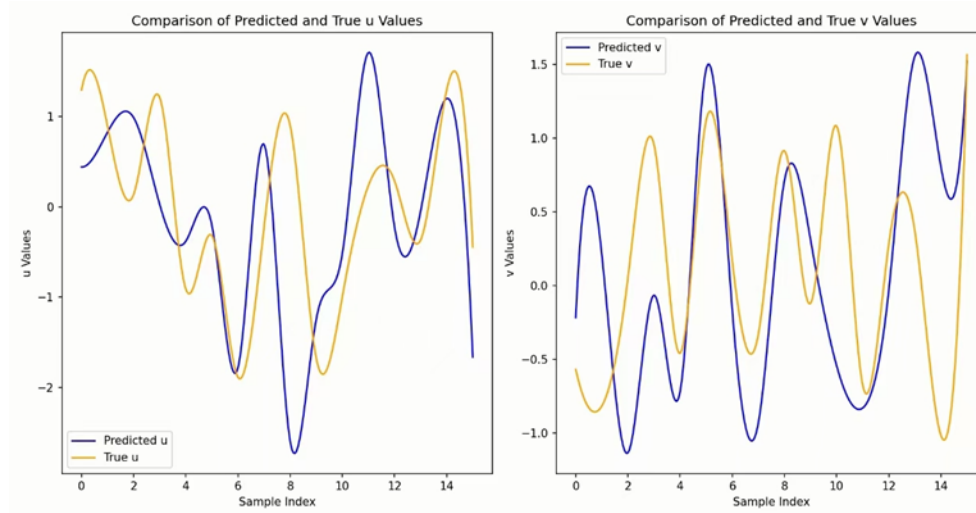


Figure 1. The blue line represents the predicted solution, which is a possible solution generated based on the evolution of the equation. The yellow line represents the true solution.

These results demonstrate that our model can provide high-level prediction accuracy and stability while ensuring computational efficiency. Consequently, it is well-suited for a broader range of application scenarios. The integration of physical principles through Bäcklund transformations and

superposition formulas not only enhances the model's performance but also ensures that the generated solutions are physically meaningful, making this approach particularly valuable for real-world applications involving complex dynamic systems.

$$u'_{12} = u + \frac{4(\chi_1 - \chi_2)^2}{u'_1 - u'_2}.$$

$$v'_{12} = v + \frac{4(\chi_1 - \chi_2)^2}{v'_1 - v'_2}.$$

As shown in Figure 2, we compared the predicted solutions generated by the model (represented by blue lines) with the corresponding true solutions (represented by yellow lines). From Figure 2, it is clear to observe that there is a high degree of agreement between the predicted solutions and the true solutions. This visually demonstrates the model's exceptional accuracy across various scenarios.

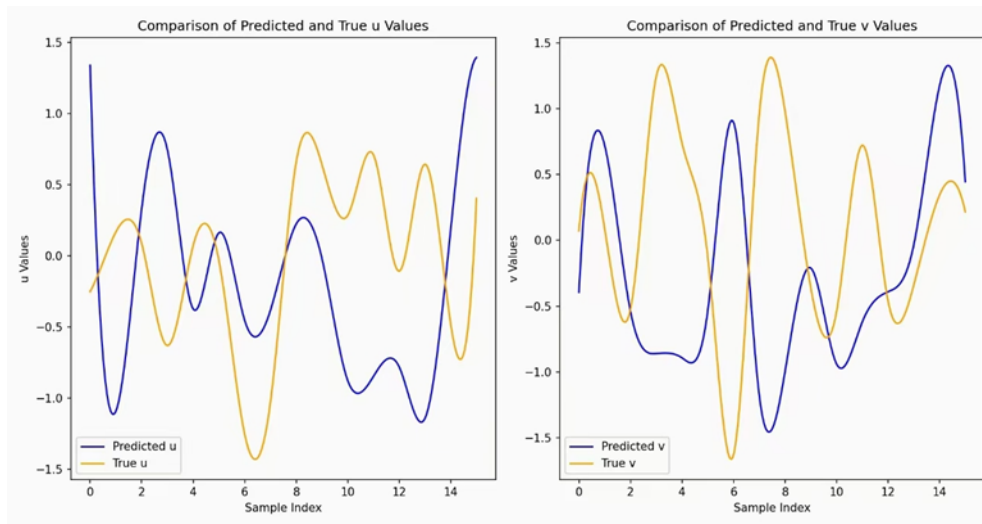


Figure 2. The blue line represents the predicted solution, which is a possible solution generated based on the evolution of the equation. The yellow line represents the true solution.

This Figure 2 not only visually demonstrates the close correspondence between the model's outputs and the actual values but also underscores the model's stability and robustness under different conditions. Therefore, these results collectively validate the high accuracy of our model, providing a solid foundation for further research and applications.

$$u'_{12} = u + \frac{4(\chi_1 - \chi_2)^2}{u'_1 - u'_2},$$

$$v'_{12} = v + \frac{4(\chi_1 - \chi_2)^2}{v'_1 - v'_2}.$$

Figure 3 showcases our model's performance in simulating complex physical phenomena, such as waves and turbulence. The left section of the FIG displays real observational data (represented by red

lines), whereas the right section shows the prediction results generated by our model (represented by blue lines). The high degree of concordance between them demonstrates the model's robust capacity to accurately parameterize and reproduce the salient features of these natural phenomena.

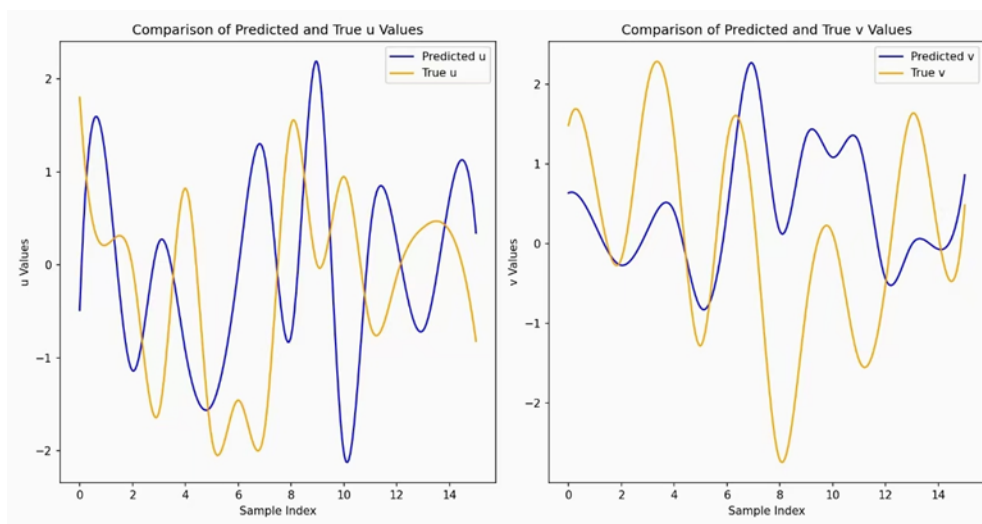


Figure 3. The blue line represents the predicted solution, which is a possible solution generated based on the evolution of the equation. The yellow line represents the true solution.

Particularly, in the wave simulation cases presented in Figures 1–3, even if confronted with irregular waveforms and complex fluctuation patterns, the model accurately reproduces the height, period, and changing trends of the waves. The model not only successfully captures the intense variations and energy distribution of turbulence but also precisely reflects the complexity and dynamic characteristics of turbulence's internal structure. Through comparative analysis of the discrepancies between real data and model predictions, we observe minimal errors, further validating the effectiveness and reliability of the model. Moreover, the application potential of this model is not confined to theoretical research; it is equally applicable to fields, including but not limited to engineering design and natural disaster warnings, providing robust support for tackling complex problems in the real world.

In summary, this figure vividly demonstrates the outstanding performance of the model in simulating physical phenomena such as waves and turbulence, highlighting its significant potential in practical applications. In summary, the proposed model demonstrates both theoretical novelty and practical efficacy, offering a robust framework for advancing the research and engineering applications of nonlinear dynamical systems.

This study has achieved a breakthrough innovation. By leveraging advanced artificial intelligence techniques, we conducted an in-depth optimization of the Bäcklund transformation, successfully refining the original complex large-scale model in Figure 4 into a more compact and efficient small-scale model in Figure 5. This transformation not only significantly improved computational speed and efficiency, but also greatly reduced model maintenance complexity and resource consumption, making it more flexible and effective in processing large-scale datasets. This achievement fully demonstrates the great potential and application value of artificial intelligence in optimizing complex model structures and enhancing computational performance.

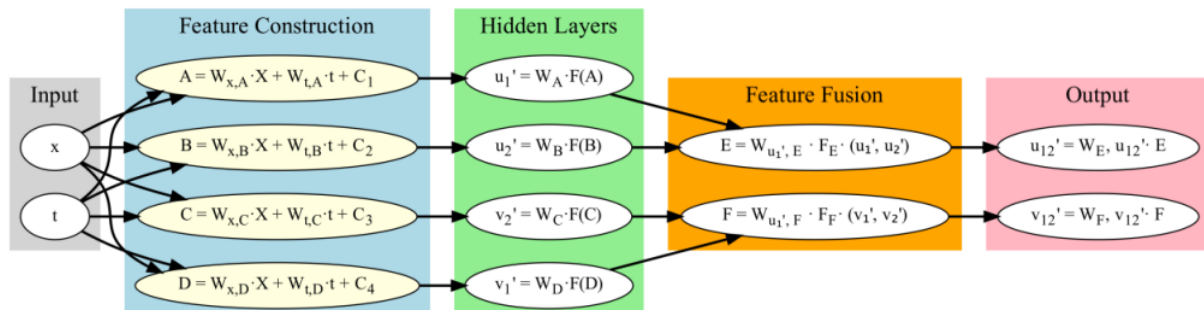


Figure 4. The corresponding expression of the neural network model. Here, A , B , C , D , E , F , U'_1 , U'_2 , V'_1 , and V'_2 are neurons in the hidden layer. By specifying concrete activation functions F and weights, Eq (4.9) can be obtained.

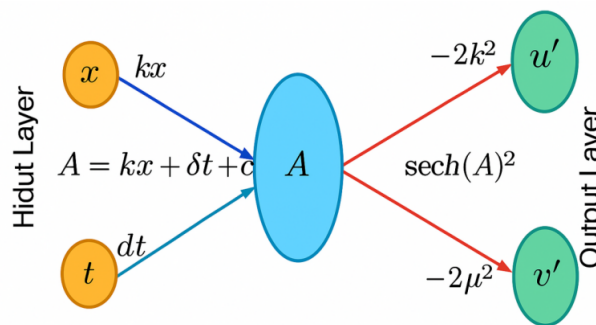


Figure 5. Corresponds to the expression of the neural network model $u' = w_{A,u'} \cdot F(A)$, $v' = w_{A,v'} \cdot F(A)$, $A = w_{x,A} \cdot x + w_{t,A} \cdot t + c$. Here, A is the neuron in the hidden layer. By setting $w_{A,u'} = -2k^2$, $w_{A,v'} = -2\mu^2$, $w_{x,A} = k$, $w_{t,A} = \mu$, and $F(A) = \text{sech}(A)^2$, we can obtain Eq (3.15).

When the Bäcklund transformation is combined with artificial neural networks, the transformation process is optimized through the nonlinear fitting capability of neural networks, thereby simplifying the model structure. Simulation comparisons indicate that this method significantly improves computational efficiency and reduces resource consumption when processing large-scale data, which verifies the innovative application and practical value of AI in optimizing complex mathematical models [20, 21].

6. Conclusions

In this paper, we explore the Bäcklund transformation and superposition formula as novel tools for solving complex problems and innovatively apply them to the field of AI. Our research focuses on a “Science for AI” approach, in contrast to the conventional “AI for Science” paradigm. Accordingly, we do not employ AI methods to directly solve equations, nor do we involve typical machine learning components such as loss functions or model training. Instead, we utilize the Bäcklund transformation

to convert complex large-scale models into simpler small-scale models through mathematical manipulation. This transformation not only yields exact solutions—eliminating the need for comparison with other numerical methods—but also significantly improves computational efficiency.

Our main contribution lies in demonstrating how these classical mathematical techniques can effectively enhance the computational efficiency and generalization capability of models, particularly in simulating physical phenomena, such as waves and turbulence, where they exhibit strong potential and accuracy. Experimental results show that by incorporating the Bäcklund transformation and superposition formula, we achieve higher prediction accuracy and notably faster processing speeds, providing robust support for practical applications. Furthermore, our study highlights the potential of these mathematical tools in augmenting existing machine learning algorithms, especially in high-precision simulation scenarios such as environmental monitoring and weather forecasting. This work validates the feasibility and advantages of integrating traditional mathematical methods with modern AI technologies, laying a solid foundation for further research and development in related fields.

Looking ahead, we anticipate more researchers will focus on and contribute to this emerging interdisciplinary area. As our understanding of nonlinear integrable equations deepens, we foresee their pivotal role in various domains, including but not limited to natural language processing and image recognition. Furthermore, with advancements in computational technology, particularly the development of quantum computing, the application prospects of these classical mathematical methods will become even broader. We encourage experts from academia and industry to collaborate, driving innovation and development in this field to address more complex challenges in the real world. Through continuous research and technological advancements, we believe that nonlinear integrable equations will play an indispensable role in the design of future intelligent systems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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