Electronic Research Archive

https://www.aimspress.com/journal/era

ERA, 33(6): 4014–4026.
DOI: 10.3934/era.2025178
Received: 25 March 2025
Revised: 26 May 2025
Accepted: 06 June 2025
Published: 24 June 2025

Research article

Group distance magic labeling of the Cartesian product of two directed cycles

Guixin Deng*, **Li Wang and Caimei Luo**

Center for Applied Mathematics of Guangxi, Nanning Normal University, Nanning 530100, China

* **Correspondence:** Email: dengguixin@live.com.

Abstract: Let $\overrightarrow{G}$ be a finite simple directed graph with n vertices, and let Γ be a finite abelian group of order n. A Γ-distance magic labeling is a bijection $\varphi : V(\overrightarrow{G}) \longrightarrow \Gamma$ for which there exists $c \in \Gamma$ such that $\sum_{y\in N^+(x)} \varphi(y) - \sum_{y\in N^-(x)} \varphi(y) = c$ for any $x \in V(\overrightarrow{G})$, where $N^+(x)$ and $N^-(x)$ denote the set of the head and the tail of x, respectively. In this paper, we obtain a necessary and sufficient condition for that there exists a Γ-distance magic labeling for the Cartesian products of two directed cycles.

Keywords: group distance magic labeling; magic constant; cartesian product; directed cycles; exponent

---

1. Introduction

Let $G = (V, E)$ be a finite undirected simple graph and $\overrightarrow{G} = (V(\overrightarrow{G}), E(\overrightarrow{G}))$ be a finite directed graph. If $x, y \in V(\overrightarrow{G})$ and there exists an arrow from x into y, then x is called a head of y and y is called a tail of x. Let $N^+(x)$ and $N^-(x)$ denote the set of the head and the tail of x, respectively. The neighborhood $N(x)$ of a vertex $x \in V(G)$ consists of the vertices adjacent to x.

Let G_1, G_2 be two graphs, and let $\overrightarrow{G_1}, \overrightarrow{G_2}$ be two directed graphs. The Cartesian product $G_1 \square G_2$ is the graph with vertex set $V(G_1) \times V(G_2)$, and two vertices (g_1, h_1) and (g_2, h_2) are adjacent in $G_1 \square G_2$ if and only if $g_1 = g_2$ and h_1 is adjacent to h_2 in G_2, or $h_1 = h_2$ and g_1 is adjacent to g_2 in G_1. Similarly, the Cartesian product $\overrightarrow{G_1} \square \overrightarrow{G_2}$ is the directed graph with vertex set $V(\overrightarrow{G_1}) \times V(\overrightarrow{G_2})$ and there is a directed arrow from (g_1, h_1) to (g_2, h_2) in $\overrightarrow{G_1} \square \overrightarrow{G_2}$ if and only if $g_1 = g_2$ and $\overrightarrow{h_1 h_2} \in E(\overrightarrow{G_2})$, or $h_1 = h_2$ and $\overrightarrow{g_1 g_2} \in E(\overrightarrow{G_1})$. Let C_n and $\overrightarrow{C_n}$ denote the cycle of length n and directed cycle of length $n \geq 3$, respectively.

The concept of group distance magic labeling of undirected graphs was introduced by Froncek in [1]. Let Γ be an abelian group and let $\varphi : V \longrightarrow \Gamma$ be a mapping. We call $w(v) := \sum_{x\in N(v)} \varphi(x)$ the weight of v with respect to φ for any $v \in V$. Then, φ is called a Γ-*distance magic labeling* if and only if

φ is bijective and the weight is independent of the choice of v. The unique weight is called the magic constant of the magic labeling. A graph G is called Γ-*distance magic* if there exists a Γ-distance magic labeling of G.

Froncek [1] showed that $C_m \Box C_n$ is $\mathbb{Z}_{mn}$-distance magic if and only if $2 \mid mn$ for any integers $m, n \geq 3$, and he asked for the full characterization full characterization of abelian groups Γ such that $C_m \Box C_n$ is Γ-distance magic. Cichacz et al. [2] answered the above question for the case that $m = n$. Recently, we generalized the idea and determined all abelian groups Γ such that $C_m \Box C_n$ admits a Γ-distance magic labeling for any $m, n \geq 3$ in [3]. The result is stated as follows.

Theorem 1.1 ([3]). *Let $m, n > 2$ be integers and let Γ be an abelian group of order mn. Then, $C_m \Box C_n$ admits a Γ-distance magic labeling if and only if $2 \mid mn$ and $\frac{l}{2d_1} \mid \exp(\Gamma)$, where $l = \mathrm{lcm}[m, n]$, $d = \gcd(m, n)$ and*

$$d_1 = \begin{cases} d, & 2 \nmid d \\ \frac{d}{2}, & 2 \mid d, 4 \nmid d \\ \frac{d}{4}, & 4 \mid d \end{cases}.$$

This type of labeling was generalized to the directed graph $\overrightarrow{G} = (V(\overrightarrow{G}), E(\overrightarrow{G}))$ in [4]. Suppose $\varphi : V(\overrightarrow{G}) \longrightarrow \Gamma$ is a map. For any $v \in V(\overrightarrow{G})$, we call $w(v) := \sum_{y \in N^+(v)} \varphi(y) - \sum_{y \in N^-(v)} \varphi(y)$ the weight of v with respect to φ. Then, φ is called a Γ-distance magic labeling of $\overrightarrow{G}$ if and only if φ is bijective and there exists $c \in \Gamma$ such that $w(v) = c$ for any $v \in V(\overrightarrow{G})$. The constant c is called the magic constant of the labeling φ. It was shown that $\overrightarrow{C_m} \Box \overrightarrow{C_n}$ is $\mathbb{Z}_{mn}$-distance magic in [4] .

Many people study the group magic labeling of directed graph, which included the directed antiprism, lexicographic product of graphs, Cartesian product of graphs, and complete multipartite graphs, see [5–7]. In this paper, we focus on the group distance magic labeling on the Cartesian product of two directed cycles $\overrightarrow{C_m} \Box \overrightarrow{C_n}$. We give a necessary and sufficient condition for that $\overrightarrow{C_m} \Box \overrightarrow{C_n}$ admits a group distance magic labeling. This is another version of Theorem 1.1.

The present paper is organized as follows. In Section 2, we obtain some necessary conditions for a group magic labeling of $\overrightarrow{C_m} \Box \overrightarrow{C_n}$ to be Γ-distance magic. We present and prove the main result in Section 3.

2. Necessity

In the remainder of this paper, we fix two integers $m, n \geq 3$ and let $l = \mathrm{lcm}[m, n]$ and $d = \gcd(m, n)$. Let Γ denote an abelian group of order mn. For convenience, we identity the vertices set of $\overrightarrow{C_m} \Box \overrightarrow{C_n}$ as the set of the equivalent classes on $\mathbb{Z} \times \mathbb{Z}$ defined by

$$(x_1, y_1)\mathcal{R}(x_2, y_2) \Leftrightarrow x_1 \equiv y_1 \pmod{m},\ x_2 \equiv y_2 \pmod{n}.$$

There is a directed edge from $(i, j)_{\mathcal{R}}$ to $(i', j')_{\mathcal{R}}$ if and only if either $i \equiv i' \pmod{m}$ and $j' - j \equiv 1 \pmod{n}$, or $j \equiv j' \pmod{m}$ and $i' - i \equiv 1 \pmod{m}$. Let $D^s = \{(i, j)_{\mathcal{R}} \in V(\overrightarrow{C_m} \Box \overrightarrow{C_n}) : j - i \equiv s \pmod{d}\}$ for $s = 0, 1, \ldots, d-1$. We have a partition of $V(\overrightarrow{C_m} \Box \overrightarrow{C_n}) = \cup_{s=0}^{d-1} D^s$, and we call $D^0, D^1, \ldots, D^{d-1}$ the diagonals of $\overrightarrow{C_m} \Box \overrightarrow{C_n}$.

We need the following basic fact, see [8].

Lemma 2.1. *Let n_i, a_i be integers, $i = 1, 2$. The following system of congruence equations*

$$\begin{cases} x \equiv a_1 \pmod{n_1} \\ x \equiv a_2 \pmod{n_2} \end{cases},$$

is solvable if and only if $\gcd(n_1, n_2) \mid a_1 - a_2$.

Lemma 2.2. *Let k be the minimal positive integer such that the following congruence equation*

$$\begin{cases} t \equiv -2k \pmod{m} \\ t \equiv 2k \pmod{n} \end{cases}$$

has a solution t_0. Then,

$$\gcd(t_0, l) = \begin{cases} d \cdot \gcd(2, l), & 2 \nmid d \\ d, & 2 \mid d, 4 \nmid d \\ \frac{d}{2}, & 4 \mid d \end{cases}.$$

Proof. If $2 \nmid d$, then $k = d$ and

$$\begin{cases} t_0 \equiv -2d \pmod{m} \\ t_0 \equiv 2d \pmod{n}. \end{cases}$$

Without loss of generality, we may assume that $2 \nmid n$. So, $\gcd(\frac{t_0}{d}, \frac{m}{d}) = \gcd(-2, \frac{m}{d}) = \gcd(2, l)$ and $\gcd(\frac{t_0}{d}, \frac{n}{d}) = \gcd(2, \frac{n}{d}) = 1$. Hence,

$$\gcd(t_0, l) = d \gcd(\frac{t_0}{d}, \frac{mn}{d^2}) = d \gcd(2, l).$$

If $d \equiv 2 \pmod{4}$, then $k = \frac{d}{2}$. So, $\gcd(t_0, l) = d \gcd(\frac{t_0}{d}, \frac{mn}{d^2}) = d$.

If $4 \mid d$, then $k = \frac{d}{4}$ and $\gcd(t_0, l) = \frac{d}{2} \gcd(\frac{2t_0}{d}, \frac{2mn}{d^2}) = \frac{d}{2}$. This finishes the proof. □

Suppose $m = \prod_{i=1}^{r} p_i^{t_i}$ and $n = \prod_{i=1}^{r} p_i^{s_i}$ are the prime factorizations of m and n. Define $f(m, n) := \prod_{i=1}^{r} p_i^{u_i}$, where

$$u_i = \begin{cases} \max\{t_i, s_i\}, & s_i \neq t_i \\ 0, & s_i = t_i \end{cases}.$$

The following lemma gives many information about the labeling of the diagonals and the magic constant for a group magic labeling of $\overrightarrow{C_m} \square \overrightarrow{C_n}$. Recall that the order of an element x of a finite abelian group, which is denoted by $o(x)$, is equal to the minimal positive integer k such that $kx = 0$.

Lemma 2.3. *Suppose that $\varphi : V(\overrightarrow{C_m} \square \overrightarrow{C_n}) \longrightarrow \Gamma$ is a Γ-distance magic labeling with magic constant c. Let $x_{ij} = \varphi((i, j)_{\mathcal{R}})$ and $a_{ij} = x_{ij} - x_{i+1,j+1}$ for any integer i, j. Let*

$$d' = \begin{cases} d \cdot \gcd(2, l), & 2 \nmid d \\ d, & 2 \mid d, 4 \nmid d \\ \frac{d}{2}, & 4 \mid d \end{cases}.$$

Then,

(1) *d' is a period of the sequence $\{a_{i+k, j+k}\}_{k=0}^{\infty}$;*

(2) *$\{x_{i+kd', j+kd'}\}_{k=0}^{\infty}$ is an arithmetic sequence of common difference h_{ij}, whose order is equal to $\frac{l}{d'}$;*

(3) *If* $(i, j)_{\mathcal{R}}$ *and* $(i', j')_{\mathcal{R}}$ *are in the same diagonal of* $\overrightarrow{C_m}\square\overrightarrow{C_n}$, *then* $h_{ij} = h_{i',j'}$*;*

(4) *The set of the labeling on the diagonal which contained* $(i, j)_{\mathcal{R}}$ *is a union of* d' *cosets of the cyclic group generated by* h_{ij}*;*

(5) *If* $2 \nmid l$*, then* $o(c) = l$*;*

(6) *If* $2 \mid l$ *and* $4 \nmid d$*, then* $f(\frac{l}{2}, \frac{d'}{2}) \mid o(c) \mid \frac{l}{2}$.

Proof. (1) By hypothesis, we have

$$w((i, j)_{\mathcal{R}}) = x_{i-1,j} - x_{i,j+1} + x_{i,j-1} - x_{i+1,j} = a_{i-1,j} + a_{i,j-1} = c,$$

for any i, j. Replacing i, j by $i+1, j-1$ respectively in the above equation, we obtain $a_{i-2,j+1} + a_{i-1,j} = c$, and thus $a_{i-1,j} = a_{i+1,j-2}$. It follows that $a_{ij} = a_{i-2k,j+2k}$ for any integers i, j, k. By Lemma 2.2, there exist integers k and t_0 such that

$$\begin{cases} t_0 \equiv -2k \pmod m \\ t_0 \equiv 2k \pmod n \end{cases},$$

and $\gcd(t_0, l) = d'$. We obtain $a_{ij} = a_{i-2k,j+2k} = a_{i+t_0,j+t_0}$. Since both l and t_0 are periods of $\{a_{i+k,j+k}\}_{k=0}^{\infty}$, it follows that $d' = \gcd(t_0, l)$ is also a period of $\{a_{i+k,j+k}\}_{k=0}^{\infty}$.

(2) Now, for any i, j, k, we have

$$\begin{aligned} x_{i+kd',j+kd'} - x_{i+(k+1)d',j+(k+1)d'} &= \sum_{r=0}^{d'-1}(x_{i+kd'+r,j+kd'+r} - x_{i+kd'+r+1,j+kd'+r+1}) \\ &= \sum_{r=0}^{d'-1} a_{i+kd'+r,j+kd'+r} \\ &= \sum_{r=0}^{d'-1} a_{i+r,j+r} \\ &= x_{ij} - x_{i+d',j+d'}. \end{aligned}$$

Thus, $\{x_{i+kd',j+kd'}\}_{k=0}^{\frac{l}{d'}-1}$ is an arithmetic sequence of common difference $h_{ij} = -\sum_{r=0}^{d'-1} a_{i+r,j+r}$. Since φ is a bijection, $x_{i+kd',j+kd'} = x_{ij} + kh_{ij} = x_{ij}$ if and only if $l \mid kd'$. It follows that $\frac{l}{d'}h_{ij} = 0$ and $rh_{ij} \neq 0$ for any positive integer $r < \frac{l}{d'}$. So, $o(h_{ij}) = \frac{l}{d'}$.

(3) Assume that $i' \equiv i + t \pmod m$ and $j' \equiv j + t \pmod n$ for some integers t. By (1), the sum of any consecutive d' terms in the sequence $\{a_{i+k,j+k}\}_{k=0}^{\infty}$ are equal. Combining this with the discussion in (2), we have

$$h_{i',j'} = -\sum_{r=0}^{d'-1} a_{i+t+r,j+t+r} = -\sum_{r=0}^{d'-1} a_{i+r,j+r} = h_{ij}.$$

(4) By (1) and (2), we have

$$\begin{aligned} \{x_{i+t,j+t} : 0 \le t \le l-1\} &= \bigcup_{r=0}^{d'-1}\{x_{i+r+kd',j+r+kd'} : 0 \le k < \frac{l}{d'}\} \\ &= \bigcup_{r=0}^{d'-1}\{x_{i+r,j+r} + kh_{ij} : 0 \le k < \frac{l}{d'}\} \end{aligned}$$

$$= \bigcup_{r=0}^{d'-1}(x_{i+r,j+r}+ < h_{ij} >).$$

This proves (4).

(5) Since $2 \nmid l$, we have $d' = d$. Recall that $a_{i-1,j+1} = a_{i-1-2k,j+1+2k}$ for any i, j, k. Setting $k = \frac{d-1}{2}$, one has

$$a_{i-1,j+1} = a_{i-d,j+d}.$$

Let t satisfy that

$$\begin{cases} t \equiv & -d \pmod m \\ t \equiv & d \pmod n). \end{cases}$$

Then, d divides t, and we obtain that $a_{i-1,j+1} = a_{i-d,j+d} = a_{i+t,j+t} = a_{ij}$ by (1). It follows that $2a_{ij} = a_{i-1,j+1} + a_{ij} = c$. Therefore, we deduce that $2a_{ij} = 2a_{i'j'} = c$ for any i, j, i', j'. Since $2 \nmid mn = |\Gamma|$, one has $a_{ij} = a_{i',j'} = a$. Thus, the sequence $\{x_{i+k,j+k}\}_{k=0}^{l-1}$ is an arithmetic sequence of common difference $-a$ with pairwise distinct terms. So, the order of $-a$ is exactly l, and $o(c) = o(2a) = \frac{o(a)}{\gcd(2,o(a))} = o(a) = l$.

(6) We have $d' \equiv 2 \pmod 4$ in this case. So,

$$a_{i-1,j+1} = a_{i-1-2\cdot\frac{d'-2}{4},j+1+2\cdot\frac{d'-2}{4}} = a_{i-\frac{d'}{2},j+\frac{d'}{2}}.$$

Let t satisfy that

$$\begin{cases} t \equiv & -\frac{d'}{2} \pmod m) \\ t \equiv & \frac{d'}{2} \pmod n). \end{cases}$$

Since $2 \mid l$ and $\frac{d'}{2}$ is odd, at least one of m, n is even and t is odd. There exists $r \in \mathbb{Z}$ such that $t = \frac{d'}{2}(2r + 1)$. So, we have $a_{i-1,j+1} = a_{i+t,j+t} = a_{i+\frac{d'}{2},j+\frac{d'}{2}}$ by (1). Hence,

$$a_{ij} + a_{i+\frac{d'}{2},j+\frac{d'}{2}} = a_{ij} + a_{i-1,j+1} = w(i, j + 1) = c.$$

Hence,

$$\begin{aligned} x_{ij} - x_{i+d',j+d'} &= \sum_{k=0}^{d'-1}(x_{i+k,j+k} - x_{i+k+1,j+k+1}) \\ &= \sum_{k=0}^{d'-1} a_{i+k,j+k} \\ &= \sum_{k=0}^{\frac{d'}{2}-1}(a_{i+k,j+k} + a_{i+k+\frac{d'}{2},j+k+\frac{d'}{2}}) \\ &= \frac{d'}{2}c. \end{aligned}$$

Combining with (2), we obtain $\frac{l}{2}c = \frac{l}{d'} \cdot \frac{d'}{2}c = \frac{l}{d'}(-h_{ij}) = 0$, and thus $o(c) \mid \frac{l}{2}$. Suppose $\frac{l}{2} = \prod_{i=1}^{r} p_i^{t_i}$, $\frac{d'}{2} = \prod_{i=1}^{r} p_i^{s_i}$, and $o(c) = \prod_{i=1}^{r} p_i^{u_i}$ are the prime factorizations of $\frac{l}{2}$, $\frac{d'}{2}$, and $o(c)$. Since $o(\frac{d'}{2}c) = \frac{o(c)}{\gcd(o(c),\frac{d'}{2})} = \frac{l}{d'} = \prod_{i=1}^{r} p_i^{t_i-s_i}$, we obtain $u_i - \min\{s_i, u_i\} = t_i - s_i$. It implies that $u_i = t_i$ if $t_i > s_i$. Therefore, $f(\frac{l}{2}, \frac{d'}{2}) \mid o(c)$. □

3. Main result

In this section, we present and prove the main result of this paper. We begin with two useful lemmas.

Lemma 3.1 ([3]). *Let B be a subgroup of an abelian group Γ such that $\frac{|\Gamma|}{|B|} = 2k$. Then, there exist $x_i, y_i, c \in \Gamma$, $i = 0, 1, \ldots, k-1$ such that $x_i + y_i = c$ and*

$$\Gamma = (\bigcup_{i=0}^{k-1}(x_i + B))\bigcup(\bigcup_{i=0}^{k-1}(y_i + B)).$$

We need the following lemma in the construction of magic labeling in our main result.

Lemma 3.2. *Let H be an abelian group. Suppose $|H| = l$ is even and d is an odd divisor of l. Suppose $c \in H$ and $f(\frac{l}{2}, d) \mid o(c) \mid \frac{l}{2}$. Then we can arrange the elements in H as a sequence $z_0, z_1, \ldots, z_{l-1}$ such that*

$$z_i - z_{i+1} + z_{i+d} - z_{i+d+1} = c$$

for all i, where the subscripts are calculated modulo l.

Proof. Let $C =< c >$ and $B =< dc >=< d_1c >$ be the cyclic group generated by c and dc, where $d_1 = \gcd(o(c), d)$. We first compute $|B| = o(dc)$. Let $l = 2^{m_0} \cdot \prod_{i=1}^{k} p_i^{m_i}$ and $d = \prod_{i=1}^{k} p_i^{n_i}$ be the prime factorization of l and d, where $p_i \geq 3$, $m_i > n_i$ for $i \in \{1, 2, \ldots, t\}$, and $m_i = n_i$ for $i \in \{t+1, t+2, \ldots, k\}$. Then, $f(\frac{l}{2}, d) = 2^{m_0-1} \prod_{i=1}^{t} p_i^{m_i}$. Since $f(\frac{l}{2}, d) \mid o(c) \mid \frac{l}{2}$, we may assume that $o(c) = 2^{m_0-1} \prod_{i=1}^{k} p_i^{s_i}$, where $s_i = m_i$ for $i \in \{1, 2, \ldots, t\}$ and $s_i \leq m_i$ for $i > t$. It follows that

$$\begin{aligned} o(dc) &= \frac{o(c)}{\gcd(o(c), d)} \\ &= \frac{2^{m_0-1} \prod_{i=1}^{k} p_i^{s_i}}{\prod_{i=1}^{k} p_i^{\min\{s_i, n_i\}}} \\ &= 2^{m_0-1} \prod_{i=1}^{k} p_i^{s_i - \min\{s_i, n_i\}} \\ &= 2^{m_0-1} \prod_{i=1}^{t} p_i^{m_i - n_i} = \frac{l}{2d}. \end{aligned}$$

We obtain $\frac{|H|}{|C|} = \frac{2d}{d_1}$. By Lemma 3.1, there exists $x_0, x_1, \ldots, x_{\frac{2d}{d_1}-1}, b \in H$ such that $H = \cup_{i=0}^{\frac{2d}{d_1}-1}(x_i + C)$, and $x_i + x_{i+\frac{d}{d_1}} = b$ for each i calculated modulo $\frac{2d}{d_1}$. Combining this with that $C = \cup_{j=0}^{d_1-1}(jc + B)$, we have

$$H = \bigcup_{i=0}^{\frac{2d}{d_1}-1}(x_i + C) = \bigcup_{i=0}^{\frac{2d}{d_1}-1}\bigcup_{j=0}^{d_1-1}(x_i + jc + B).$$

We first claim that there exist integers t_0, t_1 such that such that $t_0 + t_1 = \frac{d}{d_1}$ and $\gcd(t_0, d_1) = \gcd(t_1, d_1) = 1$.

Let $d_1 = \prod_{i=1}^{r} p_i^{e_i}$ be the prime factorization of d_1. Suppose $\frac{d}{d_1} \equiv y_i \pmod{p_i^{e_i}}$ where $0 \le y_i \le p_i^{e_i} - 1$. Let

$$(u_i, v_i) = \begin{cases} (1, y_i - 1), & p_i \nmid y_i - 1 \\ (2, y_i - 2), & p_i \mid y_i - 1. \end{cases}$$

By the Chinese Remainder Theorem, there exist t_0, q such that $t_0 \equiv u_i \pmod{p_i}$ and $q \equiv v_i \pmod{p_i}$. Since $t_0 + q \equiv y_i \equiv \frac{d}{d_1} \pmod{p_i^{e_i}}$, we have $d_1 \mid \frac{d}{d_1} - t_0 - q$, and there exists an integer k such that $kd_1 = \frac{d}{d_1} - t_0 - q$. Let $t_1 = q + kd_1$. Then, $\frac{d}{d_1} = t_0 + t_1$ and $\gcd(t_0, p_i) = \gcd(t_1, p_i) = 1$.

Now we define a sequence z_i for any integer i as follows. Write

$$i = 2qd + k\frac{d}{d_1} + s,\ 0 \le k < 2d_1, 0 \le s < \frac{d}{d_1}. \tag{3.1}$$

Then, by the division algorithm, we set

$$z_i := \begin{cases} x_i + (qd + s + kt_0)c, & 2 \mid k, 0 \le k < d_1 \\ x_i + (qd + s + (k - d_1)t_0)c, & 2 \mid k, d_1 \le k < 2d_1 \\ x_i + (qd + kt_1)c, & 2 \nmid k, 0 \le k < d_1 \\ x_i + (qd + (k - d_1)t_1)c, & 2 \nmid k, d_1 \le k < 2d_1 \end{cases}. \tag{3.2}$$

Recall that the subscript of x_i' is calculated modulo $\frac{2d}{d_1}$. We see that $i \mapsto z_i$ is well-defined and l is a period of $\{z_i\}_{i\in\mathbb{Z}}$. For i given in the form (3.1), we have

$$i + d = \begin{cases} 2qd + (k + d_1)\frac{d}{d_1} + s, & 0 \le k < d_1 \\ 2(q + 1)d + (k - d_1)\frac{d}{d_1} + s, & d_1 \le k < 2d_1 \end{cases}.$$

Hence,

$$z_{i+d} = \begin{cases} x_{i+d} + (qd + kt_1)c, & 2 \mid k, 0 \le k < d_1 \\ x_{i+d} + ((q + 1)d + (k - d_1)t_1)c, & 2 \mid k, d_1 \le k < 2d_1 \\ x_{i+d} + (qd + s + kt_0)c, & 2 \nmid k,\ 0 \le k < d_1 \\ x_{i+d} + ((q + 1)d + s + (k - d_1)t_0)c, & 2 \nmid k, d_1 \le k < 2d_1 \end{cases}. \tag{3.3}$$

Combining (3.2) and (3.3), we obtain that

$$z_i + z_{i+d} = b + (2qd + k\frac{d}{d_1} + s)c = b + ic,$$

for any i. Therefore,

$$z_i - z_{i+1} + z_{i+d} - z_{i+d+1} = -c.$$

Replacing z_i by $-z_i$, we may assume that $z_i - z_{i+1} + z_{i+d} - z_{i+d+1} = c$.

It remains to show that $H = \{z_i : 0 \le i < l\}$.

We claim that:

(i) $\{z_{i+2jd} : 0 \le j < \frac{l}{2d}\} = z_i + B$ is a coset of B for $i = 0, 1, \ldots, 2d - 1$;

(ii) $X_i = \{z_{i+j\frac{2d}{d_1}} : 0 \le j < \frac{ld_1}{2d}\} = x_i + C$ is a coset of C for $i = 0, 1, \ldots, \frac{2d}{d_1} - 1$.

Indeed, we have $z_{i+2jd} - z_i = jdc$ by (3.2) for any j. It follows that $\{z_{i+2jd} : 0 \le j < \frac{l}{2d}\} = \{z_i + jdc : 0 \le j < \frac{l}{2d}\} = z_i + B$. This proves ($i$).

For $j_1 \in \{0, 1, \ldots, \frac{ld_1}{2d} - 1\}$, write $j_1 = rd_1 + j$, where $0 \leq r < \frac{l}{2d}$, $0 \leq j < d_1$. By (i) of the claim, we have

$$\begin{aligned}\{z_{i+j_1\frac{2d}{d_1}} : 0 \leq j_1 < \frac{ld_1}{2d}\} &= \{z_{i+(rd_1+j)\frac{2d}{d_1}} : 0 \leq r < \frac{l}{2d}, 0 \leq j < d_1\} \\ &= \bigcup_{j=0}^{d_1-1}\{z_{i+j\frac{2d}{d_1}+2dr} : 0 \leq r < \frac{l}{2d}\} \\ &= \bigcup_{j=0}^{d_1-1}(z_{i+j\frac{2d}{d_1}} + B).\end{aligned}$$

The proof of (ii) is divided into two cases.

If $0 \leq i < \frac{d}{d_1}$, by (3.2), we obtain that

$$z_{i+j\frac{2d}{d_1}} = \begin{cases} x_i + (i + 2jt_0)c, & 0 \leq j < \frac{d_1+1}{2} \\ x_i + (i + (2j - d_1)t_0)c, & \frac{d_1+1}{2} \leq j < d_1 \end{cases}.$$

Hence, $\{z_{i+j\frac{2d}{d_1}} : 0 \leq j < d_1\} = \{x_i + (i + jt_0)c : 0 \leq j < d_1\}$ and

$$X_i = \bigcup_{j=0}^{d_1-1}(z_{i+j\frac{2d}{d_1}} + B) = \bigcup_{j=0}^{d_1-1}(x_i + (i + jt_0)c + B). \tag{3.4}$$

If $\frac{d}{d_1} \leq i < \frac{2d}{d_1}$, then $i + j\frac{2d}{d_1} = (2j + 1)\frac{d}{d_1} + (i - \frac{d}{d_1})$. By (3.2) again, we deduce that

$$z_{i+j\frac{2d}{d_1}} = \begin{cases} x_i + (2j + 1)t_1c, & 0 \leq j < \frac{d_1-1}{2} \\ x_i + (2j + 1 - d_1)t_1c, & \frac{d_1-1}{2} \leq j < d_1 \end{cases}.$$

In this case, we have $\{z_{i+j\frac{2d}{d_1}} : 0 \leq j < d_1\} = \{x_i + jt_1c : 0 \leq j < d_1\}$ and

$$X_i = \bigcup_{j=0}^{d_1-1}(z_{i+j\frac{2d}{d_1}} + B) = \bigcup_{j=0}^{d_1-1}(x_i + jt_1c + B). \tag{3.5}$$

Note that both $\{i + jt_0 : 0 \leq j < d_1 - 1\}$ and $\{jt_1 : 0 \leq j < d_1\}$ are complete systems of residues modulo d_1 since $\gcd(t_0, d_1) = \gcd(t_1, d_1) = 1$. Recall that $C =< c >$ and $B =< d_1c >$. So,

$$C = \bigcup_{j=0}^{d_1-1}(jc + B) = \bigcup_{j=0}^{d_1-1}((i + jt_0)c + B) = \bigcup_{j=0}^{d_1-1}(jt_1c + B). \tag{3.6}$$

Coming with (3.4), (3.5), and (3.6), we obtain that

$$X_i = \bigcup_{j=0}^{d_1-1}(x_i + jc + B) = x_i + C$$

for $i = 0, 1, \ldots, \frac{2d}{d_1} - 1$.

Finally, $\{z_i : 0 \leq i < l\} = \cup_{i=0}^{\frac{2d}{d_1}-1} X_i = \cup_{i=0}^{\frac{2d}{d_1}-1}(x_i + C) = H$. The proof is finished. □

Theorem 3.3. *$\overrightarrow{C_m}\square\overrightarrow{C_n}$ is Γ-distance magic if and only if $l_1 \mid \exp(\Gamma)$, where*

$$l_1 = \begin{cases} \frac{2l}{d}, & 4 \mid d \\ \frac{1}{2}f(l,d), & 2 \mid l,\ 4 \nmid d, 2 \mid \frac{l}{d} \\ f(l,d), & l \equiv d \equiv 2 \pmod 4 \\ l, & 2 \nmid l \end{cases}.$$

Proof. Let

$$d' = \begin{cases} d \cdot \gcd(2,l), & 2 \nmid d \\ d, & d \equiv 2 \pmod 4 \\ \frac{d}{2}, & 4 \mid d \end{cases}.$$

It is straightforward to verify that

$$f(\frac{l}{2}, \frac{d'}{2}) = \begin{cases} \frac{1}{2}f(l,d), & 2 \mid l,\ 4 \nmid d, 2 \mid \frac{l}{d} \\ f(l,d), & l \equiv d \equiv 2 \pmod 4). \end{cases}$$

So, the necessity follows from (2), (5), and (6) of Lemma 2.3.

Let H be a subgroup of Γ with order l, and let $V = V(\overrightarrow{C_m}\square\overrightarrow{C_n})$. We have the following claim.

Claim 1: Suppose there exists a labeling $\psi : V \longrightarrow H$ with the following properties.

(i) The restriction $\psi|_{D^s}$ is a bijection for $s = 0, 1, \ldots, d-1$;

(ii) There exists $c \in H$ such that the weight of any vertex with respect to ψ is c.

Then, there exists a Γ-distance magic labeling of $\overrightarrow{C_m}\square\overrightarrow{C_n}$ with magic constant c.

Proof of the Claim 1: Since $\frac{|\Gamma|}{|H|} = d$, there exist $b_0, b_1, \ldots, b_{d-1} \in \Gamma$ such that $\Gamma = \cup_{i=0}^{d-1}(b_i + H)$. We define $\varphi : V \longrightarrow \Gamma$ by

$$\varphi((i,j)_{\mathcal{R}}) := \psi((i,j)_{\mathcal{R}}) + b_s,\ \ (i,j)_{\mathcal{R}} \in D^s.$$

Then, φ is a bijection by (i), and

$$\begin{aligned} &\varphi((i-1,j)_{\mathcal{R}}) - \varphi((i,j+1)_{\mathcal{R}}) + \varphi((i,j-1)_{\mathcal{R}}) - \varphi((i+1,j)_{\mathcal{R}}) \\ = &\psi((i-1,j)_{\mathcal{R}}) - \psi((i,j+1)_{\mathcal{R}}) + \psi((i,j-1)_{\mathcal{R}}) - \psi((i+1,j)_{\mathcal{R}}) = c. \end{aligned}$$

Claim 2: We have $D^s = \{(i, i+s)_{\mathcal{R}} : 0 \le i < l\}$. Let $v = (i, i+s)_{\mathcal{R}} \in D^s$. Then,

(1) If $1 \le s \le d-2$, then $N^+(v) = \{(i-1, i+s)_{\mathcal{R}}, (i, i+s-1)_{\mathcal{R}}\}$;

(2) If $s = 0$, then $N^+(v) = \{(i-1, i)_{\mathcal{R}}, (i+k_0, i+k_0+d-1)_{\mathcal{R}}\}$;

(3) If $s = d-1$, then $N^+(v) = \{(i-k_0-1, i-k_0-1)_{\mathcal{R}}, (i, i+d-2)_{\mathcal{R}}\}$;

where $m \mid k_0$ and $k_0 \equiv -d \pmod n$.

Proof of the Claim 2: (1) is obvious. If $s = 0$, then $N^+(v) = \{(i-1, i)_{\mathcal{R}}, (i, i-1)_{\mathcal{R}}\}$. By the choice of k_0, we see that

$$\begin{cases} i \equiv i + k_0 \pmod m \\ i - 1 \equiv i + k_0 + d - 1 \equiv -d \pmod n \end{cases}.$$

So, $(i, i-1)_{\mathcal{R}} = (i+k_0, i+k_0+d-1)_{\mathcal{R}}$. The proof of (3) is similar.

We divide the proof into three cases.

Case 1: If $2 \nmid l$, then $l \mid \exp(\Gamma)$. Let $h \in \Gamma$ with order l, and let H be the cyclic subgroup of Γ generated by h. We label the $(i, i+s)_{\mathcal{R}} \mapsto x_{i,i+s} := ih$ for $(i, i+s)_{\mathcal{R}} \in D^s$. It is straightforward to check that this labeling satisfies the conditions in Claim 1.

Case 2: $4 \mid d$. Since $\frac{2l}{d} \mid \exp(\Gamma)$, there exists an element $h \in \Gamma$ with order $\frac{2l}{d}$. Let H be a subgroup of Γ such that $|H| = l$ and $h \in H$. Suppose $H = \cup_{j=0}^{\frac{d}{2}-1}(z_j + < h >)$. We first label D^{2s} by

$$(i, i+2s)_{\mathcal{R}} \mapsto x_{i,i+2s} := (-1)^s (z_r + qh),$$

where $0 \le s < \frac{d}{2}, i + s = q\frac{d}{2} + r, 0 \le r < \frac{d}{2}$.

For $(i, j)_{\mathcal{R}} \in D^{2s+1}$, we label $(i, j)_{\mathcal{R}}$ by $x_{ij} := x_{i,j-1}$, which is a translation of the labeling on D^{2s}.

It is straightforward to verify that

$$x_{i,i+2s} - x_{i+1,i+2s+1} = \begin{cases} (-1)^s (z_r - z_{r+1}), & 0 \le r < \frac{d}{2} - 1 \\ (-1)^s (z_{\frac{d}{2}-1} - z_0 - h), & r = \frac{d}{2} - 1 \end{cases}.$$

Now we compute the weight of $v = (i, i+2s+1) \in D^{2s+1}$ with respect to the labeling. Suppose $i + s = q\frac{d}{2} + r, 0 \le r < \frac{d}{2}$.

If $s < \frac{d-2}{2}$, then $(i-1, i+2s+1)_{\mathcal{R}}, (i, i+2s+2)_{\mathcal{R}} \in D^{2s+2}$ and $(i, i+2s)_{\mathcal{R}}, (i+1, i+2s+1)_{\mathcal{R}} \in D^{2s}$. Then,

$$\begin{aligned} w(v) &= x_{i-1,i+2s+1} - x_{i,i+2s+2} + x_{i,i+2s} - x_{i+1,i+2s+1} \\ &= \begin{cases} (-1)^{s+1}(z_r - z_{r+1}) + (-1)^s (z_r - z_{r+1}), & 0 \le r < \frac{d}{2} - 1 \\ (-1)^{s+1}(z_{\frac{d}{2}-1} - z_0 - h) + (-1)^s (z_{\frac{d}{2}-1} - z_0 - h), & r = \frac{d}{2} - 1 \end{cases} = 0. \end{aligned}$$

If $s = \frac{d-2}{2}$, then $N^+(v) = \{(i - k_0 - 1, i - k_0 - 1)_{\mathcal{R}}, (i, i + d - 2)_{\mathcal{R}}\}$ by (3) of Claim 2, where $m \mid k_0$ and $k_0 \equiv -d \pmod{n}$. So, $N^-(v) = \{(i - k_0, i - k_0)_{\mathcal{R}}, (i + 1, i + d - 1)_{\mathcal{R}}\}$. In this case, we have

$$x_{i-k_0-1,i-k_0-1} - x_{i-k_0,i-k_0} = \begin{cases} z_r - z_{r+1}, & 0 \le r < \frac{d}{2} - 1 \\ z_{\frac{d}{2}-1} - z_0 - h, & r = \frac{d}{2} - 1 \end{cases}.$$

Hence,

$$w(v) = x_{i-k_0-1,i-k_0-1} - x_{i-k_0,i-k_0} + x_{i,i+d-2} - x_{i+1,i+d-1} = 0.$$

Case 3: $2 \mid l$ and $4 \nmid d$. Note that $l_1 = f(\frac{l}{2}, \frac{d'}{2}) \mid \exp(\Gamma)$ and $\frac{d'}{2}$ is odd. Fix an element $c \in \Gamma$ such that $f(\frac{l}{2}, \frac{d'}{2}) \mid o(c) \mid \frac{l}{2}$. There exists a subgroup H of Γ with order l containing c. By Lemma 3.2, we can order the elements in H as a sequences $z_0, z_1, \ldots, z_{l-1}$ such that

$$z_i - z_{i+1} + z_{i+\frac{d'}{2}} - z_{i+\frac{d'}{2}+1} = c, \tag{3.7}$$

where the subscript is calculated modulo l.

It is easy to deduce that

$$z_i - z_{i+1} + z_{i+(2k+1)\frac{d'}{2}} - z_{i+(2k+1)\frac{d'}{2}+1} = c \tag{3.8}$$

from (3.7).

We divide the remains of the proof of Case 3 into two subcases. Without loss of generality we may assume that $2 \mid m$.

Subcase 3.1: $2 \mid d$ and $d = d'$. We label D^s by

$$(i, i+s)_{\mathcal{R}} \mapsto \begin{cases} z_{i+\frac{s}{2}(d+1)}, & 2 \mid s \\ z_{i+(\frac{s+d}{2})(d+1)}, & 2 \nmid s \end{cases}, \ 0 \le i \le l-1.$$

It is clear that the labeling satisfies (i) of Claim 1. Note that the labeling on D^s for odd s is a translation of the labeling of D^s for even s. So, we only need to compute the weight of the vertices on D^{2s+1}.

Let $v = (i, i + 2s + 1)_{\mathcal{R}} \in D^{2s+1}$. If $s \leq \frac{d-3}{2}$, then

$$w(v) = z_{i+s(\frac{d'}{2}+1)} - z_{i+s(\frac{d'}{2}+1)+1} + z_{i+(s+1)\frac{d'}{2}+s} - z_{i+(s+1)\frac{d'}{2}+s+1} = c$$

by Eq (3.7) and the definition of the labeling.

If $s = \frac{d-3}{2}$ and $v = (i, i + d - 1)_{\mathcal{R}}$, then $N^+(v) = \{(i - k_0 - 1, i - k_0 - 1)_{\mathcal{R}}, (i, i + d - 2)_{\mathcal{R}}\}$ by (3) of Claim 2.

Note that $(i, i + d - 2)_{\mathcal{R}} \mapsto z_{i+\frac{d-2}{2}(\frac{d}{2}+1)}$ and $(i - k_0 - 1, i - k_0 - 1)_{\mathcal{R}} \mapsto z_{i-k_0-1}$. Recalling that $2 \mid m$ and $m \mid k_0$, we obtain

$$\frac{d-2}{2}(\frac{d}{2} + 1) + k_0 + 1 \equiv \frac{d^2}{2} \pmod{m},$$

and thus $\frac{d-2}{2}(\frac{d}{2} + 1) + k_0 + 1$ is an odd multiple of $\frac{d}{2}$. By Eq (3.8), we have

$$w(v) = z_{i+\frac{d-2}{2}(\frac{d}{2}+1)} - z_{i+\frac{d-2}{2}(\frac{d}{2}+1)+1} + z_{i-k_0-1} - z_{i-k_0} = c.$$

Subcase 3.2: $2 \nmid d$ and $d' = 2d$. We label D^s in the following way. Set

$$(i, i + s)_{\mathcal{R}} \mapsto \begin{cases} z_{i+\frac{s}{2}(d+1)}, & 2 \mid s \\ z_{i+(\frac{s+d}{2})(d+1)}, & 2 \nmid s \end{cases}, \quad 0 \leq i \leq l - 1.$$

By the same discussion as in subcase 3.1, the weight of each vertex with respect to the labeling in D^j is also c, for $j = 1, 2, \ldots, d - 2$. We only write the details for $j = 0, d - 1$.

Let $v_0 = (i, i) \in D^0$ and $v_1 = (i, i+d-1) \in D^{d-1}$. By Claim 2, $N^+(v) = \{(i-1, i)_{\mathcal{R}}, (i+k_0, i+k_0+d-1)_{\mathcal{R}}\}$ and $N^+(v) = \{(i-k_0-1, i-k_0-1)_{\mathcal{R}}, (i, i+d-2)_{\mathcal{R}}\}$, where $m \mid k_0$ and $k_0 \equiv -d \pmod{n}$. By the construction of the labeling, we have

$$w(v_0) = z_{i-1+\frac{(d+1)^2}{2}} - z_{i+\frac{(d+1)^2}{2}} + z_{i+k_0+\frac{(d-1)(d+1)}{2}} - z_{i+k_0+1+\frac{(d-1)(d+1)}{2}},$$

and

$$w(v_1) = z_{i-k_0-1} - z_{i-k_0} + z_{i+(d-1)(d+1)} - z_{i+1+(d-1)(d+1)}.$$

By a simple computation, we see that

$$n_0 = \frac{(d+1)^2}{2} - 1 - k_0 - \frac{(d-1)(d+1)}{2} \equiv d \pmod{m},$$

and

$$n_1 = (d - 1)(d + 1) + k_0 + 1 \equiv d^2 \pmod{m}.$$

Since $2 \mid m$ and $2 \mid d$, both n_0 and n_1 are odd multiples of $d = \frac{d'}{2}$. We get

$$w(v_0) = w(v_1) = c$$

from (3.8).

For $v = (i, i) \in D^0$, we have $(i-1, i) \mapsto z_{i-1+\frac{d+1}{2}(d+1)}$ and $(i, i-1) = (i-k_0-1, i-k_0+d-2) \mapsto z_{i-k_0-1+\frac{d-1}{2}(d+1)}$. Again, we observe that

$$(i-1+\frac{d+1}{2}(d+1)) - (i-k_0-1+\frac{d-1}{2}(d+1)) = k_0+d+1$$

is an odd multiple of d. Hence, the weight of v is also c by Claim 2.

Finally, we construct the labeling satisfied Claim 1 in Cases 1–3. So, $\overrightarrow{C_m}\square\overrightarrow{C_n}$ is Γ-distance magic. The proof is complete. □

4. Final remark

It is natural to study the same problem on the Cartesian product of several cycles. But, it seems hard to obtain a full characterization for the existence of the group distance magic labeling product on Cartesian product of several cycles. However, we can prove an analogous result to Theorem 2.1 in [2], which describes a relation between the labeling of two graphs G_1, G_2 and the labeling of their Cartesian product $G_1\square G_2$. This is a sufficient condition.

We say the a directed graph $\overrightarrow{G}$ is locally regular if $|N^+(x)| = |N^-(x)|$ for any $x \in V(\overrightarrow{G})$.

Theorem 4.1. *Let $\overrightarrow{G_i}$ be directed graphs and let Γ_i be abelian groups, $i = 1, 2$. Suppose both $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$ are locally regular and G_i is Γ_i-distance magic. Then, $\overrightarrow{G_1}\square\overrightarrow{G_2}$ is $\Gamma_1 \oplus \Gamma_2$-distance magic.*

Proof. Let $\varphi_i : V(\overrightarrow{G_i}) \longrightarrow \Gamma_i$ be the Γ_i-distance magic labeling with magic constant c_i, $i = 1, 2$. For convenience, we identity Γ_1 as the subgroup $\{(x, 0) : x \in \Gamma_1\}$ and identity Γ_2 as the subgroup $\{(0, y) : y \in \Gamma_2\}$ of $\Gamma_1 \oplus \Gamma_2$. Define the labeling $\varphi : V(\overrightarrow{G_1}\square\overrightarrow{G_2}) \longrightarrow \Gamma_1 \oplus \Gamma_2$, as:

$$\varphi((x, y)) = \varphi_1(x) + \varphi_2(y).$$

It is clear that φ is bijective. For any $v = (x, y) \in V(\overrightarrow{G_1}\square\overrightarrow{G_2})$, we have

$$\begin{aligned}
w(v) &= \sum_{w\in N+(v)} \varphi(w) - \sum_{w\in N^-(v)} \varphi(w) \\
&= [\sum_{z\in N^+(x)} (\varphi_1(z) + \varphi_2(y)) + \sum_{z\in N^+(y)} (\varphi_1(x) + \varphi_2(z))] \\
&- [\sum_{z\in N^-(x)} (\varphi_1(z) + \varphi_2(y)) + \sum_{z\in N^-(y)} (\varphi_1(x) + \varphi_2(z))] \\
&= [\sum_{z\in N^+(x)} \varphi_1(z) - \sum_{z\in N^-(x)} \varphi_1(z)] + [\sum_{z\in N^+(y)} \varphi_1(z) - \sum_{z\in N^-(y)} \varphi_1(z)] \\
&+ (|N^+(y)| - |N^-(y)|)\varphi_1(x) + (|N^+(x)| - |N^-(x)|)\varphi_2(y) \\
&= c_1 + c_2.
\end{aligned}$$

Hence, $\overrightarrow{G_1}\square\overrightarrow{G_2}$ admits a $\Gamma_1 \oplus \Gamma_2$-distance magic labeling. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by the Guangxi Natural Science Foundation (2025GXNSFAA069810) and the National Natural Science Foundation of China (12161059).

Conflict of interest

The authors declare there is no conflicts of interest.

References

1. D. Froncek, Group distance magic labeling of Cartesian product of cycles, *Australas. J. Comb.*, **55** (2013), 167–174.
2. S. Cichacz, P. Dyrlaga, D. Froncek, Group distance magic Cartesian product of two cycles, *Discrete Math.*, **343** (2020), 111807. https://doi.org/10.1016/j.disc.2019.111807
3. X. Zeng, G. Deng, C. Luo, Characterize group distance magic labeling of Cartesian product of two cycles, *Discrete Math.*, **346** (2023), 113407. https://doi.org/10.1016/j.disc.2023.113407
4. B. Freyberg, M. Keranen, Orientable $\mathbb{Z}_n$-distance magic labeling of the Cartesian product of two cycles, *Australas. J. Comb.*, **69** (2017), 222–235.
5. S. Cichacz, B. Freyberg, D. Froncek, Orientable $\mathbb{Z}_N$-distance magic graphs, *Discuss. Math. Graph Theory*, **39** (2019), 533–546. https://doi.org/10.7151/dmgt.2094
6. P. Dyrlaga, K. Szopa, Orientable $\mathbb{Z}_n$-distance magic regular graphs, *AKCE Int. J. Graphs Comb.*, **18** (2021), 60–63. https://doi.org/10.1016/j.akcej.2019.06.005
7. B. Freyberg, M. Keranen, Orientable $\mathbb{Z}_n$-distance magic graphs via products, *Australas. J. Comb.*, **70** (2018), 319–328.
8. K. H. Rosen, *Elementary Number Theory and Its Applications*, 6th edition, Addison Wesley, 2010.