Electronic Research Archive

https://www.aimspress.com/journal/era

ERA, 33(6): 4027–4044.
DOI: 10.3934/era.2025179
Received: 28 February 2025
Revised: 26 May 2025
Accepted: 18 June 2025
Published: 25 June 2025

Research article

Positive solutions for a Caputo-type fractional differential equation with a Riemann-Stieltjes integral boundary condition

Haixia Li* **and Xin Liu**

School of Basic Teaching Department, Shandong Water Conservancy Vocational College, Rizhao 276826, China

* **Correspondence:** Email: 726596816@qq.com.

Abstract: In this paper, we study the solvability of positive solutions for a Caputo-type fractional-order Riemann-Stieltjes integral boundary value problem. Under some monotonicity conditions for the nonlinearity, we use the upper-lower solution method to obtain two existence theorems.

Keywords: Caputo-type fractional-order differential equations; integral boundary value problems; positive solutions; upper-lower solution method

1. Introduction

In this work, we study the following Caputo-type fractional integral boundary value problem:

$$\begin{cases} -{}^cD_{0+}^{\alpha}z(t) + \delta {}^cD_{0+}^{\beta}z(t) = g(t, z(t)),\ 0 < t < 1, \\ z(0) = 0,\ z(1) = \int_0^1 z(t)d\mu(t), \end{cases} \tag{1.1}$$

where ${}^cD_{0+}^{\alpha}, {}^cD_{0+}^{\beta}$ are the Caputo-type fractional derivatives with $\alpha \in (1, 2], \beta \in (0, 1]$, $2\alpha > \beta + 2$, and the functions g, μ satisfy the following conditions:

(C1) $g \in C([0, 1] \times \mathbb{R}^+, \mathbb{R}^+)$;

(C2) $\mu(t)$ is a non-negative function of bounded variation, and not a constant function on $t \in [0, 1]$; and

(C3) δ is a fixed positive constant with $\delta \in (0, \delta^*)$, where δ^* is a unique zero point of the function

$$h(t) = \frac{\alpha - 2}{\Gamma(\alpha - 1)} + \sum_{k=1}^{\infty} \frac{t^k}{\Gamma((\alpha - \beta)k + \alpha - 2)}.$$

In recent years, fractional calculus has rapidly developed in engineering, physics, electronics, chemistry, and other fields due to their profound physical background and applications. For example, in viscoelastic beam vibration control, the model for memory effects is as follows:

$$m^{c}D_{0+}^{2}u(t) + n^{c}D_{0+}^{\alpha}u(t) + ku(t) = f(t),\ 1 < \alpha < 2,$$

where ${}^{c}D_{0+}^{\alpha}$ is the Capotu-type fractional derivative, which is used to describe frequency-dependent damping characteristics. Under simply-supported boundary conditions, the vibration suppression effect can be optimized by adjusting the parameter α in the fractional-order PID controller.

As an important research field in the study of fractional-order equations, the existence of solutions for initial and boundary value problems has become popular among researchers, and a large number of excellent results have been obtained by virtue of fixed-point theorems, the upper-lower solutions technique, etc. A variety of results on this direction can be found in the literature; we refer to [1–28] and the references cited therein. In [1], the authors studied the existence and multiplicity of positive solutions for the following fractional-order integral boundary value problem:

$$\begin{cases} \left(\varphi_p\left({}^{c}D_{0+}^{\alpha}u(t)\right)\right)' + a(t)f(t,u(t)) = 0,\ t \in (0,1), \\ {}^{c}D_{0+}^{\alpha}u(0) = u'(0) = u''(0) = 0, u(1) + u'(1) = \int_0^{\eta} u(t)dt. \end{cases}$$

They provided some necessary and sufficient conditions to obtain the existence and multiplicity results via the Krasnoselskii, Schaefer, and Leggett-Williams fixed point theorems.

In [2], the authors investigated the following coupled system of nonlinear fractional integral boundary value problems:

$$\begin{cases} {}^{c}D_{0+}^{\delta}\mu(\ell) = \mathcal{F}_1(\ell, \mu(\lambda\ell), \nu(\lambda\ell)),\ \ell \in [0,1], \\ {}^{c}D_{0+}^{\varrho}\nu(\ell) = \mathcal{F}_2(\ell, \mu(\lambda\ell), \nu(\lambda\ell)),\ \ell \in [0,1], \\ \mu(0) = r(\mu),\ \mu(1) = \frac{1}{\Gamma(\delta)}\int_0^1 (1-\eta)^{\delta-1}\varphi(\eta, \mu(\eta))d\eta, \\ \nu(0) = h(\nu),\ \nu(1) = \frac{1}{\Gamma(\varrho)}\int_0^1 (1-\eta)^{\varrho-1}\psi(\eta, \nu(\eta))d\eta. \end{cases}$$

When $\mathcal{F}_i (i = 1, 2)$ and φ, ψ satisfy some Lipschitz conditions, they obtained the existence of nontrivial solutions for their system via the topological degree theory. In [3], the authors studied the following Caputo-type fractional four-point boundary value problems at resonance:

$$\begin{cases} {}^{c}D_{0+}^{\alpha}u(t) = f\left(t, u(t), {}^{c}D_{0+}^{\alpha-1}u(t)\right), t \in (0,1), \\ u(0) = Bu(\xi), u(1) = Cu(\eta). \end{cases}$$

By using the continuation theorem due to Mawhin, they obtained their results when f satisfied the Carathéodory conditions. In [4], the authors used the upper-lower solution method to study the following fractional-order integral boundary value problem:

$$\begin{cases} -D_{0+}^{\alpha}u(t) = f(t, u(t)),\ t \in (0,1), \\ u(0) = 0, u(1) = \int_0^1 u(t)\mathrm{d}A(t), \end{cases}$$

where D_{0+}^{α} is the Riemann-Liouville fractional derivative. When the nonlinearity satisfied the reverse Lipschitz condition, they obtained the existence of the extremal solutions, which can be uniformly converged from some appropriate monotone sequences.

Inspired by the works above, in this paper, we use the upper-lower solution method to study the existence of positive solutions for (1.1) when the nonlinearity satisfies some monotonicity conditions. Our innovation lies in the following aspects:

(i) Dual fractional derivatives: the differential equation in (1.1) contains two fractional derivatives, and its equivalent integral equation is obtained by using the Laplace transforms;

(ii) Comparison theorem via integral boundary conditions: we derive a comparison theorem by leveraging the integral boundary conditions; and

(iii) Existence of positive solutions: using the upper-lower solution method, we prove the existence of positive solutions for both increasing and decreasing nonlinear terms. The monotonicity conditions are easily satisfied.

2. Preliminaries

In this section, we first present some basic knowledge that will be used in the paper.

Definition 2.1 (see [29,30]). The fractional derivative of f in the Caputo sense is defined as follows:

$$ {}^{c}D_{0+}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}f^{(n)}(s)ds,\ n = [\alpha]+1, $$

where $[\alpha]$ denotes the integer part of the number α.

Definition 2.2 (see [29, 30]). The Laplace transform of a function $f(t)$ of a real variable $t \in \mathbb{R}^+$ is defined by

$$ F(s) = \mathcal{L}[f(t)] = \int_0^{+\infty} f(t)e^{-st}dt,\ s = \gamma + i\omega \in \mathbb{C},\ \gamma > 0, $$

and the inverse Laplace transform is defined by

$$ f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st}ds,\ i^2 = -1. $$

Definition 2.3 (see [29, 30]). The Laplace convolution operator of two functions, $\zeta(t)$ and $\varphi(t)$, given on $\mathbb{R}^+$, is defined for $x \in \mathbb{R}^+$ by the following integral:

$$ \zeta * \varphi = (\zeta * \varphi)(x) := \int_0^x \zeta(x-t)\varphi(t)dt. $$

Definition 2.4 (see [29,30]). The Mittag-Leffler function $E_{\alpha,\beta}(t)$ in two parameters is defined by the following:

$$ E_{\alpha,\beta}(t) := \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+\beta)},\ t \in \mathbb{C},\ \alpha,\beta > 0. $$

Lemma 2.5. Let $V \in C[0, 1]$. Then, the boundary value problem

$$ \begin{cases} -{}^{c}D_{0+}^{\alpha}z(t) + \delta {}^{c}D_{0+}^{\beta}z(t) = V(t),\ 0 < t < 1, \\ z(0) = 0,\ z(1) = \int_0^1 z(t)d\mu(t) \end{cases} \tag{2.1} $$

has a solution

$$ z(t) = \int_0^1 H(t, s)V(s)ds + \frac{g_2(t)}{g_2(1)}\int_0^1 z(t)d\mu(t), $$

where

$$H(t,s)=\frac{1}{g_2(1)}\begin{cases}g_1(1-s)g_2(t), & 0\le t\le s\le 1,\\ g_1(1-s)g_2(t)-g_1(t-s)g_2(1), & 0\le s\le t\le 1,\end{cases} \tag{2.2}$$
$$g_1(t)=t^{\alpha-1}E_{\alpha-\beta,\alpha}\left(\delta t^{\alpha-\beta}\right),\ g_2(t)=tE_{\alpha-\beta,2}\left(\delta t^{\alpha-\beta}\right).$$

Proof. Making the Laplace transform on both sides of the first equation in (2.1), using [5, (3.2)], we obtain the following:

$$-s^{\alpha}Z(s)+s^{\alpha-1}z(0)+s^{\alpha-2}z'(0)+\delta s^{\beta}Z(s)-\delta s^{\beta-1}z(0)=Y(s),$$

where $Z(s)=\mathcal{L}[z(t)]$, $Y(s)=\mathcal{L}[V(t)]$. Solving this equation, we obtain the following:

$$Z(s)=-\frac{Y(s)}{s^{\alpha}-\delta s^{\beta}}+z(0)\frac{s^{\alpha-1}}{s^{\alpha}-\delta s^{\beta}}+z'(0)\frac{s^{\alpha-2}}{s^{\alpha}-\delta s^{\beta}}-\delta z(0)\frac{s^{\beta-1}}{s^{\alpha}-\delta s^{\beta}}.$$

Making the inverse Laplace transform for this equation, we have the following:

$$z(t)=-V(t)*\mathcal{L}^{-1}[\frac{1}{s^{\alpha}-\delta s^{\beta}}]+z(0)\mathcal{L}^{-1}[\frac{s^{\alpha-1}}{s^{\alpha}-\delta s^{\beta}}]+z'(0)\mathcal{L}^{-1}[\frac{s^{\alpha-2}}{s^{\alpha}-\delta s^{\beta}}]-\delta z(0)\mathcal{L}^{-1}[\frac{s^{\beta-1}}{s^{\alpha}-\delta s^{\beta}}].$$

Note that $\frac{1}{s^{\alpha}-\delta s^{\beta}}$ can be expressed by

$$\frac{1}{s^{\alpha}-\delta s^{\beta}}=\sum_{k=0}^{\infty}\delta^k s^{-k(\alpha-\beta)-\alpha}=\sum_{k=0}^{\infty}\frac{\delta^k}{\Gamma((\alpha-\beta)k+\alpha)}\frac{\Gamma((\alpha-\beta)k+\alpha)}{s^{k(\alpha-\beta)+\alpha-1+1}}.$$

Using $\mathcal{L}^{-1}[\frac{\Gamma(\sigma+1)}{s^{\sigma+1}}]=t^{\sigma}(\sigma>-1)$, from Definition 2.4, we have the following:

$$\mathcal{L}^{-1}[\frac{1}{s^{\alpha}-\delta s^{\beta}}]=\sum_{k=0}^{\infty}\frac{\delta^k t^{k(\alpha-\beta)+\alpha-1}}{\Gamma((\alpha-\beta)k+\alpha)}=t^{\alpha-1}E_{\alpha-\beta,\alpha}(\delta t^{\alpha-\beta})=g_1(t). \tag{2.3}$$

Similarly, we have

$$\mathcal{L}^{-1}[\frac{s^{\alpha-1}}{s^{\alpha}-\delta s^{\beta}}]=\sum_{k=0}^{\infty}\frac{\delta^k t^{k(\alpha-\beta)}}{\Gamma((\alpha-\beta)k+1)}=E_{\alpha-\beta,1}(\delta t^{\alpha-\beta}),$$
$$\mathcal{L}^{-1}[\frac{s^{\alpha-2}}{s^{\alpha}-\delta s^{\beta}}]=\sum_{k=0}^{\infty}\frac{\delta^k t^{k(\alpha-\beta)+1}}{\Gamma((\alpha-\beta)k+2)}=tE_{\alpha-\beta,2}(\delta t^{\alpha-\beta})=g_2(t), \tag{2.4}$$

and

$$\mathcal{L}^{-1}[\frac{s^{\beta-1}}{s^{\alpha}-\delta s^{\beta}}]=\sum_{k=0}^{\infty}\frac{\delta^k t^{k(\alpha-\beta)+\alpha-\beta}}{\Gamma((\alpha-\beta)k+\alpha-\beta+1)}=t^{\alpha-\beta}E_{\alpha-\beta,\alpha-\beta+1}(\delta t^{\alpha-\beta}).$$

Therefore, we have

$$z(t)=-V(t)*t^{\alpha-1}E_{\alpha-\beta,\alpha}\left(\delta t^{\alpha-\beta}\right)+z(0)\cdot E_{\alpha-\beta,1}\left(\delta t^{\alpha-\beta}\right)$$
$$+z'(0)\cdot tE_{\alpha-\beta,2}\left(\delta t^{\alpha-\beta}\right)-\delta z(0)\cdot t^{\alpha-\beta}E_{\alpha-\beta,\alpha-\beta+1}\left(\delta t^{\alpha-\beta}\right),$$

where

$$V(t)*t^{\alpha-1}E_{\alpha-\beta,\alpha}\left(\delta t^{\alpha-\beta}\right)=V(t)*g_1(t)=\int_0^t g_1(t-s)V(s)ds.$$

Note that if $z(0) = 0$, then we have the following:

$$z(t) = -V(t) * t^{\alpha-1}E_{\alpha-\beta,\alpha}\left(\delta t^{\alpha-\beta}\right) + z'(0)\cdot tE_{\alpha-\beta,2}\left(\delta t^{\alpha-\beta}\right);$$

then, $z(1) = \int_0^1 z(t)d\mu(t)$ implies that

$$-\int_0^1 g_1(1-s)V(s)ds + z'(0)g_2(1) = \int_0^1 z(t)d\mu(t).$$

Consequently, we have

$$z'(0) = \frac{1}{g_2(1)}\int_0^1 z(t)d\mu(t) + \frac{1}{g_2(1)}\int_0^1 g_1(1-s)V(s)ds,$$

and

$$\begin{aligned} z(t) &= -\int_0^t g_1(t-s)V(s)ds + \frac{g_2(t)}{g_2(1)}\int_0^1 z(t)d\mu(t) + \frac{g_2(t)}{g_2(1)}\int_0^1 g_1(1-s)V(s)ds \\ &= \int_0^1 H(t,s)V(s)ds + \frac{g_2(t)}{g_2(1)}\int_0^1 z(t)d\mu(t). \end{aligned}$$

This completes the proof. □

Let $E = C[0,1]$ and $\|\cdot\| = \max_{t\in[0,1]}|\cdot|$. Then, $(E, \|\cdot\|)$ is a Banach space, and $P := \{z \in E : z(t) \geq 0, t \in [0,1]\}$ is a cone on E. By Lemma 2.5, we find that (1.1) is equivalent to the following integral equation:

$$z(t) = \int_0^1 H(t,s)g(s,z(s))ds + \frac{g_2(t)}{g_2(1)}\int_0^1 z(t)d\mu(t) := (\Theta z)(t),\ z \in E, t \in [0,1]. \tag{2.5}$$

It is easy to find that there exists $z^* \in E\backslash\{0\}$ such that $\Theta z^* = z^*$ (i.e., z^* is a solution for (1.1)).

Consider the function μ; it also satisfies the condition

(C4) $\int_0^1 \frac{g_2(t)}{g_2(1)}d\mu(t) \in [0,1)$.

Then, there exists a fixed point z^* of Θ (i.e., $\Theta z^* = z^*$). Then, from (2.5), we have

$$(\Theta z^*)(t) = z^*(t) = \int_0^1 H(t,s)g(s,z^*(s))ds + \frac{g_2(t)}{g_2(1)}\int_0^1 z^*(t)d\mu(t), \tag{2.6}$$

and

$$\int_0^1 z^*(t)d\mu(t) = \int_0^1\int_0^1 H(t,s)g(s,z^*(s))dsd\mu(t) + \int_0^1 \frac{g_2(t)}{g_2(1)}d\mu(t)\int_0^1 z^*(t)d\mu(t).$$

Then, (C4) implies that

$$\int_0^1 z^*(t)d\mu(t) = \frac{1}{1-\int_0^1 \frac{g_2(t)}{g_2(1)}d\mu(t)}\int_0^1\int_0^1 H(t,s)g(s,z^*(s))dsd\mu(t).$$

Therefore, we have

$$\begin{aligned} z^*(t) &= \int_0^1 H(t,s)g(s,z^*(s))ds + \frac{g_2(t)}{g_2(1)}\frac{1}{1-\int_0^1 \frac{g_2(t)}{g_2(1)}d\mu(t)}\int_0^1\int_0^1 H(t,s)g(s,z^*(s))dsd\mu(t) \\ &= \int_0^1 G(t,s)g(s,z^*(s))ds, \end{aligned} \tag{2.7}$$

where

$$G(t,s)=H(t,s)+\frac{g_2(t)}{g_2(1)-\int_0^1 g_2(t)d\mu(t)}\int_0^1 H(t,s)d\mu(t). \tag{2.8}$$

Note that from (2.6) and (2.7), we can also define the operator Θ as follows:

$$(\Theta z)(t)=\int_0^1 G(t,s)g(s,z(s))ds,\ z\in E, t\in[0,1]. \tag{2.9}$$

Lemma 2.6 (see [23, Lemma 2]). The functions G, H have the following properties:
(i) G, H are continuous on $[0,1]\times[0,1]$;
(ii) $G, H(t,s)\geq 0$, $t,s\in[0,1]$; and
(iii) $\Lambda_1 t(1-s)^{\alpha-1}s\leq G(t,s)\leq\Lambda_2 t(1-s)^{\alpha-1}$, $t,s\in[0,1]$, where

$$\Lambda_1=\frac{\Lambda_3}{g_2(1)-\int_0^1 g_2(t)d\mu(t)}\int_0^1(1-t)td\mu(t),\ \Lambda_2=g_1(1)+\frac{g_1(1)g_2(1)}{g_2(1)-\int_0^1 g_2(t)d\mu(t)}\int_0^1 td\mu(t).$$

Proof. Note that $\alpha\in(1,2],\beta\in(0,1]$, and from (2.3) and (2.4), $g_i(i=1,2)$ are continuous on $t\in[0,1]$. This implies that $H(t,s)$ is continuous on $t,s\in[0,1]$. From (C2)–(C4), the definition of G in (2.8) implies that it is continuous on $t,s\in[0,1]$.

From (2.4), we have

$$g_2'(t)=\sum_{k=0}^{\infty}\frac{[k(\alpha-\beta)+1]\delta^k t^{k(\alpha-\beta)}}{\Gamma((\alpha-\beta)k+2)}=\sum_{k=0}^{\infty}\frac{\delta^k t^{k(\alpha-\beta)}}{\Gamma((\alpha-\beta)k+1)},\ t\in(0,1),$$

and

$$g_2''(t)=\sum_{k=0}^{\infty}\frac{k(\alpha-\beta)\delta^k t^{k(\alpha-\beta)-1}}{\Gamma((\alpha-\beta)k+1)},\ t\in(0,1).$$

This, together with (2.4), implies that g_2, g_2' are non-decreasing on $t\in[0,1]$. On the other hand, by (2.3), we have the following:

$$g_1'(t)=\sum_{k=0}^{\infty}\frac{[k(\alpha-\beta)+\alpha-1]\delta^k t^{k(\alpha-\beta)+\alpha-2}}{\Gamma((\alpha-\beta)k+\alpha)}=\sum_{k=0}^{\infty}\frac{\delta^k t^{k(\alpha-\beta)+\alpha-2}}{\Gamma((\alpha-\beta)k+\alpha-1)},\ t\in(0,1).$$

This implies that g_1 is non-decreasing on $t\in[0,1]$. Furthermore, note that $\alpha\in(1,2]$, $\beta\in(0,1], 2\alpha>\beta+2$. We know that the function h in (C3) has the following properties:

$$h(0)=\frac{\alpha-2}{\Gamma(\alpha-1)}<0,\ \lim_{t\to+\infty}h(t)=+\infty,\ \text{and } h \text{ is strictly increasing on } t\in[0,1].$$

Therefore, there exists a unique positive number δ^* such that

$$h(\delta^*)=0.$$

This implies that

$$\begin{aligned}
g_1''(t) &= \sum_{k=0}^{\infty} \frac{[k(\alpha-\beta)+\alpha-2]\delta^k t^{k(\alpha-\beta)+\alpha-3}}{\Gamma((\alpha-\beta)k+\alpha-1)} \\
&= t^{\alpha-3}\left[\frac{\alpha-2}{\Gamma(\alpha-1)} + \sum_{k=1}^{\infty} \frac{[k(\alpha-\beta)+\alpha-2]\delta^k t^{k(\alpha-\beta)}}{\Gamma((\alpha-\beta)k+\alpha-1)}\right] \\
&= t^{\alpha-3}\left[\frac{\alpha-2}{\Gamma(\alpha-1)} + \sum_{k=1}^{\infty} \frac{\delta^k t^{k(\alpha-\beta)}}{\Gamma((\alpha-\beta)k+\alpha-2)}\right] \\
&= t^{\alpha-3} h(\delta t^{\alpha-\beta}) \\
&\le t^{\alpha-3} h(\delta^*) \\
&= 0, t \in (0,1),
\end{aligned}$$

and thus we obtain that g_1' are decreasing on $t \in [0,1]$.

Now, we prove the nonnegativity of H on $t, s \in [0,1]$. When $0 \le t \le s \le 1$, it is a non-negative function. When $0 \le s \le t \le 1$, we have

$$H_{tt}(t,s) = \frac{g_1(1-s)}{g_2(1)} g_2''(t) - g_1''(t-s) \ge 0,$$

and thus

$$H_t(t,s) \ge H_t(s,s) \ge 0.$$

Consequently, we have the following:

$$H(t,s) \ge H(s,s) = \frac{g_1(1-s)}{g_2(1)} g_2(s) \ge 0.$$

From (C2)–(C4) and (2.8), we have the following:

$$G(t,s) \ge 0, t, s \in [0,1].$$

Note that from Lemma 2 of [23], we have the following:

$$\Lambda_3(1-t)t(1-s)^{\alpha-1}s \le H(t,s) \le g_1(1)t(1-s)^{\alpha-1}, \ t, s \in [0,1],$$

where $\Lambda_3 = \min\{1/(g_2(1)\Gamma(\alpha)), (\alpha-1)g_1(1)\}$. Using $t \le g_2(t) \le tg_2(1), t \in [0,1]$, we have

$$\begin{aligned}
G(t,s) &\le g_1(1)t(1-s)^{\alpha-1} + \frac{tg_2(1)}{g_2(1) - \int_0^1 g_2(t)d\mu(t)} \int_0^1 g_1(1)t(1-s)^{\alpha-1} d\mu(t) \\
&= \Lambda_2 t(1-s)^{\alpha-1}
\end{aligned}$$

and

$$\begin{aligned}
G(t,s) &\ge \frac{t}{g_2(1) - \int_0^1 g_2(t)d\mu(t)} \int_0^1 \Lambda_3(1-t)t(1-s)^{\alpha-1} s d\mu(t) \\
&= \Lambda_1 t(1-s)^{\alpha-1} s.
\end{aligned}$$

This completes the proof. □

Using Lemma 2.6(i) and (ii) and (C1)–(C4), the continuity and non-negativity of G, g imply that $\Theta : P \to P$ is a completely continuous operator.

Definition 2.7. We say that $w \in C[0,1]$ is an upper solution of (1.1) if it satisfies the following:

$$\begin{cases} -{}^cD^{\alpha}_{0+}w(t) + \delta{}^cD^{\beta}_{0+}w(t) \geq g(t, w(t)),\ 0 < t < 1, \\ w(0) = 0,\ w(1) \geq \int_0^1 w(t)d\mu(t). \end{cases}$$

Definition 2.8. We say that $v \in C[0,1]$ is a lower solution of (1.1) if it satisfies the following:

$$\begin{cases} -{}^cD^{\alpha}_{0+}v(t) + \delta{}^cD^{\beta}_{0+}v(t) \leq g(t, v(t)),\ 0 < t < 1, \\ v(0) = 0,\ v(1) \leq \int_0^1 v(t)d\mu(t). \end{cases}$$

Lemma 2.9 (Comparison principle). Suppose that (C2)–(C4) hold; if there exists $z \in C[0,1]$ such that

$$\begin{cases} -{}^cD^{\alpha}_{0+}z(t) + \delta{}^cD^{\beta}_{0+}z(t) \geq 0,\ 0 < t < 1, \\ z(0) = 0,\ z(1) \geq \int_0^1 z(t)d\mu(t), \end{cases}$$

then $z(t) \geq 0$, $t \in [0,1]$.

Proof. Let $M = z(1) - \int_0^1 z(t)d\mu(t)$, $\overline{V}(t) = -{}^cD^{\alpha}_{0+}z(t) + \delta{}^cD^{\beta}_{0+}z(t)$, $t \in (0,1)$, and $M \geq 0, \overline{V}(t) \geq 0, t \in (0,1)$. Then, we can obtain the following boundary valve problem:

$$\begin{cases} -{}^cD^{\alpha}_{0+}z(t) + \delta{}^cD^{\beta}_{0+}z(t) = \overline{V}(t),\ 0 < t < 1, \\ z(0) = 0,\ z(1) = \int_0^1 z(t)d\mu(t) + M. \end{cases}$$

From the proof of Lemma 2.5, we have the following:

$$-\int_0^1 g_1(1-s)\overline{V}(s)ds + z'(0)g_2(1) = \int_0^1 z(t)d\mu(t) + M.$$

Consequently, we obtain

$$z'(0) = \frac{1}{g_2(1)}\int_0^1 z(t)d\mu(t) + \frac{1}{g_2(1)}\int_0^1 g_1(1-s)\overline{V}(s)ds + \frac{M}{g_2(1)},$$

and from Lemma 2.6(ii), we find

$$\begin{aligned} z(t) &= -\int_0^t g_1(t-s)\overline{V}(s)ds + \frac{g_2(t)}{g_2(1)}\int_0^1 g_1(1-s)\overline{V}(s)ds + \frac{g_2(t)}{g_2(1)}\int_0^1 z(t)d\mu(t) + \frac{g_2(t)}{g_2(1)}M \\ &= \int_0^1 H(t,s)\overline{V}(s)ds + \frac{g_2(t)}{g_2(1)}\int_0^1 z(t)d\mu(t) + \frac{g_2(t)}{g_2(1)}M \\ &\geq \frac{g_2(t)}{g_2(1)}\int_0^1 z(t)d\mu(t) + \frac{g_2(t)}{g_2(1)}M. \end{aligned}$$

Multiplying by $d\mu(t)$ on both sides of the above and integrating over $[0,1]$, from (C4), we have

$$\int_0^1 z(t)d\mu(t) \geq \int_0^1 \frac{g_2(t)}{g_2(1)}d\mu(t)\int_0^1 z(t)d\mu(t) + M\int_0^1 \frac{g_2(t)}{g_2(1)}d\mu(t),$$

and

$$\int_0^1 z(t)d\mu(t) \geq \frac{M\int_0^1 \frac{g_2(t)}{g_2(1)}d\mu(t)}{1-\int_0^1 \frac{g_2(t)}{g_2(1)}d\mu(t)}.$$

Therefore, we have the following:

$$z(t) \geq \frac{g_2(t)}{g_2(1)} \frac{M\int_0^1 \frac{g_2(t)}{g_2(1)}d\mu(t)}{1-\int_0^1 \frac{g_2(t)}{g_2(1)}d\mu(t)} + \frac{g_2(t)}{g_2(1)}M.$$

Therefore, we obtain $z(t) \geq 0, t \in [0,1]$ from the nonnegativity of M, g_2. This completes the proof. □

3. Main results

Now, we give our main results and their proofs.

Theorem 3.1. Let (C1)–(C4) and the following conditions hold:

(C5) for any constant $\rho > 0$, $g(t, \rho t) \not\equiv 0$ and

$$0 < \int_0^1 (1-s)^{\alpha-1} g(s, \rho s) ds < +\infty,$$

(C6) $g(t,x) \geq g(t,y)$ if $x \leq y$ for $t \in [0,1]$.

Then, (1.1) has a positive solution z^*, and there exist $0 < \lambda_1 < 1 < \lambda_2$ such that $\lambda_1 t \leq z^*(t) \leq \lambda_2 t, t \in [0,1]$.

Proof. First, we define a set $\overline{P} := \{z \in P : \exists 0 < l_z < L_z \text{ s.t. } l_z t \leq z(t) \leq L_z t,\ t \in [0,1]\}$. Obviously, $t \in \overline{P}$ and thus $\overline{P} \neq \emptyset$. In what follows, we prove that

$$\Theta(\overline{P}) \subset \overline{P}. \tag{3.1}$$

For any $z \in \overline{P}$, Lemma 2.6(iii), (C5), and (C6) imply that

$$(\Theta z)(t) \leq \int_0^1 \Lambda_2 t(1-s)^{\alpha-1} g(s, l_z s) ds < +\infty,$$

and

$$(\Theta z)(t) \geq \int_0^1 \Lambda_1 t(1-s)^{\alpha-1} s g(s, L_z s) ds.$$

Choose

$$l'_z = \min\{1, \int_0^1 \Lambda_1 (1-s)^{\alpha-1} s g(s, L_z s) ds\},\ L'_z = \max\{1, \int_0^1 \Lambda_2 (1-s)^{\alpha-1} g(s, l_z s) ds\};$$

then, we have

$$l'_z t \leq (\Theta z)(t) \leq L'_z t,\ t \in [0,1].$$

This implies that Θ is well-defined, and (3.1) holds. Moreover, by (C6), Θz is decreasing in z and satisfies the following:

$$\begin{cases} -{}^cD_{0+}^{\alpha}(\Theta z)(t) + \delta {}^cD_{0+}^{\beta}(\Theta z)(t) = g(t, z(t)),\ 0 < t < 1, \\ (\Theta z)(0) = 0,\ (\Theta z)(1) = \int_0^1 (\Theta z)(t) d\mu(t). \end{cases} \tag{3.2}$$

Let $e(t) = t,\ t \in [0, 1]$, and

$$m_e(t) = \min\{e(t),\ (\Theta e)(t)\},\ n_e(t) = \max\{e(t),\ (\Theta e)(t)\}.$$

If $e(t) = (\Theta e)(t)$, then e is a fixed point of Θ, and this function e is also a positive solution for (1.1); thus our theorem has been proved. If $e(t) \neq (\Theta e)(t)$, and then from (3.1), we have $m_e, n_e \in \overline{P}$ with $m_e(t) \leq n_e(t),\ t \in [0, 1]$.

Note that Θ is a decreasing operator, and

$$m_e(t) \leq e(t),\ m_e(t) \leq (\Theta e)(t),\ n_e(t) \geq e(t),\ n_e(t) \geq (\Theta e)(t).$$

Therefore, we have

$$\psi(t) := (\Theta m_e)(t) \geq (\Theta e)(t) \geq m_e(t),\ \varphi(t) := (\Theta n_e)(t) \leq (\Theta e)(t) \leq n_e(t),$$

and

$$\psi(t) \geq \varphi(t).$$

From (3.2), we have

$$\begin{aligned} -{}^cD^{\alpha}_{0+}\varphi(t) + \delta{}^cD^{\beta}_{0+}\varphi(t) - g(t, \varphi(t)) &= -{}^cD^{\alpha}_{0+}(\Theta n_e)(t) + \delta{}^cD^{\beta}_{0+}(\Theta n_e)(t) - g(t, (\Theta n_e)(t)) \\ &= g(t, n_e(t)) - g(t, (\Theta n_e)(t)) \leq 0, \end{aligned}$$

and

$$\varphi(t) = (\Theta n_e)(t) = \int_0^1 G(t, s)g(s, n_e(s))ds,$$

which implies that

$$\varphi(0) = 0,\ \varphi(1) = \int_0^1 \varphi(t)d\mu(t).$$

Using Definition 2.8, φ is a lower solution of (1.1).

For $\psi(t) = (\Theta m_e)(t)$, we have

$$\begin{aligned} -{}^cD^{\alpha}_{0+}\psi(t) + \delta{}^cD^{\beta}_{0+}\psi(t) - g(t, \psi(t)) &= -{}^cD^{\alpha}_{0+}(\Theta m_e)(t) + \delta{}^cD^{\beta}_{0+}(\Theta m_e)(t) - g(t, (\Theta m_e)(t)) \\ &= g(t, m_e(t)) - g(t, (\Theta m_e)(t)) \geq 0, \end{aligned}$$

and

$$\psi(t) = (\Theta m_e)(t) = \int_0^1 G(t, s)g(s, m_e(s))ds,$$

which indicates that

$$\psi(0) = 0,\ \psi(1) = \int_0^1 \psi(t)d\mu(t).$$

Using Definition 2.7, ψ is an upper solution of (1.1).

Now, we consider the following boundary value problem:

$$\begin{cases} -{}^cD^{\alpha}_{0+}z(t) + \delta{}^cD^{\beta}_{0+}z(t) = \widetilde{g}(t, z(t)),\ 0 < t < 1, \\ z(0) = 0,\ z(1) = \int_0^1 z(t)d\mu(t), \end{cases} \tag{3.3}$$

where

$$\widetilde{g}(t,z(t))=\begin{cases} g(t,\varphi(t)), z<\varphi, \\ g(t,z(t)), \varphi\le z\le\psi, \\ g(t,\psi(t)), z>\psi. \end{cases}$$

From Lemma 2.5, we obtain the following:

$$z(t)=\int_0^1 G(t,s)\widetilde{g}(s,z(s))ds := (\Pi z)(t).$$

Note that from (C1)–(C5), the continuity and boundedness of $G,\widetilde{g}$ imply that $\Pi : P \to P$ is a compact operator. Then, by the Schauder fixed point theorem, we know that Π has a positive fixed point, i.e., (3.3) has a positive solution.

Let z^* be a positive solution for (3.3). Then, from the definition of $\widetilde{g}$, we only need to prove that

$$\varphi(t)\le z^*(t)\le\psi(t),\ t\in[0,1], \tag{3.4}$$

which indicates that z^* is the positive solution for (1.1).

We proceed by contradiction. We divide the following cases:

Case 1. $z^*>\psi$. Then, we have the following:

$$\begin{cases} -{}^cD^{\alpha}_{0+}z^*(t)+\delta^cD^{\beta}_{0+}z^*(t)=g(t,\psi(t)),\ 0<t<1, \\ z^*(0)=0,\ z^*(1)=\int_0^1 z^*(t)d\mu(t). \end{cases}$$

Note that ψ is an upper solution; using Definition 2.7, we have

$$-{}^cD^{\alpha}_{0+}[\psi(t)-z^*(t)]+\delta^cD^{\beta}_{0+}[\psi(t)-z^*(t)]\ge g(t,\psi(t))-g(t,\psi(t))=0,$$

and

$$\psi(0)-z^*(0)=0,\ \psi(1)-z^*(1)\ge\int_0^1[\psi(t)-z^*(t)]d\mu(t).$$

Lemma 2.9 implies that $\psi(t)-z^*(t)\ge 0$ ($\psi(t)\ge z^*(t), t\in[0,1]$). This has a contradiction.

Case 2. $z^*<\varphi$. Then, we have the following:

$$\begin{cases} -{}^cD^{\alpha}_{0+}z^*(t)+\delta^cD^{\beta}_{0+}z^*(t)=g(t,\varphi(t)),\ 0<t<1, \\ z^*(0)=0,\ z^*(1)=\int_0^1 z^*(t)d\mu(t). \end{cases}$$

Note that φ is a lower solution; using Definition 2.8, we have

$$-{}^cD^{\alpha}_{0+}[z^*(t)-\varphi(t)]+\delta^cD^{\beta}_{0+}[z^*(t)-\varphi(t)]\ge g(t,\varphi(t))-g(t,\varphi(t))=0,$$

and

$$z^*(0)-\varphi(0)=0,\ z^*(1)-\varphi(1)\ge\int_0^1[z^*(t)-\varphi(t)]d\mu(t).$$

Lemma 2.9 implies that $z^*(t)-\varphi(t)\ge 0$ ($z^*(t)\ge\varphi(t), t\in[0,1]$). This also has a contradiction.

As a result, (3.4) holds, as required, and (1.1) has a positive solution z^*. Note that $\varphi, \psi \in \overline{P}$; from (3.4), we have the following:

$$z^* \in \overline{P}.$$

Hence, there exist $0 < \lambda_1 < 1 < \lambda_2$ such that $\lambda_1 t \leq z^*(t) \leq \lambda_2 t, t \in [0, 1]$. This completes the proof. □

Theorem 3.2. Let (C1)–(C4) and the following conditions hold:

(C7) $w_0, v_0 \in E$ are the upper and lower solutions of (1.1), respectively, with $v_0(t) \leq w_0(t), t \in [0, 1]$; and

(C8) $g(t, x) \geq g(t, y)$ if $x \geq y$ for $t \in [0, 1]$.

Then, there exist sequences $\{v_n\}, \{w_n\} \subset [v_0, w_0]$ such that $v_n \to v^*$, $w_n \to w^*$ as $n \to \infty$ uniformly in $[v_0, w_0]$, and v^*, w^* are positive solution of (1.1) in $[v_0, w_0]$.

Proof. We define the sequences $\{w_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ as follows:

$$\begin{cases} -^c D_{0+}^{\alpha} w_n(t) + \delta^c D_{0+}^{\beta} w_n(t) = g(t, w_{n-1}(t)),\ 0 < t < 1, \\ w_n(0) = 0,\ w_n(1) = \int_0^1 w_n(t) d\mu(t), \end{cases} \tag{3.5}$$

and

$$\begin{cases} -^c D_{0+}^{\alpha} v_n(t) + \delta^c D_{0+}^{\beta} v_n(t) = g(t, v_{n-1}(t)),\ 0 < t < 1, \\ v_n(0) = 0,\ v_n(1) = \int_0^1 v_n(t) d\mu(t). \end{cases} \tag{3.6}$$

Then, from Lemma 2.5, (3.5) and (3.6) are equivalent to the following integral equations:

$$w_n(t) = \int_0^1 H(t, s) g(s, w_{n-1}(s)) ds + \frac{g_2(t)}{g_2(1)} \int_0^1 w_n(t) d\mu(t), \tag{3.7}$$

and

$$v_n(t) = \int_0^1 H(t, s) g(s, v_{n-1}(s)) ds + \frac{g_2(t)}{g_2(1)} \int_0^1 v_n(t) d\mu(t). \tag{3.8}$$

By (C2)–(C4), (3.7) and (3.8) can also be expressed by

$$w_n(t) = \int_0^1 G(t, s) g(s, w_{n-1}(s)) ds = (\Theta w_{n-1})(t), \tag{3.9}$$

and

$$v_n(t) = \int_0^1 G(t, s) g(s, v_{n-1}(s)) ds = (\Theta v_{n-1})(t). \tag{3.10}$$

Note that $w_0 \geq v_0$, then by (3.9) and (3.10), (C8) implies that

$$w_1(t) - v_1(t) = \int_0^1 G(t, s)[g(s, w_0(s)) - g(s, v_0(s))] ds \geq 0, \text{i.e.,} w_1 \geq v_1.$$

Note that $w_n(t) - v_n(t) = \int_0^1 G(t, s)[g(s, w_{n-1}(s)) - g(s, v_{n-1}(s))] ds$; using mathematical induction, it is easy to obtain the following:

$$w_n \geq v_n, n = 0, 1, 2, \cdots. \tag{3.11}$$

Let $z_v(t) = v_1(t) - v_0(t), t \in [0, 1]$. Then, note that v_0 is a lower solution, then we have

$$\begin{aligned}
&- {}^cD^{\alpha}_{0+}z_v(t) + \delta^c D^{\beta}_{0+}z_v(t)\\
&= [-{}^cD^{\alpha}_{0+}v_1(t) + \delta^c D^{\beta}_{0+}v_1(t)] - [-{}^cD^{\alpha}_{0+}v_0(t) + \delta^c D^{\beta}_{0+}v_0(t)]\\
&\geq g(t, v_0(t)) - g(t, v_0(t)) = 0,
\end{aligned}$$

and

$$z_v(0) = v_1(0) - v_0(0) = 0,\ z_v(1) = v_1(1) - v_0(1) \geq \int_0^1 v_1(t)d\mu(t) - \int_0^1 v_0(t)d\mu(t) = \int_0^1 z_v(t)d\mu(t).$$

Lemma 2.9 implies that $z_v(t) \geq 0$ $(v_1(t) \geq v_0(t), t \in [0, 1])$.

Let $z_w(t) = w_0(t) - w_1(t), t \in [0, 1]$. Then, note that w_0 is an upper solution, then we obtain

$$\begin{aligned}
&- {}^cD^{\alpha}_{0+}z_w(t) + \delta^c D^{\beta}_{0+}z_w(t)\\
&= [-{}^cD^{\alpha}_{0+}w_0(t) + \delta^c D^{\beta}_{0+}w_0(t)] - [-{}^cD^{\alpha}_{0+}w_1(t) + \delta^c D^{\beta}_{0+}w_1(t)]\\
&\geq g(t, w_0(t)) - g(t, w_0(t)) = 0,
\end{aligned}$$

and

$$z_w(0) = w_0(0) - w_1(0) = 0,\ z_w(1) = w_0(1) - w_1(1) \geq \int_0^1 w_0(t)d\mu(t) - \int_0^1 w_1(t)d\mu(t) = \int_0^1 z_w(t)d\mu(t).$$

Lemma 2.9 implies that $z_w(t) \geq 0$ $(w_0(t) \geq w_1(t), t \in [0, 1])$.

As a result, we have the following:

$$v_0 \leq v_1 \leq w_1 \leq w_0. \tag{3.12}$$

From (3.5) and (C8), we have the following:

$$\begin{cases} -{}^cD^{\alpha}_{0+}w_1(t) + \delta^c D^{\beta}_{0+}w_1(t) = g(t, w_0(t)) \geq g(t, w_1(t)),\ 0 < t < 1,\\ w_1(0) = 0,\ w_1(1) = \int_0^1 w_1(t)d\mu(t). \end{cases}$$

By Definition 2.7, w_1 is an upper solution of (1.1). Furthermore, from (3.6) and (C8), we have the following:

$$\begin{cases} -{}^cD^{\alpha}_{0+}v_1(t) + \delta^c D^{\beta}_{0+}v_1(t) = g(t, v_0(t)) \leq g(t, v_1(t)),\ 0 < t < 1,\\ v_1(0) = 0,\ v_1(1) = \int_0^1 v_1(t)d\mu(t). \end{cases}$$

By Definition 2.8, v_1 is a lower solution of (1.1).

If w_1, v_1 are taken as the basic functions, then we can repeat the above-mentioned process, and the following conclusion can be drawn:

$$v_1 \leq v_2 \leq w_2 \leq w_1,$$

and w_2, v_2 are upper and lower solutions of (1.1), respectively. Consequently, by applying mathematical induction, we can obtain a non-decreasing sequence of lower solutions $\{v_n\}_{n=0}^{\infty}$ and a non-increasing sequence of upper solutions $\{w_n\}_{n=0}^{\infty}$, which satisfy the following:

$$v_0 \leq v_1 \leq \cdots v_n \leq \cdots \leq w_n \leq w_{n-1} \leq \cdots \leq w_1 \leq w_0.$$

It is easy for us to find that $\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ are uniformly bounded in E, and the monotone bounded theorem implies that there exist $v^*, w^* \in [v_0, w_0]$ such that

$$\lim_{n\to\infty} v_n(t) = v^*(t), \quad \lim_{n\to\infty} w_n(t) = w^*(t), \quad t \in [0,1].$$

Note that Θ is a completely continuous operator; then,

$$v^*(t) = (\Theta v^*)(t), \quad w^*(t) = (\Theta w^*)(t), \quad t \in [0,1],$$

i.e., v^*, w^* are solutions for (1.1). This completes the proof. □

4. Examples

Now, we provide some examples to illustrate our main results. Let $\alpha = 3/2$, $\beta = 1/2$, and $\mu(t) = t, t \in [0,1]$. Then, by python, we calculate $\delta^* = 0.292$, and δ can be chosen $1/5 \in (0, \delta^*)$. Moreover,

$$\int_0^1 \frac{g_2(t)}{g_2(1)} d\mu(t) \le \int_0^1 \frac{t g_2(1)}{g_2(1)} dt = \frac{1}{2} \in [0,1).$$

Hence, (C2)–(C4) hold.

Example 4.1. Let $g(t,z) = e^{-zt}$, $z \in \mathbb{R}^+$, and $t \in [0,1]$. Then, g is decreasing w.r.t. z uniformly in $t \in [0,1]$. Note that $g(t, \rho t) = e^{-\rho t^2}$ and

$$\int_0^1 (1-s)^{0.5} e^{-\rho s^2} ds \le \int_0^1 (1-s)^{0.5} ds = \frac{2}{3} < +\infty.$$

Therefore, g satisfies the conditions (C1) and (C5)-(C6). Consequently, the conclusion of Theorem 3.1 holds.

Example 4.2. Let $g(t,z) = \zeta(t) z^{\kappa}$, $z \in \mathbb{R}^+$, and $t \in [0,1]$, where $\kappa \in (0,1)$ is a given positive constant, $\zeta(t) \ge 0$. Then, g is non-decreasing w.r.t. z uniformly in $t \in [0,1]$, and (C1) and (C8) hold. In what follows, we establish the upper solution w_0 and the lower solution v_0. Let $\rho(t) = \int_0^1 G(t,s)ds$ and $\xi_\rho(t) = \int_0^1 G(t,s) g(s, \rho(s)) ds, t \in [0,1]$. Then, from Lemma 2.5, ξ_ρ satisfies the following:

$$\begin{cases} -{}^cD_{0+}^{\alpha} \xi_\rho(t) + \delta {}^cD_{0+}^{\beta} \xi_\rho(t) = g(t, \rho(t)), \ 0 < t < 1, \\ \xi_\rho(0) = 0, \ \xi_\rho(1) = \int_0^1 \xi_\rho(t) d\mu(t). \end{cases} \tag{4.1}$$

Using Lemma 2.6(iii), we obtain

$$\xi_\rho(t) \le \int_0^1 \Lambda_2 t (1-s)^{\alpha-1} g(s, \rho(s)) ds \le \frac{\int_0^1 \Lambda_2 (1-s)^{\alpha-1} g(s,\rho(s)) ds}{\int_0^1 \Lambda_1 (1-s)^{\alpha-1} s ds} \int_0^1 G(t,s) ds := \eta_{2\rho} \rho(t),$$

and

$$\xi_\rho(t) \ge \int_0^1 \Lambda_1 t (1-s)^{\alpha-1} s g(s, \rho(s)) ds \ge \frac{\int_0^1 \Lambda_1 (1-s)^{\alpha-1} s g(s,\rho(s)) ds}{\int_0^1 \Lambda_2 (1-s)^{\alpha-1} ds} \int_0^1 G(t,s) ds := \eta_{1\rho} \rho(t), t \in [0,1],$$

i.e.,

$$\eta_{1\rho}\rho(t) \le \xi_\rho(t) \le \eta_{2\rho}\rho(t), t \in [0,1]. \tag{4.2}$$

Let $v_0(t) = \vartheta_1\xi_\rho(t),\ w_0(t) = \vartheta_2\xi_\rho(t), t \in [0,1]$, where

$$0 < \vartheta_1 < \min\left\{\frac{1}{\eta_{2\rho}}, \eta_{1\rho}^{\kappa/(1-\kappa)}\right\}, \quad \vartheta_2 > \max\left\{\frac{1}{\eta_{1\rho}}, \eta_{2\rho}^{\kappa/(1-\kappa)}\right\}.$$

By (4.1), we have the following:

$$v_0(0) = 0,\ v_0(1) = \int_0^1 v_0(t)d\mu(t),\ w_0(0) = 0,\ w_0(1) = \int_0^1 w_0(t)d\mu(t).$$

Note that

$$g(t,\theta z) = \zeta(t)\theta^\kappa z^\kappa = \theta^\kappa g(t,z),\ \text{for } \theta \in [0,1]. \tag{4.3}$$

Therefore, from (4.3) and (4.2), we find the following:

$$\begin{aligned} g\left(t, v_0(t)\right) &= g\left(t, \vartheta_1\xi_\rho(t)\right) = g\left(t, \vartheta_1\frac{\xi_\rho(t)}{\rho(t)}\rho(t)\right) \\ &= \left[\vartheta_1\frac{\xi_\rho(t)}{\rho(t)}\right]^\kappa g(t,\rho(t)) \ge \left(\vartheta_1\eta_{1\rho}\right)^\kappa g(t,\rho(t)) \ge \vartheta_1 g(t,\rho(t)). \end{aligned}$$

From (4.1), we have the following:

$$-{}^cD_{0+}^\alpha v_0(t) + \delta^c D_{0+}^\beta v_0(t) = \vartheta_1[-{}^cD_{0+}^\alpha\xi_\rho(t) + \delta^c D_{0+}^\beta\xi_\rho(t)] = \vartheta_1 g(t,\rho(t)) \le g\left(t, v_0(t)\right).$$

Definition 2.8 implies that v_0 is a lower solution for (1.1).

On the other hand, by direct computation, we have the following:

$$\begin{aligned} \vartheta_2 g(t,\rho(t)) &= \vartheta_2 g\left(t, \frac{\rho(t)}{w_0(t)}w_0(t)\right) = \vartheta_2 g\left(t, \frac{\rho(t)}{\vartheta_2\xi_\rho(t)}w_0(t)\right) \\ &= \vartheta_2\left[\frac{\rho(t)}{\vartheta_2\xi_\rho(t)}\right]^\kappa g\left(t, w_0(t)\right) \ge \vartheta_2\left(\frac{1}{\vartheta_2\eta_{2\rho}}\right)^\kappa g\left(t, w_0(t)\right) \\ &\ge g\left(t, w_0(t)\right). \end{aligned}$$

From (4.1), we have the following:

$$-{}^cD_{0+}^\alpha w_0(t) + \delta^c D_{0+}^\beta w_0(t) = \vartheta_2[-{}^cD_{0+}^\alpha\xi_\rho(t) + \delta^c D_{0+}^\beta\xi_\rho(t)] = \vartheta_2 g(t,\rho(t)) \ge g\left(t, w_0(t)\right).$$

Definition 2.7 implies that w_0 is an upper solution for (1.1).

Therefore, (C7) is true, and the conclusion of Theorem 3.2 holds.

5. Conclusions

As is well-documented in the existing literature, the upper-lower solution method, when integrated with the monotone iterative technique, stands as a potent and pivotal instrument to establish the existence of solutions to nonlinear boundary value problems. In the present study, we leveraged this method to investigate the Caputo-type fractional Riemann-Stieltjes integral boundary value problem, thereby deriving a series of existence theorems for positive solutions. Our research findings unfolded in a twofold manner. First, under specific monotonicity conditions imposed on the nonlinearity, we proved the existence of positive solutions. Second, by adopting the upper and lower solutions as the initial iteration, we constructed monotone sequences that uniformly converged to the positive solutions of the problem. It is important to note that the scope of our current discussion is deliberately confined to the existence of positive solutions. A pertinent question for future exploration is whether the proposed research methodology remains effective when the nonlinear term admits sign changes.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors would like to express their heartfelt gratitude to the editors and reviewers for their constructive comments.

Conflict of interest

The authors declare no conflicts of interest.

References

1. A. Ahmadkhanlu, On the existence and multiplicity of positive solutions for a p-Laplacian fractional boundary value problem with an integral boundary condition, *Filomat*, **37** (2023), 235–250. https://doi.org/10.2298/FIL2301235A
2. A. Ali, M. Sarwar, M. B. Zada, K. Shah, Degree theory and existence of positive solutions to coupled system involving proportional delay with fractional integral boundary conditions, *Math. Methods Appl. Sci.*, **47** (2014), 10582–10594. https://doi.org/10.1002/mma.6311
3. Y. Feng, Z. Bai, Solvability of some nonlocal fractional boundary value problems at resonance in $\mathbb{R}^n$, *Fractal Fractional*, **6** (2022), 25. https://doi.org/10.3390/fractalfract6010025
4. S. Song, H. Li, Y. Zou, Monotone iterative method for fractional differential equations with integral boundary conditions, *J. Funct. Spaces*, **7** (2020), 7319098. https://doi.org/10.1155/2020/7319098
5. B. R. Sontakke, A. S. Shaikh, Properties of Caputo operator and its applications to linear fractional differential equations, *Int. J. Eng. Res. Appl.*, **5** (2015), 22–27.
6. A. Ahmadkhanlu, S. Jamshidzadeh, Existence and uniqueness of positive solutions for a Hadamard fractional integral boundary value problem, *Comput. Methods Differ. Equations*, **12** (2024), 741–748. https://doi.org/10.22034/cmde.2023.51601.2150
7. K. K. Ali, K. R. Raslan, A. A. E. Ibrahim, M. S. Mohamed, On study the fractional Caputo-Fabrizio integro differential equation including the fractional q-integral of the Riemann-Liouville type, *AIMS Math.*, **8** (2023), 18206–18222. https://doi.org/10.3934/math.2023925

8. A. E. Allaoui, L. Mbarki, Y. Allaoui, J. V. da C. Sousa, Solvability of Langevin fractional differential equation of higher-order with integral boundary conditions, *J. Appl. Anal. Comput.*, **15** (2025), 316–332. https://doi.org/10.11948/20240092

9. A. Frioui, A. Guezane-Lakoud, A. Bragdi, On a sequence of Caputo fractional differential equations with an integral condition, *Nonlinear Stud.*, **30** (2023), 63–71.

10. B. Gogoi, U. K. Saha, B. Hazarika, R. P. Agarwal, Existence of positive solutions of a fractional dynamic equation involving integral boundary conditions on time scales, *Iran. J. Sci.*, **48** (2024), 1463–1472. https://doi.org/10.1007/s40995-024-01691-z

11. A. Hamrouni, S. Beloul, Existence of solutions for fractional integro-differential equations with integral boundary conditions, *Mathematica*, **65** (2023), 249–262. https://doi.org/10.24193/mathcluj.2023.2.11

12. M. Helal, M. Kerfouf, F. Semari, Boundary value problems for fractional differential equations via Riemann-Liouville derivative and nonlinear integral conditions, *Nonlinear Stud.*, **31** (2024), 977–986.

13. K. Iatime, L. Guedda, S. Djebali, System of fractional boundary value problems at resonance, *Fractional Calculus Appl. Anal.*, **26** (2023), 1359–1383. https://doi.org/10.1007/s13540-023-00157-0

14. I. Kaddoura, Y. Awad, Stability results for nonlinear implicit ϑ-Caputo fractional differential equations with fractional integral boundary conditions, *Int. J. Differ. Equations*, **22** (2023), 5561399. https://doi.org/10.1155/2023/5561399

15. H. N. A. Khan, A. Zada, I. Khan, Analysis of a coupled system of Ψ-Caputo fractional derivatives with multipoint-multistrip integral type boundary conditions, *Qual. Theory Dyn. Syst.*, **23** (2024), 129. https://doi.org/10.1007/s12346-024-00987-0

16. S. Muthaiah, M. Murugesan, S. Ramasamy, N. G. Thangaraj, On fractional integro-differential equation involving Caputo-Hadamard derivative with Hadamard fractional integral boundary conditions, *Southeast Asian Bull. Math.*, **47** (2023), 367–380.

17. T. S. Cerdik, Solvability of a Hadamard fractional boundary value problem with multi-term integral and Hadamard fractional derivative boundary conditions, *Math. Methods Appl. Sci.*, **47** (2024), 12946–12960. https://doi.org/10.1002/mma.10475

18. H. Si, W. Jiang, G. Li, Solvability of Hilfer fractional differential equations with integral boundary conditions at resonance in $\mathbb{R}^M$, *J. Appl. Anal. Comput.*, **15** (2025), 39–55. https://doi.org/10.11948/20230410

19. S. N. Srivastava, S. Pati, S. Padhi, A. Domoshnitsky, Lyapunov inequality for a Caputo fractional differential equation with Riemann-Stieltjes integral boundary conditions, *Math. Methods Appl. Sci.*, **46** (2023), 13110–13123. https://doi.org/10.1002/mma.9238

20. R. K. Vats, K. Dhawan, V. Vijayakumar, Analyzing single and multi-valued nonlinear Caputo two-term fractional differential equation with integral boundary conditions, *Qual. Theory Dyn. Syst.*, **23** (2024), 174. https://doi.org/10.1007/s12346-024-01026-8

21. O. K. Wanassi, R. Bourguiba, D. F. M. Torres, Existence and uniqueness of solution for fractional differential equations with integral boundary conditions and the Adomian decomposition method, *Math. Methods Appl. Sci.*, **47** (2024), 3582–3595. https://doi.org/10.1002/mma.8880

22. N. Wang, Z. Zhou, Existence of solutions for fractional boundary value problems with Ψ-Caputo derivative and Stieltjes integral boundary conditions, *J. Jilin Univ. Sci.*, **61** (2023), 469–476. https://doi.org/10.13413/j.cnki.jdxblxb.2022299

23. X. Luo, Y. Xu, Existence of positive solutions for boundary value problems of fractional differential equations with parameters, *J. Guangxi Normal Univ.*, **42** (2014), 177–185. https://doi.org/10.16088/j.issn.1001-6600.2023112205

24. Y. Yang, Positive solution for a fractional switched system involving Riemann-Stieltjes integral, *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.*, **86** (2024), 57–68.

25. J. Zhang, S. Haq, A. Zada, I. L. Popa, Stieltjes integral boundary value problem involving a nonlinear multi-term Caputo-type sequential fractional integro-differential equation, *AIMS Math.*, **8** (2023), 28413–28434. https://doi.org/10.3934/math.20231454

26. N. Abdellouahab, K. Bouhali, L. Alkhalifa, K. Zennir, Existence and stability analysis of a problem of the Caputo fractional derivative with mixed conditions, *AIMS Math.*, **10** (2025), 6805–6826. https://doi.org/10.3934/math.2025312

27. B. Dehda, F. Yazid, F. S. Djeradi, K. Zennir, K. Bouhali, T. Radwan, Numerical approach based on the haar wavelet collocation method for solving a coupled system with the Caputo-Fabrizio fractional derivative, *Symmetry*, **16** (2024), 713. https://doi.org/10.3390/sym16060713

28. M. Kouidri, B. Tellab, A. Amara, K. Zennir, S. Zibar, A single and multi-valued problems involving mixed (k_1, η)-Hilfer and (k_2, ϕ)-Hilfer fractional derivatives for the fractional navier problem, *Math. Methods Appl. Sci.*, **2025** (2025). https://doi.org/10.1002/mma.10993

29. A. Kilbas, H Srivastava, J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, 2006. https://doi.org/10.3182/20060719-3-PT-4902.00008

30. I. Podlubny, *Fractional Differential Equations*, Acad. Press, 1999.