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*Research article*

# Random exponential attractor for a class of non-autonomous stochastic lattice systems

**Ailing Ban***

School of Big Data and Artificial Intelligence, Chizhou University, Chizhou 247000, China

* **Correspondence:** Email: banailing2008@163.com.

**Abstract:** The purpose of this paper was to discuss the existence of a random exponential attractor for non-autonomous coupled Klein-Gordon-Schrödinger (KGS) lattice equations with multiplicative noise. We employed the method of estimation on the tails of solutions to prove the existence of a random attractor for a continuous cocycle generated by the random KGS lattice equations on an infinite-dimensional sequence space, and used this abstract result to prove the Lipschitz continuity of the continuous cocycle. Then, we verified the existence of a random exponential attractor for the investigated system according to a known criterion.

**Keywords:** random exponential attractor; Klein-Gordon-Schrödinger lattice equations; multiplicative noise; continuous cocycle

## 1. Introduction

As is known to all, the random attractor is a core concept to describe the asymptotic behavior of infinite-dimensional random dynamical systems (RDSs), see, e.g., [1–4] for the stochastic lattice dynamical systems and [5–8] for stochastic partial differential equations. But despite its compactness, a random attractor may be infinite dimensional. This means that the long-time behavior of infinite-dimensional RDSs may not be described with a finite number of independent parameters. In addition, the rate at which the random attractor attracts trajectories may be very slow, so that the random attractor may be unstable under some small perturbations. To avoid these shortcomings, Shirikyan and Zelik [9] introduced a random exponential attractor for the autonomous RDSs, and presented its existence conditions. The random exponential attractor with a finite fractal dimension includes the random attractor, and attracts any orbit at an exponential rate. When the fractal dimension of a random attractor is less than $\frac{n}{2}$, the dimension of topological structure of the random attractor is at most $n$. This indicates that the asymptotic behavior of RDSs can be described at most by $n$ independent parameters. Compared with the reference [9], Caraballo and Sonner [10] weakened the existence conditions of the random exponential attractor. Moreover, Zhou [11] extended the random exponential attractor to non-autonomous RDSs and established an effective existence theorem of a random exponential attractor, which is applied to stochastic evolution equations, such as the first-order and second-order non-autonomous stochastic lattice systems [11, 12], the stochastic long wave-short wave equation [13], etc.

In this paper, we consider the existence of a random exponential attractor for the following non-autonomous coupled KGS lattice equations with multiplicative noise:

$$\begin{cases} d\dot{u}_j + \alpha du_j + ((Au)_j - \gamma u_j - \nu |w_j|^2)dt = g_j(t)dt + a_1 u_j \circ dW_1, \\ idw_j + (-(Aw)_j + i\beta w_j + w_j u_j)dt = h_j(t)dt + a_2 w_j \circ dW_2, \qquad j \in \mathbb{Z}, \\ u_j(\tau) = u_{j\tau}, \dot{u}_j(\tau) = u_{1,j\tau}, w_j(\tau) = w_{j\tau}, \tau \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $u_j = u_j(t) \in \mathbb{R}, w_j = w_j(t) \in \mathbb{C}$, $\alpha, \beta, \gamma, \nu$ are positive constants, $g_j(t)$ and $h_j(t)$ are given, $a_1, a_2 \in \mathbb{R}$, $i$ is the imaginary unit, the coupled linear operator $A$ is defined by $(Au)_j = 2u_j - u_{j+1} - u_{j-1}$, and $(W_1, W_2)$ are independent two-sided Brownian motion on a probability space. "$\circ$" represents the stochastic term in the sense of Stratonovich.

The system (1.1) can be regarded as the continuous non-autonomous coupled KGS equations:

$$\begin{cases} d\dot{u} + \alpha du + (Au + \gamma u - \nu |w|^2)dt = g(t)dt + a_1 u \circ dW_1, \\ idw + (-Aw + i\beta w + wu)dt = h(t)dt + a_2 w \circ dW_2, \\ u(\tau) = u_\tau, \dot{u}(\tau) = u_{1\tau}, w(\tau) = w_\tau, \end{cases} \tag{1.2}$$

where $u = (u_j)_{j\in\mathbb{Z}},\ w = (w_j)_{j\in\mathbb{Z}},\ g(t) = (g_j(t))_{j\in\mathbb{Z}},\ h(t) = (h_j(t))_{j\in\mathbb{Z}}$, and $|w|^2 = (|w_j|^2)_{j\in\mathbb{Z}}$.

The KGS system (1.2) is a very important mathematical model. It is often used to describe the interaction of a real meson field scalar $u$ and a complex scalar nucleon field $w$. $\gamma^{\frac{1}{2}}$ represents the mass of the meson, and the time-dependent functions $g(t)$ and $h(t)$ are both external force terms.

All kinds of attractors for deterministic KGS equations ($a_i = 0,\ i = 1, 2$ in (1.1)) have been widely studied, e.g, the global attractor [14–16], and the regularity of the attractor [17]. For the stochastic KGS case ($a_i \neq 0,\ i = 1, 2$ in (1.1)), some results concerning the random attractor have also been obtained, see [18–20]. However, we are aware that there are currently no results regarding the random exponential attractor for the KGS system (1.1). Motivated by the ideas and conclusions of [11, 12, 20], in the article, we focus on using the tail-estimates method to verify that the continuous cocycle for the system (1.1) satisfies the existence conditions of a random exponential attractor. This result shows that the asymptotic behavior of the investigated system (1.1) can be described with a finite number of independent parameters, which will bring convenience for numerical simulation and practical application.

## 2. Preliminaries

We review some related definitions and present the existence theorem of a random exponential attractor, which can be obtained from [8, 11], and make some assumptions about $a_1,\ a_2,\ g_j(t),\ h_j(t)$.

Let $(X, \|\cdot\|_X)$ be a separable Banach space, and $\mathcal{B}(X)$ be a Borel $\sigma-$algebra of $X$, and $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\omega : t \in \mathbb{R}\})$ be an ergodic metric dynamical system on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (see [21]), where a family of mappings $\{\theta_t\omega : t \in \mathbb{R}\}$ is defined on $\Omega : \theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t)$.

**Definition 2.1.** A continuous cocycle defined on $X$ over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\omega : t \in \mathbb{R}\})$ is a $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable mapping $\Psi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ with the following properties: for every $\tau \in \mathbb{R}, \omega \in \Omega$, and $t, r \in \mathbb{R}^+$, $(i)$ $\Psi(0, \tau, \omega, \cdot)$ is the identity $I$ on $X$; $(ii)$ $\Psi(t + r, \tau, \omega, \cdot) = \Psi(t, \tau + r, \theta_r\omega, \Psi(r, \tau, \omega, \cdot))$; $(iii)$ $\Psi(t, \tau, \omega, \cdot) : X \to X$ is continuous.

$\mathcal{D}(X)$ is endowed with the collection of all tempered families of bounded subsets of $X$, i.e., $\mathcal{D}(X) = \{D = \{D(\tau, \omega) \subseteq X : D(\tau, \omega) \neq \emptyset, \tau \in \mathbb{R}, \omega \in \Omega\}\}$, for any $\epsilon > 0, \tau \in \mathbb{R}, \omega \in \Omega$, and it holds that $\lim\limits_{t\to-\infty} e^{\epsilon t}\|D(\tau + t, \theta_t\omega)\|_X = 0$.

**Definition 2.2.** A random set $\{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}\}$ of $X$ is said to be a $\mathcal{D}(X)$−random exponential attractor for the continuous cocycle $\{\Psi(t, \tau, \omega) : t \geq 0, \tau \in \mathbb{R}, \omega \in \Omega\}$ over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\omega : t \in \mathbb{R}\})$ if there exists a full measure set $\widetilde{\Omega} \in \mathcal{F}$, such that $\mathcal{A}(\tau, \omega)$ has the following properties: for each $\tau \in \mathbb{R}$ and $\omega \in \widetilde{\Omega}$,

$(i)$ $\mathcal{A}(\tau, \omega)$ is measurable in $\omega$, and compact in $X$;

$(ii)$ there exists a random variable $\xi_\omega < \infty$, such that $\sup\limits_{\tau\in\mathbb{R}} dim_f\mathcal{A}(\tau, \omega) \leq \xi_\omega$, where $dim_f\mathcal{A}(\tau, \omega) = \limsup\limits_{\varepsilon\to0^+} \frac{\ln N_\varepsilon(\mathcal{A}(\tau,\omega))}{-\ln\varepsilon}$ is the fractal dimension of $\mathcal{A}(\tau, \omega)$, and $N_\varepsilon(\mathcal{A}(\tau, \omega))$ is the minimum number of balls with radius $\varepsilon$ required to cover $\mathcal{A}(\tau, \omega)$ in $X$;

$(iii)$ $\Psi(t, \tau - t, \theta_{-t}\omega)\mathcal{A}(\tau - t, \theta_{-t}\omega) \subseteq \mathcal{A}(\tau, \omega)$ for each $t \geq 0$;

$(iv)$ there exist a constant $b > 0$ (independent of $\omega$), $t_B(\tau, \omega) \geq 0$, and $K(\tau, \omega, \|B\|_X) > 0$ such that for any $B \in \mathcal{D}(X)$, $d_h(\Psi(t, \tau - t, \theta_{-t}\omega)B(\tau - t, \theta_{-t}\omega), \mathcal{A}(\tau, \omega)) \leq K(\tau, \omega, \|B\|_X)e^{-bt},\ t \geq t_B(\tau, \omega)$, where $d_h(E_1, E_2)$ represents the Hausdorff semi-distance between $E_1$ and $E_1$.

Set

$$l^2 = \{u = (u_j)_{j\in\mathbb{Z}} : u_j \in \mathbb{R}, \sum_{j\in\mathbb{Z}} u_j^2 < \infty\},$$

$$L^2 = \{u = (u_j)_{j\in\mathbb{Z}} : u_j \in \mathbb{C}, \sum_{j\in\mathbb{Z}} |u_j|^2 < \infty\},$$

endowed with the norms and inner products on $l^2$ and $L^2$ as: for any $u = (u_j)_{j\in\mathbb{Z}}, w = (w_j)_{j\in\mathbb{Z}} \in l^2$ or $L^2$,

$$(u, w) = \sum_{j\in\mathbb{Z}} u_j\overline{w_j}, \quad \|u\|^2 = (u, u) = \sum_{j\in\mathbb{Z}} |u_j|^2,$$

$$(u, w)_\gamma = (Bu, Bw) + \gamma(u, w), \quad \|u\|_\gamma^2 = (u, u)_\gamma = \|Bu\|^2 + \gamma\|u\|^2,$$

where $B$ is defined by $(Bu)_j = u_{j+1} - u_j$, and $\overline{w_j}$, $w_j$ are conjugate. It is evident that $\gamma\|u\|^2 \leq \|u\|_\gamma^2 \leq (4 + \gamma)\|u\|^2$. Thus, the norm $\|\cdot\|_\gamma$ is equivalent to the norm $\|\cdot\|$. Let $l_\gamma^2 = (l^2, (\cdot, \cdot)_\gamma, \|\cdot\|_\gamma)$ and $X_\gamma = l_\gamma^2 \times l^2 \times L^2$ be Hilbert spaces with the inner product $(\cdot, \cdot)_{X_\gamma}$ and the norm $\|\cdot\|_{X_\gamma}$. For $\varphi^{(i)} = (u^{(i)}, v^{(i)}, w^{(i)}) = (u_j^{(i)}, v_j^{(i)}, w_j^{(i)})_{j\in\mathbb{Z}} \in X_\gamma, i = 1, 2$,

$$(\varphi^{(1)}, \varphi^{(2)})_{X_\gamma} = (u^{(1)}, u^{(2)})_\gamma + (v^{(1)}, v^{(2)}) + (w^{(1)}, w^{(2)}),$$

$$\|\varphi\|_{X_\gamma}^2 = (\varphi, \varphi)_{X_\gamma} = \|u\|_\gamma^2 + \|v\|^2 + \|w\|^2.$$

The random variable $\eta_i(\theta_t\omega) = -\int_{-\infty}^0 e^s(\theta_t\omega_i)(s)ds,\ i = 1, 2,\ \omega \in \Omega$, is the stationary solution of the Ito equation $d\eta_i + \eta_i dt = dW_i(t, \omega_i),\ i = 1, 2$, respectively, where $t \in \mathbb{R}, \omega = (\omega_1, \omega_2) \in \Omega, W_i(t, \omega_i) =$

$\omega_i(t), i = 1,2$. It follows from [5–7] that $\eta_i(\omega)$ is tempered, and $\eta_i(\theta_t\omega)$ is continuous in $t$, $i = 1,2$, and for $\omega \in \Omega$,

$$\lim_{t\to\pm\infty}\frac{|\eta_i(\theta_t\omega)|}{t} = \lim_{t\to\pm\infty}\frac{\int_0^t \eta_i(\theta_s\omega)ds}{t} = 0,\ \lim_{t\to\pm\infty}\frac{\int_0^t |\eta_i(\theta_s\omega)|ds}{t} = \frac{1}{\sqrt{\pi}},\ i = 1,2.$$

Let

$$\begin{cases} v = \dot{u} + \delta u - a_1 u\eta_1(\theta_t\omega), \\ \widetilde{w} = e^{ia_2\eta_2(\theta_t\omega)}w, \end{cases} \tag{2.1}$$

where $\delta = \frac{\alpha\gamma}{\alpha^2+4\gamma} > 0$, and then the system (1.2) is transformed into the following random KGS system:

$$\begin{cases} \dot{\varphi} + \Theta\varphi = H(\theta_t\omega, \varphi), \\ \varphi_\tau(\omega) = (u_\tau, u_{1\tau} + \delta u_\tau - a_1 u_\tau\eta_1(\theta_\tau\omega), e^{ia_2\eta_2(\theta_\tau\omega)}w_\tau)^T, \end{cases} \tag{2.2}$$

where

$$\varphi = \begin{pmatrix} u \\ v \\ \widetilde{w} \end{pmatrix}, \qquad \Theta = \begin{pmatrix} \delta I & -I & 0 \\ A + (\gamma + \delta^2 - \delta\alpha)I & (\alpha - \delta)I & 0 \\ 0 & 0 & iA + \beta I \end{pmatrix},$$

$$H(\theta_t\omega, \varphi) = \begin{pmatrix} a_1 u\eta_1(\theta_t\omega) \\ g + v|\widetilde{w}|^2 + (2\delta u + u - \alpha u - v)a_1\eta_1(\theta_t\omega) - ua_1^2\eta_1^2(\theta_t\omega) \\ i\widetilde{w}u - ihe^{ia_2\eta_2(\theta_t\omega)} - i\widetilde{w}a_2\eta_2(\theta_t\omega) \end{pmatrix}.$$

We make the following assumptions:

(h1) $\sigma_0 > \frac{5|a_1|}{\sqrt{\pi}} + \frac{|a_1(2\delta+1-\alpha)|}{\sqrt{\gamma\pi}} + \frac{a_1^2}{2\sqrt{\gamma}} + \frac{2|a_2|}{\sqrt{\pi}}$, *where* $\sigma_0 = \min\{\sigma, \beta\}$, $\sigma = \frac{\alpha\gamma}{\alpha^2+4\gamma+\alpha\sqrt{\alpha^2+4\gamma}}$;

(h2) $\forall t \in \mathbb{R}$, $g(t) \in C_b(\mathbb{R}, l^2) : \forall\varepsilon > 0, \exists K_1(\varepsilon) \in \mathbb{N}$, *there is* $\sup_{t\in\mathbb{R}} \sum_{|j|>K_1(\varepsilon)} g_j^2(t) < \varepsilon$,

$\forall t \in \mathbb{R}$, $h(t) \in C_b(\mathbb{R}, L^2) : \forall\varepsilon > 0, \exists K_2(\varepsilon) \in \mathbb{N}$, *there is* $\sup_{t\in\mathbb{R}} \sum_{|j|>K_2(\varepsilon)} |h_j(t)|^2 < \varepsilon$;

(h3) the coefficients $a_1$ and $a_2$ in system (1.1) satisfy

$$|a_1| < \min\{\frac{\sqrt{\pi\gamma}\sigma_0}{96(5\sqrt{\gamma} + |2\delta + 1 - \alpha|)}, \frac{\sqrt[4]{\gamma}\sqrt{\sigma_0}}{4\sqrt{3}}\}, \quad |a_2| < \frac{\sqrt{\pi}\sigma_0}{192}.$$

It follows from [20] and assumptions (h1) and (h2) that for each $\tau \in \mathbb{R}, \omega \in \Omega$ and $\varphi_\tau(\omega) \in X_\gamma$, system (2.2) exists a unique solution $\varphi(\cdot, \tau, \omega, \varphi_\tau(\omega)) \in C([\tau, +\infty), X_\gamma)$, and $\varphi(\cdot)$ is continuous in $\varphi_\tau(\omega)$ and measurable in $\omega$, which generates a cocycle $\Psi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X_\gamma \to X_\gamma$ by

$$\Psi(t, \tau, \omega, \varphi_\tau) = \Psi(t + \tau, \tau, \theta_{-\tau}\omega, \varphi_\tau(\theta_{-\tau}\omega)),\ t \geq \tau,$$

on $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ with space $X_\gamma$.

## 3. Random attractor

Now, we prove that $\{\Psi(t,\tau,\omega) : t \geq 0, \tau \in \mathbb{R}, \omega \in \Omega\}$ has a random attractor. From now on, we assume that (h1)–(h3) hold.

Hereafter, all letters $k_i > 0$ ($i \in \mathbb{N}$) represent the constants independent of $(\tau, t, \omega)$.

**Lemma 3.1.** *For each $\tau \in \mathbb{R}, \omega \in \Omega$, and $D \in \mathcal{D}(X_\gamma)$, there exist $T_{D(\tau,\omega)} \geq 0$ and a tempered random variable $R(\omega)$ (independent of $\tau$), such that the solution $\varphi(s) = \varphi(s, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega))$ $(s \geq \tau - t)$ of system (2.2) with $\varphi_{\tau-t}(\theta_{-\tau}\omega) \in D(\tau - t, \theta_{-\tau}\omega)$ satisfies*

$$\|\varphi(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega))\|_{X_\gamma} \leq R(\omega), \ \forall t \geq T_{D(\tau,\omega)}, \tag{3.1}$$

*where*

$$R^2(\omega) = 2k_1 \int_{-\infty}^{0} e^{\sigma_0 s + \int_s^0 \rho(\theta_\tau \omega) d\tau} ds, \ \ k_1 = \frac{2}{\alpha}\|g\|^2 + \frac{2\nu^2}{\alpha\beta^4}\|h\|^4 + \frac{\|h\|^2}{\beta}, \tag{3.2}$$

$$\rho(\omega) = (5 + \frac{|2\delta + 1 - \alpha|}{\sqrt{\gamma}})|a_1\eta_1(\omega)| + \frac{a_1^2\eta_1^2(\omega)}{\sqrt{\gamma}} + 2|a_2\eta_2(\omega)|. \tag{3.3}$$

*Proof.* Multiplying (2.2) by $\varphi(r)$ in $X_\gamma$, and taking its real part, we get

$$Re(\dot{\varphi}, \varphi)_{X_\gamma} + Re(\Theta\varphi, \varphi)_{X_\gamma} = Re(H(\theta_{r-t}\omega, \varphi), \varphi)_{X_\gamma}. \tag{3.4}$$

Obviously, we find

$$Re(\dot{\varphi}, \varphi)_{X_\gamma} + Re(\Theta\varphi, \varphi)_{X_\gamma} \geq \frac{1}{2}\frac{d}{dt}\|\varphi(r)\|_{X_\gamma}^2 + \sigma_0\|\varphi(r)\|_{X_\gamma}^2 + \frac{\alpha}{2}\|v\|^2 + \frac{\beta}{2}\|\widetilde{w}\|^2. \tag{3.5}$$

Multiplying the third component equation of (2.2) by $\widetilde{w}$ in $L^2$, and taking its real part, we have

$$\frac{d}{dt}\|\widetilde{w}(r)\|^2 + \beta\|\widetilde{w}(r)\|^2 \leq \frac{1}{\beta}\|h\|^2. \tag{3.6}$$

Using the Gronwall lemma to (3.6) on $[\tau - t, r]$, we obtain

$$\|\widetilde{w}(r)\|^2 \leq \|\widetilde{w}_{\tau-t}\|^2 e^{-\beta(r-\tau+t)} + \frac{\|h\|^2}{\beta^2}. \tag{3.7}$$

$$\begin{aligned} Re(H(\theta_{r-\tau}\omega, \varphi), \varphi)_{X_\gamma} &= (a_1\eta_1(\theta_{r-\tau}\omega)u, u)_\gamma + (g(r), v) + (\nu(|\widetilde{w}|^2, v) \\ &+ ((2\delta + 1 - \alpha)a_1\eta_1(\theta_{r-\tau}\omega)u, v) - (a_1\eta_1(\theta_{r-\tau}\omega)v, v) - (a_1^2\eta_1^2(\theta_{r-\tau}\omega)u, v) \\ &- Im(\widetilde{w}u - he^{ia_2\eta_2(\theta_{r-\tau}\omega)} - a_2\eta_2(\theta_{r-\tau}\omega)\widetilde{w}, \widetilde{w}), \end{aligned} \tag{3.8}$$

where

$$\begin{cases} (a_1\eta_1(\theta_{r-\tau}\omega)u, u)_\gamma \leq |a_1\eta_1(\theta_{r-\tau}\omega)|\|\varphi(r)\|_{X_\gamma}^2, \\ (g(r), v) \leq \frac{\alpha}{4}\|v\|^2 + \frac{1}{\alpha}\|g\|^2, \\ (\nu|\widetilde{w}|^2, v) \leq \frac{\nu^2}{\alpha}\|w\|^4 + \frac{\alpha}{4}\|v\|^2, \\ ((2\delta + 1 - \alpha)a_1\eta_1(\theta_{r-\tau}\omega)u, v) \leq \frac{|2\delta+1-\alpha|}{2\sqrt{\gamma}}|a_1\eta_1(\theta_{r-\tau}\omega)|\|\varphi(r)\|_{X_\gamma}^2, \\ -(a_1\eta_1(\theta_{r-\tau}\omega)v, v) \leq |a_1\eta_1(\theta_{r-\tau}\omega)|\|\varphi(r)\|_{X_\gamma}^2, \\ -(a_1^2\eta_1^2(\theta_{r-\tau}\omega)u, v) \leq \frac{a_1^2\eta_1^2(\theta_{r-\tau}\omega)}{2\sqrt{\gamma}}\|\varphi(r)\|_{X_\gamma}^2, \\ Im(he^{ia_2\eta_2(\theta_{r-\tau}\omega)} + a_2\eta_2(\theta_{r-\tau}\omega)\widetilde{w}, \widetilde{w}) \leq \frac{\|h\|^2}{2\beta} + \frac{\beta}{2}\|\widetilde{w}\|^2 + |a_2\eta_2(\theta_{r-\tau}\omega)|\|\varphi(r)\|_{X_\gamma}^2. \end{cases}$$

By (3.4), (3.5), (3.7), and (3.8), we have

$$\begin{aligned}
&\frac{d}{dt}\|\varphi(r)\|^2_{X_\gamma} + (\sigma_0 - \rho(\theta_{r-t}\omega))\|\varphi\|^2_{X_\gamma} + (\sigma_0 + |a_1\eta_1(\theta_{r-t}\omega)|)\|\varphi(r)\|^2_{X_\gamma} \\
&\le \frac{2}{\alpha}\|g\|^2 + \frac{\|h\|^2}{\beta} + \frac{2\nu^2}{\alpha\beta^4}\|h\|^4 + \frac{2\nu^2}{\alpha}(\|w_{\tau-t}\|^4 + \|w_{\tau-t}\|^2\frac{\|h\|^2}{\beta^2})e^{-\beta(r-\tau+t)} \\
&\le k_1 + k_2(\|w_{\tau-t}\|^4 + \|w_{\tau-t}\|^2)e^{-\beta(r-\tau+t)}, \quad r \ge \tau - t, \qquad (3.9)
\end{aligned}$$

where $k_2 = \max\{\frac{2\nu^2}{\alpha}, \frac{2\nu^2\|h\|^2}{\alpha\beta^2}\}$.
Using the Gronwall lemma to (3.9) on $[\tau - t, r]$, we find

$$\begin{aligned}
&\|\varphi(r)\|^2_{X_\gamma} + (\sigma_0 + |a_1\eta_1(\omega)|)\int_{\tau-t}^{r} e^{-\int_s^r(\sigma_0-\rho(\theta_{l-\tau}\omega))dl}\|\varphi(s)\|^2_{X_\gamma}ds \\
&\le \|\varphi_{\tau-t}(\theta_{-\tau}\omega)\|^2_{X_\gamma}e^{-\int_{\tau-t}^r(\sigma_0-\rho(\theta_{l-\tau}\omega))dl} + k_1\int_{\tau-t}^{r} e^{-\int_s^r(\sigma_0-\rho(\theta_{l-\tau}\omega))dl}ds \\
&+ k_2(\|w_{\tau-t}\|^4 + \|w_{\tau-t}\|^2)\int_{\tau-t}^{r} e^{-\int_s^r(\sigma_0-\rho(\theta_{l-\tau}\omega)dl)}e^{-\beta(s-\tau+t)}ds.
\end{aligned}$$

Taking $r = \tau$, we find

$$\begin{aligned}
&\|\varphi(\tau)\|^2_{X_\gamma} + (\sigma_0 + |a_1\eta_1(\omega)|)\int_{\tau-t}^{\tau} e^{-\int_s^\tau(\sigma_0-\rho(\theta_{l-\tau}\omega))dl}\|\varphi(s)\|^2_{X_\gamma}ds \\
&\le \|\varphi_{\tau-t}(\theta_{-\tau}\omega)\|^2_{X_\gamma}e^{-\sigma_0 t+\int_{-t}^0\rho(\theta_l\omega)dl} + k_2(\|w_{\tau-t}\|^4 + \|w_{\tau-t}\|^2)e^{-\beta t} \\
&\times\int_{-t}^{0} e^{(\sigma_0-\beta)s+\int_s^0\rho(\theta_l\omega)dl}ds + \frac{1}{2}R^2(\omega). \qquad (3.10)
\end{aligned}$$

By $\varphi_{\tau-t}(\theta_{-\tau}\omega) \in D(\tau - t, \theta_{-t}\omega)$, we obtain

$$\lim_{t\to+\infty}(\|\varphi_{\tau-t}(\theta_{-\tau}\omega)\|^2_{X_\gamma}e^{-\sigma_0 t+\int_{-t}^0\rho(\theta_l\omega)dl} + k_2(\|w_{\tau-t}\|^4 + \|w_{\tau-t}\|^2)e^{-\beta t}\int_{-t}^{0} e^{(\sigma_0-\beta)s+\int_s^0\rho(\theta_l\omega)dl}ds) = 0.$$

Thus, we complete the proof.

We know from Lemma 3.1 that the continuous cocycle $\{\Psi(t,\tau,\omega) : t \ge 0, \tau \in \mathbb{R}, \omega \in \Omega\}$ has a tempered random absorbing set $D_0 = \{\varphi \in X_\gamma : \|\varphi\|_{X_\gamma} \le R(\omega)\} \in \mathcal{D}(X_\gamma)$, and there exists a $T_{D_0(\omega)} \ge 0$ such that for each $t \ge T_{D_0(\omega)}$, $\varphi(\tau, \tau - t, \theta_{-\tau}\omega)D_0(\theta_{-t}\omega) \subseteq D_0(\omega)$.

Next, we show the tail estimate of the solutions of system (2.2).

Taking an increasing function $\mu \in C^1(\mathbb{R}_+, [0, 1])$, defined by

$$\begin{cases} \mu(x) = 0, & 0 \le x \le 1, \\ 0 \le \mu(x) \le 1, & 1 \le x \le 2, \quad |\mu'(x)| \le \mu_0,\ \mu_0 > 0. \\ \mu(x) = 1, & x \ge 2, \end{cases}$$

**Lemma 3.2.** *For each $\tau \in \mathbb{R}$, $\omega \in \Omega$, $t \ge 0$, $J \in \mathbb{N}$, and $\varepsilon > 0$, there exists $T_\varepsilon(\omega) > 0$ such that the solution $\varphi(r) = \varphi(r, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega)) = (u(r), v(r), \widetilde{w}(r))^T$ $(r > \tau - t)$ of system (2.2) with $\varphi_{\tau-t}(\theta_{-\tau}\omega) \in D_0(\theta_{-t}\omega)$ satisfies*

$$\sum_{|j|\ge 2J}\|\varphi_j(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t})\|^2_{X_\gamma} \le \varepsilon + k_3(\frac{1}{J} + \frac{1}{J^2} + F_J)I_0(\omega),\ t > T_\varepsilon(\omega),\ J \ge 2, \qquad (3.11)$$

*where*

$$I_0(\omega) = \int_{-\infty}^{0} e^{\sigma_0 s + \int_s^0 \rho(\theta_\tau \omega) d\tau} ds, \ F_J = \sup_{r\in\mathbb{R}} (\sum_{|j|\geq J} g_j^2(r) + \sum_{|j|\geq J} |h_j(r)|^2 + (\sum_{|j|\geq \frac{J}{2}} |h_j(r)|^2)^2). \tag{3.12}$$

*Proof.* Let $\phi = (x, y, z)^T = (\phi_j)_{j\in\mathbb{Z}} = (x_j, y_j, z_j)^T_{j\in\mathbb{Z}} = (\mu(\frac{|j|}{J})u_j, \mu(\frac{|j|}{J})v_j, \mu(\frac{|j|}{J})\widetilde{w}_j)^T_{j\in\mathbb{Z}}$. Multiplying (2.2) by $\phi = (x, y, z)^T$ in $X_\gamma$, and taking its real part, we find

$$Re(\dot{\varphi}, \phi)_{X_\gamma} + Re(\Theta\varphi, \phi)_{X_\gamma} = Re(H(\theta_t\omega, \varphi), \phi)_{X_\gamma}. \tag{3.13}$$

By some computation, it follows that

$$\begin{aligned} Re(\dot{\varphi}, \phi)_{X_\gamma} &\geq \frac{1}{2}\frac{d}{dt}\sum_{j\in\mathbb{Z}} \mu(\frac{|j|}{J})\|\varphi_j\|^2_{X_\gamma} - \frac{2\mu_0}{J}(\gamma + \delta + \frac{1}{2} + |a_1\eta_1(\theta_{r-\tau}\omega)|)\|\varphi\|^2_{X_\gamma} \\ &\geq \frac{1}{2}\frac{d}{dt}\sum_{j\in\mathbb{Z}} \mu(\frac{|j|}{J})\|\varphi_j\|^2_{X_\gamma} - \frac{k_4}{J}(1 + |a_1\eta_1(\theta_{r-\tau}\omega)|)\|\varphi\|^2_{X_\gamma}. \end{aligned} \tag{3.14}$$

$$Re(\Theta\varphi, \phi)_{X_\gamma} \geq \sum_{j\in\mathbb{Z}} \mu(\frac{|j|}{J})(\sigma_0\|\varphi_j\|^2_{X_\gamma} + \frac{\alpha}{2}|v_j|^2 + \frac{\beta}{2}|w_j|^2) - \frac{k_5}{J}\|\varphi\|^2_{X_\gamma}. \tag{3.15}$$

Multiplying the third component equation of (2.2) by $z$ in $L^2$, taking the real part, and combining (3.7), we get

$$\frac{d}{dt}\sum_{j\in\mathbb{Z}} \mu(\frac{|j|}{J})|\widetilde{w}_j|^2 + \frac{3\beta}{2}\sum_{j\in\mathbb{Z}} \mu(\frac{|j|}{J})|\widetilde{w}_j|^2 \leq \frac{2}{\beta}\sum_{|j|\geq J} |h_j(r)|^2 + \frac{2\mu_0}{J}(\|\widetilde{w}_{\tau-t}\|^2 e^{-\beta(r-\tau+t)} + \frac{\|h\|^2}{\beta^2}).$$

Thus, for $J \in \mathbb{R}$, by the Gronwall lemma, we have

$$\begin{aligned} \sum_{|j|\geq 2J} |\widetilde{w}_j(r)|^2 &\leq \|\widetilde{w}_{\tau-t}\|^2 e^{-\frac{3}{2}\beta(r-\tau+t)} + \frac{2}{3\beta^2}\sum_{|j|\geq J} |h_j(r)|^2 + \frac{2\mu_0}{J}(\|w_{\tau-t}\|^2 \int_{\tau-t}^{r} e^{-\frac{3}{2}\beta(r-s)-\beta(s-\tau+t)} ds \\ &+ \frac{2}{3\beta^3}\|h\|^2) \leq k_6\|\widetilde{w}_{\tau-t}\|^2 e^{-\frac{3}{2}\beta(r-\tau+t)} + k_7\sum_{|j|\geq J} |h_j(r)|^2 + \frac{k_8}{J}. \end{aligned} \tag{3.16}$$

Then, it follows that

$$\begin{aligned} Re(H(\theta_t\omega, \varphi), \phi)_{X_\gamma} &= (a_1\eta_1(\theta_t\omega), x)_\gamma + (g, y) + (\nu|\widetilde{w}|^2, y) + (2\delta + 1 - \alpha)(a_1\eta_1(\theta_t\omega)u, y) \\ &- (a_1\eta_1(\theta_t\omega)v, y) - (a_1^2\eta_1^2(\theta_t\omega)u, y) + Im(he^{ia_2\eta_2(\theta_t\omega)} + \widetilde{w}a_2\eta_2(\theta_t\omega), z), \end{aligned} \tag{3.17}$$

where

$$\begin{cases} (a_1\eta_1(\theta_t\omega)u,x)_\gamma \le |a_1\eta_1(\theta_t\omega)|\sum_{j\in\mathbb{Z}}\mu(\frac{|j|}{J})\|\varphi_j\|^2_{X_\gamma} + \frac{4\mu_0}{J\gamma}|a_1\eta_1(\theta_t\omega)|\|\varphi\|^2_{X_\gamma}, \\ (g,y) \le \frac{1}{\alpha}\sum_{|j|\ge J} g_j(r)^2 + \frac{\alpha}{4}\sum_{j\in\mathbb{Z}}\mu(\frac{|j|}{J})v_j^2, \\ (v|\widetilde{w}|^2,y) \le \frac{\alpha}{4}\sum_{j\in\mathbb{Z}}\mu(\frac{|j|}{J})v_j^2 + k_9\|w_{\tau-t}\|^4 e^{-3\beta(r-\tau+t)} + k_{10}(\sum_{|j|\ge\frac{J}{2}}|h_j(r)|^2)^2 + \frac{k_{11}}{J^2}, \\ ((2\delta+1-\alpha)a_1\eta_1(\theta_t\omega)u,y) \le \frac{|a_1\eta_1(\theta_t\omega)(2\delta+1-\alpha)|}{2\sqrt{\gamma}}\sum_{j\in\mathbb{Z}}\mu(\frac{|j|}{J})\|\varphi_j\|^2_{X_\gamma}, \\ -(a_1\eta_1(\theta_t\omega)v,y) \le |a_1\eta_1(\theta_t\omega)|\sum_{j\in\mathbb{Z}}\mu(\frac{|j|}{J})\|\varphi_j\|^2_{X_\gamma}, \\ -(a_1^2\eta_1^2(\theta_t\omega)u,y) \le \frac{a_1^2\eta_1^2(\theta_t\omega)}{2\sqrt{\gamma}}\sum_{j\in\mathbb{Z}}\mu(\frac{|j|}{J})\|\varphi_j\|^2_{X_\gamma}, \\ Im(he^{ia_2\eta_2(\theta_t\omega)} + a_2\widetilde{w}\eta_2(\theta_t\omega),z) \le \frac{\beta}{2}\sum_{j\in\mathbb{Z}}\mu(\frac{|j|}{J})|\widetilde{w}_j|^2 + \frac{1}{2\beta}\sum_{|j|\ge J}|h_j(r)|^2 + |a_2\eta_2(\theta_t\omega)|\sum_{j\in\mathbb{Z}}\mu(\frac{|j|}{J})|\varphi_j|^2_{X_\gamma}. \end{cases}$$

Summing up (3.14), (3.15), and (3.17), we have

$$\begin{aligned} &\frac{d}{dt}\sum_{j\in\mathbb{Z}}\mu(\frac{|j|}{J})\|\varphi_j\|^2_{X_\gamma} + (\sigma_0 - \rho(\theta_{r-\tau}\omega))\sum_{j\in\mathbb{Z}}\mu(\frac{|j|}{J})\|\varphi_j\|^2_{X_\gamma} \\ &\le \frac{k_{12}}{J}(1+|a_1\eta_1(\theta_{r-\tau}\omega)|)\|\varphi\|^2_{X_\gamma} + k_{13}\|w_{\tau-t}\|^4 e^{-3\beta(r-\tau+t)} + \frac{k_{14}}{J^2} + k_{15}F_J. \end{aligned}$$

Using the Gronwall lemma on $[\tau-t,\tau]$, for $J\ge 2$, we obtain that

$$\begin{aligned} \sum_{j\in\mathbb{Z}}\mu(\frac{|j|}{J})\|\varphi_j\|^2_{X_\gamma} \le{}& \|\varphi(\tau-t)\|^2_{X_\gamma}e^{-\int_{-t}^0(\sigma_0-\rho(\theta_l\omega))dl} + k_{16}\|w_{\tau-t}\|^4 e^{-3\beta t}\int_{-t}^0 e^{(\sigma_0-3\beta)s+\int_s^0\rho(\theta_l\omega)dl}ds \\ &+ \frac{1}{J}\int_{\tau-t}^{\tau} k_{12}(1+|a_1\eta_1(\theta_{s-\tau}\omega)|)\|\varphi(s)\|^2_{X_\gamma}e^{-\int_s^\tau(\sigma_0-\rho(\theta_{l-\tau}\omega))dl}ds \\ &+ k_{17}F_J\int_{-\infty}^0 e^{\sigma_0 s+\int_s^0\rho(\theta_l\omega)dl}ds + \frac{k_{18}}{J^2}\int_{-\infty}^0 e^{\sigma_0 s+\int_s^0\rho(\theta_l\omega)dl}ds. \end{aligned} \tag{3.18}$$

By (3.10), we obtain

$$\begin{aligned} &k_{12}(1+|a_1\eta_1(\theta_{s-\tau}\omega)|)\|\varphi(s)\|^2_{X_\gamma}e^{-\int_s^\tau(\sigma_0-\rho(\theta_{l-\tau}\omega))dl} \le k_{19}R^2(\omega) + k_{20}(\|\varphi_{\tau-t}(\theta_{-\tau}\omega)\|^2_{X_\gamma} \\ &\times e^{-\int_t^0(\sigma_0-\rho(\theta_l\omega))dl} + k_{21}(\|w_{\tau-t}\|^4 + \|w_{\tau-t}\|^2)e^{-\beta t}\int_{-t}^0 e^{(\sigma_0-\beta)s+\int_s^0\rho(\theta_l\omega)dl}ds). \end{aligned} \tag{3.19}$$

Putting (3.19) into (3.18), we get

$$\sum_{j\in\mathbb{Z}}\mu(\frac{|j|}{J})\|\varphi_j\|^2_{X_\gamma} \le M_1(t,\omega) + k_{22}I_0(\omega)(\frac{1}{J} + \frac{1}{J^2} + F_J),$$

where

$$M_1(t,\omega) = k_{23}R^2(\theta_{-t}\omega)e^{-\int_{-t}^0(\sigma_0-\rho(\theta_l\omega))dl} + k_{24}R^2(\theta_{-t}\omega)e^{-3\beta t}\int_{-t}^0 e^{(\sigma_0-3\beta)s+\int_s^0\rho(\theta_l\omega)dl}ds$$
$$+ k_{25}(R^4(\theta_{-t}\omega) + R^2(\theta_{-t}\omega))e^{-\beta t}\int_{-t}^0 e^{(\sigma_0-\beta)s+\int_s^0\rho(\theta_l\omega)dl}ds.$$

By (h1), we get $\lim_{t\to+\infty} M_1(t,\omega) = 0$, and therefore, for any $\varepsilon > 0, J \geq 2$, there exists

$$T_\varepsilon(\omega) : T_\varepsilon(\omega) = \min\{t : M_1(t,\omega) \leq \varepsilon\} \geq 0, \tag{3.20}$$

such that (3.11) holds. Lemma 3.2 is proved.

It follows from Lemma 3.1, Lemma 3.2 and Theorem 3.6 in [3] that we obtain:

**Theorem 3.1.** *The cocycle* $\{\Psi(t,\tau,\omega) : t \geq 0, \tau \in \mathbb{R}, \omega \in \Omega\}$ *of the system* (2.2) *has a random attractor* $\mathcal{K} = \{\mathcal{K}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ in $\mathcal{D}(X_\gamma)$.

## 4. Random exponential attractor

Now, we verify that $\Psi$ possesses a random exponential attractor.

For each $\tau \in \mathbb{R}, \omega \in \Omega$, and the above $T_\varepsilon(\omega)$ defined by (3.20), set

$$\mathcal{S}(\tau,\omega) = \overline{\bigcup_{s\geq\max\{T_{D_0(\omega)},T_\varepsilon(\omega)\}} \varphi(\tau,\tau-s,\theta_{-\tau}\omega, D_0(\theta_{-s}\omega))} \subseteq D_0(\omega) \subset X_\gamma. \tag{4.1}$$

By (3.11) and (4.1), for any $\varphi \in \mathcal{S}(\tau,\omega)$ and $J \geq 2$, we have

$$\sum_{|j|\geq 2J} |\varphi_j|^2_{X_\gamma} \leq \varepsilon + k_3 I_0(\omega)(\frac{1}{J} + \frac{1}{J^2} + F_J). \tag{4.2}$$

For each $\tau \in \mathbb{R}$, $\omega \in \Omega$, $t \geq 0$, $\varphi_{i,\tau-t}(\theta_{-\tau}\omega) \in \mathcal{S}(\tau-t,\theta_{-t}\omega)$, $i = 1,2$. Let $\varphi_i(r) = \varphi(r,\tau-t,\theta_{-\tau}\omega,\varphi_{i,\tau-t}(\theta_{-\tau}\omega))$, $i = 1,2$, $\psi(r) = \varphi_1(r) - \varphi_2(r) = (\xi,\zeta,\varsigma)^T$, $r \geq \tau - t$, and then

$$\dot{\psi} + \Theta\psi = H(\theta_{r-\tau}\omega,\varphi_1) - H(\theta_{r-\tau}\omega,\varphi_2). \tag{4.3}$$

By Lemma 3.1 and (4.2), we have $\|\varphi_i(r)\|_{X_\gamma} \leq R(\theta_{r-\tau}\omega)$, $i = 1,2$.

**Lemma 4.1.** *For each* $\tau \in \mathbb{R}, \omega \in \Omega, t \geq 0$, *and* $\varphi_{i,\tau-t}(\theta_{-\tau}\omega) \in \mathcal{S}(\tau-t,\theta_{-t}\omega), i = 1,2$, *there exist positive random variables* $C_0(\omega)$ *and* $C_1(\omega)$, *such that*

$$\|\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{1,\tau-t}(\theta_{-\tau}\omega)) - \varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{2,\tau-t}(\theta_{-\tau}\omega))\|_{X_\gamma}$$
$$\leq e^{\int_{-t}^0 C_0(\theta_s\omega)ds}\|\psi(\tau-t)\|_{X_\gamma}, \tag{4.4}$$

*and*

$$\sum_{|j|>4J}|(\varphi_j(r,\tau-t,\theta_{-\tau}\omega,\varphi_{1,\tau-t}(\theta_{-\tau}\omega))-\varphi_j(r,\tau-t,\theta_{-\tau}\omega,\varphi_{2,\tau-t}(\theta_{-\tau}\omega)))_j|^2_{X_\gamma}$$
$$\leq (e^{-\frac{\sigma_0}{2}t+\frac{1}{2}\int_{-t}^{0}\rho(\theta_s\omega)ds}+\frac{\delta_J}{2}e^{\int_{-t}^{0}C_1(\theta_s\omega)ds})^2\|\psi(\tau-t)\|^2_{X_\gamma},\quad \forall J\in\mathbb{N}, \tag{4.5}$$

*where* $\delta_J=\frac{2}{\sqrt[4]{2\sigma_0}}\sqrt{\frac{1}{J}+\frac{1}{J^2}+F_J}$.

*Proof.* Multiplying (4.3) by $\psi(r)$ in $X_\gamma$, and taking its real part, we find

$$Re(\dot{\psi},\psi)_{X_\gamma}+Re(\Theta\psi,\psi)_{X_\gamma}=Re(H(\theta_{r-\tau}\omega,\varphi_1)-H(\theta_{r-\tau}\omega,\varphi_2),\psi)_{X_\gamma}. \tag{4.6}$$

By computation, we get

$$Re(\dot{\psi},\psi)_{X_\gamma}+Re(\Theta\psi,\psi)_{X_\gamma}\geq \frac{1}{2}\frac{d}{dt}\|\psi(r)\|^2_{X_\gamma}+\sigma_0\|\psi(r)\|^2_{X_\gamma}+\frac{\alpha}{2}|\zeta|^2+\frac{\beta}{2}|\varsigma|^2, \tag{4.7}$$

and

$$\begin{aligned}&Re(H(\theta_{r-\tau}\omega,\varphi_1)-H(\theta_{r-\tau}\omega,\varphi_2),\psi)_{X_\gamma}\\&=(a_1\eta_1(\theta_{r-\tau}\omega)\xi,\xi)_\gamma+(\nu(|\widetilde{w}_1|^2-|\widetilde{w}_2|^2),\zeta)+((2\delta+1-\alpha)a_1\eta_1(\theta_{r-\tau}\omega)\xi,\zeta)-(a_1\eta_1(\theta_{r-\tau}\omega)\zeta,\zeta)\\&-(a_1^2\eta_1^2(\theta_{r-\tau}\omega)\xi,\zeta)-Im((\widetilde{w}_1u_1-\widetilde{w}_2u_2),\varsigma)+Im(a_2\eta_2(\theta_{r-\tau}\omega)\varsigma,\varsigma),\end{aligned}$$

where

$$\begin{cases}(a_1\eta_1(\theta_{r-\tau}\omega)\xi,\xi)_\gamma\leq|a_1\eta_1(\theta_{r-\tau}\omega)|\|\psi(r)\|^2_{X_\gamma},\\(\nu(|\widetilde{w}_1|^2-|\widetilde{w}_2|^2),\zeta)\leq\frac{2\nu^2}{\alpha}R^2(\theta_{r-\tau}\omega)\|\psi(r)\|^2_{X_\gamma}+\frac{\alpha}{2}|\zeta|^2,\\((2\delta+1-\alpha)a_1\eta_1(\theta_{r-\tau}\omega)\xi,\zeta)\leq\frac{|2\delta+1-\alpha|}{2\sqrt{\gamma}}|a_1\eta_1(\theta_{r-\tau}\omega)|\|\psi(r)\|^2_{X_\gamma},\\-(a_1\eta_1(\theta_{r-\tau}\omega)\zeta,\zeta)\leq|a_1\eta_1(\theta_{r-\tau}\omega)|\|\psi(r)\|^2_{X_\gamma},\\-(a_1^2\eta_1^2(\theta_{r-\tau}\omega)\xi,\zeta)\leq\frac{a_1^2\eta_1^2(\theta_{r-\tau}\omega)}{2\sqrt{\gamma}}\|\psi(r)\|^2_{X_\gamma},\\-Im((\widetilde{w}_1u_1-\widetilde{w}_2u_2),\varsigma)\leq\frac{1}{\gamma\beta}R^2(\theta_{r-\tau}\omega)\|\psi(r)\|^2_{X_\gamma}+\frac{\beta}{2}|\varsigma|^2,\\Im(a_2\eta_2(\theta_{r-\tau}\omega)\varsigma,\varsigma)\leq|a_2\eta_2(\theta_{r-\tau}\omega)|\|\psi(r)\|^2_{X_\gamma}.\end{cases}$$

Thus, we obtain

$$\frac{d}{dt}\|\psi(r)\|^2_{X_\gamma}+2\sigma_0\|\psi(r)\|^2_{X_\gamma}\leq 2C_0(\theta_{r-\tau}\omega)\|\psi(r)\|^2_{X_\gamma}, \tag{4.8}$$

where

$$C_0(\omega)=(2+\frac{|2\delta+1-\alpha|}{2\sqrt{\gamma}})|a_1\eta_1(\omega)|+\frac{a_1^2\eta_1^2(\omega)}{2\sqrt{\gamma}}+(\frac{2\nu^2}{\alpha}+\frac{1}{\gamma\beta})R^2(\omega)+|a_2\eta_2(\omega)|. \tag{4.9}$$

By the Gronwall lemma, integrating (4.8) on $[\tau-t,\tau]$, we get that

$$\|\psi(\tau)\|^2_{X_\gamma}\leq e^{2\int_{-t}^{0}C_0(\theta_s\omega)ds}\|\psi(\tau-t)\|^2_{X_\gamma}.$$

Thus, (4.4) holds.

Let $M\in\mathbb{N}$ and $\widetilde{\psi}=(\mu(\frac{|j|}{M})\psi_j)_{j\in\mathbb{Z}}=(\mu(\frac{|j|}{M})\xi_j,\mu(\frac{|j|}{M})\zeta_j,\mu(\frac{|j|}{M})\varsigma_j)^T_{j\in\mathbb{Z}}=(\widetilde{\xi}_j,\widetilde{\zeta}_j,\widetilde{\varsigma}_j)^T_{j\in\mathbb{Z}}=(\widetilde{\xi},\widetilde{\zeta},\widetilde{\varsigma})^T$.

Multiplying (4.3) by $\widetilde{\psi}$ in $X_\gamma$, and taking its real part, we get

$$Re(\dot{\psi}, \widetilde{\psi})_{X_\gamma} + Re(\Theta\psi, \widetilde{\psi})_{X_\gamma} = Re(H(\theta_{r-\tau}\omega, \varphi_1) - H(\theta_{r-\tau}\omega, \varphi_2), \widetilde{\psi})_{X_\gamma}. \tag{4.10}$$

By computation, we obtain

$$\begin{aligned}
&Re(\dot{\psi}, \widetilde{\psi})_{X_\gamma} + Re(\Theta\psi, \widetilde{\psi})_{X_\gamma} \\
&\geq \frac{1}{2}\frac{d}{dt}\sum_{j\in\mathbb{Z}}\mu(\frac{|j|}{M})\|\psi_j(r)\|^2_{X_\gamma} + \sigma_0\sum_{j\in\mathbb{Z}}\mu(\frac{|j|}{M})\|\psi_j(r)\|^2_{X_\gamma} + \frac{\alpha}{2}\sum_{j\in\mathbb{Z}}\mu(\frac{|j|}{M})|\zeta_j(r)|^2 \\
&+ \frac{\beta}{2}\sum_{j\in\mathbb{Z}}\mu(\frac{|j|}{M})|\varsigma_j(r)|^2 - \frac{1}{M}(k_5 + k_4(1 + |a_1\eta_1(\theta_{r-\tau}\omega)|))\sum_{j\in\mathbb{Z}}\mu(\frac{|j|}{M})\|\psi_j(r)\|^2_{X_\gamma},
\end{aligned}$$

and

$$\begin{aligned}
&Re(H(\theta_{r-\tau}\omega, \varphi_1) - H(\theta_{r-\tau}\omega, \varphi_2), \widetilde{\psi})_{X_\gamma} \\
&= (a_1\eta_1(\theta_{r-\tau}\omega)\xi, \widetilde{\xi})_\gamma + (\nu(|\widetilde{w}_1|^2 - |\widetilde{w}_2|^2), \widetilde{\zeta}) + ((2\delta + 1 - \alpha)a_1\eta_1(\theta_{r-\tau}\omega)\xi, \widetilde{\zeta}) - (a_1\eta_1(\theta_{r-\tau}\omega)\zeta, \widetilde{\zeta}) \\
&- (a_1^2\eta_1^2(\theta_{r-\tau}\omega)\xi, \widetilde{\zeta}) - Im((\widetilde{w}_1u_1 - \widetilde{w}_2u_2), \widetilde{\varsigma}) + Im(a_2\eta_2(\theta_{r-\tau}\omega)\varsigma, \widetilde{\varsigma}),
\end{aligned}$$

where

$$\begin{cases}
(a_1\eta_1(\theta_{r-\tau}\omega)\xi, \widetilde{\xi})_\gamma \leq |a_1\eta_1(\theta_{r-\tau}\omega)|\sum\limits_{j\in\mathbb{Z}}\mu(\frac{|j|}{M})\|\psi_j(r)\|^2_{X_\gamma} + \frac{4\mu_0}{M}|a_1\eta_1(\theta_{r-\tau}\omega)|\|\psi(r)\|^2_{X_\gamma}, \\
(\nu(|\widetilde{w}_1|^2 - |\widetilde{w}_2|^2), \widetilde{\zeta}) \leq \frac{\nu^2}{\alpha}\sum\limits_{j\in\mathbb{Z}}\mu(\frac{|j|}{M})(|\widetilde{w}_{1,j}|^2 + |\widetilde{w}_{2,j}|^2)\|\psi_j(r)\|^2_{X_\gamma} + \frac{\alpha}{2}|\zeta_j(r)|^2, \\
((2\delta + 1 - \alpha)a_1\eta_1(\theta_{r-\tau}\omega)\xi, \widetilde{\zeta}) \leq \frac{|2\delta+1-\alpha|}{2\sqrt{\gamma}}|a_1\eta_1(\theta_{r-\tau}\omega)|\sum\limits_{j\in\mathbb{Z}}\mu(\frac{|j|}{M})\|\psi_j(r)\|^2_{X_\gamma}, \\
-(a_1\eta_1(\theta_{r-\tau}\omega)\zeta, \widetilde{\zeta}) \leq |a_1\eta_1(\theta_{r-\tau}\omega)|\sum\limits_{j\in\mathbb{Z}}\mu(\frac{|j|}{M})\|\psi_j(r)\|^2_{X_\gamma}, \\
-(a_1^2\eta_1^2(\theta_{r-\tau}\omega)\xi, \widetilde{\zeta}) \leq \frac{a_1^2\eta_1^2(\theta_{r-\tau}\omega)}{2\sqrt{\gamma}}\sum\limits_{j\in\mathbb{Z}}\mu(\frac{|j|}{M})\|\psi_j(r)\|^2_{X_\gamma}, \\
-Im((\widetilde{w}_1u_1 - \widetilde{w}_2u_2), \widetilde{\varsigma}) \leq \sum\limits_{j\in\mathbb{Z}}\mu(\frac{|j|}{M})(\frac{1}{\beta}(|u_{1,j}|^2 + |\widetilde{w}_{2,j}|^2)\|\psi_j(r)\|^2_{X_\gamma} + \frac{\beta}{2}|\varsigma_j|^2), \\
Im(a_2\eta_2(\theta_{r-\tau}\omega)\varsigma, \widetilde{\varsigma}) \leq |a_2\eta_2(\theta_{r-\tau}\omega)|\sum\limits_{j\in\mathbb{Z}}\mu(\frac{|j|}{M})\|\psi_j(r)\|^2_{X_\gamma}.
\end{cases}$$

By (4.2), for $J \geq 2$, $\sum\limits_{j\geq 2J}(|\widetilde{w}_{1,j}|^2 + |\widetilde{w}_{2,j}|^2) \leq 2\varepsilon + 2k_3(\frac{1}{J} + \frac{1}{J^2} + F_J)I_0(\theta_{r-\tau}\omega)$ and $\sum\limits_{j\geq 2J}(|u_{1,j}|^2 + |\widetilde{w}_{2,j}|^2) \leq 2\varepsilon + 2k_3(\frac{1}{J} + \frac{1}{J^2} + F_J)I_0(\theta_{r-\tau}\omega)$. Let a positive constant $\varepsilon = \varepsilon_0$ be small enough, such that $(\frac{2}{\beta} + \frac{2\nu^2}{\alpha})\varepsilon_0 \leq \frac{\sigma_0}{2}$. Thus, we get that for $J \geq 2$, $M \geq 2J$,

$$\begin{aligned}
&\frac{d}{dt}\sum_{j\in\mathbb{Z}}\mu(\frac{|j|}{M})\|\psi_j(r)\|^2_{X_\gamma} + (\sigma_0 - \rho(\theta_{r-\tau}\omega))\sum_{j\in\mathbb{Z}}\mu(\frac{|j|}{M})\|\psi_j(r)\|^2_{X_\gamma} \\
&\leq (\frac{M_2(\theta_{r-\tau}\omega)}{M} + k_{26}(\frac{1}{J} + \frac{1}{J^2} + F_J)I_0(\theta_{r-\tau}\omega))\|\psi(r)\|^2_{X_\gamma} \\
&\leq k_{27}(\frac{1}{J} + \frac{1}{J^2} + F_J)(M_2(\theta_{r-\tau}\omega) + I_0(\theta_{r-\tau}\omega))\|\psi(\tau - t)\|^2_{X_\gamma}e^{2\int_{\tau-t}^{r}C_0(\theta_{s-\tau}\omega)ds},
\end{aligned} \tag{4.11}$$

where

$$M_2(\omega) = \frac{2}{M}(k_5 + k_4(1 + |a_1\eta_1(\theta_{r-\tau}\omega)|) + 4\mu_0|a_1\eta_1(\theta_{r-\tau}\omega)|). \tag{4.12}$$

Multiplying (4.11) with $e^{\sigma_0-\rho(\theta_{r-\tau}\omega)}$, and integrating the inequality on $[\tau-t,\tau]$, we obtain

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}\mu(\frac{|j|}{M})\|\psi_j(\tau)\|^2_{X_\gamma}\le\sum_{j\in\mathbb{Z}}\mu(\frac{|j|}{M})\|\psi_j(\tau-t)\|^2_{X_\gamma}e^{2\int_{\tau-t}^{\tau}(\sigma_0+\rho(\theta_{s-\tau}\omega))ds}+(\frac{1}{J}+\frac{1}{J^2}+F_J)\\&\times\|\psi(\tau-t)\|^2_{X_\gamma}\int_{\tau-t}^{\tau}k_{23}(M_2(\theta_{r-\tau}\omega)+I_0(\theta_{r-\tau}\omega))e^{2\int_{\tau-t}^{r}C_0(\theta_{s-\tau}\omega)ds}e^{\int_r^{\tau}(-\sigma_0+\rho(\theta_{s-\tau}\omega))ds}dr\\&\le e^{\int_{-t}^{0}(-\sigma_0+\rho(\theta_s\omega))ds}\|\psi(\tau-t)\|^2_{X_\gamma}+(\frac{1}{J}+\frac{1}{J^2}+F_J)\|\psi(\tau-t)\|^2_{X_\gamma}\\&\times e^{\int_{-t}^{0}(2C_0(\theta_s\omega)+\rho(\theta_s\omega))ds}\int_{-t}^{0}k_{27}(M_2(\theta_r\omega)+I_0(\theta_r\omega))e^{\sigma_0 r}dr.\end{aligned}\tag{4.13}$$

Since for all $p\ge0,\ \sqrt{p}\le e^p$, it follows that

$$\begin{aligned}&\int_{-t}^{0}k_{27}(M_2(\theta_r\omega)+I_0(\theta_r\omega))e^{\sigma_0 r}dr\\&\le(\int_{-t}^{0}k_{27}^2(M_2(\theta_r\omega)+I_0(\theta_r\omega))^2dr)^{\frac12}(\int_{-t}^{0}e^{2\sigma_0 r}dr)^{\frac12}\\&\le\frac{1}{\sqrt{2\sigma_0}}e^{\int_{-t}^{0}k_{27}^2(M_2(\theta_r\omega)+I_0(\theta_r\omega))^2dr}.\end{aligned}$$

By (4.13), for $J\ge2,\ M\ge2J$,

$$\begin{aligned}&\sum_{|j|>4J}\|\psi_j(\tau)\|^2_{X_\gamma}\le\sum_{j\in\mathbb{Z}}\mu(\frac{|j|}{M})\|\psi_j(\tau)\|^2_{X_\gamma}\\&\le(e^{-\sigma_0 t+\int_{-t}^{0}\rho(\theta_s\omega)ds}+(\frac{\delta_J}{2})^2e^{\int_{-t}^{0}2C_1(\theta_s\omega)ds})\|\psi(\tau-t)\|^2_{X_\gamma},\end{aligned}\tag{4.14}$$

where

$$C_1(\omega)=C_0(\omega)+\frac12\rho(\omega)+\frac12k_{27}^2(M_2(\omega)+I_0(\omega))^2.\tag{4.15}$$

Thus, (4.5) holds, and we have completed the proof.

**Lemma 4.2.** *Let $a_1,a_2$ in (1.1) be suitably small and satisfy (h3), then*

$$0\le\mathbf{E}[\frac12\rho(\omega)]\le\frac{\sigma_0}{64},\ \ 0\le\mathbf{E}[C_1(\omega)],\ \mathbf{E}[C_1^2(\omega)]<\infty.$$

*Proof.* It follows from [5,7] that the Ornstein-Uhlenbeck process $\eta_i(\theta_t\omega),\ i=1,2$, satisfies

$$\mathbf{E}[|\eta_i(\theta_s\omega)|^r]=\frac{\Gamma(\frac{1+r}{2})}{\sqrt{\pi}},\ \forall r\ge0,\begin{cases}\mathbf{E}[e^{\epsilon\int_\tau^{\tau+t}|\eta_i(\theta_s\omega)|^2ds}]\le e^{\epsilon t},\ 0<2\epsilon\le1,\\\mathbf{E}[e^{\epsilon\int_\tau^{\tau+t}|\eta_i(\theta_s\omega)|ds}]\le e^{\epsilon t},\ 0<\epsilon^2\le1,\end{cases}\tau\in\mathbb{R},\ t\ge0,\ i=1,2,$$

where $\Gamma(\cdot)$ is the Gamma function.
Thus, we have

$$\mathbf{E}[\eta_1^4(\omega)]=\frac34,\ \mathbf{E}[\eta_i^2(\omega)]=\frac12,\ \mathbf{E}[\eta_i(\omega)]=\frac{1}{\sqrt{\pi}},\ i=1,2,\tag{4.16}$$

and

$$\mathbf{E}[\frac{1}{2}\rho(\omega)] = \frac{|a_1|}{2\sqrt{\pi}}(5 + \frac{|2\delta + 1 - \alpha|}{\sqrt{\gamma}}) + \frac{a_1^2}{4\sqrt{\gamma}} + \frac{|a_2|}{\sqrt{\pi}} \le \frac{\sigma_0}{64}. \tag{4.17}$$

By (4.9), (4.12), and (4.15), we have

$$\begin{aligned} C_1^2(\omega) &\le 4C_0^2(\omega) + \rho^2(\omega) + 4k_{27}^4 M_2^4(\omega) + 4k_{27}^4 I_0^4(\omega) \\ &\le k_{28}\eta_1^4(\omega) + k_{29}\eta_1^2(\omega) + k_{30}\eta_2^2(\omega) + k_{31}I_0^2(\omega) + k_{32}I_0^4(\omega) + k_{33}. \end{aligned}$$

By the Hölder inequality, we obtain

$$\begin{aligned} \mathbf{E}[I_0^4(\omega)] &= \mathbf{E}(\int_{-\infty}^0 e^{\sigma_0 s + \int_s^0 \rho(\theta_\tau\omega)d\tau} ds)^4 \\ &\le (\int_{-\infty}^0 e^{\sigma_0 s} ds)^3 \int_{-\infty}^0 e^{\sigma_0 s}\mathbf{E}[e^{4\int_s^0 \rho(\theta_\tau\omega)d\tau}]ds \\ &\le \frac{1}{\sigma_0^3}(\frac{1}{\sigma_0 - 8|a_1|(5 + \frac{|2\delta+1-\alpha|}{\sqrt{\gamma}})} + \frac{1}{\sigma_0 - 16|a_2|} + \frac{1}{\sigma_0 - \frac{4a_1^2}{\sqrt{\gamma}}}) \\ &< \infty. \end{aligned}$$

Thus,

$$0 \le \mathbf{E}[I_0^2(\omega)] \le \frac{1}{2}(1 + \mathbf{E}[I_0^4(\omega)]) < \infty,$$

$$\mathbf{E}[C_1^2(\omega)] \le k_{28}\mathbf{E}[\eta_1^4(\omega)] + k_{29}\mathbf{E}[\eta_1^2(\omega)] + k_{30}\mathbf{E}[\eta_2^2(\omega)] + k_{31}\mathbf{E}[I_0^2(\omega)] + k_{32}\mathbf{E}[I_0^4(\omega)] + k_{33} < \infty,$$

$$0 \le \mathbf{E}[C_1(\omega)] \le \frac{1}{2}(1 + \mathbf{E}[C_1^2(\omega)]) < \infty.$$

Lemma 4.2 is proved.

**Lemma 4.3.** *For each $\tau \in \mathbb{R},\ \omega \in \Omega$,*

$$\begin{cases} \lim\limits_{t\to 0^+} \sup\limits_{\varphi\in\mathcal{S}(\tau,\omega)} \|\Psi(t,\tau,\omega)\varphi - \varphi\|_{X_\gamma} = 0, \\ \lim\limits_{t\to 0^+} \sup\limits_{\varphi\in\mathcal{S}(\tau-t,\theta_{-t}\omega)} \|\Psi(0,\tau-t,\theta_{-t}\omega)\varphi - \varphi\|_{X_\gamma} = 0. \end{cases}$$

*Proof.* For $\varphi \in \mathcal{S}(\tau,\omega)$ and $t \ge 0$,

$$\begin{aligned} \|H(\theta_{r-\tau}\omega,\varphi)\|_{X_\gamma}^2 &= |a_1\eta_1(\theta_{r-\tau}\omega)|^2(\|Bu\|^2 + \gamma\|u\|^2) + \|i\widetilde{w}u - ihe^{ia_2\eta_2(\theta_{r-\tau}\omega)} - i\widetilde{w}a_2\eta_2(\theta_{r-\tau}\omega)\|^2 \\ &+ \|g + \nu|\widetilde{w}|^2 + (2\delta u + u - \alpha u - v)a_1\eta_1(\theta_{r-\tau}\omega) - ua_1^2\eta_1^2(\theta_{r-\tau}\omega)\|^2 \\ &\le k_{34}(|\eta_1(\theta_{r-\tau}\omega)|^2 + |\eta_1(\theta_{r-\tau}\omega)|^4 + |\eta_2(\theta_{r-\tau}\omega)|^2)R^2(\theta_{r-\tau}\omega) + k_{35}R^4(\theta_{r-\tau}\omega) + k_{36}, \end{aligned}$$

$$\begin{aligned} \|\Theta\varphi(r)\|_{X_\gamma}^2 &\le \|\delta u - v\|_\gamma^2 + \|Au + (\gamma + \delta^2 - \delta\alpha)u + (\alpha - \delta)v\|^2 + \|iA\widetilde{w} + \beta\widetilde{w}\|^2 \\ &\le k_{37}R^2(\theta_{r-\tau}\omega), \end{aligned}$$

and therefore,

$$\|\Psi(t,\tau,\omega)\varphi - \varphi\|_{X_\gamma}^2 \le t\int_\tau^{\tau+t} \|H(\theta_{r-\tau}\omega,\varphi) - \Theta\varphi(r)\|_{X_\gamma}^2 dr$$

$$\leq 2t\int_0^t [k_{34}(|\eta_1(\theta_r\omega)|^2 + |\eta_1(\theta_r\omega)|^4 + |\eta_2(\theta_r\omega)|^2)R^2(\theta_r\omega) + k_{36}]dr$$
$$+ 2t\int_0^t (k_{35}R^4(\theta_r\omega) + k_{37}R^2(\theta_r\omega))dr \to 0 \ \ as \ \ t \to 0.$$

Thus, $\lim\limits_{t\to 0^+} \sup\limits_{\varphi\in\mathcal{S}(\tau,\omega)} \|\Psi(t,\tau,\omega)\varphi - \varphi\|_{X_\gamma} = 0.$

Similarly, $\lim\limits_{t\to 0^+} \sup\limits_{\varphi\in\mathcal{S}(\tau-t,\theta_{-t}\omega)} \|\Psi(0,\tau-t,\theta_{-t}\omega)\varphi - \varphi\|_{X_\gamma} = 0.$

We have completed the proof.

According to Theorems 2.1–2.4 in [11] and Theorem 2.1 in [12], noting that the phase space contains complex components, we obtain the following result.

**Theorem 4.1.** *The cocycle* $\{\Psi(t,\tau,\omega) : \ t \geq 0, \tau \in \mathbb{R}, \omega \in \Omega\}$ *has a random exponential attractor* $\{\mathcal{A}(\tau,\omega) : \ \tau \in \mathbb{R}, \omega \in \Omega\}$ *with the following properties: for each* $\tau \in \mathbb{R}, \omega \in \Omega$,
(*i*) $\mathcal{K}(\tau,\omega) \subseteq \mathcal{A}(\tau,\omega) \subseteq \mathcal{S}(\tau,\omega)$ *is a compact set of* $X_\gamma$ *and measurable in* $\omega$;
(*ii*) $\Psi(t,\tau,\omega)\mathcal{A}(\tau,\omega) \subseteq \mathcal{A}(t+\tau,\omega)$, $\forall t \geq 0$;
(*iii*) *there exists* $J_0 \in \mathbb{N}$ *such that*

$$dim_f\mathcal{K}(\tau,\omega) \leq dim_f\mathcal{A}(\tau,\omega) \leq \frac{6(8J_0+1)\ln(\frac{2\sqrt{6(8J_0+1)}}{\delta_J}+1)}{\ln\frac{4}{3}} < \infty;$$

(*iv*) *for any* $D = \mathcal{D}(X_\gamma)$, *there exist* $t_{D(\tau,\omega)} \geq 0$ *and a tempered random variable* $b_{D(\tau,\omega)} > 0$, *such that for any* $t \geq t_{D(\tau,\omega)}$ *they satisfy*

$$d_h(\Psi(t,\tau,\omega)D(\tau,\omega), \mathcal{A}(\tau+t,\theta_t\omega)) \leq b_{D(\tau,\omega)}e^{-\frac{\sigma_0\ln\frac{4}{3}}{64\ln 2}t};$$

(*v*) *for every* $\tau \in \mathbb{R}, \omega \in \Omega$, $\lim\limits_{t\to 0} d_h(\mathcal{A}(\tau+t,\theta_t\omega), \mathcal{A}(\tau,\omega)) = 0.$

*Proof.* Taking $t = \frac{16\ln 2}{\sigma_0}$ in (4.5), by Lemma 4.2,

$$0 < e^{-\frac{2}{\ln\frac{3}{2}}(\frac{16\ln 2}{\sigma_0})^2(\mathbf{E}[C_1^2(\omega)]+\frac{\sigma_0}{2}\mathbf{E}[C_1(\omega)])} < \infty.$$

By (h2) and $\lim\limits_{J\to+\infty}(\frac{1}{J}+\frac{1}{J^2}) = 0$, there exists a finite integer $J_0 \in \mathbb{N}$ satisfying

$$0 < \delta_{J_0} = \frac{2}{\sqrt[4]{2\sigma_0}}\sqrt{\frac{1}{J_0}+\frac{1}{J_0^2}+F_{J_0}} \leq \min\{\frac{1}{16}, \ e^{-\frac{2}{\ln\frac{3}{2}}(\frac{16\ln 2}{\sigma_0})^2(\mathbf{E}[C_1^2(\omega)]+\frac{\sigma_0}{2}\mathbf{E}[C_1(\omega)])}\}.$$

According to Theorem 2.1 in [12], and then combining Lemmas 4.1–4.3 and Theorem 3.1, we prove Theorem 4.1.

**Remark 4.1.** We notice that there are some restrictive conditions for the random term coefficients $a_1$ and $a_2$ in order to prove the boundedness of the expectation of the random variables $\rho(\omega), C_1(\omega)$, and $C_1^2(\omega)$. However, using the same method, we can also prove that the existence of a random exponential attractor for the coupled KGS lattice equations with additive noise:

$$\begin{cases} d\dot{u}_j + \alpha du_j + ((Au)_j - \gamma u_j - \nu|w_j|^2)dt = g_j(t)dt + a_jdW_1, \\ idw_j + (-(Aw)_j + i\beta w_j + w_ju_j)dt = h_j(t)dt + b_jdW_2, \qquad j \in \mathbb{Z}, \\ u_j(\tau) = u_{j\tau}, \dot{u}_j(\tau) = u_{1,j\tau}, w_j(\tau) = w_{j\tau}, \tau \in \mathbb{R}, \end{cases} \tag{4.18}$$

where $a = (a_j)_{j\in\mathbb{Z}} \in l^2, b = (b_j)_{j\in\mathbb{Z}} \in l^2$, and $g_j, h_j$ satisfy (h2). For the system (4.18), we do not need to make the restrictive conditions on the $a_j$ and $b_j$ because the additive noise term is independent of the state variables $u_j$ and $w_j$ but the multiplicative noise term depends on $u_j$ and $w_j$.

## Use of AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The author declares that there is no conflict of interest.

## References

1. B. Wang, R. Wang, Asymptotic behavior of stochastic Schrödinger lattice systems driven by nonlinear noise, *Stochastic Anal. Appl.*, **38** (2020), 213–237. http://doi.org/10.1080/07362994.2019.1679646
2. P. W. Bates, H. Lisei, K. Lu, Attractors for stochastic lattice dynamical systems, *Stochastics Dyn.*, **6** (2006), 1–21. https://doi.org/10.1142/S0219493706001621
3. P. W. Bates, K. Lu, B. Wang, Attractors of non-autonomous stochastic lattice systems in weighted spaces, *Phys. D: Nonlinear Phenom.*, **289** (2014), 32–50. http://doi.org/10.1016/j.physd.2014.08.004
4. T. Caraballo, F. Morillas, J. Valero, Attractors of stochastic lattice dynamical systems with a multiplicative noise and non-Lipschitz nonlinearities, *J. Differ. Equations*, **253** (2012), 667–693. http://doi.org/10.1016/j.jde.2012.03.020
5. X. Fan, Attractors for a damped stochastic wave equation of Sine-Gordon type with sublinear multiplicative noise, *Stochastic Anal. Appl.*, **24** (2006), 767–793. http://doi.org/10.1080/07362990600751860
6. L. Arnold, *Random Dynamical Systems*, Springer, 1998. https://doi.org/10.1007/978-3-662-12878-7
7. X. Fan, Random attractors for damped stochastic wave equations with multiplicative noise, *Int. J. Math.*, **19** (2008), 421–437. https://doi.org/10.1142/S0129167X08004741
8. B. Wang, Sufficient and necessary criteria for existence of pullback attractors for non-compact random dynamical systems, *J. Differ. Equations*, **253** (2012), 1544–1583. http://doi.org/10.1016/j.jde.2012.05.015

9. A. Shirikyan, S. Zelik, Exponential attractors for random dynamical systems and applications, *Stochastic Partial Differ. Equations: Anal. Comput.*, **1** (2013), 241–281. https://doi.org/10.1007/s40072-013-0007-1
10. T. Caraballo, S. Sonner, Random pullback exponential attractors: General existence results for random dynamical systems in Banach spaces, *Discrete Contin. Dyn. Syst.*, **37** (2017), 6383–6403. http://doi.org/10.3934/dcds.2017277
11. S. Zhou, Random exponential attractor for cocycle and application to non-autonomous stochastic lattice systems with multiplicative white noise, *J. Differ. Equations*, **263** (2017), 2247–2279. http://doi.org/10.1016/j.jde.2017.03.044
12. H. Su, S. Zhou, L. Wu, Random exponential attractor for second-order nonautonomous stochastic lattice systems with multiplicative white noise, *Stochastics Dyn.*, **19** (2019), 1950044. http://doi.org/10.1142/S0219493719500448
13. X. Tan, F. Yin, G. Fan, Random exponential attractor for stochastic discrete long wave-short wave resonance equation with multiplicative white noise, *Discrete Contin. Dyn. Syst. Ser. B*, **25** (2020), 3153–3170. http://doi.org/10.3934/dcdsb.2020055
14. P. Biler, Attractors for the system of Schrödinger and Klein-Gordon equations with Yukawa coupling, *SIAM J. Math. Anal.*, **21** (1990), 1190–1212. http://doi.org/10.1137/0521065
15. K. Lu, B. Wang, Global attractors for the Klein-Gordon-Schrödinger equation in unbounded domains, *J. Differ. Equations*, **170** (2001), 281–316. http://doi.org/10.1006/jdeq.2000.3827
16. A. Y. Abdallah, Asymptotic behavior of the Klein-Gordon-Schrödinger lattice dynamical systems, *Commun. Pure Appl. Anal.*, **5** (2006), 55–69. http://doi.org/10.3934/cpaa.2006.5.55
17. S. Missaoui, Regularity of the attractor for a coupled Klein-Gordon-Schrödinger system in $\mathbb{R}^3$ nonlinear KGS system, *Commun. Pure. Appl. Anal.*, **21** (2022), 567–584. http://doi.org/10.3934/cpaa.2021189
18. W. Yan, S. Ji, Y. Li, Random attractors for stochastic discrete Klein-Gordon-Schrödinger equations, *Phys. Lett. A*, **373** (2009), 1268–1275. https://doi.org/10.1016/j.physleta.2009.02.019
19. J. Shu, Random attractors for stochastic discrete Klein-Gordon-Schrödinger equations driven by fractional Brownian motions, *Discrete Contin. Dyn. Syst. Ser. B*, **22** (2017), 1587–1599. http://doi.org/10.3934/dcdsb.2017077
20. A. Ban, K. Zhou, Asymptotic behavior of solutions of a stochastic lattice systems, *Acta Math. Appl. Sin.*, **45** (2022), 821–837.
21. I. Chueshov, *Monotone Random systems Theory and Applications*, Springer, 2002. https://doi.org/10.1007/b83277