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*Research article*

# Space-time decay rate for the 3D compressible quantum magnetohydrodynamic model

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**Abstract:** This paper investigates the space-time decay properties of solutions to the three-dimensional compressible quantum magnetohydrodynamic (QMHD) model. By employing weighted Sobolev space techniques, we establish the optimal decay rate for the $k$-th order spatial derivatives of solutions with $k \in [0, 4]$, which concides with the heat equation. Specifically, we prove that the decay rate in the weighted space $H^2_\gamma(\mathbb{R}^3)$ is given by $t^{-\frac{3}{4}-\frac{k}{2}+\gamma}$ for the spatial derivatives of order $k$. The key contribution lies in developing a unified framework that connects the weighted energy estimates with time decay analysis, which enables us to simultaneously capture both the spatial regularity and temporal decay characteristics of the solution. This result generalizes the previous decay estimates and provides a new description of the solution's asymptotic behavior in quantum magnetohydrodynamic systems.

**Keywords:** compressible vQMHD model; weighted estimates; space-time decay rates

## 1. Introduction

In this paper, we consider the space-time decay rates for derivatives of strong solutions to the following 3D compressible viscous quantum magnetohydrodynamic (vQMHD) model in $(x, t) \in \mathbb{R}^3 \times \mathbb{R}_+$:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) - \mu\Delta u - (\mu + \lambda)\nabla \operatorname{div} u + \nabla P(\rho) - \frac{\vartheta^2}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) = (\nabla \times B) \times B, \\ B_t - \nabla \times (u \times B) = -\nabla \times (\nu\nabla \times B), \quad \operatorname{div} B = 0, \end{cases} \tag{1.1}$$

where the symbol $\otimes$ represents the Kronecker tensor product. The unknown functions $\rho = \rho(x, t)$, $u = (u_1, u_2, u_3)(x, t)$, and $B = (B_1, B_2, B_3)(x, t)$ represent the density, velocity field, and magnetic field, respectively. The pressure $P = P(\rho) = a\rho^\gamma$ satisfies $a > 0$ and $\gamma \geq 1$. The viscosity coefficients $\mu$ and

$\lambda$ adhere to the physical constraints:

$$\mu > 0, \quad \frac{2}{3}\mu + \lambda \geq 0,$$

while $\nu > 0$ denotes the magnetic diffusivity. The quantum effect parameter $\vartheta > 0$ corresponds to the scaled Planck constant. The Bohm quantum potential term $\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}$ satisfies the identity:

$$2\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) = \nabla\Delta\rho - 4\,\mathrm{div}(\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}).$$

The system (1.1) is complemented by initial data:

$$(\rho, u, B)\big|_{t=0} = (\rho_0(x), u_0(x), B_0(x)), \tag{1.2}$$

with the initial perturbation vanishing at spatial infinity:

$$\lim_{|x|\to\infty}(\rho_0 - 1, u_0, B_0)(x) = 0. \tag{1.3}$$

### *1.1. History of the problem*

Quantum fluid models provide a fundamental framework for semiconductor simulation and quantum plasma dynamics, with applications spanning quantum semiconductors [1], Bose-Einstein condensates [2], and Bohmian mechanics [3]. The quantum magnetohydrodynamic (QMHD) system, originally derived by Hass [4] via the Wigner–Maxwell formalism, reduces to classical MHD equations when quantum effects are neglected.

Extensive research has been conducted on decay properties of compressible fluid systems. For comprehensive surveys, see [5–11]. We highlight key advances relevant to our work. Pu and Guo [10] established optimal decay rates for full MHD solutions near equilibrium in $\mathbb{R}^3$ via spectral methods. Subsequent work by Pu and Xu [12] demonstrated the decay rate for the QMHD system:

$$\|\nabla^k(\rho-1)(t)\|_{H^{5-k}} + \|\nabla^k u(t)\|_{H^{4-k}} + \|\nabla^k B(t)\|_{H^{4-k}} \leq C(1+t)^{-\frac{3+2k}{4}}, \quad k = 0, 1. \tag{1.4}$$

The method is based on spectral analysis and nonlinear energy estimates. Employing the energy methods from [13], Pu and Xu [14] extended these results to higher-order derivatives under initial perturbations in $(H^{N+2}\cap\dot{H}^{-s})\times(H^{N+1}\cap\dot{H}^{-s})\times(H^{N}\cap\dot{H}^{-s})$ for $N \geq 3$, $s \in [0, \frac{3}{2})$. Recent progress by Xi, Pu, and Guo [15] via Fourier splitting yielded refined estimates:

$$\|\nabla^k(\rho-1)(t)\|_{H^{5-k}} + \|\nabla^k u(t)\|_{H^{4-k}} + \|\nabla^k B(t)\|_{H^{4-k}} \leq C(1+t)^{-\frac{3+2k}{4}}, \quad k = 0, 1, 2, 3. \tag{1.5}$$

Wang and Zhang [16] further established decay rates for higher-order derivatives:

$$\left\|\nabla^4(\rho-1)(t)\right\|_{H^1} + \left\|\nabla^4 u(t)\right\|_{L^2} + \left\|\nabla^4 B(t)\right\|_{L^2} \leq C(1+t)^{-\frac{11}{4}}, \tag{1.6}$$

$$\left\|\nabla^5(\rho-1)(t)\right\|_{L^2} + \left\|\nabla^4 u^h(t)\right\|_{L^2} + \left\|\nabla^4 B^h(t)\right\|_{L^2} \leq C(1+t)^{-\frac{13}{4}}. \tag{1.7}$$

It should be mentioned that due to the presence of the Bohm potential, the fifth-order spatial derivative of the density is faster than the ones of the velocity and magnetic, which is different from the MHD system [10].

While temporal decay rates are well-established, the *weighted* spatial decay properties remain unexplored. Motivated by Weng [17] where space-time decay estimates for the incompressible viscous resistive MHD and Hall-MHD equations were established, we investigate the space-time decay estimates for the viscous QMHD system. More precisely, we resolve this issue by proving that solutions to the viscous QMHD system exhibit the weighted $L^2$-decay rate $t^{-\frac{3}{4}-\frac{k}{2}+\gamma}$ in $H^2_\gamma(\mathbb{R}^3)$ for spatial derivatives of order $k \in [0,4]$, where $H^2_\gamma(\mathbb{R}^3)$ is the spatial-time weighted Sobolev space defined in the following notation. Compared to Wang and Zhang [16], the new difficulties lie in closing the uniform spatial-time weighted energy estimates. Our analysis combines weighted energy estimates with time-decay frameworks, providing new insights into the spatial-temporal asymptotics of quantum magnetohydrodynamic systems. The results are novel and contribute meaningfully to the analysis of quantum fluid models, with potential applications in semiconductor simulation and plasma dynamics.

### *1.2. Notation*

We first introduce the notation and function spaces used throughout this paper. Let $H^k(\mathbb{R}^3)$ denote the standard Sobolev space with norm $\|\cdot\|_{H^k}$, and $L^p(\mathbb{R}^3)$ for $1 \le p \le \infty$ represent the usual Lebesgue spaces. The notation $\|(A,B)\|_X := \|A\|_X + \|B\|_X$ will denote the sum of norms in space $X$. Constants independent of time $t$ are generically denoted by $C$, and we write $A \lesssim B$ if $A \le CB$ for some constant $C > 0$.

For $\gamma \in \mathbb{R}$ and $2 \le p < \infty$, the weighted Lebesgue space $L^p_\gamma(\mathbb{R}^3)$ is defined by:

$$L^p_\gamma(\mathbb{R}^3) := \left\{ f \in L^p(\mathbb{R}^3) : \|f\|^p_{L^p_\gamma} = \int_{\mathbb{R}^3} |x|^{p\gamma}|f(x)|^p dx < \infty \right\}.$$

The associated weighted Sobolev space $H^k_\gamma(\mathbb{R}^3)$ is given by:

$$H^k_\gamma(\mathbb{R}^3) := \left\{ f \in L^2_\gamma(\mathbb{R}^3) : \|f\|^2_{H^k_\gamma} = \sum_{|\alpha|\le k} \|\partial^\alpha f\|^2_{L^2_\gamma} < \infty \right\}.$$

We denote $L^2(\mathbb{R}^3) := L^2_0(\mathbb{R}^3)$ and $H^k(\mathbb{R}^3) := H^k_0(\mathbb{R}^3)$. The notation $\nabla^k$ with integer $k \ge 0$ represents any $k$-th order spatial derivative. A function belongs to the Schwartz class $\mathcal{S}(\mathbb{R}^3)$ if it satisfies:

$$\sup_{x\in\mathbb{R}^3} |x^\alpha \partial^\beta f(x)| < \infty \quad \forall \alpha, \beta \in \mathbb{N}^3.$$

### *1.3. Main results*

We recall the following key proposition from prior work, which serves as the foundation for our analysis:

**Proposition 1.1** ( [16, Theorem 1.1])**.** Suppose the initial data $(\rho_0 - 1, u_0, B_0) \in H^5(\mathbb{R}^3) \times H^4(\mathbb{R}^3) \times H^4(\mathbb{R}^3)$ satisfies:

$$\|\rho_0 - 1\|_{H^5} + \|u_0\|_{H^4} + \|B_0\|_{H^4} \le \varepsilon, \tag{1.8}$$

for sufficiently small $\varepsilon > 0$. Then the system (1.1)–(1.3) admits a unique global solution $(\rho, u, B)$ satisfying:

$$\|(\rho - 1, u, B)(t)\|^2_{H^4} + \vartheta^2 \|\nabla\rho(t)\|^2_{H^4} + \int_0^t \|\nabla(u, B, \vartheta\rho)(s)\|^2_{H^4} ds \tag{1.9}$$

$$\leq C\left(\|\rho_0-1\|_{H^5}^2+\|u_0\|_{H^4}^2+\|B_0\|_{H^4}^2\right). \tag{1.10}$$

Moreover, if $(\rho_0-1,u_0,B_0)\in L^1(\mathbb{R}^3)$, then it holds that

$$\|\nabla^k(\rho-1)(t)\|_{H^{5-k}}+\|\nabla^k u(t)\|_{H^{4-k}}+\|\nabla^k B(t)\|_{H^{4-k}}\leq C(1+t)^{-\frac{3+2k}{4}}, \tag{1.11}$$

for $k=0,1,2,3,4$.

Our main result establishes the weighted decay properties:

**Theorem 1.2.** *Under the assumptions of Proposition 1.1, the global solution $(\rho,u,B)$ satisfies:*

$$\|\nabla^k(\rho-1)(t)\|_{H^{5-k}_\gamma}+\|\nabla^k u(t)\|_{H^{4-k}_\gamma}+\|\nabla^k B(t)\|_{H^{4-k}_\gamma}\lesssim (1+t)^{-\frac{3+2k}{4}+\gamma}, \tag{1.12}$$

*for $k=0,1,2,3,4$ and $0\leq\gamma<\infty$.*

**Remark 1.3.** Theorem 1.2 establishes the space-time decay rate $t^{-\frac{3}{4}-\frac{k}{2}+\gamma}$ for $k$-th order derivatives in $H^2_\gamma(\mathbb{R}^3)$. The proof strategy involves three main steps:

1) **Base Case ($k=0$)**: We derive weighted energy estimates by handling critical terms like:

$$-\frac{\vartheta^2}{4}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla\Delta\varrho\cdot u dx,$$

through integration by parts and interpolation techniques. This yields the key identity:

$$\begin{aligned}-\frac{\vartheta^2}{4}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla\Delta\varrho\cdot u dx=&\frac{\vartheta^2}{8}\frac{d}{dt}\|\nabla\varrho\|_{L^2_\gamma}^2+\frac{\vartheta^2}{4}\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\cdot\Delta\varrho\cdot u dx\\&+\frac{\vartheta^2}{4}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla(\varrho\operatorname{div}u+u\cdot\nabla\varrho)\cdot\nabla\varrho dx\\&-\frac{\vartheta^2}{4}\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\cdot\nabla\varrho\cdot\operatorname{div}u dx.\end{aligned} \tag{1.13}$$

2) **General Case**: For $k\geq 1$, we extend (1.13) to higher derivatives through induction. Particularly, we can prove that

$$\begin{aligned}-\frac{\vartheta^2}{4}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^{k+1}\Delta\varrho\cdot\nabla^k u dx=&\frac{\vartheta^2}{8}\frac{d}{dt}\|\nabla^{k+1}\varrho\|_{L^2_\gamma}^2+\frac{\vartheta^2}{4}\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\cdot\nabla^k\Delta\varrho\cdot\nabla^k u dx\\&+\frac{\vartheta^2}{4}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^{k+1}(\varrho\operatorname{div}u+u\cdot\nabla\varrho)\cdot\nabla^{k+1}\varrho dx\\&-\frac{\vartheta^2}{4}\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\cdot\nabla^{k+1}\varrho\cdot\nabla^k\operatorname{div}u dx.\end{aligned} \tag{1.14}$$

3) **Decay Analysis**: The energy functional $\mathcal{E}(t):=\|\nabla\varrho\|_{L^2_\gamma}^2+\|\varrho\|_{L^2_\gamma}^2+\|u\|_{L^2_\gamma}^2+\|B\|_{L^2_\gamma}^2$ satisfies:

$$\frac{d}{dt}\mathcal{E}(t)\leq C_0t^{-\frac{5}{4}}\mathcal{E}(t)+C_1t^{-\frac{3}{4\gamma}}\mathcal{E}(t)^{\frac{2\gamma-1}{2\gamma}}+C_2t^{-\frac{3}{2\gamma}}\mathcal{E}(t)^{\frac{\gamma-1}{\gamma}}, \tag{1.15}$$

leading to the optimal decay rate through nonlinear Gronwall-type arguments. Detailed proofs for higher-order cases ($k=1,2,3,4$) are provided in subsequent lemmas.

## 2. Reformulation and preliminaries

We reformulate the original system (1.1) in perturbation form by setting $\varrho = \rho - 1$:

$$\begin{cases} \partial_t\varrho + \operatorname{div} u = G_1, \\ \partial_t u - \mu\Delta u - (\mu+\lambda)\nabla \operatorname{div} u + \nabla\varrho - \frac{\vartheta^2}{4}\nabla\Delta\varrho = G_2, \\ \partial_t B - \nu\Delta B = G_3, \end{cases} \tag{2.1}$$

with nonlinear terms defined as:

$$\begin{aligned} G_1 &:= -\varrho \operatorname{div} u - u\cdot\nabla\varrho, \\ G_2 &:= -u\cdot\nabla u + f(\varrho)(\mu\Delta u + (\mu+\lambda)\nabla \operatorname{div} u) - F(\varrho)\nabla\varrho \\ &\quad - \frac{\vartheta^2}{4} f(\varrho)\nabla\Delta\varrho + \frac{\vartheta^2}{4}\left(\frac{|\nabla\varrho|^2\nabla\varrho}{(1+\varrho)^3} - \frac{\nabla\varrho\Delta\varrho}{(1+\varrho)^2} - \frac{\nabla\varrho\cdot\nabla^2\varrho}{(1+\varrho)^2}\right) \\ &\quad + g(\varrho)((\nabla\times B)\times B), \\ G_3 &:= \nabla\times(u\times B), \end{aligned}$$

where the coefficient functions satisfy:

$$|g(\varrho)| \le C, \quad |f(\varrho)| \le C|\varrho|, \quad \text{and} \quad |F(\varrho)| \le C|\varrho|. \tag{2.2}$$

The initial data satisfy the far-field condition:

$$\lim_{|x|\to\infty}(\varrho_0, u_0, B_0)(x) = (0,0,0). \tag{2.3}$$

**Lemma 2.1** (Gagliardo–Nirenberg Inequalities). *For $f \in H^k(\mathbb{R}^3)$ with $0 \le j \le i \le k$, the following estimates hold:*

*1) General case ($1 \le p, q, r \le \infty$):*

$$\|\nabla^i f\|_{L^p} \lesssim \|\nabla^j f\|_{L^q}^{1-\alpha}\|\nabla^k f\|_{L^r}^{\alpha},$$

*where $\alpha \in [\frac{i}{k}, 1]$ satisfies:*

$$\frac{i}{3} - \frac{1}{p} = \left(\frac{j}{3} - \frac{1}{q}\right)(1-\alpha) + \left(\frac{k}{3} - \frac{1}{r}\right)\alpha.$$

*2) Special case ($p = q = r = 2$):*

$$\|\nabla^i f\|_{L^2} \lesssim \|\nabla^j f\|_{L^2}^{\frac{k-i}{k-j}}\|\nabla^k f\|_{L^2}^{\frac{i-j}{k-j}}.$$

*3) Sobolev embeddings:*

$$\begin{aligned} &\|f\|_{L^6} \lesssim \|\nabla f\|_{L^2}, \\ &\|f\|_{L^3} \lesssim \|f\|_{L^2}^{1/2}\|\nabla f\|_{L^2}^{1/2}, \\ &\|f\|_{L^\infty} \lesssim \|\nabla f\|_{H^1} \quad (f \in H^2(\mathbb{R}^3)). \end{aligned}$$

*Proof.* *The proof can be found in [18].*

**Lemma 2.2** (Nonlinear Gronwall Inequality)**.** *Let* $\mathrm{F} \in C^1([1,\infty);[0,\infty))$ *satisfy:*

$$\frac{d}{dt}\mathrm{F}(t) \leq C_0 t^{-\alpha_0}\mathrm{F}(t) + \sum_{i=1}^{2} C_i t^{-\alpha_i}\mathrm{F}(t)^{\beta_i} + C_3 t^{\gamma_2 - 1},$$

*with initial bound* $\mathrm{F}(1) \leq K_0$*, where the parameters satisfy:*

- $\alpha_0 > 1$, $\alpha_i < 1$, $\beta_i < 1$ $(i = 1, 2)$
- $\gamma_i := \frac{1-\alpha_i}{1-\beta_i} > 0$ $(i = 1, 2)$.

*Assume that* $\gamma_1 \geq \gamma_2$*, and then there exists a positive* $C^*$ *depending on* $\alpha_0, \alpha_1, \beta, \alpha_2, \beta_2, K_0, C_0, C_1, C_2, C_3, C_4$ *such that:*

$$\mathrm{F}(t) \leq C^* t^{\gamma_1} \quad \forall t \geq 1. \tag{2.4}$$

*Proof.* *The proof can be found in [19].*

**Lemma 2.3** (Convolution Estimate)**.** *For* $r_1, r_2 > 0$ *and* $\varepsilon > 0$*, the temporal integral satisfies:*

$$\int_0^t (1+t-s)^{-r_1}(1+s)^{-r_2}\,ds \lesssim (1+t)^{-\min\{r_1, r_2, r_1+r_2-1-\varepsilon\}}.$$

## 3. Proof of Theorem 1.2

First, we establish the decay properties of solutions in weighted $L^2_\gamma$ norms. To do this, by setting $\mathrm{E}(t) := \|\nabla\varrho\|^2_{L^2_\gamma} + \|\varrho\|^2_{L^2_\gamma} + \|u\|^2_{L^2_\gamma} + \|B\|^2_{L^2_\gamma}$ and making careful energy estimates, we can show that $\mathrm{E}(t)$ satisfies a Gronwall-type inequality (see (3.37) for details). Then, applying Lemma 2.2 for this inequality and using an interpolation trick, we can prove the following decay rates of solutions in weighted $L^2_\gamma$ norms, which is stated in the following lemma.

**Lemma 3.1.** *Under the assumptions of Proposition 1.1, there exists* $T_1 > 0$ *such that the global solution satisfies:*

$$\|\varrho(t)\|_{H^1_\gamma} + \|u(t)\|_{L^2_\gamma} + \|B(t)\|_{L^2_\gamma} \leq C t^{-\frac{3}{4}+\gamma}, \quad \forall \gamma \geq 0,\ t > T_1, \tag{3.1}$$

*where* $C > 0$ *is time-independent.*

*Proof.* We begin by reformulating the momentum equation through perturbation analysis:

$$\begin{aligned} u_t - \mu\Delta u - (\mu+\lambda)\nabla \operatorname{div} u + \nabla\varrho - \frac{\vartheta^2}{4} g(\varrho)\nabla\Delta\varrho \\ = G_4 + \frac{\vartheta^2}{4}\left(\frac{|\nabla\varrho|^2\nabla\varrho}{(1+\varrho)^3} - \frac{\nabla\varrho\Delta\varrho}{(1+\varrho)^2} - \frac{\nabla\varrho\cdot\nabla^2\varrho}{(1+\varrho)^2}\right), \end{aligned}$$

where the nonlinear term $G_4$ contains critical interactions:

$$G_4 := -u\cdot\nabla u + f(\varrho)(\mu\Delta u + (\mu+\lambda)\nabla \operatorname{div} u)$$

$$- F(\varrho)\nabla\varrho + g(\varrho)((\nabla\times B)\times B).$$

Multiplying the system by $|x|^{2\gamma}(\varrho, u, B)$ and integrating over $\mathbb{R}^3$ yields:

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\||x|^\gamma\varrho\|_{L^2}^2 + \langle |x|^{2\gamma}\varrho, \operatorname{div} u\rangle = \langle |x|^{2\gamma}\varrho, G_1\rangle, \\ \frac{1}{2}\frac{d}{dt}\left(\||x|^\gamma u\|_{L^2}^2 + \frac{\vartheta^2}{4}\||x|^\gamma\nabla\varrho\|_{L^2}^2\right) + \mu\||x|^\gamma\nabla u\|_{L^2}^2 + (\mu+\lambda)\||x|^\gamma \operatorname{div} u\|_{L^2}^2 \\ \quad -\langle |x|^{2\gamma}\varrho, \operatorname{div} u\rangle = \langle |x|^{2\gamma}u, G_4\rangle + \sum_{i=1}^7 M_i, \\ \frac{1}{2}\frac{d}{dt}\||x|^\gamma B\|_{L^2}^2 + \nu\||x|^\gamma\nabla B\|_{L^2}^2 = \langle |x|^{2\gamma}B, G_3\rangle + M_8. \end{cases} \tag{3.2}$$

Thus, by summing up (3.2), we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|x|^{2\gamma}(\frac{\vartheta^2}{4}|\nabla\varrho|^2 + |\varrho|^2 + |u|^2 + |B|^2)\,\mathrm{d}x + \mu\int_{\mathbb{R}^3}|x|^{2\gamma}|\nabla u|^2\,\mathrm{d}x \\ &\quad + (\mu+\lambda)\int_{\mathbb{R}^3}|x|^{2\gamma}|\mathrm{div}u|^2\,\mathrm{d}x + \nu\int_{\mathbb{R}^3}|x|^{2\gamma}|\nabla B|^2\,\mathrm{d}x \\ &= \int_{\mathbb{R}^3}|x|^{2\gamma}\varrho\cdot G_1\,\mathrm{d}x + \int_{\mathbb{R}^3}|x|^{2\gamma}B\cdot G_3\,\mathrm{d}x + \int_{\mathbb{R}^3}|x|^{2\gamma}u\cdot G_4\,\mathrm{d}x \\ &+ M_1 + M_2 + M_3 + M_4 + M_5 + M_6 + M_7 + M_8 \end{aligned} \tag{3.3}$$

where the functions $M_1, M_2, M_3, M_4, M_5, M_6, M_7$ and $M_8$ are defined by

$$M_1 := -\mu\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})u\cdot\nabla u\,\mathrm{d}x, \tag{3.4}$$

$$M_2 := -(\mu+\lambda)\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})u\cdot\mathrm{div}u\,\mathrm{d}x, \tag{3.5}$$

$$M_3 := \int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})u\cdot\varrho\,\mathrm{d}x, \tag{3.6}$$

$$M_4 := -\frac{\vartheta^2}{4}\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\triangle\varrho\cdot u\,\mathrm{d}x, \tag{3.7}$$

$$M_5 := \frac{\vartheta^2}{4}\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\nabla\varrho\cdot\mathrm{div}u\,\mathrm{d}x, \tag{3.8}$$

$$M_6 := -\frac{\vartheta^2}{4}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla(\varrho\cdot\mathrm{div}u + u\cdot\nabla\varrho)\cdot\nabla\varrho\,\mathrm{d}x, \tag{3.9}$$

$$M_7 := \frac{\vartheta^2}{4}\int_{\mathbb{R}^3}|x|^{2\gamma}\left(\frac{|\nabla\varrho|^2\nabla\varrho}{(1+\varrho)^3} - \frac{\nabla\varrho\Delta\varrho}{(1+\varrho)^2} - \frac{\nabla\varrho\cdot\nabla^2\varrho}{(1+\varrho)^2}\right)\cdot u\,\mathrm{d}x, \tag{3.10}$$

and

$$M_8 := \nu\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})B\cdot\nabla B\,\mathrm{d}x. \tag{3.11}$$

For terms $M_1$–$M_6$ and $M_8$, by virtue of Cauchy's inequality and Hölder's inequality, we obtain

$$\begin{aligned} |M_1| + |M_2| &\lesssim \mu\|\nabla u\|_{L^2_\gamma}\|u\|_{L^2_{\gamma-1}} + (\mu+\lambda)\|\operatorname{div} u\|_{L^2_\gamma}\|u\|_{L^2_{\gamma-1}} \\ &\lesssim \varepsilon\mu\|\nabla u\|_{L^2_\gamma}^2 + \varepsilon(\mu+\lambda)\|\operatorname{div} u\|_{L^2_\gamma}^2 + \|u\|_{L^2_{\gamma-1}}^2, \end{aligned} \tag{3.12}$$

$$|M_3| \lesssim \|\varrho\|_{L^2_\gamma}\|u\|_{L^2_{\gamma-1}}, \tag{3.13}$$

$$\begin{aligned}|M_4| &\lesssim \left|\int_{\mathbb{R}^3}|x|^{2\gamma-2}\cdot\nabla\varrho\cdot u\,\mathrm{d}x\right| + \left|\int_{\mathbb{R}^3}|x|^{2\gamma-1}\cdot\nabla\varrho\cdot\nabla u\,\mathrm{d}x\right|\\ &\lesssim \|\nabla\varrho\|^2_{L^2_{\gamma-1}} + \|u\|^2_{L^2_{\gamma-1}} + \varepsilon\|\nabla u\|^2_{L^2_\gamma},\end{aligned} \tag{3.14}$$

$$|M_5| \lesssim \|\nabla\varrho\|^2_{L^2_{\gamma-1}} + \varepsilon\|\nabla u\|^2_{L^2_\gamma}, \tag{3.15}$$

$$\begin{aligned}|M_6| &\lesssim \|\nabla\varrho\|_{L^\infty}\|\nabla(\varrho,u)\|^2_{L^2_\gamma} + \|\nabla^2(\varrho,u)\|_{L^3}\|\nabla\varrho\|_{L^2_\gamma}\|(\varrho,u)\|_{L^6_\gamma}\\ &\lesssim \|\nabla\varrho\|_{H^2}\|\nabla(\varrho,u)\|^2_{L^2_\gamma} + \|\nabla(\varrho,u)\|_{H^2}\|\nabla\varrho\|_{L^2_\gamma}(\|\nabla(\varrho,u)\|_{L^2_\gamma} + \|(\varrho,u)\|_{L^2_{\gamma-1}})\\ &\lesssim \|\nabla(\varrho,u)\|_{H^2}\|\nabla\varrho\|^2_{L^2_\gamma} + \|(\varrho,u)\|^2_{L^2_{\gamma-1}} + \varepsilon\|\nabla u\|^2_{L^2_\gamma},\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}|M_8| &\lesssim \|\nabla B\|_{L^2_\gamma}\|B\|_{L^2_{\gamma-1}}\\ &\lesssim \varepsilon\|\nabla B\|^2_{L^2_\gamma} + \|B\|^2_{L^2_{\gamma-1}}.\end{aligned} \tag{3.17}$$

where $\varepsilon$ is a small positive constant to be determined later. For the term $M_7$ , using integration by parts, Hölder's inequality, Cauchy's inequality, and the Sobolev inequality, we have

$$\begin{aligned}|M_7| &\lesssim \|\nabla\varrho\|_{L^2_\gamma}\|\nabla\varrho\|^2_{L^6}\|u\|_{L^6_\gamma} + \|\nabla\varrho\|_{L^2_\gamma}\|\nabla^2\varrho\|_{L^3}\|u\|_{L^6_\gamma}\\ &\lesssim \|\nabla\varrho\|^2_{L^2_\gamma} + \|u\|^2_{L^2_{\gamma-1}} + \varepsilon\|\nabla u\|^2_{L^2_\gamma},\end{aligned} \tag{3.18}$$

where $\varepsilon$ is a sufficiently small constant and we have used the fact that

$$\|u\|_{L^6_\gamma} \lesssim \|\nabla(|x|^\gamma u)\|_{L^2_\gamma} \lesssim \|\nabla u\|_{L^2_\gamma} + \|u\|_{L^2_{\gamma-1}}. \tag{3.19}$$

Next, we deal with the term$|\int_{\mathbb{R}^3}|x|^{2\gamma}\varrho\cdot G_1\,\mathrm{d}x|$. To begin with, we notice that

$$\begin{aligned}\left|\int_{\mathbb{R}^3}|x|^{2\gamma}\varrho\cdot G_1\,\mathrm{d}x\right| &\lesssim \left|\int_{\mathbb{R}^3}|x|^{2\gamma}(u\cdot\nabla\varrho)\cdot\varrho\,\mathrm{d}x\right| + \left|\int_{\mathbb{R}^3}|x|^{2\gamma}\varrho^2\mathrm{div}u\,\mathrm{d}x\right|\\ &:= M_9 + M_{10}\end{aligned} \tag{3.20}$$

For the term $M_9$, by using the Hölder, Cauchy and Sobolev inequalities, we get

$$\begin{aligned}|M_9| &\lesssim \|\nabla\varrho\|_{L^3}\|\varrho\|_{L^2_\gamma}\|u\|_{L^6_\gamma}\\ &\lesssim \|\nabla\varrho\|^2_{H^1}\|\varrho\|^2_{L^2_\gamma} + \|u\|^2_{L^2_{\gamma-1}} + \varepsilon\|\nabla u\|^2_{L^2_\gamma},\end{aligned} \tag{3.21}$$

For the term $M_{10}$, we have from integration by parts that

$$\begin{aligned}|M_{10}| &= \left|\int_{\mathbb{R}^3}u\cdot\nabla(|x|^{2\gamma}\varrho^2)\,\mathrm{d}x\right|\\ &\lesssim \|\nabla\varrho\|_{L^3}\|\varrho\|_{L^2_\gamma}\|u\|_{L^6_\gamma} + \|\varrho\|_{L^\infty}\|\varrho\|_{L^2_\gamma}\|u\|_{L^2_{\gamma-1}}\\ &\lesssim \|\nabla\varrho\|^2_{H^1}\|\varrho\|^2_{L^2_\gamma} + \|u\|^2_{L^2_{\gamma-1}} + \varepsilon\|\nabla u\|^2_{L^2_\gamma}.\end{aligned} \tag{3.22}$$

Substituting the estimates (3.21) and (3.22) into (3.20), we get

$$\left|\int_{\mathbb{R}^3} |x|^{2\gamma}\varrho \cdot G_1 \,\mathrm{d}x\right| \lesssim \|\nabla\varrho\|_{H^1}^2\|\varrho\|_{L^2_\gamma}^2 + \|u\|_{L^2_{\gamma-1}}^2 + \varepsilon\|\nabla u\|_{L^2_\gamma}^2. \tag{3.23}$$

For the term $\left|\int_{\mathbb{R}^3} |x|^{2\gamma}B \cdot G_3 \,\mathrm{d}x\right|$, it holds that

$$\begin{aligned}\left|\int_{\mathbb{R}^3} |x|^{2\gamma}B \cdot G_3 \,\mathrm{d}x\right| \lesssim& \left|\int_{\mathbb{R}^3} |x|^{2\gamma}B \cdot u \cdot \nabla B \,\mathrm{d}x\right| + \left|\int_{\mathbb{R}^3} |x|^{2\gamma}\,|B|^2 \cdot \nabla u \,\mathrm{d}x\right| \\ &+ \left|\int_{\mathbb{R}^3} |x|^{2\gamma}\,|B|^2 \,\mathrm{div}u \,\mathrm{d}x\right| + \left|\int_{\mathbb{R}^3} |x|^{2\gamma}B \cdot u \cdot \mathrm{div}B \,\mathrm{d}x\right| \\ :=& M_{11} + M_{12} + M_{13} + M_{14}.\end{aligned} \tag{3.24}$$

By using the Hölder, Cauchy and Sobolev inequalities, we have

$$\begin{aligned}|M_{11}| + |M_{14}| &\lesssim \|\nabla B\|_{L^3}\|B\|_{L^2_\gamma}\|u\|_{L^6_\gamma} \\ &\lesssim \|\nabla B\|_{H^1}^2\|B\|_{L^2_\gamma}^2 + \|u\|_{L^2_{\gamma-1}}^2 + \varepsilon\|\nabla u\|_{L^2_\gamma}^2,\end{aligned} \tag{3.25}$$

$$\begin{aligned}|M_{12}| + |M_{13}| &\lesssim \|\nabla u\|_{L^3}\|B\|_{L^2_\gamma}\|B\|_{L^6_\gamma} \\ &\lesssim \|\nabla u\|_{H^1}^2\|B\|_{L^2_\gamma}^2 + \|B\|_{L^2_{\gamma-1}}^2 + \varepsilon\|\nabla B\|_{L^2_\gamma}^2.\end{aligned} \tag{3.26}$$

Then we have

$$\left|\int_{\mathbb{R}^3} |x|^{2\gamma}B \cdot G_3 \,\mathrm{d}x\right| \lesssim \|\nabla(u,B)\|_{H^1}^2\|B\|_{L^2_\gamma}^2 + \|(u,B)\|_{L^2_{\gamma-1}}^2 + \varepsilon\|\nabla(u,B)\|_{L^2_\gamma}^2. \tag{3.27}$$

Finally, we deal with the term $\left|\int_{\mathbb{R}^3} |x|^{2\gamma}u \cdot G_4 \,\mathrm{d}x\right|$. From (3.2) , we have

$$\begin{aligned}\left|\int_{\mathbb{R}^3} |x|^{2\gamma}u \cdot G_4 \,\mathrm{d}x\right| \lesssim& \left|\int_{\mathbb{R}^3} |x|^{2\gamma}(u \cdot \nabla u) \cdot u \,\mathrm{d}x\right| + \mu\left|\int_{\mathbb{R}^3} |x|^{2\gamma}f(\varrho)\Delta u \cdot u \,\mathrm{d}x\right| \\ &+ (\mu+\lambda)\left|\int_{\mathbb{R}^3} |x|^{2\gamma}f(\varrho)\nabla\mathrm{div}u \cdot u \,\mathrm{d}x\right| + \left|\int_{\mathbb{R}^3} |x|^{2\gamma}F(\varrho)\nabla\varrho \cdot u \,\mathrm{d}x\right| \\ &+ \left|\int_{\mathbb{R}^3} |x|^{2\gamma}g(\varrho)[B \cdot \nabla B - \frac{1}{2}\nabla(|B|^2)] \cdot u \,\mathrm{d}x\right| := \sum_{i=15}^{19} M_i.\end{aligned} \tag{3.28}$$

The terms on right-hand side of the above equation can be estimated as follows. Using the Hölder, Cauchy and Sobolev inequalities and the uniform bound (1.8), we have

$$\begin{aligned}|M_{15}| &\lesssim \|\nabla u\|_{L^3}\|u\|_{L^2_\gamma}\|u\|_{L^6_\gamma} \\ &\lesssim \varepsilon\|\nabla u\|_{L^2_\gamma}^2 + \|u\|_{L^2_{\gamma-1}}^2 + \|\nabla u\|_{H^1}^2\|u\|_{L^2_\gamma}^2,\end{aligned} \tag{3.29}$$

and

$$\begin{aligned}|M_{16}| \lesssim& \|\varrho\|_{L^\infty}\|\nabla u\|_{L^2_\gamma}^2 + \|\nabla\varrho\|_{L^3}\|\nabla u\|_{L^2_\gamma}\|u\|_{L^6_\gamma} \\ &+ \|\varrho\|_{L^\infty}\|\nabla u\|_{L^2_\gamma}\|u\|_{L^2_{\gamma-1}} \\ \lesssim& (\|\varrho\|_{L^\infty} + \|\nabla\varrho\|_{L^3})\left(\|\nabla u\|_{L^2_\gamma}^2 + \|u\|_{L^2_{\gamma-1}}^2\right) \\ \lesssim& \|\nabla\varrho\|_{H^1}\|\nabla u\|_{L^2_\gamma}^2 + \|u\|_{L^2_{\gamma-1}}^2.\end{aligned} \tag{3.30}$$

Similarly, we also have

$$|M_{17}| \lesssim \|\nabla\varrho\|_{H^1}\|\nabla u\|_{L^2_\gamma}^2 + \|u\|_{L^2_{\gamma-1}}^2. \tag{3.31}$$

For $M_{18}$, and $M_{19}$, similar to $M_9$, we have

$$|M_{18}| \lesssim \|\nabla\varrho\|_{L^3}\|\varrho\|_{L^2_\gamma}\|u\|_{L^6_\gamma} \lesssim \|\nabla\varrho\|_{H^1}^2\|\varrho\|_{L^2_\gamma}^2 + \|u\|_{L^2_{\gamma-1}}^2 + \varepsilon\|\nabla u\|_{L^2_\gamma}^2, \tag{3.32}$$

and

$$|M_{19}| \lesssim \|\nabla B\|_{L^3}\|B\|_{L^2_\gamma}\|u\|_{L^6_\gamma} \lesssim \varepsilon\|\nabla u\|_{L^2_\gamma}^2 + \|u\|_{L^2_{\gamma-1}}^2 + \|\nabla B\|_{H^1}^2\|B\|_{L^2_\gamma}^2. \tag{3.33}$$

Thus by substituting (3.29)–(3.33) into (3.28) and using the smallness of $\varepsilon$, we get

$$\left|\int_{\mathbb{R}^3} |x|^{2\gamma} u \cdot G_4 \,\mathrm{d}x\right| \lesssim \|\nabla(\varrho, u, B)\|_{H^1}\|(\varrho, u, B)\|_{L^2_\gamma}^2 + \|u\|_{L^2_{\gamma-1}}^2 + \|\nabla\varrho\|_{H^1}\|\nabla u\|_{L^2_\gamma}^2. \tag{3.34}$$

Consequently, plugging the estimates (3.12)–(3.18), (3.23), (3.27), and (3.34) into (3.3) and using the uniform bound (1.8) and the smallness of $\varepsilon$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\vartheta^2}{4}\|\nabla\varrho\|_{L^2_\gamma}^2 + \|\varrho\|_{L^2_\gamma}^2 + \|u\|_{L^2_\gamma}^2 + \|B\|_{L^2_\gamma}^2\right) + \mu\|\nabla u\|_{L^2_\gamma}^2 + (\mu+\lambda)\|\mathrm{div} u\|_{L^2_\gamma}^2 + \nu\|\nabla B\|_{L^2_\gamma}^2 \\
&\quad\lesssim \|\nabla(\varrho, u, B)\|_{H^2}\|\nabla\varrho, \varrho, u, B\|_{L^2_\gamma}^2 + \|\varrho\|_{L^2_\gamma}\|u\|_{L^2_{\gamma-1}} + \|(\nabla\varrho, \varrho, u, B)\|_{L^2_{\gamma-1}}^2 \\
&\quad\quad + \varepsilon\|\nabla u\|_{L^2_\gamma}^2 + \varepsilon\mu\|\nabla u\|_{L^2_\gamma}^2 + \varepsilon(\mu+\lambda)\|\operatorname{div} u\|_{L^2_\gamma}^2 + \varepsilon\|\nabla B\|_{L^2_\gamma}^2 + \|\nabla\varrho\|_{H^1}\|\nabla u\|_{L^2_\gamma}^2.
\end{aligned} \tag{3.35}$$

Using the inequality $\|f\|_{L^2_{\gamma-1}} \lesssim \|f\|_{L^2_\gamma}^{\frac{\gamma-1}{\gamma}}\|f\|_{L^2}^{\frac{1}{\gamma}}$, by leveraging the uniform-in-time bounds established in Proposition 1.1, the assumption of small $\varepsilon$, and a sufficiently large positive time $T_1$, we can derive the desired conclusion:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\vartheta^2}{4}\|\nabla\varrho\|_{L^2_\gamma}^2 + \|\varrho\|_{L^2_\gamma}^2 + \|u\|_{L^2_\gamma}^2 + \|B\|_{L^2_\gamma}^2\right) + \mu\|\nabla u\|_{L^2_\gamma}^2 + (\mu+\lambda)\|\mathrm{div} u\|_{L^2_\gamma}^2 + \nu\|\nabla B\|_{L^2_\gamma}^2 \\
&\lesssim \|\nabla(\varrho, u, B)\|_{H^2}\|\nabla\varrho, \varrho, u, B\|_{L^2_\gamma}^2 + \|\varrho\|_{L^2_\gamma}\|u\|_{L^2_\gamma}^{\frac{\gamma-1}{\gamma}}\|u\|_{L^2}^{\frac{1}{\gamma}} \\
&\quad + \|(\nabla\varrho, \varrho, u, B)\|_{L^2_\gamma}^{\frac{2(\gamma-1)}{\gamma}}\|(\nabla\varrho, \varrho, u, B)\|_{L^2}^{\frac{2}{\gamma}}
\end{aligned} \tag{3.36}$$

for all $t \geq T_1$. Denoting $\mathrm{E}(t) := \|\nabla\varrho\|_{L^2_\gamma}^2 + \|\varrho\|_{L^2_\gamma}^2 + \|u\|_{L^2_\gamma}^2 + \|B\|_{L^2_\gamma}^2$, we can obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}(t) \leq C_0 t^{-\frac{5}{4}}\mathrm{E}(t) + C_1 t^{-\frac{3}{4\gamma}}\mathrm{E}(t)^{\frac{2\gamma-1}{2\gamma}} + C_2 t^{-\frac{3}{2\gamma}}\mathrm{E}(t)^{\frac{\gamma-1}{\gamma}}, \tag{3.37}$$

where $\alpha_0 = \frac{5}{4}, \alpha_1 = \frac{3}{4\gamma}, \beta_1 = \frac{2\gamma-1}{2\gamma}, \alpha_2 = \frac{3}{2\gamma}, \beta_2 = \frac{\gamma-1}{\gamma}, C_3 = 0$. To assure that $\alpha_1 < 1, \alpha_2 < 1$, we require $\gamma > \frac{3}{2}$. Hence, $\gamma_1 = \frac{1-\alpha_1}{1-\beta_1} = 2\gamma - \frac{3}{2} > \gamma_2 = \frac{1-\alpha_2}{1-\beta_2} = \gamma - \frac{3}{2}$. Thus, by the virtue of Lemma 2.2, we deduce from (3.37) that,

$$\mathrm{E}(t) \leq Ct^{\gamma_1} = Ct^{-\frac{3}{2}+2\gamma}. \tag{3.38}$$

Applying the interpolation inequality into (3.38) yields

$$\|(\nabla\varrho, \varrho, u, B)(t)\|_{L^2_{\gamma_0}} \leq C\|(\nabla\varrho, \varrho, u, B)(t)\|_{L^2}^{1-\frac{\gamma_0}{\gamma}}\|(\nabla\varrho, \varrho, u, B)(t)\|_{L^2_\gamma}^{\frac{\gamma_0}{\gamma}} \leq Ct^{-\frac{3}{4}+\gamma_0}, \tag{3.39}$$

for all $\gamma_0 \in [0, \gamma]$, and thus completes the proof. □

Similarly, we can establish the space-time decay behavior of the first-order spatial derivatives of the solution.

**Lemma 3.2** (First-Order Derivative Decay)**.** *Under the assumptions of Proposition 1.1, there exists* $T_2 > 0$ *such that the global solution satisfies:*

$$\|\nabla\varrho(t)\|_{H^1_\gamma} + \|\nabla u(t)\|_{L^2_\gamma} + \|\nabla B(t)\|_{L^2_\gamma} \leq Ct^{-\frac{5}{4}+\gamma}, \quad \forall t > T_2, \tag{3.40}$$

*where* $C > 0$ *is time-independent.*

*Proof.* Since the proof is similar to that of Lemma (3.1), we omit the details for simplicity. □

Similarly, we have the following space-time decay behavior of the second-order spatial derivatives of the solution.

**Lemma 3.3.** *Under the assumptions of Proposition 1.1, there exists a large time* $T_3$ *such that the global solution* $(\varrho, u, B)$ *satisfies the estimates*

$$\|\nabla^2\varrho(t)\|_{H^1_\gamma} + \|\nabla^2 u(t)\|_{L^2_\gamma} + \|\nabla^2 B(t)\|_{L^2_\gamma} \leq Ct^{-\frac{7}{4}+\gamma}, \tag{3.41}$$

*where* $C$ *is a positive constant independent of time.*

*Proof.* Since the proof is similar to that of Lemma (3.1), we omit the details for simplicity. □

Next, we establish the space-time decay rate for the third-order spatial derivative of the solution.

**Lemma 3.4.** *Under the assumptions of Proposition 1.1, there exists a large time* $T_4$*, and then the global solution* $(\varrho, u, B)$ *has the estimates*

$$\|\nabla^3\varrho(t)\|_{H^1_\gamma} + \|\nabla^3 u(t)\|_{L^2_\gamma} + \|\nabla^3 B(t)\|_{L^2_\gamma} \leq Ct^{-\frac{9}{4}+\gamma}, \tag{3.42}$$

*where* $C$ *is a positive constant independent of time.*

*Proof.* Since the proof is similar to that of Lemma (3.1), we omit the details for simplicity. □

Finally, we establish the space-time decay rate for the fourth-order spatial derivative of the solution. Compared to the proofs of Lemmas 3.1–3.4, the new difficulties lie in dealing with the trouble terms including the highest-order derivatives. We will employ integration by parts, making full use of the equations and making careful energy estimates to overcome these difficulties.

**Lemma 3.5.** *Under the assumptions of Proposition 1.1, there exists a large time* $T_5$*, and then the global solution* $(\varrho, u, B)$ *has the estimates*

$$\|\nabla^4\varrho(t)\|_{H^1_\gamma} + \|\nabla^4 u(t)\|_{L^2_\gamma} + \|\nabla^4 B(t)\|_{L^2_\gamma} \leq Ct^{-\frac{11}{4}+\gamma}, \tag{3.43}$$

*where* $C$ *is a positive constant independent of time.*

*Proof.* First, we rewrite $(2.1)_2$ as follows:

$$u_t - \mu\Delta u - (\mu+\lambda)\nabla\mathrm{div}\, u + \nabla\varrho - \frac{\vartheta^2}{4}g(\varrho)\nabla\Delta\varrho = G_4 + \frac{\vartheta^2}{4}\left(\frac{|\nabla\varrho|^2\nabla\varrho}{(1+\varrho)^3} - \frac{\nabla\varrho\Delta\varrho}{(1+\varrho)^2} - \frac{\nabla\varrho\cdot\nabla^2\varrho}{(1+\varrho)^2}\right).$$

In the above equation:

$$G_4 = -u \cdot \nabla u + f(\varrho)\,(\mu\Delta u + (\mu+\lambda)\nabla \operatorname{div} u) - F(\varrho)\nabla\varrho + g(\varrho)\,((\nabla \times B)\times B)\,.$$

Then multiplying $\nabla^4(2.1)_1, \nabla^4(2.1)_2$ and $\nabla^4(2.1)_3$ by $|x|^{2\gamma}\nabla^4\varrho, |x|^{2\gamma}\nabla^4 u$ and $|x|^{2\gamma}\nabla^4 B$, respectively, and integrating over $\mathbb{R}^3$, then summing up, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|x|^{2\gamma}(\frac{\vartheta^2}{4}|\nabla^5\varrho|^2 + |\nabla^4\varrho|^2 + |\nabla^4 u|^2 + |\nabla^4 B|^2)\,\mathrm{d}x + \mu\int_{\mathbb{R}^3}|x|^{2\gamma}|\nabla^5 u|^2\,\mathrm{d}x\\
&\quad + (\mu+\lambda)\int_{\mathbb{R}^3}|x|^{2\gamma}|\nabla^4 \mathrm{div} u|^2\,\mathrm{d}x + \nu\int_{\mathbb{R}^3}|x|^{2\gamma}|\nabla^5 B|^2\,\mathrm{d}x\\
&= \int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^4\varrho\cdot\nabla^4 G_1\,\mathrm{d}x + \int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^4 B\cdot\nabla^4 G_3\,\mathrm{d}x + \int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^4 u\cdot\nabla^4 G_4\,\mathrm{d}x\\
&+ J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8,
\end{aligned}\tag{3.44}$$

where the functions $J_1, J_2, J_3, J_4, J_5, J_6, J_7$ and $J_8$ are defined by

$$J_1 := -\mu\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\nabla^5 u\cdot\nabla^4 u\,\mathrm{d}x,\tag{3.45}$$

$$J_2 := -(\mu+\lambda)\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\nabla^4 u\cdot\nabla^4 \mathrm{div} u\,\mathrm{d}x,\tag{3.46}$$

$$J_3 := \int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\nabla^4 u\cdot\nabla^4\varrho\,\mathrm{d}x,\tag{3.47}$$

$$J_4 := -\frac{\vartheta^2}{4}\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\nabla^4\triangle\varrho\cdot\nabla^4 u\,\mathrm{d}x,\tag{3.48}$$

$$J_5 := \frac{\vartheta^2}{4}\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\nabla^5\varrho\cdot\nabla^4 \mathrm{div} u\,\mathrm{d}x,\tag{3.49}$$

$$J_6 := -\frac{\vartheta^2}{4}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^5(\varrho\cdot\mathrm{div} u + u\cdot\nabla\varrho)\cdot\nabla^5\varrho\,\mathrm{d}x,\tag{3.50}$$

$$J_7 := \frac{\vartheta^2}{4}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^4 u\nabla^4\left(\frac{|\nabla\varrho|^2\nabla\varrho}{(1+\varrho)^3} - \frac{\nabla\varrho\Delta\varrho}{(1+\varrho)^2} - \frac{\nabla\varrho\cdot\nabla^2\varrho}{(1+\varrho)^2}\right)\mathrm{d}x,\tag{3.51}$$

and

$$J_8 := \nu\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\nabla^4 B\cdot\nabla^5 B\,\mathrm{d}x.\tag{3.52}$$

For terms $J_1$–$J_5$ and $J_8$, by virtue of Cauchy's inequality and Hölder's inequality, we obtain

$$\begin{aligned}
|J_1| + |J_2| &\lesssim \mu\|\nabla^5 u\|_{L^2_\gamma}\|\nabla^4 u\|_{L^2_{\gamma-1}} + (\mu+\lambda)\|\nabla^4\operatorname{div} u\|_{L^2_\gamma}\|\nabla^4 u\|_{L^2_{\gamma-1}}\\
&\lesssim \varepsilon\mu\|\nabla^5 u\|^2_{L^2_\gamma} + \varepsilon(\mu+\lambda)\|\nabla^4\operatorname{div} u\|^2_{L^2_\gamma} + \|\nabla^4 u\|^2_{L^2_{\gamma-1}},
\end{aligned}\tag{3.53}$$

$$|J_3| \lesssim \|\nabla^4\varrho\|_{L^2_\gamma}\|\nabla^4 u\|_{L^2_{\gamma-1}},\tag{3.54}$$

$$
\begin{aligned}
|J_4| \lesssim & \left|\int_{\mathbb{R}^3} |x|^{2\gamma-2} \cdot \nabla^5 \varrho \cdot \nabla^4 u \,\mathrm{d}x\right| + \left|\int_{\mathbb{R}^3} |x|^{2\gamma-1} \cdot \nabla^5 \varrho \cdot \nabla^5 u \,\mathrm{d}x\right| \\
\lesssim & \|\nabla^5 \varrho\|^2_{L^2_{\gamma-1}} + \|\nabla^4 u\|^2_{L^2_{\gamma-1}} + \varepsilon \|\nabla^5 u\|^2_{L^2_\gamma},
\end{aligned} \tag{3.55}
$$

$$
|J_5| \lesssim \|\nabla^5 \varrho\|^2_{L^2_{\gamma-1}} + \varepsilon \|\nabla^5 u\|^2_{L^2_\gamma}, \tag{3.56}
$$

$$
\begin{aligned}
|J_8| \lesssim & \|\nabla^5 B\|_{L^2_\gamma} \|\nabla^4 B\|_{L^2_{\gamma-1}} \\
\lesssim & \varepsilon \|\nabla^5 B\|^2_{L^2_\gamma} + \|\nabla^4 B\|^2_{L^2_{\gamma-1}}.
\end{aligned} \tag{3.57}
$$

To bound the term $J_6$, we first notice that

$$
\begin{aligned}
|J_6| =& \left|\frac{\vartheta^2}{4} \int_{\mathbb{R}^3} |x|^{2\gamma} \nabla^5 (\varrho \operatorname{div} u + u \cdot \nabla \varrho) \cdot \nabla^5 \varrho \,\mathrm{d}x\right| \\
=& \left|\frac{\vartheta^2}{4} \int_{\mathbb{R}^3} |x|^{2\gamma} \nabla^5 (\varrho \operatorname{div} u) \cdot \nabla^5 \varrho \,\mathrm{d}x\right| + \left|\frac{\vartheta^2}{4} \int_{\mathbb{R}^3} |x|^{2\gamma} \nabla^5 (u \cdot \nabla \varrho) \cdot \nabla^5 \varrho \,\mathrm{d}x\right| \\
:=& \sum_{i=1}^{2} J_{6,i}.
\end{aligned} \tag{3.58}
$$

For the term $J_{6,1}$, we have

$$
\begin{aligned}
J_{6,1} \lesssim & \left|\int_{\mathbb{R}^3} |x|^{2\gamma} \varrho \cdot \nabla^6 u \cdot \nabla^5 \varrho \mathrm{d}x\right| + \left|\int_{\mathbb{R}^3} |x|^{2\gamma} \nabla \varrho \cdot \nabla^5 u \cdot \nabla^5 \varrho \mathrm{d}x\right| \\
& + \left|\int_{\mathbb{R}^3} |x|^{2\gamma} \nabla^2 \varrho \cdot \nabla^4 u \cdot \nabla^5 \varrho \mathrm{d}x\right| + \left|\int_{\mathbb{R}^3} |x|^{2\gamma} \nabla^3 \varrho \cdot \nabla^3 u \cdot \nabla^5 \varrho \mathrm{d}x\right| \\
& + \left|\int_{\mathbb{R}^3} |x|^{2\gamma} \nabla^4 \varrho \cdot \nabla^2 u \cdot \nabla^5 \varrho \mathrm{d}x\right| + \left|\int_{\mathbb{R}^3} |x|^{2\gamma} \nabla^5 \varrho \cdot \nabla u \cdot \nabla^5 \varrho \mathrm{d}x\right| \\
& := \sum_{i=1}^{6} J_{6,1}^i.
\end{aligned} \tag{3.59}
$$

By virtue of Hölder's inequality, Cauchy's inequality, Lemma 2.1, and Proposition 1.1, we have

$$
\begin{aligned}
|J_{6,1}^1| \ \lesssim\ & t^{-\frac{13}{2}+2\gamma} + t^{-\frac{5}{4}} \|\nabla^4 u\|^2_{L^2_\gamma} + \|\nabla \varrho\|_{H^2} \|\nabla^5 \varrho\|^2_{L^2_\gamma} \\
& + \|\nabla^5 \varrho\|^2_{L^2_{\gamma-1}} + \varepsilon \|\nabla^5 u\|^2_{L^2_\gamma},
\end{aligned} \tag{3.60}
$$

and

$$
\begin{aligned}
|J_{6,1}^2| + |J_{6,1}^6| \ \lesssim\ & \|\nabla(\varrho, u)\|_{L^\infty} \|\nabla^5 (\varrho, u)\|_{L^2_\gamma} \|\nabla^5 \varrho\|_{L^2_\gamma} \\
\lesssim\ & \|\nabla(\varrho, u)\|_{H^2} \|\nabla^5 \varrho\|^2_{L^2_\gamma} + \varepsilon \|\nabla^5 u\|^2_{L^2_\gamma}.
\end{aligned} \tag{3.61}
$$

Taking the summation of the term $J_{6,1}^3$ and the term $J_{6,1}^5$ , we can arrive that

$$
\begin{aligned}
|J_{6,1}^3| + |J_{6,1}^5| \ \lesssim\ & \|\nabla^2 (\varrho, u)\|_{L^3} \|\nabla^4 (\varrho, u)\|_{L^6_\gamma} \|\nabla^5 \varrho\|_{L^2_\gamma} \\
\lesssim\ & \|\nabla^2 (\varrho, u)\|_{L^3} \left(\|\nabla^5 (\varrho, u)\|_{L^2_\gamma} + \|\nabla^4 (\varrho, u)\|_{L^2_{\gamma-1}}\right) \|\nabla^5 \varrho\|_{L^2_\gamma} \\
\lesssim\ & \|\nabla (\varrho, u)\|_{H^2} \|\nabla^5 \varrho\|^2_{L^2_\gamma} + \|\nabla^4 (\varrho, u)\|^2_{L^2_{\gamma-1}} + \varepsilon \|\nabla^5 u\|^2_{L^2_\gamma}.
\end{aligned} \tag{3.62}
$$

For the fourth term in the right–hand side of (3.59), using integration by parts, Hölder's inequality, Cauchy's inequality, the Sobolev inequality, Lemma 2.1, and Proposition 1.1, we have

$$\begin{aligned} \left|J_{6,1}^4\right| &\lesssim \|\nabla^3\varrho\|_{L^3}\left\|\nabla^5\varrho\right\|_{L^2_\gamma}\|\nabla^3 u\|_{L^6_\gamma} \\ &\lesssim t^{-\frac{13}{2}+2\gamma}+t^{-4}\|\nabla^5\varrho\|^2_{L^2_\gamma}. \end{aligned} \tag{3.63}$$

Combining the estimates (3.60)–(3.63) into (3.59), we get

$$\left|J_{6,1}\right| \lesssim t^{-\frac{13}{2}+2\gamma}+t^{-\frac{5}{4}}\|\nabla^5 u\|^2_{L^2_\gamma}+\|\nabla^4(\nabla\varrho,\varrho,u)\|^2_{L^2_{\gamma-1}}+\varepsilon\|\nabla^5 u\|^2_{L^2_\gamma}+\|\nabla(\varrho,u)\|_{H^2}\|\nabla^5\varrho\|^2_{L^2_\gamma}. \tag{3.64}$$

For the term $J_{6,2}$, we have

$$\begin{aligned} J_{6,2} \lesssim &\left|\int_{\mathbb{R}^3}|x|^{2\gamma}u\cdot\nabla^6\varrho\cdot\nabla^5\varrho\mathrm{d}x\right|+\left|\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla u\cdot\nabla^5\varrho\cdot\nabla^5\varrho\mathrm{d}x\right| \\ &+\left|\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^2 u\cdot\nabla^4\varrho\cdot\nabla^5\varrho\mathrm{d}x\right|+\left|\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^3 u\cdot\nabla^3\varrho\cdot\nabla^5\varrho\mathrm{d}x\right| \\ &+\left|\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^4 u\cdot\nabla^2\varrho\cdot\nabla^5\varrho\mathrm{d}x\right|+\left|\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^5 u\cdot\nabla\varrho\cdot\nabla^5\varrho\mathrm{d}x\right| \\ &:=\sum_{i=1}^{6}J_{6,2}^i. \end{aligned} \tag{3.65}$$

Next, we estimate six terms in the right–hand side of the above equation. For the first term, it follows from integration by parts that

$$|J_{6,2}^1| \lesssim \|\nabla u\|_{H^2}\|\nabla^5\varrho\|^2_{L^2_\gamma}+\|\nabla^5\varrho\|^2_{L^2_{\gamma-1}}. \tag{3.66}$$

Similar to (3.61)–(3.62), we can obtain

$$|J_{6,2}^2|+|J_{6,2}^6| \lesssim \|\nabla(\varrho,u)\|_{H^2}\|\nabla^5\varrho\|^2_{L^2_\gamma}+\varepsilon\|\nabla^5 u\|^2_{L^2_\gamma}. \tag{3.67}$$

and

$$|J_{6,2}^3|+|J_{6,2}^5| \lesssim \|\nabla(\varrho,u)\|_{H^2}\|\nabla^5\varrho\|^2_{L^2_\gamma}+\|\nabla^4(\varrho,u)\|^2_{L^2_{\gamma-1}}+\varepsilon\|\nabla^5 u\|^2_{L^2_\gamma}. \tag{3.68}$$

For the fourth term in the right–hand side of (3.65), using integration by parts, Hölder's inequality, Cauchy's inequality, the Sobolev inequality, Lemma 2.1, and Proposition 1.1, we have

$$\begin{aligned} \left|J_{6,2}^4\right| &\lesssim \|\nabla^3 u\|_{L^3}\left\|\nabla^5\varrho\right\|_{L^2_\gamma}\|\nabla^3\varrho\|_{L^6_\gamma} \\ &\lesssim t^{-\frac{13}{2}+2\gamma}+t^{-4}\|\nabla^5\varrho\|^2_{L^2_\gamma}. \end{aligned} \tag{3.69}$$

Combining the estimates (3.66)–(3.69) into (3.65), we get

$$\left|J_{6,2}\right| \lesssim \|\nabla(\varrho,u)\|_{H^2}\|\nabla^5\varrho\|^2_{L^2_\gamma}+\|\nabla^4(\nabla\varrho,\varrho,u)\|^2_{L^2_{\gamma-1}}+\varepsilon\|\nabla^5 u\|^2_{L^2_\gamma}+t^{-\frac{13}{2}+2\gamma}. \tag{3.70}$$

Thus, it is easy to deduce that

$$
\begin{aligned}
|J_6| &\lesssim \|\nabla(\varrho,u)\|_{H^2}\|\nabla^5\varrho\|^2_{L^2_\gamma}+\|\nabla^4(\nabla\varrho,\varrho,u)\|^2_{L^2_{\gamma-1}}+t^{-\frac{13}{2}+2\gamma}\\
&\quad+t^{-\frac{5}{4}}\|\nabla^5 u\|^2_{L^2_\gamma}+\varepsilon\|\nabla^5 u\|^2_{L^2_\gamma}.
\end{aligned}\tag{3.71}
$$

For the term $J_7$ , using integration by parts, Hölder's inequality, Cauchy's inequality and Lemma 2.1, we have

$$
\begin{aligned}
|J_7| &= \left|\frac{\vartheta^2}{4}\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma}\nabla^4 u)\nabla^3\left(\frac{|\nabla\varrho|^2\nabla\varrho}{(1+\varrho)^3}-\frac{\nabla\varrho\Delta\varrho}{(1+\varrho)^2}-\frac{\nabla\varrho\cdot\nabla^2\varrho}{(1+\varrho)^2}\right)\mathrm{d}x\right|\\
&\lesssim \left|\frac{\vartheta^2}{4}\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\nabla^3\left(\frac{|\nabla\varrho|^2\nabla\varrho}{(1+\varrho)^3}-\frac{\nabla\varrho\Delta\varrho}{(1+\varrho)^2}-\frac{\nabla\varrho\cdot\nabla^2\varrho}{(1+\varrho)^2}\right)\nabla^4 u\,\mathrm{d}x\right|\\
&\quad+\left|\frac{\vartheta^2}{4}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^3\left(\frac{|\nabla\varrho|^2\nabla\varrho}{(1+\varrho)^3}-\frac{\nabla\varrho\Delta\varrho}{(1+\varrho)^2}-\frac{\nabla\varrho\cdot\nabla^2\varrho}{(1+\varrho)^2}\right)\nabla^5 u\,\mathrm{d}x\right|\\
&\lesssim \|\nabla\varrho\|_{L^\infty}\left\|\nabla^4 u\right\|_{L^2_\gamma}\left\|\nabla^5\varrho\right\|_{L^2_{\gamma-1}}+\left\|\nabla^2\varrho\right\|_{L^\infty}\left\|\nabla^4 u\right\|_{L^2_\gamma}\left\|\nabla^4\varrho\right\|_{L^2_{\gamma-1}}\\
&\quad+\left\|\nabla^3\varrho\right\|_{L^\infty}\left\|\nabla^4 u\right\|_{L^2_\gamma}\left\|\nabla^3\varrho\right\|_{L^2_{\gamma-1}}+\|\nabla\varrho\|_{L^\infty}\left\|\nabla^5 u\right\|_{L^2_\gamma}\left\|\nabla^5\varrho\right\|_{L^2_\gamma}\\
&\quad+\left\|\nabla^2\varrho\right\|_{L^\infty}\left\|\nabla^5 u\right\|_{L^2_\gamma}\left\|\nabla^4\varrho\right\|_{L^2_\gamma}+\left\|\nabla^3\varrho\right\|_{L^\infty}\left\|\nabla^5 u\right\|_{L^2_\gamma}\left\|\nabla^3\varrho\right\|_{L^2_\gamma}\\
&\lesssim \left\|\nabla^2\varrho\right\|_{H^3}\left\|\nabla^4(\nabla\varrho,\varrho,u)\right\|^2_{L^2_\gamma}+\left\|\nabla^2\varrho\right\|_{H^2}\left\|\nabla^4(\nabla\varrho,\varrho)\right\|^2_{L^2_{\gamma-1}}+\varepsilon\left\|\nabla^5 u\right\|^2_{L^2_\gamma}+t^{-\frac{13}{2}+2\gamma}.
\end{aligned}\tag{3.72}
$$

For the term $|\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^4\varrho\cdot\nabla^4 G_1\,\mathrm{d}x|$, we have

$$
\begin{aligned}
\left|\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^4\varrho\nabla^4 G_1\,\mathrm{d}x\right| &\lesssim \left|\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^4(\varrho\mathrm{div}u)\cdot\nabla^4\varrho\,\mathrm{d}x\right|+\left|\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^4(u\cdot\nabla\varrho)\nabla^4\varrho\,\mathrm{d}x\right|\\
&:= J_9+J_{10}.
\end{aligned}\tag{3.73}
$$

For the term $J_9$, using Hölder's inequality, we have

$$
\begin{aligned}
|J_9| &\lesssim \left|\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^4\varrho\cdot\nabla u\cdot\nabla^4\varrho\,\mathrm{d}x\right|+\left|\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^3\varrho\cdot\nabla^2 u\cdot\nabla^4\varrho\,\mathrm{d}x\right|\\
&\quad+\left|\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^2\varrho\cdot\nabla^3 u\cdot\nabla^4\varrho\,\mathrm{d}x\right|+\left|\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla\varrho\cdot\nabla^4 u\cdot\nabla^4\varrho\,\mathrm{d}x\right|\\
&\quad+\left|\int_{\mathbb{R}^3}|x|^{2\gamma}\varrho\cdot\nabla^5 u\cdot\nabla^4\varrho\,\mathrm{d}x\right|\\
&:= \sum_{i=1}^5 J_{9,i}.
\end{aligned}\tag{3.74}
$$

Using Hölder's inequality, Cauchy's inequality, the Sobolev inequality and the lower bound of density, we can obtain

$$
\begin{aligned}
\left|J_{9,1}\right|+\left|J_{9,4}\right| &\lesssim \|\nabla(\varrho,u)\|_{L^\infty}\left\|\nabla^4(\varrho,u)\right\|_{L^2_\gamma}\left\|\nabla^4\varrho\right\|_{L^2_\gamma}\\
&\lesssim \left\|\nabla^2(\varrho,u)\right\|_{H^1}\left\|\nabla^4(\varrho,u)\right\|^2_{L^2_\gamma},
\end{aligned}\tag{3.75}
$$

$$
\begin{aligned}
\left|J_{9,2}\right| + \left|J_{9,3}\right| &\lesssim \left\|\nabla^2(\varrho, u)\right\|_{L^\infty} \left\|\nabla^3(\varrho, u)\right\|_{L^2_\gamma} \left\|\nabla^4\varrho\right\|_{L^2_\gamma} \\
&\lesssim \left\|\nabla^3(\varrho, u)\right\|_{H^1} \left\|\nabla^4\varrho\right\|^2_{L^2_\gamma} + \left\|\nabla^3(\varrho, u)\right\|_{H^1} \left\|\nabla^3(\varrho, u)\right\|^2_{L^2_\gamma},
\end{aligned}
\tag{3.76}
$$

$$
\begin{aligned}
\left|J_{9,5}\right| &\lesssim \|\varrho\|_{L^\infty} \left\|\nabla^5 u\right\|_{L^2_\gamma} \left\|\nabla^4\varrho\right\|_{L^2_\gamma} \\
&\lesssim \|\nabla\varrho\|^2_{H^1} \left\|\nabla^4\varrho\right\|^2_{L^2_\gamma} + \varepsilon \left\|\nabla^5 u\right\|^2_{L^2_\gamma}.
\end{aligned}
\tag{3.77}
$$

Substituting (3.75)–(3.77) into (3.74) and using the uniform bound (1.8), we get

$$
|J_9| \lesssim \|\nabla(\varrho, u)\|_{H^3} \left\|\nabla^4(\varrho, u)\right\|^2_{L^2_\gamma} + \left\|\nabla^3(\varrho, u)\right\|_{H^1} \left\|\nabla^3(\varrho, u)\right\|^2_{L^2_\gamma} + \varepsilon \left\|\nabla^5 u\right\|^2_{L^2_\gamma}. \tag{3.78}
$$

For the term $J_{10}$, its method is consistent with that of the term $J_9$, so it is easy to get

$$
|J_{10}| \lesssim \|\nabla(\varrho, u)\|_{H^3} \left\|\nabla^4(\nabla\varrho, \varrho, u)\right\|^2_{L^2_\gamma} + \left\|\nabla^3(\varrho, u)\right\|_{H^1} \left\|\nabla^3(\varrho, u)\right\|^2_{L^2_\gamma}. \tag{3.79}
$$

Substituting (3.78) and (3.79) into (3.73), we get

$$
\begin{aligned}
\left|\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^4\varrho\nabla^4 G_1 \,\mathrm{d}x\right| &\lesssim \|\nabla(\varrho, u)\|_{H^3} \left\|\nabla^4(\nabla\varrho, \varrho, u)\right\|^2_{L^2_\gamma} \\
&\quad + \left\|\nabla^3(\varrho, u)\right\|_{H^1} \left\|\nabla^3(\varrho, u)\right\|^2_{L^2_\gamma} + \varepsilon \left\|\nabla^5 u\right\|^2_{L^2_\gamma}.
\end{aligned}
\tag{3.80}
$$

Next, for the term $\left|\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^4 u\nabla^4 G_4 \,\mathrm{d}x\right|$, we have

$$
\begin{aligned}
\left|\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^4 u\nabla^4 G_4 \,\mathrm{d}x\right| &\lesssim \left|\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^4 u\nabla^4(u\cdot\nabla u) \,\mathrm{d}x\right| \\
&\quad + \left|\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^4 u\nabla^4(f(\varrho)[\mu\Delta u + (\mu + \lambda\nabla\mathrm{div}u)]) \,\mathrm{d}x\right| \\
&\quad + \left|\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^4 u\nabla^4(F(\varrho)\nabla\varrho) \,\mathrm{d}x\right| \\
&\quad + \left|\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^4 u\nabla^4(g(\varrho)[B\cdot\nabla B - \frac{1}{2}\nabla(|B|^2)]) \,\mathrm{d}x\right| \\
&:= \sum_{i=11}^{14} J_i.
\end{aligned}
\tag{3.81}
$$

By using the Leibniz formula, we have

$$
\begin{aligned}
|J_{11}| &\lesssim \left|\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^4 u\cdot\nabla^4 u\cdot\nabla u \,\mathrm{d}x\right| + \left|\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^3 u\cdot\nabla^2 u\cdot\nabla^4 u \,\mathrm{d}x\right| \\
&\quad + \left|\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^3 u\cdot\nabla^2 u\cdot\nabla^4 u \,\mathrm{d}x\right| + \left|\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^4 u\cdot\nabla^4 u\cdot\nabla u \,\mathrm{d}x\right| \\
&\quad + \left|\int_{\mathbb{R}^3} |x|^{2\gamma} u\cdot\nabla^4 u\cdot\nabla^5 u \,\mathrm{d}x\right| \\
&:= \sum_{i=1}^{5} J_{11,i}.
\end{aligned}
\tag{3.82}
$$

Using Hölder's inequality, Cauchy's inequality, the Sobolev inequality and the lower bound of density, we can obtain

$$\left|J_{11,1}\right| + \left|J_{11,4}\right| \lesssim \left\|\nabla^2 u\right\|_{H^1} \left\|\nabla^4 u\right\|_{L^2_\gamma}^2 . \tag{3.83}$$

$$\left|J_{11,2}\right| + \left|J_{11,3}\right| \lesssim \left\|\nabla^3 u\right\|_{H^1} \left\|\nabla^4 u\right\|_{L^2_\gamma}^2 + \left\|\nabla^3 u\right\|_{H^1} \left\|\nabla^3 u\right\|_{L^2_\gamma}^2 , \tag{3.84}$$

$$\begin{aligned} \left|J_{11,5}\right| &\lesssim \|u\|_{L^\infty} \left\|\nabla^5 u\right\|_{L^2_\gamma} \left\|\nabla^4 u\right\|_{L^2_\gamma} \\ &\lesssim \|\nabla u\|_{H^1}^2 \left\|\nabla^4 u\right\|_{L^2_\gamma}^2 + \varepsilon \left\|\nabla^5 u\right\|_{L^2_\gamma}^2 . \end{aligned} \tag{3.85}$$

Therefore, we have

$$|J_{11}| \lesssim \|\nabla u\|_{H^3} \left\|\nabla^4 u\right\|_{L^2_\gamma}^2 + \left\|\nabla^3 u\right\|_{H^1} \left\|\nabla^3 u\right\|_{L^2_\gamma}^2 + \varepsilon \left\|\nabla^5 u\right\|_{L^2_\gamma}^2 . \tag{3.86}$$

Next, we use Hölder's inequality, Cauchy's inequality, the Sobolev inequality and mathematical induction to get

$$|J_{12}| \lesssim \|\nabla \varrho\|_{H^1}^2 \left\|\nabla^5 u\right\|_{L^2_\gamma}^2 + \left\|\nabla^4 u\right\|_{L^2_{\gamma-1}}^2 + \left\|\nabla^2 \varrho\right\|_{H^1}^2 (\left\|\nabla^4 u\right\|_{L^2_\gamma}^2 + \left\|\nabla^3 u\right\|_{L^2_{\gamma-1}}^2), \tag{3.87}$$

$$\begin{aligned} |J_{13}| \lesssim{}& \varepsilon \left\|\nabla^5 u\right\|_{L^2_\gamma}^2 + \left\|\nabla^4 u\right\|_{L^2_{\gamma-1}}^2 \\ &+ \|\nabla \varrho\|_{H^1}^2 (\left\|\nabla^4 \varrho\right\|_{L^2_\gamma}^2 + \left\|\nabla^3 \varrho\right\|_{L^2_{\gamma-1}}^2), \end{aligned} \tag{3.88}$$

$$\begin{aligned} |J_{14}| \lesssim{}& \varepsilon \left\|\nabla^5 u\right\|_{L^2_\gamma}^2 + \left\|\nabla^4 u\right\|_{L^2_{\gamma-1}}^2 + \|\nabla(\varrho, B)\|_{H^1}^2 (\left\|\nabla^4 (\varrho, B)\right\|_{L^2_\gamma}^2 + \left\|\nabla^3 (\varrho, B)\right\|_{L^2_{\gamma-1}}^2) \\ &+ \|\nabla B\|_{H^1}^2 \left\|\nabla^2 B\right\|_{L^2_{\gamma-2}}^2 . \end{aligned} \tag{3.89}$$

Substituting (3.86)–(3.89) into (3.81), and using Hölder's inequality, Cauchy's inequality, and the Sobolev inequality, we can obtain

$$\begin{aligned} \left|\int_{\mathbb{R}^3} |x|^{2\gamma} \nabla^4 u \nabla^4 G_4 \, \mathrm{d}x\right| \lesssim{}& \varepsilon \left\|\nabla^5 u\right\|_{L^2_\gamma}^2 + \left\|\nabla^4 u\right\|_{L^2_{\gamma-1}}^2 + \|\nabla(\varrho, u, B)\|_{H^3} \left\|\nabla^4 (\varrho, u, B)\right\|_{L^2_\gamma}^2 \\ &+ \|\nabla(\varrho, B)\|_{H^2}^2 \left\|\nabla^3 (\varrho, u, B)\right\|_{L^2_{\gamma-1}}^2 + \left\|\nabla^3 u\right\|_{H^1} \left\|\nabla^3 u\right\|_{L^2_\gamma}^2 \\ &+ \|\nabla B\|_{H^1}^2 \left\|\nabla^2 B\right\|_{L^2_{\gamma-2}}^2 + \|\nabla \varrho\|_{H^1}^2 \left\|\nabla^5 u\right\|_{L^2_\gamma}^2 . \end{aligned} \tag{3.90}$$

Finally, for the term $\left|\int_{\mathbb{R}^3} |x|^{2\gamma} \nabla^4 B \nabla^4 G_3 \, \mathrm{d}x\right|$, we have

$$\begin{aligned} \left|\int_{\mathbb{R}^3} |x|^{2\gamma} \nabla^4 B \nabla^4 G_3 \, \mathrm{d}x\right| \lesssim{}& \left|\int_{\mathbb{R}^3} |x|^{2\gamma} \nabla^4 B \nabla^4 (u \cdot \nabla B) \, \mathrm{d}x\right| + \left|\int_{\mathbb{R}^3} |x|^{2\gamma} \nabla^4 B \nabla^4 (B \cdot \nabla u) \, \mathrm{d}x\right| \\ &+ \left|\int_{\mathbb{R}^3} |x|^{2\gamma} \nabla^4 B \nabla^4 (B \mathrm{div} u) \, \mathrm{d}x\right| + \left|\int_{\mathbb{R}^3} |x|^{2\gamma} \nabla^4 B \nabla^4 (u \mathrm{div} B) \, \mathrm{d}x\right| \\ :={}& \sum_{i=15}^{18} J_i . \end{aligned} \tag{3.91}$$

By using the Leibniz formula, we have

$$
\begin{aligned}
|J_{15}| \lesssim & \left|\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^4 B\cdot\nabla^4 u\cdot\nabla B\,\mathrm{d}x\right| + \left|\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^4 B\cdot\nabla^3 u\cdot\nabla^2 B\,\mathrm{d}x\right| \\
& + \left|\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^4 B\cdot\nabla^2 u\cdot\nabla^3 B\,\mathrm{d}x\right| + \left|\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^4 B\cdot\nabla^4 B\cdot\nabla u\,\mathrm{d}x\right| \\
& + \left|\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^4 B\cdot u\cdot\nabla^5 B\,\mathrm{d}x\right| \\
:= & \sum_{i=1}^{5} J_{15,i}.
\end{aligned} \tag{3.92}
$$

Using Hölder's inequality, Cauchy's inequality, the Sobolev inequality and the lower bound of density, we can obtain

$$
\left|J_{15,1}\right| + \left|J_{15,4}\right| \lesssim \left\|\nabla^2(u,B)\right\|_{H^1}\left\|\nabla^4(u,B)\right\|_{L^2_\gamma}^2, \tag{3.93}
$$

$$
\left|J_{15,2}\right| + \left|J_{15,3}\right| \lesssim \left\|\nabla^3(u,B)\right\|_{H^1}\left\|\nabla^3(u,B)\right\|_{L^2_\gamma}^2 + \left\|\nabla^3(u,B)\right\|_{H^1}\left\|\nabla^4 B\right\|_{L^2_\gamma}^2, \tag{3.94}
$$

$$
\begin{aligned}
\left|J_{15,5}\right| & \lesssim \|u\|_{L^\infty}\left\|\nabla^5 B\right\|_{L^2_\gamma}\left\|\nabla^4 B\right\|_{L^2_\gamma} \\
& \lesssim \|\nabla u\|_{H^1}^2\left\|\nabla^4 B\right\|_{L^2_\gamma}^2 + \varepsilon\left\|\nabla^5 B\right\|_{L^2_\gamma}^2.
\end{aligned} \tag{3.95}
$$

Therefore, we have

$$
|J_{15}| \lesssim \|\nabla(u,B)\|_{H^3}\left\|\nabla^4(u,B)\right\|_{L^2_\gamma}^2 + \left\|\nabla^3(u,B)\right\|_{H^1}\left\|\nabla^3(u,B)\right\|_{L^2_\gamma}^2 + \varepsilon\left\|\nabla^5 B\right\|_{L^2_\gamma}^2. \tag{3.96}
$$

Similar to the $J_{15}$, we have

$$
|J_{16}|, |J_{17}| \lesssim \|\nabla(u,B)\|_{H^3}\left\|\nabla^4(u,B)\right\|_{L^2_\gamma}^2 + \left\|\nabla^3(u,B)\right\|_{H^1}\left\|\nabla^3(u,B)\right\|_{L^2_\gamma}^2 + \varepsilon\left\|\nabla^5 u\right\|_{L^2_\gamma}^2 \tag{3.97}
$$

and

$$
|J_{18}| \lesssim \|\nabla(u,B)\|_{H^3}\left\|\nabla^4(u,B)\right\|_{L^2_\gamma}^2 + \left\|\nabla^3(u,B)\right\|_{H^1}\left\|\nabla^3(u,B)\right\|_{L^2_\gamma}^2 + \varepsilon\left\|\nabla^5 B\right\|_{L^2_\gamma}^2. \tag{3.98}
$$

Substituting (3.96)–(3.98) into (3.91), we get

$$
\begin{aligned}
\left|\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^4 B\nabla^4 G_3\,\mathrm{d}x\right| \lesssim & \|\nabla(u,B)\|_{H^3}\left\|\nabla^4(u,B)\right\|_{L^2_\gamma}^2 + \left\|\nabla^3(u,B)\right\|_{H^1}\left\|\nabla^3(u,B)\right\|_{L^2_\gamma}^2 \\
& + \varepsilon\left\|\nabla^5 B\right\|_{L^2_\gamma}^2 + \varepsilon\left\|\nabla^5 u\right\|_{L^2_\gamma}^2.
\end{aligned} \tag{3.99}
$$

In the end, substituting (3.53)–(3.57), (3.71), (3.72), (3.80), (3.90) and (3.99) into (3.44), and using the uniform bound and the smallness of $\varepsilon$, we can conclude that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\vartheta^2}{4}\|\nabla^5\varrho\|_{L^2_\gamma}^2+\|\nabla^4\varrho\|_{L^2_\gamma}^2+\|\nabla^4u\|_{L^2_\gamma}^2+\|\nabla^4B\|_{L^2_\gamma}^2\right)+\mu\|\nabla^5u\|_{L^2_\gamma}^2\\
&\quad+(\mu+\lambda)\|\nabla^4\mathrm{div}u\|_{L^2_\gamma}^2+\nu\|\nabla^5B\|_{L^2_\gamma}^2\\
&\lesssim\|\nabla(\varrho,u,B)\|_{H^3}\left\|\nabla^4(\nabla\varrho,\varrho,u,B)\right\|_{L^2_\gamma}^2+\left\|\nabla^2\varrho\right\|_{H^3}\left\|\nabla^4(\nabla\varrho,\varrho,u)\right\|_{L^2_\gamma}^2+\|\nabla^4\varrho\|_{L^2_\gamma}\|\nabla^4u\|_{L^2_{\gamma-1}}\\
&\quad+\left\|\nabla^4(\nabla\varrho,\varrho,u,B)\right\|_{L^2_{\gamma-1}}^2+\|\nabla\varrho\|_{H^1}\left\|\nabla^5u\right\|_{L^2_\gamma}^2+\varepsilon\mu\|\nabla^5u\|_{L^2_\gamma}^2+\varepsilon(\mu+\lambda)\|\nabla^4\operatorname{div}u\|_{L^2_\gamma}^2\\
&\quad+t^{-\frac{5}{4}}\|\nabla^5u\|_{L^2_\gamma}^2+t^{-\frac{13}{2}+2\gamma}+\varepsilon\|\nabla^5u\|_{L^2_\gamma}^2+\varepsilon\|\nabla^5B\|_{L^2_\gamma}^2+\|\nabla(\varrho,B)\|_{H^2}^2\left\|\nabla^3(\varrho,u,B)\right\|_{L^2_{\gamma-1}}^2\\
&+\left\|\nabla^3(\varrho,u,B)\right\|_{H^1}\left\|\nabla^3(\varrho,u,B)\right\|_{L^2_\gamma}^2+\|\nabla B\|_{H^1}^2\left\|\nabla^2B\right\|_{L^2_{\gamma-2}}^2.
\end{aligned}
\tag{3.100}
$$

Using the inequality $\|\nabla^kf\|_{L^2_{\gamma-1}}\le\|\nabla^kf\|_{L^2_\gamma}^{\frac{\gamma-1}{\gamma}}\|\nabla^kf\|_{L^2}^{\frac{1}{\gamma}}$, and by leveraging the uniform-in-time bounds established in Proposition (1.1), the assumption of small $\varepsilon$, and a sufficiently large positive time $T_5$, we can derive the desired conclusion:

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\vartheta^2}{4}\|\nabla^5\varrho\|_{L^2_\gamma}^2+\|\nabla^4\varrho\|_{L^2_\gamma}^2+\|\nabla^4u\|_{L^2_\gamma}^2+\|\nabla^4B\|_{L^2_\gamma}^2\right)+\mu\|\nabla^5u\|_{L^2_\gamma}^2\\
&\quad+(\mu+\lambda)\|\nabla^4\mathrm{div}u\|_{L^2_\gamma}^2+\nu\|\nabla^5B\|_{L^2_\gamma}^2\\
&\lesssim\|\nabla(\varrho,u,B)\|_{H^3}\left\|\nabla^4(\nabla\varrho,\varrho,u,B)\right\|_{L^2_\gamma}^2+\left\|\nabla^2\varrho\right\|_{H^3}\left\|\nabla^4(\nabla\varrho,\varrho,u)\right\|_{L^2_\gamma}^2\\
&\quad+\|\nabla^4\varrho\|_{L^2_\gamma}\|\nabla^4u\|_{L^2_\gamma}^{\frac{\gamma-1}{\gamma}}\|\nabla^4u\|_{L^2}^{\frac{1}{\gamma}}+\left\|\nabla^4(\nabla\varrho,\varrho,u,B)\right\|_{L^2_\gamma}^{\frac{2(\gamma-1)}{\gamma}}+\left\|\nabla^4(\nabla\varrho,\varrho,u,B)\right\|_{L^2}^{\frac{2}{\gamma}}+t^{-\frac{13}{2}+2\gamma}
\end{aligned}
\tag{3.101}
$$

for all t $\geq T_5$. Then, we are able to define the temporal energy functional as follows:

$$
\tilde{\mathrm{E}}(t):=\|\nabla^5\varrho\|_{L^2_\gamma}^2+\|\nabla^4\varrho\|_{L^2_\gamma}^2+\|\nabla^4u\|_{L^2_\gamma}^2+\|\nabla^4B\|_{L^2_\gamma}^2. \tag{3.102}
$$

By direct calculation, we can obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathrm{E}}(t)\le C_0t^{-\frac{5}{4}}\tilde{\mathrm{E}}(t)+C_1t^{-\frac{11}{4\gamma}}\tilde{\mathrm{E}}(t)^{\frac{2\gamma-1}{2\gamma}}+C_2t^{-\frac{11}{2\gamma}}\tilde{\mathrm{E}}(t)^{\frac{\gamma-1}{\gamma}}+C_3t^{-\frac{13}{2}+\gamma}. \tag{3.103}
$$

Here $\tilde{\alpha}_0=\frac{5}{4}$, $\tilde{\alpha}_1=-\frac{11}{4\gamma}$, $\tilde{\beta}_1=\frac{2\gamma-1}{2\gamma}$, $\tilde{\alpha}_2=\frac{11}{2\gamma}$, $\tilde{\beta}_2=\frac{\gamma-1}{\gamma}$ . To assure that $\tilde{\alpha}_1<1$, $\tilde{\alpha}_2<1$, we require $\gamma>\frac{11}{2}$. Hence, $\tilde{\gamma}_1=\frac{1-\tilde{\alpha}_1}{1-\beta_1}=2\gamma-\frac{11}{2}>\tilde{\gamma}_2=\frac{1-\tilde{\alpha}_2}{1-\beta_2}=\gamma-\frac{11}{2}$. Thus, with the help of Lemma 2.2, we deduce from (3.272) that

$$
\tilde{\mathrm{E}}(t)\le Ct^{\tilde{\gamma}_1}=Ct^{-\frac{11}{2}+2\gamma}. \tag{3.104}
$$

Applying the interpolation inequality into (3.273) yields

$$
\|\nabla^4(\nabla\varrho,\varrho,u,B)(t)\|_{L^2_{\gamma_0}}\le C\|\nabla^4(\nabla\varrho,\varrho,u,B)(t)\|_{L^2}^{1-\frac{\gamma_0}{\gamma}}\|\nabla^4(\nabla\varrho,\varrho,u,B)(t)\|_{L^2_\gamma}^{\frac{\gamma_0}{\gamma}}\le Ct^{-\frac{11}{4}+\gamma_0}, \tag{3.105}
$$

for all $\gamma_0\in[0,\gamma]$, and this completes the proof. □

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare there are no conflicts of interest.

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