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Research article

Well-posedness for a coupled system of gKdV equations in analytic spaces

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Abstract: In this article, a Cauchy problem for a coupled system of the generalized Korteweg-de Vries equations (gKdV) is considered. In the periodic case, it is shown that the system is locally well-posed in a large class of analytic functions and conditions for which weak solutions extend holomorphically in a symmetric strip of the complex plane around the x-axis at large times. In addition, the uniform analyticity radius of the solution does not change as time progresses. Also, information about the regularity of the solution in the time variable is obtained.

Keywords: nonlinear equations; gKdV system; local well-posedness; energy and industry; analytic spaces

1. Introduction

Equation

$$\partial_t u + a\partial_x u + \partial_x^3 u + g(u)\partial_x u = 0,$$

includes and models important phenomena in the propagation of nonlinear waves applied in energy and industry named the Korteweg-de Vries equation for $g(u) = u$ and the modified Korteweg-de Vries equation for $g(u) = \pm u^2$, which describes the propagation of one-dimensional nonlinear waves in media with and without dissipation. Similar issues were previously considered in articles [1–6]. Article [7] considers the problem for the equation

$$\partial_t u + \partial_x u + \partial_x^3 u + g(u)\partial_x u + b(x)u = 0, \tag{1.1}$$

is considered. It is assumed that the function b is non-negative and lies in the space $L^2(0, L)$. The function g is such that $g(0) = 0$ and satisfies a certain condition. It was shown that the problem (1.1) has a unique solution in the space

$$C([0,T]; L^2(0,L)) \cap L^2(0,T; H^1(0,L)).$$

If it is additionally known that the support of the function b contains an open non-empty subset $(0,L)$, then the decay rate for such solutions is obtained. Earlier in the article [8], for problem (1.1), similar results were obtained for $p = 1$, and stronger conditions on b are satisfied than in the article [7].

In [9] it was shown that in the case $g(u) = u^4$, if $u_0 \in L^2(0,L)$ has a sufficiently small norm, problem (1.1) has a solution in the space

$$C([0,T]; L^2(0,L)) \cap L^2(0,T; H^1(0,L)),\ T > 0.$$

In [10], the questions of existence, uniqueness of solutions, and their decrease at large times for the initial boundary value problem in the case of more general equations were considered; see [11–13].

For $u = u(x,t)$ and $v = v(x,t)$, a coupled system is considered in the Hamiltonian form.

$$\begin{cases} \partial_t u + \partial_x^3 u + \mu\partial_x \left(G_u(u,v)\right) = 0 \\ \partial_t v + \partial_x^3 v + \mu\partial_x \left(G_v(u,v)\right) = 0, \end{cases} \tag{1.2}$$

where u and v are real-valued functions, G_u and G_v are the derivatives of a smooth function G with respect to u and v, respectively, and $\mu > 0$, which we normalize to be ± 1.

For $\mu = 1$ and

$$G(u,v) = Au^3 + Bv^3 + Cu^2v + Duv^2,$$

with A, B, C, and D being real constants, the system was considered in [14]. If

$$G(u,v) = u^2v$$

system (1.2) is a special case of the system treated in [15].

We consider the IVP for a coupled system of gKdV equations.

$$\begin{cases} \partial_t u + \partial_x^3 u + \mu\partial_x(f(u,v)) = 0 \\ \\ \partial_t v + \partial_x^3 v + \mu\partial_x(g(u,v)) = 0, t > 0, x \in \mathbb{T} \\ \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)), \end{cases} \tag{1.3}$$

where, for $k \geq 1$, we take

$$\begin{cases} f(u,v) = au^{2k+1} + bu^k v^{k+1} + \frac{k+2}{k} cu^{k+1}v^k + du^{k-1}v^{k+2}. \\ \\ g(u,v) = av^{2k+1} + bv^k u^{k+1} + \frac{k+2}{k} dv^{k+1}u^k + cv^{k-1}u^{k+2}. \end{cases} \tag{1.4}$$

Here a, b, c, and d are nonnegative real constants.

The initial data $u_0(x)$ and $v_0(x)$ belong to a class of 2π-periodic analytic functions $G^{\lambda,s}$ that can be extended holomorphically in a symmetric strip

$$S_\lambda = \{a + ib : |b| < \lambda\}, \lambda > 0,$$

of the complex plane around the x-axis.

$$G^{\lambda,s}(\mathbb{T}) = \left\{ w \in L^2(\mathbb{T}) : \|w\|^2_{G^{\lambda,s}(\mathbb{T})} = \sum_{n\in\mathbb{Z}} |n|^{2s} e^{2\lambda|n|} |\hat{w}(n)|^2 < \infty \right\}.$$

For more detail on this type of space, please see [16–19].

2. Locally well-posed

Our first goal is to study the local well-posedness of (1.3) in $G^{\lambda,s}(\mathbb{T})\times G^{\lambda,s}(\mathbb{T})$. One consequence of this result is that the radius of the uniform analysis of the solution (u,v) during its local lifetime is the same as the radius of the initial data, (u_0,v_0).

Theorem 2.1. *Assume that $s\geq 1/2,\ \lambda>0$. Taking (u_0,v_0) small in*

$$G^{\lambda,s}(\mathbb{T})\times G^{\lambda,s}(\mathbb{T}),$$

then (1.3) is locally well-posed in

$$C\left([-1,1],G^{\lambda,s}(\mathbb{T})\right)\times C\left([-1,1],G^{\lambda,s}(\mathbb{T})\right).$$

2.1. Existence

By Fourier transform with respect to x in (1.3), and inverse Fourier transform reduces the system (1.3) to be given as

$$u(x,t)=W(t)u_0(x)-\int_0^t W(t-\tau)(w_1(x,\tau)+w_2(x,\tau)+w_3(x,\tau)+w_4(x,\tau))d\tau, \tag{2.1}$$

$$v(x,t)=W(t)v_0(x)-\int_0^t W(t-\tau)(w_5(x,\tau)+w_6(x,\tau)+w_7(x,\tau)+w_8(x,\tau))d\tau. \tag{2.2}$$

Where $W(t)=e^{-t\partial_x^3}$ and

$$\begin{aligned}
w_1 &= a\partial_x u^{2k+1},\\
w_2 &= b\partial_x u^k v^{k+1},\\
w_3 &= \frac{k+2}{k}c\partial_x(u^{k+1}v^k),\\
w_4 &= d\partial_x(u^{k-1}v^{k+2}),\\
w_5 &= a\partial_x v^{2k+1},\\
w_6 &= b\partial_x(v^k u^{k+1}),\\
w_7 &= \frac{k+2}{k}d\partial_x(v^{k+1}u^k),\\
w_8 &= c\partial_x(v^{k-1}u^{k+2}).
\end{aligned}$$

Localizing in t by using a cut-off function $\varpi(t)\in C_0^\infty(-2,2)$ where $0\leq\varpi\leq 1$ and $\varpi(t)=1$ for $|t|<1$. Multiplying (2.1) and (2.2) by ϖ to get

$$\varpi(t)u=\varpi(t)W(t)u_0(x)-\varpi(t)\int_0^t W(t-\tau)(w_1(x,\tau)+w_2(x,\tau)+w_3(x,\tau)+w_4(x,\tau))d\tau,$$

$$\varpi(t)v=\varpi(t)W(t)v_0(x)-\varpi(t)\int_0^t W(t-\tau)(w_5(x,\tau)+w_6(x,\tau)+w_7(x,\tau)+w_8(x,\tau))d\tau.$$

Then

$$\varpi(t)u=\varpi(t)\sum_{n\in\mathbb{Z}}e^{i\left(nx+n^3t\right)}\hat{u}_0(n)$$

$$+ i\varpi(t)\sum_{n\in\mathbb{Z}} e^{i\left(nx+n^3t\right)}\int_{-\infty}^{\infty}\frac{e^{i\left(\xi-n^3\right)t}-1}{\xi-n^3}$$
$$(\hat{w}_1(n,\xi)+\hat{w}_2(n,\xi)+\hat{w}_3(n,\xi)+\hat{w}_4(n,\xi))d\xi,$$

$$\begin{aligned}\varpi(t)v =&\varpi(t)\sum_{n\in\mathbb{Z}} e^{i\left(nx+n^3t\right)}\hat{v}_0(n)\\ &+ i\varpi(t)\sum_{n\in\mathbb{Z}} e^{i\left(nx+n^3t\right)}\int_{-\infty}^{\infty}\frac{e^{i\left(\xi-n^3\right)t}-1}{\xi-n^3}\\ &(\hat{w}_5(n,\xi)+\hat{w}_6(n,\xi)+\hat{w}_7(n,\xi)+\hat{w}_8(n,\xi))d\xi.\end{aligned}$$

Now, we limit our attention to the zero-mean data (see [17], page 238).

$$\hat{u}_0(0)=\frac{1}{2\pi}\int_{\mathbb{T}}u_0(x)dx=0,$$

$$\hat{v}_0(0)=\frac{1}{2\pi}\int_{\mathbb{T}}v_0(x)dx=0.$$

We introduce the RHS of (2.1) by T_1u and (2.2) by T_2u,

$$\begin{aligned}T_1u =&\varpi(t)\sum_{n\in\mathbb{Z}^*} e^{i\left(nx+n^3t\right)}\hat{u}_0(n)\\ &+ i\sum_{j=1}^{\infty}\frac{i^jt^j}{j!}\varpi(t)\sum_{n\in\mathbb{Z}^*} e^{i\left(nx+n^3t\right)}\int_{\mathbb{R}}\varpi\left(\xi-n^3\right)\left(\xi-n^3\right)^{j-1}\\ &(\hat{w}_1(n,\xi)+\hat{w}_2(n,\xi)+\hat{w}_3(n,\xi)+\hat{w}_4(n,\xi))d\xi\\ &+ i\varpi(t)\sum_{n\in\mathbb{Z}^*} e^{inx}\int_{\mathbb{R}}\frac{(1-\varpi)\left(\xi-n^3\right)}{\xi-n^3}e^{i\xi t}\\ &(\hat{w}_1(n,\xi)+\hat{w}_2(n,\xi)+\hat{w}_3(n,\xi)+\hat{w}_4(n,\xi))d\xi\\ &- i\varpi(t)\sum_{n\in\mathbb{Z}^*} e^{i\left(nx+n^3t\right)}\int_{\mathbb{R}}\frac{(1-\varpi)\left(\xi-n^3\right)}{\xi-n^3}\\ &(\hat{w}_1(n,\xi)+\hat{w}_2(n,\xi)+\hat{w}_3(n,\xi)+\hat{w}_4(n,\xi))d\xi,\end{aligned}$$

$$\begin{aligned}T_2v =&\varpi(t)\sum_{n\in\mathbb{Z}^*} e^{i\left(nx+n^3t\right)}\hat{v}_0(n)\\ &+ i\sum_{j=1}^{\infty}\frac{i^jt^j}{j!}\varpi(t)\sum_{n\in\mathbb{Z}^*} e^{i\left(nx+n^3t\right)}\int_{\mathbb{R}}\varpi\left(\xi-n^3\right)\left(\xi-n^3\right)^{j-1}\\ &(\hat{w}_5(n,\xi)+\hat{w}_6(n,\xi)+\hat{w}_7(n,\xi)+\hat{w}_8(n,\xi))d\xi\\ &+ i\varpi(t)\sum_{n\in\mathbb{Z}^*} e^{inx}\int_{\mathbb{R}}\frac{(1-\varpi)\left(\xi-n^3\right)}{\xi-n^3}e^{i\xi t}\end{aligned}$$

$$
\begin{aligned}
&(\hat{w}_5(n,\xi)+\hat{w}_6(n,\xi)+\hat{w}_7(n,\xi)+\hat{w}_8(n,\xi))d\xi \\
&- i\varpi(t)\sum_{n\in\mathbb{Z}^*} e^{i\left(nx+n^3t\right)}\int_{\mathbb{R}} \frac{(1-\varpi)\left(\xi-n^3\right)}{\xi-n^3} \\
&(\hat{w}_5(n,\xi)+\hat{w}_6(n,\xi)+\hat{w}_7(n,\xi)+\hat{w}_8(n,\xi))d\xi,
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{w}_1(n,\xi) &= a\widehat{\partial_x u^{2k+1}} \simeq a(\underbrace{n\hat{u}*\hat{u}*\cdots*\hat{u}}_{2k+1})(n,\xi) \\
\hat{w}_2(n,\xi) &= b\partial_x\widehat{(u^k v^{k+1})} \simeq b(\underbrace{n\hat{u}*\hat{u}*\cdots*\hat{u}}_{k})(\underbrace{\hat{v}*\hat{v}*\cdots*\hat{v}}_{k+1})(n,\xi) \\
\hat{w}_3(n,\xi) &= \frac{k+2}{k}c\partial_x\widehat{(u^{k+1}v^k)} \simeq \frac{k+2}{k}c(\underbrace{n\hat{u}*\hat{u}*\cdots*\hat{u}}_{k+1})(\underbrace{\hat{v}*\hat{v}*\cdots*\hat{v}}_{k})(n,\xi) \\
\hat{w}_4(n,\xi) &= d\partial_x\widehat{(u^{k-1}v^{k+2})} \simeq d(\underbrace{n\hat{u}*\hat{u}*\cdots*\hat{u}}_{k-1})(\underbrace{\hat{v}*\hat{v}*\cdots*\hat{v}}_{k+2})(n,\xi) \\
\hat{w}_5(n,\xi) &= a\widehat{\partial_x v^{2k+1}} \simeq a(\underbrace{n\hat{v}*\hat{v}*\cdots*\hat{v}}_{2k+1})(n,\xi) \\
\hat{w}_6(n,\xi) &= b\partial_x\widehat{(v^k u^{k+1})} \simeq b(\underbrace{n\hat{v}*\hat{v}*\cdots*\hat{v}}_{k})(\underbrace{\hat{u}*\hat{u}*\cdots*\hat{u}}_{k+1})(n,\xi) \\
\hat{w}_7(n,\xi) &= \frac{k+2}{k}c\partial_x\widehat{(v^{k+1}u^k)} \simeq \frac{k+2}{k}c(\underbrace{n\hat{v}*\hat{v}*\cdots*\hat{v}}_{k+1})(\underbrace{\hat{u}*\hat{u}*\cdots*\hat{u}}_{k})(n,\xi) \\
\hat{w}_8(n,\xi) &= d\partial_x\widehat{(v^{k-1}u^{k+2})} \simeq d(\underbrace{n\hat{v}*\hat{v}*\cdots*\hat{v}}_{k-1})(\underbrace{\hat{u}*\hat{u}*\cdots*\hat{u}}_{k+2})(n,\xi),
\end{aligned}
$$

and

$$\mathbb{Z}^* = \mathbb{Z}\backslash\{0\}.$$

We are now ready to solve the system $(T_1u, T_2v) = (u, v)$. We define the required spaces. This can be considered a periodic version of the spaces in [20].

For $\lambda > 0$, $s \geq 0$, let us define

$$X_{\lambda,s}(\mathbb{T}\times\mathbb{R}) = X_{\lambda,s} = \left\{w \in L^2(\mathbb{T}\times\mathbb{R}) : \|w\|^2_{X_{\lambda,s}} < \infty\right\},$$

where

$$\|w\|^2_{X_{\lambda,s}} = \sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}\left(1+\left|\tau-n^3\right|\right)|n|^{2s}e^{2\lambda|n|}|\hat{w}(n,\tau)|^2 d\tau.$$

As in [20], it is clear that the space $X_{\lambda,s}(\mathbb{T}\times\mathbb{R})$ is the natural periodic extension of the space $X_{\lambda,s,1/2}$. In the periodic case, the constant b should be taken as $b = 1/2$ to show the multi-linear estimates in X_s, which is consistent with the condition $\lambda = 0$; please see [21]. Because if $b = 1/2$, we have no continuous embedding $X_{\lambda,s}(\mathbb{T}\times\mathbb{R}) \hookrightarrow C\left([0,T], G^{\lambda,s}(\mathbb{T})\right)$ used in [20].

Definition 2.1. *For $\lambda > 0$, $s \geq 0$, let*

$$Y_{\lambda,s}(\mathbb{T}\times\mathbb{R}) = Y_{\lambda,s} = \left\{w \in L^2(\mathbb{T}\times\mathbb{R}) : \|w\|^2_{Y_{\lambda,s}} < \infty\right\},$$

where

$$\|w\|_{Y_{\lambda,s}} = \|w\|_{X_{\lambda,s}} + \left(\sum_{n\in\mathbb{Z}} |n|^{2s} e^{2\lambda|n|} \left[\int_{\mathbb{R}} |\hat{w}(n,\tau)| d\tau\right]^2\right)^{\frac{1}{2}}.$$

Definition 2.2. *For $\lambda > 0$, $s \geq 0$, let*

$$\mathcal{Y}_{\lambda,s}(\mathbb{T}\times\mathbb{R}) = \mathcal{Y}_{\lambda,s} = \left\{(u,v) \in L^2(\mathbb{T}\times\mathbb{R}) \times L^2(\mathbb{T}\times\mathbb{R}) : \|(u,v)\|^2_{\mathcal{Y}_{\lambda,s}} < \infty\right\},$$

where

$$\mathcal{Y}_{\lambda,s} = Y_{\lambda,s} \times Y_{\lambda,s} = X_{\lambda,s} \times X_{\lambda,s} + Z_{\lambda,s} \times Z_{\lambda,s},$$

with norm

$$\begin{aligned} \|(u,v)\|_{\mathcal{Y}_{\lambda,s}} &= \|u\|_{Y_{\lambda,s}} + \|v\|_{Y_{\lambda,s}} = \|(u,v)\|_{\mathcal{X}_{\lambda,s}} + \|(u,v)\|_{\mathcal{Z}_{\lambda,s}}, \\ \|(u,v)\|_{\mathcal{X}_{\lambda,s}} &= \|u\|_{X_{\lambda,s}} + \|v\|_{X_{\lambda,s}} \\ &= \left(\sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \left(1 + \left|\tau - n^3\right|\right) |n|^{2s} e^{2\lambda|n|} |\hat{u}(n,\tau)|^2 d\tau\right) \\ &\quad + \left(\sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \left(1 + \left|\tau - n^3\right|\right) |n|^{2s} e^{2\lambda|n|} |\hat{v}(n,\tau)|^2 d\tau\right), \end{aligned}$$

and

$$\begin{aligned} \|(u,v)\|_{\mathcal{Z}_{\lambda,s}} &= \|u\|_{Z_{\lambda,s}} + \|v\|_{Z_{\lambda,s}} \\ &= \left(\sum_{n\in\mathbb{Z}} |n|^{2s} e^{2\lambda|n|} \left[\int_{\mathbb{R}} |\hat{u}(n,\tau)| d\tau\right]^2\right)^{\frac{1}{2}} \\ &\quad + \left(\sum_{n\in\mathbb{Z}} |n|^{2s} e^{2\lambda|n|} \left[\int_{\mathbb{R}} |\hat{v}(n,\tau)| d\tau\right]^2\right)^{\frac{1}{2}}. \end{aligned}$$

The spaces $\mathcal{Y}_{\lambda,s}$ possess the following important property.

Lemma 2.1. *We have*

$$\mathcal{Y}_{\lambda,s}(\mathbb{T}\times\mathbb{R}) \hookrightarrow C\left([-T,T], G^{\lambda,s}(\mathbb{T})\right) \times C\left([-T,T], G^{\lambda,s}(\mathbb{T})\right), \ \forall T > 0.$$

Proof. Let $T > 0$ be given. Recalling that the norm of

$$(u,v) \in C\left([-T,T], G^{\lambda,s}(\mathbb{T})\right) \times C\left([-T,T], G^{\lambda,s}(\mathbb{T})\right),$$

is defined as

$$|(u,v)|_{C_{T,\lambda,s}} = \sup_{|t|\leq T} \|u(\cdot,t)\|_{G^{\lambda,s}(\mathbb{T})} + \sup_{|t|\leq T} \|v(\cdot,t)\|_{G^{\lambda,s}(\mathbb{T})},$$

we have

$$\begin{aligned} \|u(\cdot,t)\|_{G^{\lambda,s}(\mathbb{T})} &= \left(\sum_{n\in\mathbb{Z}} |n|^{2s} e^{2\lambda|n|} \left|\frac{1}{2\pi} \int_{\mathbb{R}} e^{i\tau t} \hat{u}(n,\tau) d\tau\right|^2\right)^{\frac{1}{2}} \\ &\leq \frac{1}{2\pi} \left(\sum_{n\in\mathbb{Z}} |n|^{2s} e^{2\lambda|n|} \left(\int_{\mathbb{R}} |\hat{u}(n,\tau)| d\tau\right)^2\right)^{\frac{1}{2}} \\ &\leq \frac{1}{2\pi} \|u\|_{Y_{\lambda,s}}, \end{aligned}$$

and

$$\|v(\cdot,t)\|_{G^{\lambda,s}(\mathbb{T})} = \left(\sum_{n\in\mathbb{Z}} |n|^{2s} e^{2\lambda|n|} \left|\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\tau t}\hat{v}(n,\tau)d\tau\right|^2\right)^{\frac{1}{2}}$$
$$\leq \frac{1}{2\pi}\left(\sum_{n\in\mathbb{Z}} |n|^{2s} e^{2\lambda|n|}\left(\int_{\mathbb{R}} |\hat{v}(n,\tau)|d\tau\right)^2\right)^{\frac{1}{2}}$$
$$\leq \frac{1}{2\pi}\|v\|_{Y_{\lambda,s}}.$$

Thus,

$$|(u,v)|_{C_{T,\lambda,s}} = \sup_{|t|\leq T}\|u(\cdot,t)\|_{G^{\lambda,s}(\mathbb{T})} + \sup_{|t|\leq T}\|v(\cdot,t)\|_{G^{\lambda,s}(\mathbb{T})} \leq \frac{1}{\pi}\|(u,v)\|_{\mathcal{Y}_{\lambda,s}}.$$

□

Next, computing the $\mathcal{Y}_{\lambda,s}$ norm of T_1u and T_2v, we have

Lemma 2.2. *[17] For $s \geq 1/2$, $\exists c_\varpi > 0$ where*

$$\|T_1u\|_{Y_{\lambda,s}} \leq c_\varpi\left(\|w_1\|_{Z_{\lambda,s}} + \|w_2\|_{Z_{\lambda,s}} + \|w_3\|_{Z_{\lambda,s}} + \|w_4\|_{Z_{\lambda,s}} + \|u_0\|_{G^{\lambda,s}}\right), \tag{2.3}$$

and

$$\|T_2v\|_{Y_{\lambda,s}} \leq c_\varpi\left(\|w_5\|_{Z_{\lambda,s}} + \|w_6\|_{Z_{\lambda,s}} + \|w_7\|_{Z_{\lambda,s}} + \|w_8\|_{Z_{\lambda,s}} + \|v_0\|_{G^{\lambda,s}}\right), \tag{2.4}$$

for all $u, v \in Y_{\lambda,s}$, where

$$\|w_i\|_{Z_{\lambda,s}} = \left(\sum_{n\in\mathbb{Z}^*} |n|^{2s} e^{2\lambda|n|}\int_{\mathbb{R}} \frac{|\hat{w}_i(n,\xi)|^2}{1+|\xi-n^3|}d\xi\right)^{\frac{1}{2}}$$
$$+\left(\sum_{n\in\mathbb{Z}^*} |n|^{2s} e^{2\lambda|k|}\left(\int_{\mathbb{R}} \frac{|\hat{w}_i(n,\xi)|}{1+|\xi-n^3|}d\xi\right)^2\right)^{\frac{1}{2}}, \quad i = 1,2,\cdots 8.$$

Proposition 2.1. *For $s \geq 1/2$ and $u_1, u_2, \ldots, u_{k+2} \in Y_{\lambda,s}$ and $v_1, v_2, \ldots, v_{k+2} \in Y_{\lambda,s}$, we have*

$$\|w_1\|_{Z_{\lambda,s}} = \|a\partial_x(u_1\cdot u_2\cdot\ldots\cdot u_{2k+1})\|_{Z_{\lambda,s}} \lesssim \|u_1\|_{Y_{\lambda,s}}\|u_2\|_{Y_{\lambda,s}}\ldots\|u_{2k+1}\|_{Y_{\lambda,s}}, \tag{2.5}$$

$$\begin{aligned}\|w_2\|_{Z_{\lambda,s}} &= \|b\partial_x(u_1\cdot u_2\cdot\ldots\cdot u_k)(v_1\cdot v_2\cdot\ldots\cdot v_{k+1})\|_{Z_{\lambda,s}}\\ &\lesssim \|u_1\|_{Y_{\lambda,s}}\|u_2\|_{Y_{\lambda,s}}\ldots\|u_k\|_{Y_{\lambda,s}}\|v_1\|_{Y_{\lambda,s}}\|v_2\|_{Y_{\lambda,s}}\ldots\|v_{k+1}\|_{Y_{\lambda,s}},\end{aligned} \tag{2.6}$$

$$\begin{aligned}\|w_3\|_{Z_{\lambda,s}} &= \left\|\frac{k+2}{k}c\partial_x(u_1\cdot u_2\cdot\ldots\cdot u_{k+1})(v_1\cdot v_2\cdot\ldots\cdot v_k)\right\|_{Z_{\lambda,s}}\\ &\lesssim \|u_1\|_{Y_{\lambda,s}}\|u_2\|_{Y_{\lambda,s}}\ldots\|u_{k+1}\|_{Y_{\lambda,s}}\|v_1\|_{Y_{\lambda,s}}\|v_2\|_{Y_{\lambda,s}}\ldots\|v_k\|_{Y_{\lambda,s}},\end{aligned} \tag{2.7}$$

$$\|w_4\|_{Z_{\lambda,s}} = \|d\partial_x(u_1\cdot u_2\cdot\ldots\cdot u_{k-1})(v_1\cdot v_2\cdot\ldots\cdot v_{k+2})\|_{Z_{\lambda,s}}$$

$$\lesssim \|u_1\|_{Y_{\lambda,s}}\|u_2\|_{Y_{\lambda,s}}\ldots\|u_{k-1}\|_{Y_{\lambda,s}}\|v_1\|_{Y_{\lambda,s}}\|v_2\|_{Y_{\lambda,s}}\ldots\|v_{k+2}\|_{Y_{\lambda,s}}, \tag{2.8}$$

$$\|w_5\|_{Z_{\lambda,s}} = \|a\partial_x(v_1\cdot v_2\cdot\ldots\cdot v_{2k+1})\|_{Z_{\lambda,s}} \lesssim \|v_1\|_{Y_{\lambda,s}}\|v_2\|_{Y_{\lambda,s}}\ldots\|v_{2k+1}\|_{Y_{\lambda,s}}, \tag{2.9}$$

$$\begin{aligned}\|w_6\|_{Z_{\lambda,s}} &= \|a\partial_x(v_1\cdot v_2\cdot\ldots\cdot v_k)(u_1\cdot u_2\cdot\ldots\cdot u_{k+1})\|_{Z_{\lambda,s}}\\ &\lesssim \|v_1\|_{Y_{\lambda,s}}\|v_2\|_{Y_{\lambda,s}}\ldots\|v_k\|_{Y_{\lambda,s}}\|u_1\|_{Y_{\lambda,s}}\|u_2\|_{Y_{\lambda,s}}\ldots\|u_{k+1}\|_{Y_{\lambda,s}},\end{aligned} \tag{2.10}$$

$$\begin{aligned}\|w_7\|_{Z_{\lambda,s}} &= \left\|\frac{k+2}{k}d\partial_x(v_1\cdot v_2\cdot\ldots\cdot v_{k+1})(u_1\cdot u_2\cdot\ldots\cdot u_k)\right\|_{Z_{\lambda,s}}\\ &\lesssim \|v_1\|_{Y_{\lambda,s}}\|v_2\|_{Y_{\lambda,s}}\ldots\|v_{k+1}\|_{Y_{\lambda,s}}\|u_1\|_{Y_{\lambda,s}}\|u_2\|_{Y_{\lambda,s}}\ldots\|u_k\|_{Y_{\lambda,s}},\end{aligned} \tag{2.11}$$

$$\begin{aligned}\|w_8\|_{Z_{\lambda,s}} &= \|c\partial_x(v_1\cdot v_2\cdot\ldots\cdot v_{k-1})(u_1\cdot u_2\cdot\ldots\cdot u_{k+2})\|_{Z_{\lambda,s}}\\ &\lesssim \|v_1\|_{Y_{\lambda,s}}\|v_2\|_{Y_{\lambda,s}}\ldots\|v_{k-1}\|_{Y_{\lambda,s}}\|u_1\|_{Y_{\lambda,s}}\|u_2\|_{Y_{\lambda,s}}\ldots\|u_{k+2}\|_{Y_{\lambda,s}}.\end{aligned} \tag{2.12}$$

Proof. The operator Λ given by

$$\widehat{\Lambda u}(n,\xi) = e^{\lambda|n|}\hat{u}(n,\xi), \quad \widehat{\Lambda v}(n,\xi) = e^{\lambda|n|}\hat{v}(n,\xi),$$

satisfies

$$\begin{aligned}\|u\|_{Y_{\lambda,s}} &= \|\Lambda u\|_{Y_s}, \quad \text{for all } u\in Y_{\lambda,s},\\ \|v\|_{Y_{\lambda,s}} &= \|\Lambda v\|_{Y_s}, \quad \text{for all } v\in Y_{\lambda,s}.\end{aligned}$$

Inequality (2.5): For any $u_1, u_2\ldots u_{2k+1}\in Y_{\lambda,s}$ it satisfies the relation

$$\|\partial_x(u_1\cdot u_2\cdots u_{2k+1})\|_{Z_{\lambda,s}} = \|\Lambda(\partial_x(u_1\cdot u_2\cdots u_{2k+1}))\|_{Z_s}.$$

Then, to establish (2.5), it suffices to prove that

$$\|\Lambda(\partial_x(u_1\cdot u_2\cdots u_{2k+1}))\|_{Z_s} \le \|(\Lambda u_1)(\Lambda u_2)\cdots(\Lambda u_{2k+1})\|_{Z_s}. \tag{2.13}$$

By Proposition 1 of [13], we have

$$\|\Lambda(\partial_x(u_1\cdot u_2\cdots u_{2k+1}))\|_{Z_s} \le \|\Lambda u_1\|_{Y_s}\|\Lambda u_1\|_{Y_s}\cdot\ldots\cdot\|\Lambda u_{2k+1}\|_{Y_s}.$$

Let us prove (2.13), we have

$$\begin{aligned}&\|\Lambda(\partial_x(u_1\cdot u_2\cdots u_{2k+1}))\|_{Z_s}\\ &= \left(\sum_{n\in\mathbb{Z}^*}|n|^{2s}\int_{\mathbb{R}}\frac{|\Lambda(\partial_x(u_1\widehat{u_2\cdots}u_{2k+1}))(n,\xi)|^2}{1+\left|\xi-n^3\right|}d\xi\right)^{\frac{1}{2}}\\ &+\left(\sum_{n\in\mathbb{Z}^*}|n|^{2s}\left(\int_{\mathbb{R}}\frac{|\Lambda(\partial_x(u_1\widehat{u_2\cdots}u_{2k+1}))(n,\xi)|}{1+\left|\xi-n^3\right|}d\xi\right)^2\right)^{\frac{1}{2}}\\ &= \left(\sum_{n\in\mathbb{Z}^*}|n|^{2s}\int_{\mathbb{R}}\frac{\left|e^{\lambda|n|}n\left(\widehat{u_1}*\widehat{u_2}*\cdots*\widehat{u_{2k+1}}\right)(n,\xi)\right|^2}{1+\left|\xi-n^3\right|}d\xi\right)^{\frac{1}{2}}\\ &+\left(\sum_{n\in\mathbb{Z}^*}|n|^{2s}\left(\int_{\mathbb{R}}\frac{\left|e^{\lambda|n|}n\left(\widehat{u_1}*\widehat{u_2}*\cdots*\widehat{u_{2k+1}}\right)(n,\xi)\right|}{1+\left|\xi-n^3\right|}d\xi\right)^2\right)^{\frac{1}{2}}.\end{aligned}$$

We have

$$\left(\sum_{n\in\mathbb{Z}^*}|n|^{2s}\int_{\mathbb{R}}\frac{\left|e^{\lambda|n|}n\left(\widehat{u_1}*\widehat{u_2}*\cdots*\widehat{u_{2k+1}}\right)(n,\xi)\right|^2}{1+\left|\xi-n^3\right|}d\xi\right)^{\frac{1}{2}}$$

$$=\left(\sum_{n\in\mathbb{Z}^*}|n|^{2s}\int_{\mathbb{R}}|\,e^{\lambda|n|}\sum_{n_1\in\mathbb{Z}}\int_{\mathbb{R}_1}\cdots\sum_{n_{2k}\in\mathbb{Z}}\int_{\mathbb{R}_{2k}}\sum_{n_{2k+1}\in\mathbb{Z}}\int_{\mathbb{R}_{2k+1}}n_1\widehat{u_1}(n-n_1,\xi-\xi_1)\widehat{u_2}(n_1-n_2,\xi_1-\xi_2)\ldots\right.$$

$$\left.\times\,\widehat{u_{2k}}(n_{2k-1}-n_{2k},\xi_{2k-1}-\xi_{2k})\widehat{u_{2k+1}}(n_{2k},\xi_{2k})d\xi_1d\xi_2\ldots d\xi_{2k+1}\right|^2\left(1+\left|\xi-n^3\right|\right)^{-1}d\xi\Bigg)^{\frac{1}{2}}$$

$$=\left(\sum_{n\in\mathbb{Z}^*}|n|^{2s}\int_{\mathbb{R}}|\,e^{\lambda|n|}\sum_{n_1\in\mathbb{Z}}\int_{\mathbb{R}_1}\cdots\sum_{n_{2k}\in\mathbb{Z}}\int_{\mathbb{R}_{2k}}\sum_{n_{2k+1}\in\mathbb{Z}}\int_{\mathbb{R}_{2k+1}}n_1e^{\lambda|n-n_1|}\widehat{u_1}(n-n_1,\xi-\xi_1)e^{\lambda|n_1-n_2|}\widehat{u_2}(n_1-n_2,\xi_1-\xi_2)\ldots\right.$$

$$\left.\times\,e^{\lambda|n_{2k-1}-n_{2k}|}\widehat{u_{2k}}(n_{2k-1}-n_{2k},\xi_{2k-1}-\xi_{2k})e^{\lambda|n_{2k}|}\widehat{u_{2k+1}}(n_{2k},\xi_{2k})d\xi_1d\xi_2\ldots d\xi_{2k+1}\right|^2\left(1+\left|\xi-n^3\right|\right)^{-1}d\xi\Bigg)^{\frac{1}{2}}$$

$$\leq\left(\sum_{n\in\mathbb{Z}^*}|n|^{2s}\int_{\mathbb{R}}|\sum_{n_1\in\mathbb{Z}}\int_{\mathbb{R}_1}\cdots\sum_{n_{2k}\in\mathbb{Z}}\int_{\mathbb{R}_{2k}}\sum_{n_{2k+1}\in\mathbb{Z}}\int_{\mathbb{R}_{2k+1}}\partial_x\widehat{\Lambda u_1}(n-n_1,\xi-\xi_1\widehat{\Lambda u_2}(n_1-n_2,\xi_1-\xi_2)\ldots\right.$$

$$\left.\times\,\widehat{\Lambda u_{2k}}(n_{2k-1}-n_{2k},\xi_{2k-1}-\xi_{2k})\widehat{\Lambda u_{2k+1}}(n_{2k},\xi_{2k})d\xi_1d\xi_2\ldots d\xi_{2k+1}\right|^2\left(1+\left|\xi-n^3\right|\right)^{-1}d\xi\Bigg)^{\frac{1}{2}}$$

$$\leq\left(\sum_{n\in\mathbb{Z}^*}|n|^{2s}\int_{\mathbb{R}}|\left(\partial_x\widehat{\Lambda u_1}*\widehat{\Lambda u_2}*\cdots*\widehat{\Lambda u_{2k+1}}\right)(n,\xi)|^2\left(1+\left|\xi-n^3\right|\right)^{-1}d\xi\right)^{\frac{1}{2}}.$$

Similarly, we obtain

$$\left(\sum_{n\in\mathbb{Z}^*}|n|^{2s}\left(\int_{\mathbb{R}}\frac{\left|e^{\lambda|n|}n\left(\widehat{u_1}*\widehat{u_2}*\cdots*\widehat{u_{2k+1}}\right)(n,\xi)\right|}{1+\left|\xi-n^3\right|}d\xi\right)^2\right)^{\frac{1}{2}}$$

$$\leq\left(\sum_{n\in\mathbb{Z}^*}|n|^{2s}\left(\int_{\mathbb{R}}\frac{|\left(\partial_x\widehat{\Lambda u_1}*\widehat{\Lambda u_2}*\cdots*\widehat{\Lambda u_{2k+1}}\right)(n,\xi)|}{1+\left|\xi-n^3\right|}d\xi\right)^2\right)^{\frac{1}{2}}.$$

Then (2.13) is met.

Inequality (2.6): For any $u_1,u_2,\ldots,u_k\in Y_{\lambda,s}$ and $v_1,v_2,\ldots,v_{k+1}\in Y_{\lambda,s}$ it satisfies the relation

$$\|\partial_x((u_1u_2\cdots u_k)(v_1v_2\cdots v_{k+1}))\|_{Z_{\lambda,s}}=\|\Lambda(\partial_x((u_1u_2\cdots u_k)(v_1v_2\cdots v_{k+1})))\|_{Z_s}.$$

Then, to show (2.5), it suffices to prove that

$$\|\Lambda(\partial_x((u_1u_2\cdots u_k)(v_1v_2\cdots v_{k+1})))\|_{Z_s}\leq\|(\Lambda u_1)(\Lambda u_2)\cdots(\Lambda u_k)(\Lambda v_1)(\Lambda v_2)\cdots(\Lambda v_{k+1})\|_{Z_s}. \tag{2.14}$$

By Proposition 1 of [13], we have

$$\|\Lambda(\partial_x((u_1u_2\cdots u_k)(v_1v_2\cdots v_{k+1})))\|_{Z_s}\lesssim\|\Lambda u_1\|_{Y_s}\|\Lambda u_1\|_{Y_s}\cdots\|\Lambda u_k\|_{Y_s}\|\Lambda v_1\|_{Y_s}\|\Lambda v_2\|_{Y_s}\cdots\|\Lambda v_{k+1}\|_{Y_s}.$$

Let us prove (2.14), we have

$$
\begin{aligned}
&\|\Lambda(\partial_x((u_1u_2\cdots u_k)(v_1v_2\cdots v_{k+1})))\|_{Z_s}\\
&=\left(\sum_{n\in\mathbb{Z}^*}|n|^{2s}\int_{\mathbb{R}}\frac{|\Lambda(\partial_x((u_1u_2\cdots \widehat{u_k)(}v_1v_2\cdots v_{k+1})))(n,\xi)|^2}{1+\left|\xi-n^3\right|}d\xi\right)^{\frac{1}{2}}\\
&+\left(\sum_{n\in\mathbb{Z}^*}|n|^{2s}\left(\int_{\mathbb{R}}\frac{|\Lambda(\partial_x((u_1u_2\cdots \widehat{u_k)(}v_1v_2\cdots v_{k+1})))(n,\xi)|}{1+\left|\xi-n^3\right|}d\xi\right)^2\right)^{\frac{1}{2}}\\
&=\left(\sum_{n\in\mathbb{Z}^*}|n|^{2s}\int_{\mathbb{R}}\frac{\left|e^{\lambda|n|}n((\widehat{u_1}*\widehat{u_2}*\cdots*\widehat{u_k})\,(\widehat{v_1}*\widehat{v_2}*\cdots*\widehat{v_{k+1}}))(n,\xi)\right|^2}{1+\left|\xi-n^3\right|}d\xi\right)^{\frac{1}{2}}\\
&+\left(\sum_{n\in\mathbb{Z}^*}|n|^{2s}\left(\int_{\mathbb{R}}\frac{\left|e^{\lambda|n|}n((\widehat{u_1}*\widehat{u_2}*\cdots*\widehat{u_k})\,(\widehat{v_1}*\widehat{v_2}*\cdots*\widehat{v_{k+1}}))(n,\xi)\right|}{1+\left|\xi-n^3\right|}d\xi\right)^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

Next, we analyze the first part of the last sum.

$$
\begin{aligned}
&\left(\sum_{n\in\mathbb{Z}^*}|n|^{2s}\int_{\mathbb{R}}\frac{\left|e^{\lambda|n|}n((\widehat{u_1}*\widehat{u_2}*\cdots*\widehat{u_k})\,(\widehat{v_1}*\widehat{v_2}*\cdots*\widehat{v_{k+1}}))(n,\xi)\right|^2}{1+\left|\xi-n^3\right|}d\xi\right)^{\frac{1}{2}}\\
&=\left(\sum_{n\in\mathbb{Z}^*}|n|^{2s}\int_{\mathbb{R}}|\;e^{\lambda|n|}\sum_{n_1\in\mathbb{Z}}\int_{\mathbb{R}_1}\dots\sum_{n_{2k}\in\mathbb{Z}}\int_{\mathbb{R}_{2k}}\sum_{n_{2k+1}\in\mathbb{Z}}\int_{\mathbb{R}_{2k+1}}n_1\widehat{u_1}(n-n_1,\xi-\xi_1\widehat{u_2}(n_1-n_2,\xi_1-\xi_2)\dots\right.\\
&\quad\left.\times\,\widehat{v_k}(n_{2k-1}-n_{2k},\xi_{2k-1}-\xi_{2k})\widehat{v_{k+1}}(n_{2k},\xi_{2k})d\xi_1d\xi_2\dots d\xi_{2k+1}\right|^2\left(1+\left|\xi-n^3\right|\right)^{-1}d\xi\Bigg)^{\frac{1}{2}}\\
&\le\left(\sum_{n\in\mathbb{Z}^*}|n|^{2s}\int_{\mathbb{R}}|\;e^{\lambda|n|}\sum_{n_1\in\mathbb{Z}}\int_{\mathbb{R}_1}\dots\sum_{n_{2k}\in\mathbb{Z}}\int_{\mathbb{R}_{2k}}\sum_{n_{2k+1}\in\mathbb{Z}}\int_{\mathbb{R}_{2k+1}}n_1e^{\lambda|n-n_1|}\widehat{u_1}(n-n_1,\xi-\xi_1e^{\lambda|n_1-n_2|}\widehat{u_2}(n_1-n_2,\xi_1-\xi_2)\dots\right.\\
&\quad\left.\times\,e^{\lambda|n_{2k-1}-n_{2k}|}\widehat{v_k}(n_{2k-1}-n_{2k},\xi_{2k-1}-\xi_{2k})e^{\lambda|n_{2k}|}\widehat{v_{k+1}}(n_{2k},\xi_{2k})d\xi_1d\xi_2\dots d\xi_{2k+1}\right|^2\left(1+\left|\xi-n^3\right|\right)^{-1}d\xi\Bigg)^{\frac{1}{2}}\\
&\le\left(\sum_{n\in\mathbb{Z}^*}|n|^{2s}\int_{\mathbb{R}}|\sum_{n_1\in\mathbb{Z}}\int_{\mathbb{R}_1}\dots\sum_{n_{2k}\in\mathbb{Z}}\int_{\mathbb{R}_{2k}}\sum_{n_{2k+1}\in\mathbb{Z}}\int_{\mathbb{R}_{2k+1}}\partial_x\widehat{\Lambda u_1}(n-n_1,\xi-\xi_1)\widehat{\Lambda u_2}(n_1-n_2,\xi_1-\xi_2)\dots\right.\\
&\quad\left.\times\,\widehat{\Lambda v_k}(n_{2k-1}-n_{2k},\xi_{2k-1}-\xi_{2k})\widehat{\Lambda v_{k+1}}(n_{2k},\xi_{2k})d\xi_1d\xi_2\dots d\xi_{2k+1}\right|^2\left(1+\left|\xi-n^3\right|\right)^{-1}d\xi\Bigg)^{\frac{1}{2}}\\
&\le\left(\sum_{n\in\mathbb{Z}^*}|n|^{2s}\int_{\mathbb{R}}|\left(\partial_x\widehat{\Lambda u_1}*\widehat{\Lambda u_2}*\cdots*\widehat{\Lambda v_k}*\widehat{\Lambda v_{k+1}}\right)(n,\xi)|^2\left(1+\left|\xi-n^3\right|\right)^{-1}d\xi\right)^{\frac{1}{2}}.
\end{aligned}
$$

Similarly, we obtain

$$\left(\sum_{n\in\mathbb{Z}^*}|n|^{2s}\left(\int_{\mathbb{R}}\frac{\left|e^{\lambda|n|}n\left(\widehat{u_1}*\widehat{u_2}*\cdots*\widehat{v_k}\widehat{v_{k+1}}\right)(n,\xi)\right|}{1+\left|\xi-n^3\right|}d\xi\right)^2\right)^{\frac{1}{2}}$$

$$\leq\left(\sum_{n\in\mathbb{Z}^*}|n|^{2s}\left(\int_{\mathbb{R}}\frac{|\left(\partial_x\widehat{\Lambda u_1}*\widehat{\Lambda u_2}*\cdots*\widehat{\Lambda v_k}*\widehat{\Lambda v_{k+1}}\right)(n,\xi)|}{1+\left|\xi-n^3\right|}d\xi\right)^2\right)^{\frac{1}{2}}.$$

Adding the above two inequalities gives (2.14). □

By Lemma 2.2 and Proposition 2.1, we have

Proposition 2.2. *For $s\geq 1/2, \exists c_{\varpi}>0$ such that*

$$\|T_1u\|_{Y_{\lambda,s}}\leq c_{\varpi}(\|u\|^{2k+1}_{Y_{\lambda,s}}+\|u\|^{k}_{Y_{\lambda,s}}\|v\|^{k+1}_{Y_{\lambda,s}}+\|u\|^{k+1}_{Y_{\lambda,s}}\|v\|^{k}_{Y_{\lambda,s}}+\|u\|^{k-1}_{Y_{\lambda,s}}\|v\|^{k+2}_{Y_{\lambda,s}})+c_{\varpi}\|u_0\|_{G^{\lambda,s}(\mathbb{T})},\quad u\in Y_{\lambda,s},$$

$$\|T_2v\|_{Y_{\lambda,s}}\leq c_{\varpi}(\|v\|^{2k+1}_{Y_{\lambda,s}}+\|v\|^{k}_{Y_{\lambda,s}}\|u\|^{k+1}_{Y_{\lambda,s}}+\|v\|^{k+1}_{Y_{\lambda,s}}\|u\|^{k}_{Y_{\lambda,s}}+\|v\|^{k-1}_{Y_{\lambda,s}}\|u\|^{k+2}_{Y_{\lambda,s}})+c_{\varpi}\|v_0\|_{G^{\lambda,s}(\mathbb{T})},\quad v\in Y_{\lambda,s},$$

and

$$\begin{aligned}\|T_1u-T_1u^*\|_{Y_{\lambda,s}}\leq c_{\varpi}\Bigg(&\sum_{\ell=0}^{2k+1}\|u\|^{(2k+1)-\ell}_{Y_{\lambda,s}}\|u^*\|^{\ell}_{Y_{\lambda,s}}+\sum_{\ell=0}^{k}\|u\|^{k-\ell}_{Y_{\lambda,s}}\|u^*\|^{\ell}_{Y_{\lambda,s}}\sum_{\ell=0}^{k+1}\|v\|^{(k+1)-\ell}_{Y_{\lambda,s}}\|v^*\|^{\ell}_{Y_{\lambda,s}}\\&+\sum_{\ell=0}^{k+1}\|u\|^{(k+1)-\ell}_{Y_{\lambda,s}}\|u^*\|^{\ell}_{Y_{\lambda,s}}\sum_{\ell=0}^{k}\|v\|^{k-\ell}_{Y_{\lambda,s}}\|v^*\|^{\ell}_{Y_{\lambda,s}}\\&+\sum_{\ell=0}^{k-1}\|u\|^{(k-1)-\ell}_{Y_{\lambda,s}}\|u^*\|^{\ell}_{Y_{\lambda,s}}\sum_{\ell=0}^{k+2}\|u\|^{(k+2)-\ell}_{Y_{\lambda,s}}\|u^*\|^{\ell}_{Y_{\lambda,s}}\Bigg)\|u-u^*\|_{Y_{\lambda,s}},\end{aligned}$$

$$\begin{aligned}\|T_2u-T_2u^*\|_{Y_{\lambda,s}}\leq c_{\varpi}\Bigg(&\sum_{\ell=0}^{2k+1}\|v\|^{(2k+1)-\ell}_{Y_{\lambda,s}}\|v^*\|^{\ell}_{Y_{\lambda,s}}+\sum_{\ell=0}^{k}\|v\|^{k-\ell}_{Y_{\lambda,s}}\|v^*\|^{\ell}_{Y_{\lambda,s}}\sum_{\ell=0}^{k+1}\|u\|^{(k+1)-\ell}_{Y_{\lambda,s}}\|u^*\|^{\ell}_{Y_{\lambda,s}}\\&+\sum_{\ell=0}^{k+1}\|v\|^{(k+1)-\ell}_{Y_{\lambda,s}}\|v^*\|^{\ell}_{Y_{\lambda,s}}\sum_{\ell=0}^{k}\|u\|^{k-\ell}_{Y_{\lambda,s}}\|u^*\|^{\ell}_{Y_{\lambda,s}}\\&+\sum_{\ell=0}^{k-1}\|v\|^{(k-1)-\ell}_{Y_{\lambda,s}}\|v^*\|^{\ell}_{Y_{\lambda,s}}\sum_{\ell=0}^{k+2}\|u\|^{(k+2)-\ell}_{Y_{\lambda,s}}\|u^*\|^{\ell}_{Y_{\lambda,s}}\Bigg)\|u-u^*\|_{Y_{\lambda,s}},\end{aligned}$$

and $u,u^*,v,v^*\in Y_{\lambda,s}$

In the next, we find that the map (T_1,T_2) is a contraction.

Proposition 2.3. *Suppose that $s\geq 1/2$. For a small (u_0,v_0), we have*

$$\|u_0\|_{G^{\lambda,s}(\mathbb{T})}\leq\frac{3^{2k}-4}{2\cdot 3^{2k+1}c_{\varpi}^{\frac{2k+1}{2k}}(2k+1)^{\frac{2k+1}{2k}}},$$

$$\|v_0\|_{G^{\lambda,s}(\mathbb{T})} \le \frac{3^{2k}-4}{2\cdot 3^{2k+1} c_{\varpi}^{\frac{2k+1}{2k}} (2k+1)^{\frac{2k+1}{2k}}}.$$

Choosing the ball

$$\mathbb{B}(0,r) = \left\{u, v \in Y_{\lambda,s} : \|(u,v)\|_{\mathcal{Y}_{\lambda,s}} = \|u\|_{Y_{\lambda,s}} + \|v\|_{Y_{\lambda,s}} \le r\right\},$$

with

$$r = \frac{1}{3c_{\varpi}^{\frac{1}{2k}}(2k+1)^{\frac{1}{2k}}},$$

thus

$$(T_1, T_2) : \mathbb{B}(0,r) \to \mathbb{B}(0,r),$$

is a contraction.

Proof. By Proposition 2.2, we have

$$\|T_1 u\|_{Y_{\lambda,s}} \le c_{\varpi}(\|u\|^{2k+1}_{Y_{\lambda,s}} + \|u\|^{k}_{Y_{\lambda,s}}\|v\|^{k+1}_{Y_{\lambda,s}} + \|u\|^{k+1}_{Y_{\lambda,s}}\|v\|^{k}_{Y_{\lambda,s}} + \|u\|^{k-1}_{Y_{\lambda,s}}\|v\|^{k+2}_{Y_{\lambda,s}}) + c_{\varpi}\|u_0\|_{G^{\lambda,s}(\mathbb{T})},$$

$$\|T_2 v\|_{Y_{\lambda,s}} \le c_{\varpi}(\|v\|^{2k+1}_{Y_{\lambda,s}} + \|v\|^{k}_{Y_{\lambda,s}}\|u\|^{k+1}_{Y_{\lambda,s}} + \|v\|^{k+1}_{Y_{\lambda,s}}\|u\|^{k}_{Y_{\lambda,s}} + \|v\|^{k-1}_{Y_{\lambda,s}}\|u\|^{k+2}_{Y_{\lambda,s}}) + c_{\varpi}\|v_0\|_{G^{\lambda,s}(\mathbb{T})}.$$

Adding the inequalities, we find that

$$\begin{aligned}
\|(T_1 u, T_2 v)\|_{\mathcal{Y}_{\lambda,s}} &\le c_{\varpi}(\|(u,v)\|^{2k+1}_{\mathcal{Y}_{\lambda,s}} + \|(u,v)\|^{k}_{\mathcal{Y}_{\lambda,s}}\|(u,v)\|^{k+1}_{\mathcal{Y}_{\lambda,s}} + \|(u,v)\|^{k+1}_{\mathcal{Y}_{\lambda,s}}\|u\|^{k}_{\mathcal{Y}_{\lambda,s}} + \|(u,v)\|^{k-1}_{\mathcal{Y}_{\lambda,s}}\|(u,v)\|^{k+2}_{\mathcal{Y}_{\lambda,s}}) \\
&\quad + c_{\varpi}\left(\|u_0\|_{G^{\lambda,s}(\mathbb{T})} + \|v_0\|_{G^{\lambda,s}(\mathbb{T})}\right) \\
&\le 4c_{\varpi}\left(\frac{1}{3c_{\varpi}^{\frac{1}{2k}}(2k+1)^{\frac{1}{2k}}}\right)^{2k+1} + c_{\varpi}\left(\frac{3^{2k}-4}{3^{2k+1}c_{\varpi}^{\frac{2k+1}{k}}(2k+1)^{\frac{2k+1}{2k}}}\right) \\
&= \frac{c_{\varpi}\cdot 3^{2k}}{3^{2k+1}c_{\varpi}^{\frac{2k+1}{2k}}(2k+1)^{\frac{2k+1}{2k}}} = \frac{1}{3c_{\varpi}^{\frac{1}{2k}}(2k+1)^{\frac{1}{2k}}} = r.
\end{aligned}$$

Thus (T_1, T_2) maps $\mathbb{B}(0,r)$ into $\mathbb{B}(0,r)$ is a contraction, since

$$\begin{aligned}
\|T_1 u - T_1 u^*\|_{Y_{\lambda,s}} &\le c_{\varpi}\Bigg(\sum_{\ell=0}^{2k}\|u\|^{2k-\ell}_{Y_{\lambda,s}}\|u^*\|^{\ell}_{Y_{\lambda,s}} + \sum_{\ell=0}^{k-1}\|u\|^{(k-1)-\ell}_{Y_{\lambda,s}}\|u^*\|^{\ell}_{Y_{\lambda,s}}\sum_{\ell=0}^{k+1}\|v\|^{(k+1)-\ell}_{Y_{\lambda,s}}\|v^*\|^{\ell}_{Y_{\lambda,s}} \\
&\quad + \sum_{\ell=0}^{k}\|u\|^{k-\ell}_{Y_{\lambda,s}}\|u^*\|^{\ell}_{Y_{\lambda,s}}\sum_{\ell=0}^{k}\|v\|^{k-\ell}_{Y_{\lambda,s}}\|v^*\|^{\ell}_{Y_{\lambda,s}} \\
&\quad + \sum_{\ell=0}^{k-2}\|u\|^{(k-2)-\ell}_{Y_{\lambda,s}}\|u^*\|^{\ell}_{Y_{\lambda,s}}\sum_{\ell=0}^{k+2}\|u\|^{(k+2)-\ell}_{Y_{\lambda,s}}\|u^*\|^{\ell}_{Y_{\lambda,s}}\Bigg)\|u-u^*\|_{Y_{\lambda,s}}, \quad \forall u, u^* \in Y_{\lambda,s} \\
\|T_2 v - T_2 v^*\|_{Y_{\lambda,s}} &\le c_{\varpi}\Bigg(\sum_{\ell=0}^{2k}\|v\|^{2k-\ell}_{Y_{\lambda,s}}\|v^*\|^{\ell}_{Y_{\lambda,s}} + \sum_{\ell=0}^{k-1}\|v\|^{(k-1)-\ell}_{Y_{\lambda,s}}\|v^*\|^{\ell}_{Y_{\lambda,s}}\sum_{\ell=0}^{k+1}\|u\|^{(k+1)-\ell}_{Y_{\lambda,s}}\|u^*\|^{\ell}_{Y_{\lambda,s}} \\
&\quad + \sum_{\ell=0}^{k}\|v\|^{k-\ell}_{Y_{\lambda,s}}\|v^*\|^{\ell}_{Y_{\lambda,s}}\sum_{\ell=0}^{k}\|u\|^{k-\ell}_{Y_{\lambda,s}}\|u^*\|^{\ell}_{Y_{\lambda,s}} \\
&\quad + \sum_{\ell=0}^{k-2}\|v\|^{(k-2)-\ell}_{Y_{\lambda,s}}\|v^*\|^{\ell}_{Y_{\lambda,s}}\sum_{\ell=0}^{k+2}\|v\|^{(k+2)-\ell}_{Y_{\lambda,s}}\|v^*\|^{\ell}_{Y_{\lambda,s}}\Bigg)\|v-v^*\|_{Y_{\lambda,s}}, \quad \forall v, v^* \in Y_{\lambda,s}.
\end{aligned}$$

Adding the inequalities, we find that

$$\begin{aligned}
\|(T_1u - T_1u^*, T_2v - T_2v^*)\|_{\mathcal{Y}_{\lambda,s}} &\le c_\varpi 4\left(\sum_{\ell=0}^{2k} r^{2k-\ell} r^\ell\right)\|(u-u^*, v-v^*)\|_{\mathcal{Y}_{\lambda,s}}\\
&= c_\varpi 4 r^{2k}(2k+1)\|(u-u^*, v-v^*)\|_{\mathcal{Y}_{\lambda,s}}\\
&= c_\varpi 4\left(\frac{1}{3c_\varpi^{\frac{1}{2k}}(2k+1)^{\frac{1}{2k}}}\right)^{2k}(2k+1)\|(u-u^*, v-v^*)\|_{\mathcal{Y}_{\lambda,s}}\\
&= \frac{4}{3^{2k}}\|(u-u^*, v-v^*)\|_{\mathcal{Y}_{\lambda,s}}.
\end{aligned}$$

□

2.2. Uniqueness and continuous dependence of the initial data

We prove the uniqueness in

$$C\left([-1,1], G^{\lambda,s}(\mathbb{T})\right) \times C\left([-1,1], G^{\lambda,s}(\mathbb{T})\right),$$

by the next standard argument.

Lemma 2.3. *Let*

$$(u,v),(u^*,v^*) \in C\left([-1,1], G^{\lambda,S}(\mathbb{T})\right),$$

be solutions to (1.3) with

$$(u_0,v_0) = (u_0^*, v_0^*),$$

in

$$G^{\lambda,s}(\mathbb{T}) \times G^{\lambda,s}(\mathbb{T}),$$

and

$$s \ge 1/2.$$

Then $(u,v) = (u^*,v^*)$.

The continuity of the data-to-solution map is given in the following.

Lemma 2.4. *Let* (u,v) *and* (u^*,v^*) *be solutions to (1.3) corresponding to* (u_0,v_0) *and* (u_0^*,v_0^*) *respectively with the norms*

$$\|u_0\|_{G^{\lambda,s}(\mathbb{T})} + \|v_0\|_{G^{\lambda,s}(\mathbb{T})}, \|u_0^*\|_{G^{\lambda,s}(\mathbb{T})} + \|v_0^*\|_{G^{\lambda,s}(\mathbb{T})},$$

small and $s \ge 1/2$. *Then*

$$|(u-u^*, v-v^*)|_{c,\lambda,s} \lesssim c_\varpi\left(\|(u_0-u_0^*)\|_{G^{\lambda,s}(\mathbb{T})} + \|(v_0-v_0^*)\|_{G^{\lambda,s}(\mathbb{T})}\right).$$

The proof of Lemma 2.3 and Lemma 2.4 follows a standard argument.

3. Uniform radius of analyticity

Let
$$(u_0, v_0) \in G^{\lambda,s}(\mathbb{T}) \times G^{\lambda,s}(\mathbb{T}),$$
then by the definition of $G^{\lambda,s}(\mathbb{T}) \times G^{\lambda,s}(\mathbb{T})$, we have for some $L > 0$ and $n \in \mathbb{Z}$
$$|\hat{u}_0(n)| \leq Le^{-\lambda|n|}, \tag{3.1}$$
$$|\hat{v}_0(n)| \leq Le^{-\lambda|n|}. \tag{3.2}$$
By (3.1) and (3.1), we have
$$\varpi_1(x) = \sum_{n=0}^{\infty} \hat{u}_0(n)e^{inx} \in C^{\vartheta}(\mathbb{T}),$$
$$\varpi_2(x) = \sum_{n=0}^{\infty} \hat{u}_0(n)e^{inx} \in C^{\vartheta}(\mathbb{T}).$$
Furthermore,
$$\hat{\varpi}_1(n) = \hat{u}_0(n), \hat{\varpi}_2(n) = \hat{v}_0(n), \forall n \in \mathbb{Z}.$$
Since $(u_0, v_0) \in L^2(\mathbb{T}) \times L^2(\mathbb{T})$, we conclude that $(u_0, v_0) \in C^{\vartheta}(\mathbb{T}) \times C^{\vartheta}(\mathbb{T})$.

The next analytic continuation result is stated, and its proof is similar to that in [18].

Lemma 3.1. *Let*
$$(u_0, v_0) \in G^{\lambda,s}(\mathbb{T}) \times G^{\lambda,s}(\mathbb{T}),$$
then (u_0, v_0) *has an analytic extension in a symmetric strip around the real axis, and its width is equal to* λ.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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