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Research article

Existence of globally bounded solutions for a predator-prey system with indirect taxis-driven pursuit-evasion and parabolic-elliptic signal

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Abstract: In this paper, we consider a cross-diffusive model for pursuit-evasion processes which was described as

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u(u+1)^{r-1}\nabla w) + u(\lambda_1 - \mu_1 u^{r_1-1} + av), & x\in\Omega, t>0,\\ v_t = \Delta v + \xi\nabla\cdot(v\nabla z) + v(\lambda_2 - \mu_2 v^{r_2-1} - bu), & x\in\Omega, t>0,\\ w_t = \Delta w - w + v, & x\in\Omega, t>0,\\ 0 = \Delta z - z + u, & x\in\Omega, t>0 \end{cases}$$

in a smooth bounded domain $\Omega\subset\mathbb{R}^N$ $(N\ge 1)$ under homogeneous Neumann boundary conditions. For nonnegative initial data with appropriate regularity, the unique globally bounded classical solution is obtained provided that $\min\{(r_1-1)(r_2-1),(r_1-r)(r_2-1)\} > \frac{(N-2)_+}{N}$, where $r,\lambda_i,\mu_i,\chi,\xi > 0$ and $r_i > 1 (i=1,2)$. This generalizes the results of Li et al. [1], Zheng et al. [2], and Zheng and Zhang [3].

Keywords: pursuit-evasion; global existence; parabolic-elliptic; boundedness

1. Introduction

In this paper, we consider a predator-prey model with pursuit-evasion (parabolic-elliptic)

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u(u+1)^{r-1}\nabla w) + u(\lambda_1 - \mu_1 u^{r_1-1} + av), & x\in\Omega, t>0,\\ v_t = \Delta v + \xi\nabla\cdot(v\nabla z) + v(\lambda_2 - \mu_2 v^{r_2-1} - bu), & x\in\Omega, t>0,\\ w_t = \Delta w - w + v, & x\in\Omega, t>0,\\ 0 = \Delta z - z + u, & x\in\Omega, t>0 \end{cases} \tag{1.1}$$

supplemented with Neumann boundary conditions

$$\frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = \frac{\partial w}{\partial\nu} = \frac{\partial z}{\partial\nu} = 0, \quad x\in\partial\Omega,\ t>0 \tag{1.2}$$

and the initial conditions

$$u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ w(x,0) = w_0(x), \quad x \in \Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $\frac{\partial}{\partial\nu}$ denotes the outward normal derivative on $\partial\Omega$, r, λ_i, μ_i, ξ, χ are positive constants and $r_i > 1 (i = 1, 2)$. The initial data (u_0, v_0, w_0) satisfies

$$\begin{cases} u_0 \in C^0(\bar{\Omega}) \text{ with } u_0 \geq 0 \text{ in } \bar{\Omega}, \\ v_0 \in C^0(\bar{\Omega}) \text{ with } v_0 \geq 0 \text{ in } \bar{\Omega}, \\ w_0 \in C^0(\bar{\Omega}) \text{ with } w_0 \geq 0 \text{ in } \bar{\Omega}. \end{cases} \tag{1.4}$$

This coupled nonlinear framework governs the spatiotemporal evolution of interacting predator and prey population densities through advective-reaction mechanisms. Obviously, the predator-prey populations interact with each other. Otherwise the individuals always move toward a chemical signal that is produced by opposing species. In fact, the unknown functions $u = u(x,t)$ and $v = v(x,t)$ denote densities of the predator and prey population, respectively, while $w = w(x,t)$ and $z = z(x,t)$ represent concentrations of respectively emitted chemicals produced by $v(x,t)$ and $u(x,t)$. The cross-trophic coupling demonstrates an asymmetric energetic transfer structure: prey populations experience consumption-driven suppression through predator-induced mortality, while predator demographic rates exhibit nonlinear amplification governed by prey resource availability. The model in [4] posits predator-prey interactions mediated by species-specific chemical cues (e.g., pheromones/scent marks) through chemosensory signaling dynamics.

In grounding our analytical approach, we survey existing literature pertinent to the coupled system (1.1). The most well-known chemotaxis model is the Keller-Segel system, which was proposed by Keller and Segel [5] in 1970s, which can be described as follows:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v), \\ v_t = \Delta v + u - v, \end{cases} \tag{1.5}$$

where $u(x,t)$ and $v(x,t)$ denote the concentrations of cells and the chemoattractant, respectively, and $\chi > 0$ represents the chemoattractant sensitivity. This PDE framework captures autochemotactic phenomena in biosystems, where microorganisms exhibit directed migration along self-generated chemical gradients, coupled with nonlinear signal transduction mechanisms governed by Fickian diffusion processes. Substantial analytical investigations have focused on this logistic-modified chemotaxis system and its extensions, particularly addressing solution regularity, finite-time blow-up dynamics, and asymptotic stabilization under parameter constraints. For example, Winkler [6] studied the fully parabolic Keller-Segel system and obtained the global bounded solution and the large time behavior. Moreover, Winkler [7] considered the Neumann initial boundary value problem for a fully parabolic system and proved that for some initial data, the corresponding solution would blow up in finite time, and the essentially explicit blow-up criterion was obtained. For related comprehensive exposition of the model's theoretical foundations, readers are referred to the seminal works in [8–12] and their extended bibliographic networks.

Significant scholarly interest has also been directed toward two-species predator-prey systems. In fact, assuming that predators and preys exert species-characteristic substances such as phenomenons or scent marks, Tyutyunov et al. [4] proposed the pursuit-evasion model

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla w) + f(u,v), & x\in\Omega, t>0,\\ v_t = \Delta v + \xi\nabla\cdot(v\nabla z) + g(u,v), & x\in\Omega, t>0,\\ \tau w_t = \Delta w - w + v, & x\in\Omega, t>0,\\ \tau z_t = \Delta z - z + u, & x\in\Omega, t>0, \end{cases} \tag{1.6}$$

where $\tau\in\{0,1\}$, $f(u,v)$ and $g(u,v)$ represent the local kinetics. For the degenerated parabolic-elliptic coupling ($\tau = 0$), analysis establishes that the two-dimensional Cauchy problem with vanishing source terms ($f = g = 0$) admits a unique nonnegative weak solution exhibiting uniform boundedness, as rigorously demonstrated in Goudon and Urrutia's foundational work [13]. A functional-analytic framework proposed by Amorim et al. [14] was a seminal contribution, where Lebesgue space uniform estimates for the two-dimensional system with competitive Lotka-Volterra kinetics ($f = u(\lambda - u + av)$, $g = v(\mu - v - bu)$) were established, thereby demonstrating global well-posedness and uniform boundedness of solutions. Recent analytical advances have rigorously established globally defined weak solutions with uniform asymptotic stabilization within 1D bounded domains [15,16], critically advancing the theoretical framework for cross-trophic interaction systems. For the parabolic-elliptic system (1.6) with competitive Lotka-Volterra kinetics ($f = u(\lambda - u + av)$, $g = v(\mu - v - bu)$), Li et al. [1] established globally bounded classical solutions in spatial dimensions $N \le 3$ under the interspecific competition constraint $b > 0$. Furthermore, under critical parameter constraints governing taxis sensitivities (χ, ξ) and growth rates (μ, λ), the system exhibits asymptotic convergence to either spatially homogeneous coexistence equilibria or prey-depleted steady states in the long-time regime. The theoretical understanding of the fully parabolic coupling ($\tau = 1$ in (1.6)) remains markedly underdeveloped compared to its parabolic-elliptic counterpart ($\tau = 0$ in (1.6)). In a seminal contribution, Qiu et al. [17] established rigorous existence theorems for globally bounded classical solutions to two-chemical-mediated predator-prey chemotaxis systems in spatial dimensions $N \le 3$. Their pioneering analysis further characterizes asymptotic stabilization properties through detailed energy estimates. Under the critical parameter regime $\tau = 1$ with competitive Lotka-Volterra kinetics ($f = u(\lambda - u + av)$ with $a < 2$; $g = v(\mu - v - bu)$), Qi and Ke [18] established the existence of globally bounded classical solutions when the chemotactic sensitivity constraint

$$\frac{N(2-a)}{2(C_{\frac{N}{2}+1})^{\frac{1}{\frac{N}{2}+1}}(N-2)_+} > \max\{\chi,\xi\}$$

holds for $C_{\frac{N}{2}+1} > 0$, significantly advancing the analysis of fully parabolic predator-prey taxis systems. Furthermore, subject to the critical parameter inequality $b\mu < \lambda$ and precise upper bounds on chemotactic sensitivities χ, ξ, the system asymptotically converges to a spatially homogeneous positive equilibrium in $L^\infty(\Omega)$ as $t\to\infty$. In the dynamical regime where $b\mu > \lambda$, subject to quantitative upper bounds on predator taxis sensitivity χ (without a priori constraints on prey taxis ξ), all uniformly bounded solutions undergo asymptotic collapse to the prey-extinction equilibrium $(\mu, 0, 0, \mu)$ under the $L^\infty(\Omega)$ topology as $t\to\infty$. For the other related models, one can refer to [3, 19–21] and the references therein.

Building upon these theoretical advancements, we focus on the indirect taxis-mediated pursuit-evasion framework (1.1) under parabolic-elliptic coupling. Our objective centers on establishing the existence of globally bounded classical solutions to the initial-boundary value problem (1.1)–(1.4) through rigorous analytical criteria governing taxis sensitivity parameters.

Main results. This work establishes that for arbitrary $N \geq 1$, provided the parameters satisfy $r_1 > 1$, $r_2 > 1$, and $\min\{(r_1-1)(r_2-1),(r_1-r)(r_2-1)\} > \frac{(N-2)_+}{N}$ holds for $r > 0$, then the solution quadruple (u, v, w, z) associated with system (1.1)–(1.4) exhibits global temporal existence and boundedness. The mathematical framework employs an innovative variable upper limit integration methodology to achieve the following regularity results.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be a bounded domain with smooth boundary. Assume that the initial data (u_0, v_0, w_0) fulfills* (1.4)*, and r, λ_i, μ_i, χ, ξ, a, b are positive constants as well as $r_1, r_2 > 1$. If*

$$\min\{(r_1-1)(r_2-1),(r_1-r)(r_2-1)\} > \frac{(N-2)_+}{N},$$

then the system (1.1)–(1.4) *admits a unique globally defined classical solution satisfying*

$$\begin{cases} u \in C^0(\bar{\Omega}\times[0,\infty)) \cap C^{2,1}(\bar{\Omega}\times(0,\infty)), \\ v \in C^0(\bar{\Omega}\times[0,\infty)) \cap C^{2,1}(\bar{\Omega}\times(0,\infty)), \\ w \in C^0(\bar{\Omega}\times[0,\infty)) \cap C^{2,1}(\bar{\Omega}\times(0,\infty)), \\ z \in C^0(\bar{\Omega}\times[0,\infty)) \cap C^{2,0}(\bar{\Omega}\times(0,\infty)). \end{cases} \tag{1.7}$$

Furthermore, the solution quadruple (u, v, w, z) remains uniformly bounded in $\Omega\times(0,\infty)$, and there exists a positive constant $C > 0$ satisfying

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|z(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq C \text{ for all } t > 0.$$

Remark 1.1. *For any $N \geq 1$ with $r = 1$, Theorem 1.1's condition can be reduced to $(r_1-1)(r_2-1) > \frac{(N-2)_+}{N}$, which the well-posedness result in this case was obtained in [2,3]. Moreover, we also generalize the previous results of [1] with $r_1 = r_2 = 2$ and $N \leq 3$.*

We begin by introducing key notations employed throughout. The symbols C and C_i ($i = 1, 2, 3, \cdots$) will denote generic positive constants, with values potentially varying across instances. Furthermore, for notational brevity, spatial integrals $\int_\Omega u(x)dx$ are abbreviated as $\int_\Omega u$, while temporal-spatial dependencies $u(x, t)$ are simplified to u where unambiguous.

The methodological architecture of this analysis proceeds as follows: Section 2 establishes the local well-posedness of classical solutions and synthesizes prerequisite technical lemmas. Section 3 derives foundational a priori estimate through elementary energy functional techniques, concurrently furnishing the complete demonstration of the central theorem within this unified framework.

2. Preliminaries

In this section, we first give the local existence and uniqueness of classical solutions to system (1.1). As the proof is quite standard, readers can refer to [3, 22] for detailed proof.

Lemma 2.1. *Suppose that $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded domain with a smooth boundary. Assume that the initial data (u_0, v_0, w_0) satisfies* (1.4)*. Then system* (1.1) *has a unique nonnegative classical* *solution*

$$\begin{cases} u \in C_0(\overline{\Omega} \times [0, T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max})), \\ v \in C_0(\overline{\Omega} \times [0, T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max})), \\ w \in C_0(\overline{\Omega} \times [0, T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max})), \\ z \in C_0(\overline{\Omega} \times [0, T_{max})) \cap C^{2,0}(\overline{\Omega} \times (0, T_{max})), \end{cases} \tag{2.1}$$

where T_{max} denotes the maximal existence time. Moreover, if $T_{max} < +\infty$, then

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)} \to \infty \quad as \quad t \nearrow T_{max}.$$

The Gagliardo-Nirenberg inequality is central to our proof of the main theorem.

Lemma 2.2. *(Gagliardo-Nirenberg inequality [9]) Let Ω be a bounded Lipschitz domain, p, q, r, s≥ 1, j, m $\in \mathbb{N}_0$ and $\alpha \in [\frac{j}{m}, 1]$ satisfying*

$$\frac{1}{p} = \frac{j}{N} + \left(\frac{1}{r} - \frac{m}{N}\right)\alpha + \frac{1-\alpha}{q}.$$

Then there are positive constants C_{GN} and G'_{GN} such that for all functions $w \in L^q(\Omega)$ with $\nabla w \in L^r(\Omega)$, $w \in L^s(\Omega)$,

$$\|D^j w\|_{L^p(\Omega)} \leq C_{GN}\|D^m w\|^{\alpha}_{L^r(\Omega)}\|w\|^{1-\alpha}_{L^q(\Omega)} + C'_{GN}\|w\|_{L^s(\Omega)}.$$

We need to describe the following lemma, which will be used in the proof of Theorem 1.1.

Lemma 2.3. *(see [23]) Suppose that $\gamma \in (1, \infty)$, $g \in L^\gamma((0, T); L^\gamma(\Omega))$, and c is a solution of the following initial problem:*

$$\begin{cases} c_t = \Delta c - c + g, \\ \dfrac{\partial c}{\partial \nu} = 0, \\ c(x, 0) = c_0(x). \end{cases}$$

Then there exists a positive constant C_γ such that if $s_0 \in [0, T)$ and $c(\cdot, s_0) \in W^{2,\gamma}(\Omega)$ with $\frac{\partial c(\cdot, s_0)}{\partial \nu} = 0$, one has

$$\begin{aligned} &\int_{s_0}^{T} e^{\gamma s}\|\Delta c(\cdot, s)\|^{\gamma}_{L^\gamma(\Omega)} ds \\ \leq &C_\gamma \left(\int_{s_0}^{T} e^{\gamma s}\|g(\cdot, s)\|^{\gamma}_{L^\gamma(\Omega)} ds + e^{\gamma s_0}(\|c_0(\cdot, s_0)\|^{\gamma}_{L^\gamma(\Omega)} + \|\Delta c_0(\cdot, s_0)\|^{\gamma}_{L^\gamma(\Omega)}) \right). \end{aligned}$$

We will employ the semigroup theory to prove Theorem 1.1.

Lemma 2.4. *(see [6]) Let $(e^{t\Delta})_{t\geq 0}$ be the Neumann heat semigroup in Ω, and let $\lambda_1 > 0$ denote the first nonzero eigenvalue of $-\Delta$ in Ω under Neumann boundary conditions. Then we have the following properties with constants k_1, k_2, and k_3 depending only on Ω.*
(i) If $p, q \in [1, +\infty]$, then

$$\|e^{t\Delta}\varphi\|_{L^p(\Omega)} \leq k_3(1 + t^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})})e^{-\lambda_1 t}\|\varphi\|_{L^q(\Omega)}, \quad for\ all\ t > 0 \tag{2.2}$$

is valid for all $\varphi \in L^q(\Omega)$ *fulfilling* $\int_\Omega \varphi dx = 0$.
(ii) If $1 \le q \le p \le \infty$, *then*

$$\|\nabla e^{t\Delta}\varphi\|_{L^p(\Omega)} \le k_2(1 + t^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})})e^{-\lambda_1 t}\|\varphi\|_{L^q(\Omega)}, \quad \textit{for all } t > 0 \tag{2.3}$$

is true for each $\varphi \in L^q(\Omega)$.
(iii) If $1 < q \le p < \infty$, *then*

$$\|e^{t\Delta}\nabla \cdot \varphi\|_{L^p(\Omega)} \le k_1(1 + t^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})})e^{-\lambda_1 t}\|\varphi\|_{L^q(\Omega)}, \quad \textit{for all } t > 0 \tag{2.4}$$

holds for all $\varphi \in (C_0^\infty(\Omega))^n$. *Accordingly, for all* $t > 0$, *the operator* $e^{t\Delta}\nabla\cdot$ *admits a unique extension to all of* $\varphi \in (L^p(\Omega))^n$, *with norm controlled according to* (2.4).

We also need the following ODE theory in the proof of our main result.

Lemma 2.5. *(see [22]) Let* $y(t) \ge 0$ *be a solution of the problem*

$$\begin{cases} y'(t) + Ay^p \le B, & t > 0, \\ y(0) = y_0 \end{cases}$$

with $A > 0$, $p > 0$, *and* $B \ge 0$. *Then for any* $t > 0$, *we have*

$$y(t) \le \max\left\{ y_0, \left(\frac{B}{A}\right)^{\frac{1}{p}} \right\}.$$

3. A priori estimate

In this section, we first give the L^1 estimate of u and v by using Young's inequality.

Lemma 3.1. *Invoking the hypotheses of Lemma 2.1, we rigorously establish the existence of a positive constant* $C > 0$ *ensuring that solutions to system* (1.1) *satisfy*

$$\int_\Omega u + \int_\Omega v \le C \quad \textit{for all } \ t \in (0, T_{max}). \tag{3.1}$$

Proof. Integrating by parts in the first and second equations of (1.1) respectively, one has

$$\frac{d}{dt}\int_\Omega u + \int_\Omega u = (\lambda_1 + 1)\int_\Omega u - \mu_1\int_\Omega u^{r_1} + a\int_\Omega uv \quad \text{for all } \ t \in (0, T_{max}) \tag{3.2}$$

and

$$\frac{d}{dt}\int_\Omega v + \int_\Omega v = (\lambda_2 + 1)\int_\Omega v - \mu_2\int_\Omega v^{r_2} - b\int_\Omega uv \quad \text{for all } \ t \in (0, T_{max}). \tag{3.3}$$

$b\times$(3.2) + $a\times$(3.3) and by Young's inequality, for $r_1 > 1$ and $r_2 > 1$, we can find $C_1 > 0$ fulfilling

$$\begin{aligned} &b\frac{d}{dt}\int_\Omega u + b\int_\Omega u + a\frac{d}{dt}\int_\Omega v + a\int_\Omega v \\ &\le -\frac{\mu_1 b}{2}\int_\Omega u^{r_1} - \frac{\mu_2 a}{2}\int_\Omega v^{r_2} + C_1 \quad \text{for all } \ t \in (0, T_{max}). \end{aligned} \tag{3.4}$$

Thus a standard ODE comparison argument implies

$$\int_\Omega u + \int_\Omega v \le C_2 \quad \text{for all } \ t \in (0, T_{max}) \tag{3.5}$$

with a constant $C_2 > 0$. The proof of Lemma 3.1 is completed. □

By applying Lemma 3.1 to equations about z and w, we rigorously demonstrate that these components exhibit enhanced regularity properties transcending basic L^1-boundedness.

Lemma 3.2. *Invoking the hypotheses posited in Lemma 2.1, the solution vector to system* (1.1) *fulfills*

$$\int_\Omega z^{l_0} + \int_\Omega w^{l_0} \le C \quad \textit{for all } t \in (0, T_{max}), \tag{3.6}$$

where $C > 0$ *and* $l_0 \in \left[1, \frac{N}{(N-2)_+}\right)$.

Proof. Utilizing Lemma 3.1 in conjunction with the equations of z and w, we invoke the seminal work of Brézis [23], along with the Minkowski inequality and Sobolev embedding theorem, to straightforwardly demonstrate Lemma 3.2. For the detailed proof, readers are referred to [3]. □

Establishing the global existence and uniform boundedness of classical solutions to system (1.1) necessitates deriving $L^p(\Omega)$-norm estimates as a foundational analytical prerequisite.

Lemma 3.3. *Let* $\Omega \subset \mathbb{R}^N$ $(N \ge 1)$ *be a bounded domain. Suppose that* $r_1 > 1$, $r_2 > 1$, $r > 0$. *If* $\min\{(r_1 - 1)(r_2 - 1), (r_1 - r)(r_2 - 1)\} > \frac{(N-2)_+}{N}$, *then one has a positive constant* $C > 0$ *such that*

$$\int_\Omega u^p + \int_\Omega v^p \le C \quad \textit{for all } \ t \in (0, T_{max}). \tag{3.7}$$

Proof. Let $p > 1$. Multiplying the second equation in (1.1) by v^{p-1} and integrating by parts, we have

$$\frac{1}{p}\frac{d}{dt}\int_\Omega v^p = \int_\Omega \Delta v \cdot v^{p-1} + \xi \int_\Omega v^{p-1}\nabla\cdot(v\nabla z) + \int_\Omega v^p(\lambda_2 - \mu_2 v^{r_2-1} - bu), \tag{3.8}$$

for all $t \in (0, T_{max})$.

Then, in light of $\Delta z = z - u$ and $\frac{\partial v}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0$, for all $t \in (0, T_{max})$, we can infer that

$$\int_\Omega \Delta v \cdot v^{p-1} = -(p-1)\int_\Omega v^{p-2}|\nabla v|^2 \tag{3.9}$$

and

$$\begin{aligned}\xi \int_\Omega v^{p-1}\nabla\cdot(v\nabla z) =& -\xi\frac{(p-1)}{p}\int_\Omega \nabla z\nabla v^p \\ =&\xi\frac{(p-1)}{p}\int_\Omega v^p\nabla z \\ =&\xi\frac{p-1}{p}\int_\Omega v^p z - \xi\frac{p-1}{p}\int_\Omega v^p u.\end{aligned} \tag{3.10}$$

Combining (3.8)–(3.10), we have

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega v^p+(p-1)\int_\Omega v^{p-2}|\nabla v|^2\\ =&\xi\frac{p-1}{p}\int_\Omega v^p z-\xi\frac{p-1}{p}\int_\Omega v^p u+\lambda_2\int_\Omega v^p-\mu_2\int_\Omega v^{p+r_2-1}-b\int_\Omega v^p u.\end{aligned} \tag{3.11}$$

Using Young's inequality, there exist constants $C_1, C_2 > 0$ such that

$$\xi\frac{p-1}{p}\int_\Omega v^p z\le\frac{\mu_2}{4}\int_\Omega v^{p+r_2-1}+C_1\int_\Omega z^{\frac{p+r_2-1}{r_2-1}} \tag{3.12}$$

and

$$\lambda_2\int_\Omega v^p\le\frac{\mu_2}{4}\int_\Omega v^{p+r_2-1}+C_2 \tag{3.13}$$

for all $t\in(0,T_{max})$.

Combining (3.12), (3.13) with (3.11), we can obtain

$$\frac{1}{p}\frac{d}{dt}\int_\Omega v^p\le-\frac{\mu_2}{2}\int_\Omega v^{p+r_2-1}+C_1\int_\Omega z^{\frac{p+r_2-1}{r_2-1}}+C_3\quad\text{for all }\ t\in(0,T_{max}) \tag{3.14}$$

with a positive constant $C_3 > 0$.

Let $s=\frac{p+r_2-1}{r_2-1}$, $r'=p+r_1-1$; by using the Gagliardo-Nirenberg inequality and Lemma 3.2, there exist positive constants C_4 and C_5 such that

$$\begin{aligned}C_1\|z\|^s_{L^s(\Omega)}&\le C_4(\|z\|^{s\theta}_{W^{2,r'}(\Omega)}\|z\|^{s(1-\theta)}_{L^{l_0}(\Omega)}+C_1\|z\|^s_{L^{l_0}(\Omega)})\\&\le C_5(\|z\|^{s\theta}_{W^{2,r'}(\Omega)}+1)\quad\text{for all }\ t\in(0,T_{max}),\end{aligned} \tag{3.15}$$

where $\theta=\frac{\frac{1}{l_0}-\frac{1}{s}}{\frac{2}{N}+\frac{1}{l_0}-\frac{1}{r'}}$, $l_0\in\left[1,\frac{N}{(N-2)_+}\right)$.

By using the L^p theory of elliptic equations, we have

$$\|z\|^{r'}_{W^{2,r'}(\Omega)}\le C_6\|u\|^{r'}_{L^{r'}(\Omega)}\quad\text{for all }\ t\in(0,T_{max}) \tag{3.16}$$

with $C_6 > 0$.

Due to

$$r_1>1,\, r_2>1,\, (r_1-1)(r_2-1)>\tfrac{(N-2)_+}{N}\ \text{ and }\ (r_1-r)(r_2-1)>\tfrac{(N-2)_+}{N},$$

we can conclude that

$$\frac{p+r_1-1}{r_1-1}<p+r_2-1 \tag{3.17}$$

and

$$\frac{p+r_1-1}{r_1-r}<p+r_2-1. \tag{3.18}$$

Furthermore, for any exponent $p > 1$, there exists a parameter $l_0\in\left[1,\frac{N}{(N-2)_+}\right)$, chosen within any prescribed proximity to $\frac{N}{(N-2)_+}$ such that

$$\frac{p+r_2-1}{r_2-1}<(p+r_1-1)\cdot l_0. \tag{3.19}$$

Based on the above estimate (3.19), we can derive that $s\theta \le r'$. Using the Young's inequality, there exist positive constants C_7, C_8, and C_9 such that

$$\begin{aligned} C_5(\|z\|^{s\theta}_{W^{2,r'}(\Omega)} + 1) \le& C_7\|z\|^{r'}_{W^{2,r'}(\Omega)} + C_8 \\ \le& C_6C_7\varepsilon\|u\|^{r'}_{L^{r'}(\Omega)} + C_9 \quad \text{for all } t \in (0, T_{max}). \end{aligned} \tag{3.20}$$

Taking $\varepsilon = \frac{\mu_1}{16C_6C_7}$, we have

$$C_1\|z\|^{s}_{L^{s}(\Omega)} \le \frac{\mu_1}{16}\|u\|^{r'}_{L^{r'}(\Omega)} + C_9 \quad \text{for all } t \in (0, T_{max}). \tag{3.21}$$

Therefore, we obtain

$$\frac{1}{p}\frac{d}{dt}\int_\Omega v^p \le -\frac{\mu_2}{2}\int_\Omega v^{p+r_2-1} + \frac{\mu_1}{16}\int_\Omega u^{p+r_1-1} + C_{10} \quad \text{for all } t \in (0, T_{max}) \tag{3.22}$$

with constant $C_{10} > 0$.

Let $H = \frac{p+r_1-1}{r_1-r}$; we can add $\frac{H}{p}\int_\Omega v^p$ on both sides of (3.22) such that

$$\frac{1}{p}\frac{d}{dt}\int_\Omega v^p + \frac{H}{p}\int_\Omega v^p \le -\frac{\mu_2}{2}\int_\Omega v^{p+r_2-1} + \frac{\mu_1}{16}\int_\Omega u^{p+r_1-1} + \frac{H}{p}\int_\Omega v^p + C_{10} \tag{3.23}$$

for all $t \in (0, T_{max})$.

Employing the Young's inequality, since $r_2 > 1$, then there is a positive constant $C_{11} > 0$ such that

$$\frac{H}{p}\int_\Omega v^p \le \frac{\mu_2}{4}\int_\Omega v^{p+r_2-1} + C_{11} \ \text{ for all } t \in (0, T_{max}). \tag{3.24}$$

Combining (3.23) with (3.24), there exists $C_{12} > 0$ such that

$$\frac{1}{p}\frac{d}{dt}\int_\Omega v^p + \frac{H}{p}\int_\Omega v^p \le -\frac{\mu_2}{4}\int_\Omega v^{p+r_2-1} + \frac{\mu_1}{16}\int_\Omega u^{p+r_1-1} + C_{12} \ \text{ for all } t \in (0, T_{max}). \tag{3.25}$$

According to a standard ODE comparison argument, let s_0 be the same as Lemma 2.3; there exists $C_{13} > 0$ such that

$$\int_\Omega v^p \le e^{-Ht}\left\{\int_{s_0}^t pe^{Hs}\Big(-\frac{\mu_2}{4}\int_\Omega v^{p+r_2-1} + \frac{\mu_1}{16}\int_\Omega u^{p+r_1-1} + C_{12}\Big) + C_{13}\right\} \tag{3.26}$$

for all $t \in (s_0, T_{max})$.

Next, multiplying both sides of the first equation in (1.1) by u^{p-1}, integrating by parts on Ω, we see that

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p = \int_\Omega \Delta u \cdot u^{p-1} - \chi\int_\Omega u^{p-1}\nabla\cdot(u(u+1)^{r-1}\nabla w) + \int_\Omega u^p(\lambda_1 - \mu_1 v^{r_1-1} + av) \tag{3.27}$$

for all $t \in (0, T_{max})$.

In light of $\frac{\partial v}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0$, for all $t \in (0, T_{max})$, we can obtain

$$\int_\Omega \Delta u \cdot u^{p-1} = -(p-1)\int_\Omega u^{p-2}|\nabla u|^2. \tag{3.28}$$

Combining (3.27) and (3.28) with $w_t = \Delta w - w + v$, we can deduce that

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + (p-1)\int_\Omega u^{p-2}|\nabla u|^2 \\
=& -\chi\int_\Omega \nabla\cdot(u(1+u)^{r-1}\nabla w)u^{p-1} + \int_\Omega u^p(\lambda_1 - \mu_1 u^{r_1-1} + av) \\
=&\chi(p-1)\int_\Omega u^{p-1}(1+u)^{r-1}\nabla w\cdot\nabla u + \int_\Omega u^p(\lambda_1 - \mu_1 u^{r_1-1} + av) \\
=&\chi(p-1)\int_\Omega \nabla\int_0^u s^{p-1}(1+s)^{r-1}ds\cdot\nabla w + \int_\Omega u^p(\lambda_1 - \mu_1 u^{r_1-1} + av) \\
\le&\chi(p-1)\int_\Omega\int_0^u (1+s)^{p+r-2}ds|\Delta w| + \int_\Omega u^p(\lambda_1 - \mu_1 u^{r_1-1} + av) \\
\le&\frac{\chi(p-1)}{p+r-1}\int_\Omega (1+u)^{p+r-1}|\Delta w| + \lambda_1\int_\Omega u^p - \mu_1\int_\Omega u^{p+r_1-1} + a\int_\Omega u^p v \\
\le& -\frac{3}{4}\mu_1\int_\Omega u^{p+r_1-1} + \lambda_1\int_\Omega u^p + C_{14}\int_\Omega |\Delta w|^H + a\int_\Omega u^p v + C_{15}
\end{aligned} \tag{3.29}$$

for all $t \in (0, T_{max})$ and constants $C_{14} > 0$ and $C_{15} > 0$, where $H = \frac{p+r_1-1}{r_1-r}$.

Employing Young's inequality, since $r_1 > 1$, we have

$$\lambda_1\int_\Omega u^p \le \frac{\mu_1}{4}\int_\Omega u^{p+r_1-1} + C_{16} \tag{3.30}$$

and

$$a\int_\Omega u^p v \le \frac{\mu_1}{4}\int_\Omega u^{p+r_1-1} + C_{17}\int_\Omega v^{\frac{p+r_1-1}{r_1-1}} \tag{3.31}$$

for all $t \in (0, T_{max})$ with constants $C_{16}, C_{17} > 0$, then we can obtain

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p \le -\frac{\mu_1}{4}\int_\Omega u^{p+r_1-1} + C_{14}\int_\Omega |\Delta w|^H + C_{17}\int_\Omega v^{\frac{p+r_1-1}{r_1-1}} + C_{18} \tag{3.32}$$

for all $t \in (0, T_{max})$ with $C_{18} > 0$.

In light of Young's inequality, we have

$$\frac{H}{p}\int_\Omega u^p \le \frac{\mu_1}{8}\int_\Omega u^{p+r_1-1} + C_{19} \tag{3.33}$$

for all $t \in (0, T_{max})$ with $C_{19} > 0$, so we have

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \frac{H}{p}\int_\Omega u^p \le -\frac{\mu_1}{8}\int_\Omega u^{p+r_1-1} + C_{14}\int_\Omega |\Delta w|^H + C_{17}\int_\Omega v^{\frac{p+r_1-1}{r_1-1}} + C_{19} \tag{3.34}$$

for all $t \in (0, T_{max})$.

Recalling Lemma 2.3, let s_0 be the same as Lemma 2.3; we know that

$$\begin{aligned}
&\int_{s_0}^{t} pe^{Hs}C_{14}\int_\Omega |\Delta w|^H ds\\
=&pC_{14}\int_{s_0}^{t} e^{Hs}\|\Delta w\|_{L^H(\Omega)}^H ds\\
\le&pC_{20}\left[\int_{s_0}^{t} e^{Hs}\|v\|_{L^H(\Omega)}^H ds + e^{Hs_0}(\|w_0(\cdot,s_0)\|_{L^H(\Omega)}^H + \|\Delta w_0(\cdot,s_0)\|_{L^H(\Omega)}^H)\right]\\
\le&pC_{21}\left(\int_{s_0}^{t} e^{Hs}\|v\|_{L^H(\Omega)}^H ds + 1\right)
\end{aligned} \tag{3.35}$$

for all $t \in (s_0, T_{max})$ with constants $C_{20} > 0$ and $C_{21} > 0$.

Based on the standard ODE comparison argument, there exist constants $C_{22} > 0$, $C_{23} > 0$, and $C_{24} > 0$ such that

$$\begin{aligned}
\int_\Omega u^p \le& e^{-Ht}\left\{\int_{s_0}^{t} pe^{Hs}\left(-\frac{\mu_1}{8}\int_\Omega u^{p+r_1-1} + C_{22}\int_\Omega v^H\right.\right.\\
&\left.\left.+C_{17}\int_\Omega v^{\frac{p+r_1-1}{r_1-1}} + C_{23}\right) + C_{24}\right\} \quad \text{for all } t \in (s_0, T_{max}).
\end{aligned} \tag{3.36}$$

According to (3.17) and (3.18), employing Young's inequality, we have

$$C_{22}\int_\Omega v^H \le \frac{\mu_2}{8}\int_\Omega v^{p+r_2-1} + C_{25} \tag{3.37}$$

and

$$C_{17}\int_\Omega v^{\frac{p+r_1-1}{r_1-1}} \le \frac{\mu_2}{16}\int_\Omega v^{p+r_2-1} + C_{26} \tag{3.38}$$

for all $t \in (0, T_{max})$ with constants $C_{25}, C_{26} > 0$. Thus, the estimate (3.36) can be rewritten as

$$\int_\Omega u^p \le e^{-Ht}\left\{\int_{s_0}^{t} pe^{Hs}\left(-\frac{\mu_1}{8}\int_\Omega u^{p+r_1-1} + \frac{3\mu_2}{16}\int_\Omega v^{p+r_2-1} + C_{27}\right)ds + C_{28}\right\}. \tag{3.39}$$

Combing (3.39) with (3.26), there are some constants $C_i > 0$ ($i = 29, 30, 31$) such that

$$\begin{aligned}
\int_\Omega u^p + \int_\Omega v^p &\le e^{-Ht}\left\{\int_{s_0}^{t} pe^{Hs}(-\frac{\mu_1}{16}\int_\Omega u^{p+r_1-1} - \frac{\mu_2}{16}\int_\Omega v^{p+r_2-1} + C_{29})ds + C_{30}\right\}\\
&\le e^{-Ht}\left\{C_{29}p\int_{s_0}^{t} e^{Hs}ds + C_{30}\right\}\\
&\le C_{31} \quad \text{for all } t \in (s_0, T_{max}).
\end{aligned} \tag{3.40}$$

Then the proof of Lemma 3.3 is finished. □

Next, we will complete the proof of global existence and boundedness of solutions to (1.1) by using a Moser-type iteration.

Lemma 3.4. *Let* (u, v, w, z) *be a classical nonnegative solution of* (1.1) *in* $\Omega \times (0, T_{max})$*, there exists a constant* $C > 0$ *satisfying*

$$\sup_{t\in(0,T_{max})} (\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|z(\cdot,t)\|_{W^{1,\infty}(\Omega)}) \le C. \tag{3.41}$$

Proof. In light of $w_t = \Delta w - w + v$ in $\Omega \times (0, T_{max})$, by means of an associated variation-of-constants formula, we can represent $w(\cdot, t)$ for each $t \in (0, T_{max})$ according to

$$w(\cdot,t) = e^{(\Delta-1)t}w_0 + \int_0^t e^{(\Delta-1)(t-s)}v(\cdot,s)ds.$$

We use Lemma 2.4 to derive that

$$\begin{aligned}\|\nabla w(\cdot,t)\|_{L^\infty(\Omega)} \le& \|\nabla e^{(\Delta-1)t}w_0\|_{L^\infty(\Omega)} + \int_0^t \|\nabla e^{(\Delta-1)(t-s)}v(\cdot,s)\|_{L^\infty(\Omega)}ds \\ \le& C_1\int_0^t [1+(t-s)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{2N}-\frac{1}{\infty})}]e^{-\lambda(t-s)}\|v(\cdot,s)\|_{L^{2N}(\Omega)}ds + C_2\end{aligned} \tag{3.42}$$

with $C_1 > 0$, $C_2 > 0$, and for all $t \in (0, T_{max})$.

Since $-\frac{1}{2} - \frac{N}{2}(\frac{1}{2N} - \frac{1}{\infty}) > -1$, we can draw that

$$\|\nabla w(\cdot,t)\|_{L^\infty(\Omega)} \le C_3 \quad \text{for all } t \in (0,T_{max}) \tag{3.43}$$

with $C_3 > 0$.

Noting that $\Delta z - z + u = 0$ and using Lemma 3.3 and the L^p theory of elliptic equation, for $p > N$, there exist constants $C_4 > 0$ and $C_5 > 0$ such that

$$\|z\|_{W^{2,p}(\Omega)} \le C_4\|u\|_{L^p(\Omega)} \le C_5 \quad \text{for all } t \in (0,T_{max}). \tag{3.44}$$

Applying the Sobolev embedding theorem, there exist constants $C_6 > 0$ and $C_7 > 0$ such that

$$\|z\|_{W^{1,\infty}} \le C_6\|z\|_{W^{2,p}} \le C_7 \quad \text{for all } t \in (0,T_{max}). \tag{3.45}$$

Then we can get

$$\sup_{t\in(0,T_{max})} (\|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|z(\cdot,t)\|_{W^{1,\infty}(\Omega)}) \le C_8 \quad \text{for all } t \in (0,T_{max}) \tag{3.46}$$

with a constant $C_8 > 0$.

Set $p = P_k = 2^k$, $V_k = \max\{1, \sup_{t\in(0,T_{max})} \int_\Omega v^{P_k}\}$ for all $k \in \mathbb{N}$; by the proof of Lemma 3.3, there exists a positive constant $C_9 > 0$ such that

$$\begin{aligned}&\frac{1}{P_k}\frac{d}{dt}\int_\Omega v^{P_k} + (P_k-1)\int_\Omega v^{P_k-2}|\nabla v|^2 + \int_\Omega v^{P_k} \\ =& -\xi(P_k-1)\int_\Omega v^{P_k-1}\nabla v\nabla z + \int_\Omega v^{P_k}(\lambda_2 + 1 - \mu_2 v^{r_2-1} - bu) \\ \le& (P_k-1)C_9\int_\Omega v^{P_k-1}|\nabla v| + (\lambda_2+1)\int_\Omega v^{P_k} \\ =& (P_k-1)C_9\int_\Omega v^{\frac{P_k-2}{2}}|\nabla v|v^{\frac{P_k}{2}} + (\lambda_2+1)\int_\Omega v^{P_k} \quad \text{for all } t \in (0,T_{max}).\end{aligned} \tag{3.47}$$

By using Young's inequality, there are constants $C_{10} > 0$ and $C_{11} > 0$ such that

$$\begin{aligned}
&\frac{1}{P_k}\frac{d}{dt}\int_\Omega v^{P_k} + \frac{(P_k-1)}{2}\int_\Omega v^{P_k-2}|\nabla v|^2 + \int_\Omega v^{P_k} \\
\leq &(C_{10}\frac{(P_k-1)}{2} + \lambda_2 + 1)\int_\Omega v^{P_k} \\
\leq &C_{11}P_k\int_\Omega v^{P_k} \quad \text{for all } \; t \in (0, T_{max}),
\end{aligned} \tag{3.48}$$

which can be written as

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega v^{P_k} + \frac{P_k(P_k-1)}{2}\int_\Omega v^{P_k-2}|\nabla v|^2 + P_k\int_\Omega v^{P_k} \\
\leq &C_{11}P_k^2\int_\Omega v^{P_k} \\
= &C_{11}P_k^2\|v^{\frac{P_k}{2}}\|^2_{L^2(\Omega)} \quad \text{for all } \; t \in (0, T_{max}).
\end{aligned} \tag{3.49}$$

Applying the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}
&C_{11}P_k^2\|v^{\frac{P_k}{2}}\|^2_{L^2(\Omega)} \\
\leq &C_{12}P_k^2(\|\nabla v^{\frac{P_k}{2}}\|^{\frac{2N}{N+2}}_{L^2(\Omega)}\|v^{\frac{P_k}{2}}\|^{\frac{4}{N+2}}_{L^1(\Omega)} + \|v^{\frac{P_k}{2}}\|^2_{L^1(\Omega)})
\end{aligned} \tag{3.50}$$

for all $t \in (0, T_{max})$ with a constant $C_{12} > 0$.

Using Young's inequality again, there exists a positive constant $C_{13} > 0$ such that

$$C_{12}P_k^2\|\nabla v^{\frac{P_k}{2}}\|^{\frac{2N}{N+2}}_{L^2(\Omega)}\|v^{\frac{P_k}{2}}\|^{\frac{4}{N+2}}_{L^1(\Omega)} \leq \frac{1}{2}\|\nabla v^{\frac{P_k}{2}}\|^2_{L^2(\Omega)} + C_{13}P_k^{N+2}\|v^{\frac{P_k}{2}}\|^2_{L^1(\Omega)} \tag{3.51}$$

for all $t \in (0, T_{max})$. Combining (3.51) with (3.50), we have

$$\begin{aligned}
C_{11}P_k^2\int_\Omega v^{P_k} \leq &\frac{1}{2}\|\nabla v^{\frac{P_k}{2}}\|^2_{L^2(\Omega)} + C_{13}P_k^{N+2}\|v^{\frac{P_k}{2}}\|^2_{L^1(\Omega)} + C_{12}P_k^2\|v^{\frac{P_k}{2}}\|^2_{L^1(\Omega)} \\
\leq &\frac{1}{2}\|\nabla v^{\frac{P_k}{2}}\|^2_{L^2(\Omega)} + C_{14}P_k^{N+2}\|v^{\frac{P_k}{2}}\|^2_{L^1(\Omega)} \\
\leq &\frac{1}{2}\|\nabla v^{\frac{P_k}{2}}\|^2_{L^2(\Omega)} + C_{14}P_k^{N+2}V_{k-1}^2 \\
\leq &\frac{1}{2}\|\nabla v^{\frac{P_k}{2}}\|^2_{L^2(\Omega)} + C_{15}^kV_{k-1}^2 \quad \text{for all } \; t \in (0, T_{max}),
\end{aligned} \tag{3.52}$$

where $C_{14} > 0$ and $C_{15} = \max\{1, C_{14}\}2^{N+2}$.

Then, substituting (3.52) into (3.49), we can obtain

$$\frac{d}{dt}\int_\Omega v^{P_k} + \int_\Omega v^{P_k} \leq C_{15}^kV_{k-1}^2 \quad \text{for all } t \in (0, T_{max}). \tag{3.53}$$

Applying Lemma 2.5, we have

$$\int_\Omega v^{P_k} \leq \max\left\{\int_\Omega v_0^{P_k}, C_{15}^kV_{k-1}^2\right\} \quad \text{for all } t \in (0, T_{max}). \tag{3.54}$$

If $\int_\Omega v^{P_k} \le \int_\Omega v_0^{P_k}$, for any $k \in \mathbb{N}$, we can take $k \to \infty$ (see [20]), then there exists $C_{16} > 0$ such that

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \le C_{16} \quad \text{for all } t \in (0, T_{max}). \tag{3.55}$$

If $\int_\Omega v^{P_k} \le C_{15}^k V_{k-1}^2$ for all $t \in (0, T_{max})$, we have

$$\begin{aligned}\int_\Omega v^{P_k} \le & C_{15}^k V_{k-1}^2 \\ \le & C_{15}^{k+2(k-1)} V_{k-2}^{2^2} \\ \le & C_{15}^{k+2(k-1)+2^2(k-2)} V_{k-3}^{2^3} \\ & \cdots \\ \le & C_{15}^{k+\sum_{j=2}^{k}(j-1)\cdot 2^{k-j+1}} V_0^{2^k} \quad \text{for all } t \in (0, T_{max}).\end{aligned} \tag{3.56}$$

Since

$$\frac{k + \sum_{j=2}^{k}(j-1)\cdot 2^{k-j+1}}{P_k} = \frac{2(2^k-1)-k}{2^k} \le \frac{2^{k+1}}{2^k} = 2 \quad \text{for all } k \ge 1, \tag{3.57}$$

we have

$$\|v(\cdot,t)\|_{L^{P_k}(\Omega)} \le (C_{15}V_0)^2 \quad \text{for all } k \ge 1,\ t \in (0, T_{max}). \tag{3.58}$$

Combining (3.58) with (3.55) and making $k \to \infty$ in (3.58), there exists a constant $C_{17} = \max\{C_{16}, (C_{15}V_0)^2\}$ such that

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \le C_{17} \quad \text{for all } t \in (0, T_{max}). \tag{3.59}$$

By the proof of Lemma 3.3, there exist $C_{18} > 0$ and $\lambda_3 > 0$ such that

$$\begin{aligned}& \frac{1}{p}\frac{d}{dt}\int_\Omega u^p + (p-1)\int_\Omega u^{p-2}|\nabla u|^2 + \int_\Omega u^p \\ =& \chi(p-1)\int_\Omega u^{p-1}(1+u)^{r-1}\nabla w \nabla u + \int_\Omega u^p(\lambda_1 + 1 - \mu_1 u^{r_1-1} + av) \\ \le& \chi(p-1)\int_\Omega u^{p-1}(1+u)^{r-1}|\nabla w||\nabla u| + (\lambda_3+1)\int_\Omega u^p - \mu_1\int_\Omega u^{p+r_1-1} \\ \le& C_{18}\chi(p-1)\int_\Omega u^{p-1}(1+u)^{r-1}|\nabla u| + (\lambda_3+1)\int_\Omega u^p \\ =& C_{18}\chi(p-1)\int_\Omega u^{\frac{p-2}{2}}|\nabla u| u^{\frac{p}{2}}(1+u)^{r-1} + (\lambda_3+1)\int_\Omega u^p \\ \le& \frac{p-1}{2}\int_\Omega u^{p-2}|\nabla u|^2 + C_{19}p\int_\Omega u^p(1+u^{2r-2}) + (\lambda_3+1)\int_\Omega u^p \\ \le& \frac{p-1}{2}\int_\Omega u^{p-2}|\nabla u|^2 + C_{19}p\int_\Omega u^{p+2r-2} + C_{20}p\int_\Omega u^p \quad \text{for all } t \in (0, T_{max}),\end{aligned} \tag{3.60}$$

which implies that

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p+\frac{(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2+\int_\Omega u^p\\ \le& C_{19}p\int_\Omega u^{p+2r-2}+C_{20}p\int_\Omega u^p \quad \text{for all } t\in(0,T_{max}).\end{aligned} \tag{3.61}$$

Below, we focus solely on the scenario where $u \ge 1$, as the case for $0 \le u < 1$ is analogous and less complex. When $r \le 1$, there exists a constant $C_{21} > 0$independent of p such that

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p+\frac{(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2+\int_\Omega u^p \le C_{21}p\int_\Omega u^p \quad \text{for all } t\in(0,T_{max}). \tag{3.62}$$

Similarly, there exists a positive constant $C_{22} > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C_{22} \quad \text{for all } \ t\in(0,T_{max}). \tag{3.63}$$

Otherwise, if $r > 1$, there is a constant $C_{23} > 0$ such that

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p+\frac{(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2+\int_\Omega u^p \le C_{23}p\int_\Omega u^{p+2r-2} \quad \text{for all } t\in(0,T_{max}). \tag{3.64}$$

Let $p = P_i = 2^i$, $U_i = \max\{1, \sup\limits_{t\in(0,T_{max})} \int_\Omega u^{P_i}\}$ and

$$i > \max\{[\log_2 2Nr]+1, [\log_2 4N(r-1)]+1\},$$

we have

$$\frac{1}{P_i}\frac{d}{dt}\int_\Omega u^{P_i}+\frac{P_i-1}{2}\int_\Omega u^{P_i-2}|\nabla u|^2+\int_\Omega u^{P_i} \le C_{23}P_i\int_\Omega u^{P_i+2r-2} \tag{3.65}$$

for all $t \in (0, T_{max})$. By using the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}&C_{23}P_i^2\int_\Omega u^{P_i+2r-2}\\ =&C_{23}P_i^2\|u^{\frac{P_i}{2}}\|^{\frac{2(P_i+2r-2)}{P_i}}_{L^{\frac{2(P_i+2r-2)}{P_i}}(\Omega)}\\ \le& C_{24}P_i^2(\|\nabla u^{\frac{P_i}{2}}\|^{\frac{2(P_i+2r-2)}{P_i}\theta}_{L^2(\Omega)}\|u^{\frac{P_i}{2}}\|^{\frac{2(P_i+2r-2)}{P_i}(1-\theta)}_{L^1(\Omega)}+\|u^{\frac{P_i}{2}}\|^{\frac{2(P_i+2r-2)}{P_i}}_{L^1(\Omega)})\end{aligned} \tag{3.66}$$

for all $t \in (0, T_{max})$, where $\theta = \frac{N-\frac{NP_i}{2(P_i+2r-2)}}{1+\frac{N}{2}}$ and $\frac{2(P_i+2r-2)}{P_i}\frac{N-\frac{NP_i}{2(P_i+2r-2)}}{1+\frac{N}{2}} = \frac{2N(P_i+4r-4)}{P_i(2+N)} < 2$, $C_{24} > 0$. Moreover, we have

$$\begin{aligned}&C_{24}P_i^2\|\nabla u^{\frac{P_i}{2}}\|^{\frac{2(P_i+2r-2)}{P_i}\theta}_{L^2(\Omega)}\|u^{\frac{P_i}{2}}\|^{\frac{2(P_i+2r-2)}{P_i}(1-\theta)}_{L^1(\Omega)}\\ \le& \|\nabla u^{\frac{P_i}{2}}\|^2_{L^2(\Omega)}+C_{25}P_i^{\frac{P_i(2+N)}{P_i+2N(1-r)}}\|u^{\frac{P_i}{2}}\|^{\frac{2(P_i+2r-2)+2N(1-r)}{P_i+2N(1-r)}}_{L^1(\Omega)} \quad \text{for all } \ t\in(0,T_{max})\end{aligned} \tag{3.67}$$

with a constant $C_{25} > 0$.

Combining (3.66) and (3.67) with (3.65), we get

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u^{P_i}+\frac{P_i(P_i-1)}{2}\int_\Omega u^{P_i-2}|\nabla u|^2+P_i\int_\Omega u^{P_i}\\&\le\|\nabla u^{\frac{P_i}{2}}\|^2_{L^2(\Omega)}+C_{25}P_i^{\frac{P_i(2+N)}{P_i+2N(1-r)}}\|u^{\frac{P_i}{2}}\|_{L^1(\Omega)}^{\frac{2(P_i+2r-2)+2N(1-r)}{P_i+2N(1-r)}}+C_{24}P_i^2\|u^{\frac{P_i}{2}}\|_{L^1(\Omega)}^{\frac{2(P_i+2r-2)}{P_i}}\end{aligned}\tag{3.68}$$

for all $t\in(0,T_{max})$, where $\frac{P_i(2+N)}{P_i+2N(1-r)}<2(2+N)$ and $\frac{2(P_i+2r-2)+2N(1-r)}{P_i+2N(1-r)}>\frac{2(P_i+2r-2)}{P_i}$.

Define

$$M_i=\frac{2(P_i+2r-2)-2N(r-1)}{P_i-2N(r-1)}.$$

We can easily obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^{P_i}+P_i\int_\Omega u^{P_i}&\le C_{26}P_i^{2(2+N)}\|u^{\frac{P_i}{2}}\|_{L^1(\Omega)}^{\frac{2(P_i+2r-2)+2N(1-r)}{P_i+2N(1-r)}}\\&\le C_{27}^i(\int_\Omega u^{\frac{P_i}{2}})^{M_i}=C_{27}^i(U_{i-1})^{M_i}\quad\text{for all } t\in(0,T_{max}),\end{aligned}\tag{3.69}$$

where $C_{26}>0$ and $C_{27}=\max\{1,C_{26}\}2^{2(2+N)}$ are independent of P_i.

Employing Lemma 2.5, we have

$$\int_\Omega u^{P_i}\le\max\left\{\int_\Omega u_0^{P_i},C_{27}^i(U_{i-1})^{M_i}\right\}\quad\text{for all } t\in(0,T_{max}).\tag{3.70}$$

If $\int_\Omega u^{P_i}\le\int_\Omega u_0^{P_i}$, for any $i\in\mathbb{N}$, we take $i\to\infty$, there is a constant $C_{28}>0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)}\le C_{28}\quad\text{for all }\ t\in(0,T_{max}).\tag{3.71}$$

Otherwise, if $\int_\Omega u^{P_i}\le C_{27}^i(U_{i-1})^{M_i}$, using the similar method as in (3.56), we have

$$\int_\Omega u^{P_i}\le C_{27}^{i+\sum\limits_{j=q+2}^{i}(j-1)\cdot\prod\limits_{k=j}^{i}M_k}(U_q)^{\prod\limits_{k=q+1}^{i}M_k}\tag{3.72}$$

for all $i>\max\{[\log_2 2Nr]+1,[\log_2 4N(r-1)]+1,q+2\}$, where $q=[\log_2(1+2N(r-1))]+1$, and $M_k=2(1+\varepsilon_k)$ satisfies

$$\varepsilon_k=\frac{(N+2)(r-1)}{P_k+2N(1-r)}\le\frac{C_{29}}{2^k}$$

for all $k\ge[\log_2(1+2N(r-1))]+1$ with constant $C_{29}>0$.

Due to $\ln(1+x)\le x(x\ge0)$, there is a constant $C_{30}>0$ such that

$$\begin{aligned}\prod_{k=j}^{i}M_k=&2^{i+1-j}\exp\{\sum_{k=j}^{i}\ln(1+\varepsilon_j)\}\\\le&2^{i+1-j}\exp\{\sum_{k=j}^{i}\varepsilon_j\}\\\le&2^{i+1-j}\exp\{C_{30}\}\end{aligned}\tag{3.73}$$

for all $i > \max\{[\log_2 2Nr]+1, [\log_2 4N(r-1)]+1, q+2\}$ and $j \in \{q+2, \cdots, i\}$.

We obtain

$$\begin{aligned}\frac{i+\sum\limits_{j=q+2}^{i}(j-1)\cdot\prod\limits_{k=j}^{i}M_k}{2^i} \leq & \frac{i+\sum\limits_{j=q+2}^{i}(j-1)2^{i+1-j}\exp\{C_{30}\}}{2^i} \\ \leq & \frac{i}{2^i}+2\exp\{C_{30}\}\sum_{j=q+2}^{i}\frac{j}{2^j} \\ \leq & \frac{1}{2}+2\exp\{C_{30}\}\cdot(2-\frac{1}{2}-\frac{2}{2^2}) \\ = & \frac{4\exp\{C_{30}\}+1}{2}\end{aligned} \tag{3.74}$$

as well as

$$\frac{\prod\limits_{k=q+1}^{i}M_k}{2^i} \leq \frac{\exp\{C_{30}\}}{2}.$$

Then we have

$$\|u(\cdot,t)\|_{L^{p_i}(\Omega)} \leq C_{27}^{\frac{4\exp\{C_{30}\}+1}{2}}(U_q)^{\frac{\exp\{C_{30}\}}{2}} \quad \text{for all } t \in (0, T_{max}). \tag{3.75}$$

Taking $i \to \infty$, we have

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq C_{31} \quad \text{for all } t \in (0, T_{max}), \tag{3.76}$$

where $C_{31} = \max\left\{C_{22}, C_{28}, C_{27}^{\frac{4\exp\{C_{30}\}+1}{2}}(U_q)^{\frac{\exp\{C_{30}\}}{2}}\right\}$.

This completes the proof of Lemma 3.4. □

Remark 3.1. *To establish the L^∞ estimates for u and v, one may also reference the relevant general results from the works of Tao and Winkler [20] and Tang et al. [24].*

The proof of Theorem 1.1. By applying the extensibility criterion in Lemma 2.1 and combining it with the estimates from Lemmas 3.1–3.4, it can be concluded that $T_{max} = \infty$. Consequently, the classical solution (u, v, w, z) to the model defined by (1.1)–(1.4) exists globally in time and remains bounded. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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