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Research article

Bifurcation analysis of an SIS model with a modified nonlinear incidence rate

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Abstract: A modified nonlinear incidence rate in an SIS epidemic model was investigated. When a new disease emerged or an old one resurged, the infectivity was initially high. Subsequently, the psychological effect attenuated the infectivity. Eventually, due to the crowding effect, the infectivity reached a saturation state. The nonlinearity of the functional form of the infection incidence was modified to enhance its biological plausibility. The stability of the associated equilibria was examined, and the basic reproduction number and the critical value that governed the dynamics of the model were deduced. Bifurcation analyses were presented, encompassing backward bifurcation, saddle-node bifurcation, Bogdanov-Takens bifurcation of codimension two, and Hopf bifurcation. Numerical simulations were conducted to validate our findings.

Keywords: SIS model; stability; saddle-node bifurcation; Bogdanov-Takens bifurcation; Hopf bifurcation

1. Introduction

For thousands of years, people have suffered the ravages of infectious diseases. Construction of a mathematical model for epidemic diseases and exploration of its dynamics can offer a theoretical foundation for the formulation of prevention and control strategies. [1] investigated positivity, stability, and endemic steady-state attainability for a class of epidemic models using linear algebraic tools. [2] studied the nonnegativity and local and global stability properties of the solutions of a newly proposed epidemic model. Bacterial-transmitted diseases, such as encephalitis, gonorrhea, and bacillary dysentery, have no immunity after rehabilitation and can be reinfected. In 1932, Kermack and Mckendrick [3] presented the SIS(Susceptible-Infectious-Susceptible) model for this kind of disease, and the infected person (I) became susceptible (S) again after rehabilitation.

The incidence rate of infectious diseases constitutes one of the most crucial elements in characterizing the infectious capacity of diseases within the model. In the classical epidemic model, the incidence rate was postulated as the bilinear function $f(I)S = kIS$, where k denotes the probability of a susceptible individual becoming infected upon contact with an infected one. This implies that as the number of infected individuals increases, the likelihood of susceptible individuals becoming infected also increases. Such a scenario might prevail when the number of infective individuals I is relatively small. However, when the number of infective individuals I becomes exceedingly large, the bilinear function kIS fails to provide a rational description of the spread of infectious diseases. This is attributable, in part, to the advancements in modern medicine and improved public awareness of protection against infectious diseases.

In 1978, Capasso and Serio [4] proposed a nonlinear saturated incidence rate $f(I)S = \frac{kIS}{1+\alpha I}$ for modeling the epidemics of cholera in Bari in 1973. Here, $\frac{1}{1+\alpha I}$ represents the inhibitory effect of the protective measures adopted by susceptible individuals, and $\frac{kI}{1+\alpha I}$ is known as an incidence function which will approach a saturation level $\frac{k}{\alpha}$ as the number of infectious individuals I increases.

In [5,6], a generalized form of the nonmonotonic incidence rate $f(I)S = \frac{kIS}{1+\beta I+\alpha I^2}$ was taken into account. When the number I approaches infinity, the incidence function $f(I) = \frac{kI}{1+\beta I+\alpha I^2}$ will tend to zero, implying that the psychological effect or the inhibitory effect is overly strong. As a result, this model might not be capable of appropriately describing some specific infectious diseases like influenza.

With the aim of devising a more reasonable incidence function, which increases initially when a disease breaks out, then declines due to the psychological effect, and finally attains a saturation level because of the crowding effect, Lu et al. [7] put forward a generalized nonmonotone and saturated incidence rate in the form of $\frac{kI^2S}{1+\beta I+\alpha I^2}$. Nevertheless, in their research, the parameter β is capable of taking a negative value, which lacks a realistic biological interpretation. Moreover, it might seem rather forced to endow the incidence rate with the combined characteristics of monotonicity, nonmonotonicity, and saturation properties. In reality, in 2000, P. van den Driessche [8] et al. put forward an epidemic model whose incidence rate was given by $(k_1 I + k_2 I^2)S$. In doing so, they accounted for the dependence of the contact rate on either the fraction of infective individuals or the level of severity of the infection among the infected individuals. M. H. Alharbi [9] formulated an ODE(Ordinary Differential Equation) model by splitting the total host population into three disjoint epidemiological classes, susceptible individuals $S(t)$, asymptomatic infectious individuals $I_a(t)$, and symptomatic infectious individuals $I_s(t)$, and proposed several reasonable optimal control strategies for the control and the prevention of the disease.

Inspired by the aforementioned aspects, we are going to take into account the following SIS model, which comes with a modified nonlinear incidence rate.

$$\begin{cases} \frac{dS}{dt} = b - dS - \frac{k_1 I + k_2 I^2}{1 + aI^2} S + \mu I, \\ \frac{dI}{dt} = \frac{k_1 I + k_2 I^2}{1 + aI^2} S - (d + \varepsilon + \mu) I, \end{cases} \tag{1.1}$$

where $S(t)$ and $I(t)$ signify the quantities of susceptible and infective individuals at time t. It is assumed that for all $t \geq 0$, both $S(t) \geq 0$ and $I(t) \geq 0$. The basic postulates in this paper are presented as follows:

1) The positive constant b represents the rate of population recruitment.

2) The positive constant d represents the natural death rate of $S(t)$ and $I(t)$.

3) The modified nonlinear incidence rate is $\dfrac{k_1 I + k_2 I^2}{1 + aI^2} S$, where $k_1 \geq 0, k_2 > 0, a > 0$.
4) The positive constant μ stands for the recovery rate of infective individuals.
5) The positive constant ε stands for the death rate induced by the disease of $I(t)$.

Remark 1. *In light of the analysis of previous research, we propose a more reasonable incidence rate here.*

$$f(I)S = \frac{k_1 I + k_2 I^2}{1 + aI^2} S,$$

where $f(I)$ combines monotonicity, nonmonotonicity, and saturation. When $k_1 = 0$, $f(I)$ monotonically increases to the saturation level $\frac{k_2}{a}$ as I approaches infinity. If $k_1 > 0$, $f(I)$ is nonmonotonic, first rising and then falling to $\frac{k_2}{a}$ as I goes to infinity. The basic reproduction number and parameters have clear biological meanings. This incidence rate also enriches the model with more diverse dynamical behaviors.

As far as we know, this is the pioneer study to incorporate this modified nonlinear incidence rate into an SIS model. In subsequent sections, we will first analyze the qualitative features of equilibria like existence and topological types. Then, we will explore bifurcation behaviors such as transcritical, saddle-node, Hopf, and Bogdanov-Takens bifurcations. Finally, a summary and discussion will be presented.

2. Qualitative analysis of equilibria

In this segment, we conduct an analysis on the equilibria of system (1.1) as well as their corresponding topological types. For convenience, we perform the following transformation.

$$S = \sqrt{\frac{d + \varepsilon + \mu}{k_2}} x, \quad I = \sqrt{\frac{d + \varepsilon + \mu}{k_2}} y, \quad t = \frac{1}{d + \varepsilon + \mu} \tau,$$

and the notations

$$\Lambda = \frac{b}{d + \varepsilon + \mu} \sqrt{\frac{k_2}{d + \varepsilon + \mu}}, \quad m = \frac{d}{d + \varepsilon + \mu}, \quad q = \frac{k_1}{\sqrt{k_2(d + \varepsilon + \mu)}},$$

$$p = a\frac{d + \varepsilon + \mu}{k_2}, \quad w = \frac{\mu}{d + \varepsilon + \mu}.$$

We transform system (1.1) (for the sake of simplicity, we continue to use t to represent τ into the form presented below).

$$\begin{cases} \dfrac{dx}{dt} = \Lambda - mx - \dfrac{qy + y^2}{1 + py^2} x + wy, \\ \dfrac{dy}{dt} = \dfrac{qy + y^2}{1 + py^2} x - y, \end{cases} \tag{2.1}$$

where

$$\Lambda > 0, \quad 0 < m < 1, \quad q \geq 0, \quad p > 0, \quad 0 < w < 1.$$

2.1. Existence of equilibria

Consequently, the equilibrium points of system (2.1) comply with the subsequent equations.

$$\begin{cases} \Lambda - mx - \dfrac{qy + y^2}{1 + py^2} x + wy = 0, \\ \dfrac{qy + y^2}{1 + py^2} x - y = 0. \end{cases}$$

Obviously, a disease-free equilibrium $E_0(\frac{\Lambda}{m}, 0)$ of (2.1) always exists. For the purpose of finding the endemic equilibrium, we are required to solve the subsequent equations.

$$x = \frac{1 + py^2}{q + y}, \quad Ay^2 + By + C = 0, \tag{2.2}$$

where $A = mp + 1 - w, B = q - qw - \Lambda, C = m - \Lambda q$. Obviously, we have $A > 0$. The discriminant of (2.2) is

$$\Delta = (q - qw - \Lambda)^2 - 4(mp + 1 - w)(m - \Lambda q).$$

From (2.2), there are at most two endemic equilibria $E_1(x_1, y_1)$ and $E_2(x_2, y_2)$ in the model (2.1), and they can merge into a unique endemic equilibrium $E_*(x_*, y_*)$ when $\Delta = 0$, where

$$\begin{aligned} y_1 &= \frac{\Lambda + qw - q - \sqrt{\Delta}}{2(mp + 1 - w)}, \quad x_1 = \frac{1 + py_1^2}{q + y_1}, \\ y_2 &= \frac{\Lambda + qw - q + \sqrt{\Delta}}{2(mp + 1 - w)}, \quad x_2 = \frac{1 + py_2^2}{q + y_2}, \\ y_* &= \frac{\Lambda + qw - q}{2(mp + 1 - w)}, \quad x_* = \frac{1 + py_*^2}{q + y_*}, \end{aligned}$$

and

$$y_1 + y_2 = \frac{\Lambda + qw - q}{mp + 1 - w}, \quad y_1 y_2 = \frac{m - \Lambda q}{mp + 1 - w}.$$

From $\Delta = 0$, we derive $p = \dfrac{(q - qw - \Lambda)^2 - 4(1 - w)(m - \Lambda q)}{4m(m - \Lambda q)}$. Let

$$p_* = \frac{(q - qw - \Lambda)^2 - 4(1 - w)(m - \Lambda q)}{4m(m - \Lambda q)},$$

from condition $0 < m < 1, 0 < w < 1$, and it can be concluded that $p_* > 0$ if, and only if,

$$\Lambda q < m < \frac{(q - qw - \Lambda)^2}{4(1 - w)} + \Lambda q.$$

Therefore, the following lemma is derived.

Lemma 2. *There always exists a disease-free equilibrium* $E_0(\frac{\Lambda}{m}, 0)$ *in system* (2.1). *Moreover,*
1) Model (2.1) *has no endemic equilibria if, and only if, one of the following conditions holds*
1.1) $C \geq 0, -\dfrac{B}{2A} \leq 0$, *i.e.,* $m \geq \Lambda q$ *and* $\Lambda \leq q - qw$.

1.2) $p_* > 0, -\frac{B}{2A} > 0, \Delta < 0$, *i.e.,* $\Lambda q < m < \frac{(q - qw - \Lambda)^2}{4(1-w)} + \Lambda q, \Lambda > q - qw$, *and* $p > p_*$.

1.3) $p_* < 0$ *(which means* $p > p_*$*, i.e.,* $\Delta < 0$*) and* $-\frac{B}{2A} > 0$*, i.e.,* $m \geq \frac{(q - qw - \Lambda)^2}{4(1-w)} + \Lambda q$ *and* $\Lambda > q - qw$.

2) There exists a unique endemic equilibrium $E_*(x_*, y_*)$ *if, and only if, one of the following conditions holds:*

2.1) $p_* > 0, \Delta = 0, -\frac{B}{2A} > 0$*, i.e.,* $\Lambda q < m < \frac{(q - qw - \Lambda)^2}{4(1-w)} + \Lambda q, \Lambda > q - qw$*, and* $p = p_*$.

2.2) $C = 0, -\frac{B}{2A} > 0$*, i.e.,* $m = \Lambda q$ *and* $\Lambda > q - qw$.

2.3) $C < 0$*, i.e.,* $m < \Lambda q$.

3) There exist two different endemic equilibria $E_1(x_1, y_1)$ *and* $E_2(x_2, y_2)$ *if, and only if, the following conditions hold:*

$\Lambda q < m < \frac{(q - qw - \Lambda)^2}{4(1-w)} + \Lambda q, \Lambda > q - qw$*, and* $p < p_*$*, where* $0 < y_1 < y_* < y_2$.

Remark 3. *[10] The basic reproduction number (*R_0*) is the average number of secondary infections produced by an infection throughout the period of infection. System* (2.1) *always exists a disease-free equilibrium* $E_0(\frac{\Lambda}{m}, 0)$*. Hence,* R_0 *can be calculated using the next-generation matrix approach [11]. Let* $\mathcal{X} = (y, x)^T$*, then model* (2.1) *can be written as*

$$\frac{d\mathcal{X}}{dt} = \mathcal{F}(\mathcal{X}) - \mathcal{V}(\mathcal{X}),$$

where

$$\mathcal{F}(\mathcal{X}) = \begin{pmatrix} \frac{qy + y^2}{1 + py^2} x \\ 0 \end{pmatrix}, \quad \mathcal{V}(\mathcal{X}) = \begin{pmatrix} y \\ -\Lambda + mx + \frac{qy + y^2}{1 + py^2} x - wy \end{pmatrix}.$$

Furthermore, we can calculate that

$$F = \begin{pmatrix} \frac{\Lambda q}{m} \end{pmatrix}_{1\times 1}, \quad V = \begin{pmatrix} 1 \end{pmatrix}_{1\times 1}.$$

So, we can get

$$R_0 = \rho(FV^{-1}) = \frac{\Lambda q}{m}.$$

We note that $\Delta = 0$ (i.e., $p = p_*$) if, and only if, $R_0 = 1 - \frac{(q - qw - \Lambda)^2}{4m(mp + 1 - w)}$. We denote the critical value $1 - \frac{(q - qw - \Lambda)^2}{4m(mp + 1 - w)}$ by R_a, and it can be obtained that $0 < R_a < 1$. Moreover, $R_a > 0$ is equivalent to $p > p_0$, where

$$p_0 = \frac{(q - qw - \Lambda)^2 - 4m(1 - w)}{4m^2}.$$

Note that $p > p_0$, i.e., $k_1 > k_1^*$, where

$$k_1^* = \frac{bk_2}{d + \varepsilon} - \frac{1}{d + \varepsilon}\sqrt{\frac{(d + \varepsilon)k_2 + ad(d + \varepsilon + \mu)}{d}}.$$

Hence, Lemma 2 can be written as the following result:

Theorem 4. *With regard to model* (2.1)*, the existence of a disease-free equilibrium* $E_0(\frac{\Lambda}{m}, 0)$ *is always ensured. Moreover,*
1) there are no endemic equilibria if one of the following conditions holds: $R_0 \leq 1$*,* $\Lambda \leq q - qw$ *or* $R_0 < 1$*,* $\Lambda > q - qw, R_0 < R_a$*;*
2) there exists a unique endemic equilibrium E_* *if one of the following conditions holds:* $R_0 = 1$*,* $\Lambda > q - qw$*, or* $R_0 > 1$*;*
3) there exist two endemic equilibria E_1 *and* E_2 *if* $R_0 < 1, \Lambda > q - qw, R_0 > R_a$*.*

Remark 5. *Based on Theorem* 4 *and system* (1.1)*, it should be noted that the inequality* $\Lambda \leq q - qw$ *is equivalent to* $k_1 \geq k_{10}$*, where* $k_{10} = \frac{bk_2}{d + \varepsilon}$*. This implies that when the ratio* $\frac{k_1}{k_2}$*, which represents the proportion of linear over nonlinear infections, is greater than or equal to* $\frac{b}{d + \varepsilon}$ *and* $R_0 \leq 1$ *(meaning that, on average, an infected individual generates less than or equal to one new infected individual throughout its infectious period), the infection will not spread, that is, the disease cannot infiltrate the population.*

2.2. Topological types of equilibria

We will explore the topological type as well as the global properties of the disease-free equilibrium.

Theorem 6. *For system* (2.1)*, the equilibrium* $E_0(\frac{\Lambda}{m}, 0)$ *is*
$R_0 < 1$*: an attracting node;*
$R_0 > 1$*: a hyperbolic saddle point;*
$R_0 = 1$ *and* $\Lambda \neq q - qw$*: a saddle node of com-dimension 1;*
$R_0 = 1$ *and* $\Lambda = q - qw$*: a repelling node of com-dimension 2.*

Proof. The Jacobian matrix of the system (2.1) at equilibrium $E(x, y)$ is given by

$$J(E) = \begin{bmatrix} -m - \frac{qy + y^2}{1 + py^2} & -\frac{(q + 2y)(1 + py^2) - (qy + y^2)2py}{(1 + py^2)^2}x + w \\ \frac{qy + y^2}{1 + py^2} & \frac{(q + 2y)(1 + py^2) - (qy + y^2)2py}{(1 + py^2)^2}x - 1 \end{bmatrix}. \tag{2.3}$$

By substituting x with $\frac{\Lambda}{m}$ and y with 0 in (2.3), respectively, we obtain

$$J(E_0) = \begin{bmatrix} -m & -\frac{\Lambda q}{m} + w \\ 0 & \frac{\Lambda q}{m} - 1 \end{bmatrix}.$$

Matrix $J(E_0)$ has two eigenvalues $\lambda_1 = -m$ and $\lambda_2 = \frac{\Lambda q}{m} - 1 = R_0 - 1$. Then, if $R_0 < 1$, the disease-free equilibrium E_0 is an attracting node. If $R_0 > 1$, disease-free equilibrium E_0 is a hyperbolic saddle point. If $R_0 = 1$, the second eigenvalue is zero. To determine the type of E_0, first, we linearize the system (2.1) at E_0, let $u = x - \frac{\Lambda}{m}, v = y$ and, using Taylor expansion, the following system is obtained.

$$\begin{cases} \frac{du}{dt} = -mu + (w - 1)v - quv - \frac{\Lambda}{m}v^2 - uv^2 + pv^3 + o(|u, v|^4), \\ \frac{dv}{dt} = quv + \frac{\Lambda}{m}v^2 + uv^2 - pv^3 + o(|u, v|^4). \end{cases} \tag{2.4}$$

Next, we diagonalize the linear part of the system (2.4), let $X = v, Y = u - \frac{w-1}{m}v$, and system (2.4) is transformed into

$$\begin{cases} \frac{dX}{dt} = \frac{\Lambda + qw - q}{m}X^2 + qXY + (\frac{w-1}{m} - p)X^3 + X^2Y + o(|X, Y|^4), \\ \frac{dY}{dt} = -mY - \frac{w+m-1}{m}(\frac{\Lambda + qw - q}{m}X^2 + qXY + (\frac{w-1}{m} - p)X^3 + X^2Y) \\ \quad + o(|X, Y|^4). \end{cases} \tag{2.5}$$

Let $\tau = -mt$ (for simplicity, we still use t for τ), and we change system (2.5) to the following form:

$$\begin{cases} \frac{dX}{dt} = -\frac{\Lambda + qw - q}{m^2}X^2 - \frac{q}{m}XY - \frac{w-1-mp}{m^2}X^3 - \frac{1}{m}X^2Y + o(|X, Y|^4), \\ \frac{dY}{dt} = Y + \frac{w+m-1}{m^2}(\frac{\Lambda + qw - q}{m}X^2 + qXY + \frac{w-1-mp}{m}X^3 + X^2Y) + o(|X, Y|^4). \end{cases}$$

The system satisfies the condition for the existence of a local center manifold. Assuming that the center manifold of the system is $Y = h(X)$, it can be obtained that

$$\begin{aligned} \frac{dX}{dt} = &- \frac{\Lambda + qw - q}{m^2}X^2 + \Big(\frac{q(w+m-1)(\Lambda + qw - q)}{m^4} - \frac{w-1-mp}{m^2}\Big)X^3 \\ &+ \Big(\frac{q(w+m-1)(w-1-mp)}{m^4} + \frac{(w+m-1)(\Lambda + qw - q)}{m^4}\Big)X^4 \\ &+ \frac{(w+m-1)(w-1-mp)}{m^4}X^5 + o(X^6). \end{aligned} \tag{2.6}$$

Hence, E_0 is a saddle node of com-dimension 1 when $\Lambda \neq q - qw$ [12].

If $R_0 = 1$ and $\Lambda = q - qw$, model (2.6) becomes

$$\frac{dX}{dt} = -\frac{w-1-mp}{m^2}X^3 + o(X^4).$$

Obviously, the coefficient is $-\frac{w-1-qp\Lambda}{m^2} > 0$; therefore, according to Theorem 7.1 in [13], E_0 is a repelling node of com-dimension 2. □

Theorem 7. *For the system* (2.1)*, if* $R_0 < 1$*, the disease-free equilibrium* E_0 *is globally asymptotically stable, that is, the disease dies out.*

Proof. Set $N(t) = x(t) + y(t)$ and sum up the first equation and the second equation of system (2.1). Then, we obtain

$$\frac{dN}{dt} + mN = \Lambda + (m + w - 1)y,$$

We know that $m + w - 1 < 0$, which implies

$$\frac{dN}{dt} + mN \leq \Lambda.$$

By the theory of ODEs, we have

$$N(t) \leq N(0)e^{-mt} + \frac{\Lambda}{m}(1 - e^{-mt}) \leq N(0) + \frac{\Lambda}{m},$$

where $N(0)$ is the initial value of $N(t)$ at $t = 0$. Therefore, the positive invariant set of system (2.1) is

$$\Omega = \left\{(x, y)|x(t) + y(t) \leq N(0) + \frac{\Lambda}{m}, x(t) \geq 0, y(t) \geq 0\right\}.$$

In the case where $R_0 < 1$, as per Theorem 4, it is known that the system lacks an endemic equilibrium. Furthermore, according to Theorem 6, when $R_0 < 1$, the disease-free equilibrium E_0 is an attracting node. Given that the disease-free equilibrium E_0 is located on the boundary of Ω, it can be deduced from the Poincaré-Bendixson theorem that for every positive solution of system (2.1), it will approach the equilibrium E_0 as t tends to $+\infty$. □

Remark 8. *It can be observed that by choosing the parameters* $\Lambda = 0.3$*,* $m = 0.3$*,* $q = 0.5$*,* $w = 0.6$*, and* $p = 1$*, the condition of Theorem* 7 *is fulfilled. In other words, when* $R_0 < 1$*, the disease-free equilibrium* E_0 *is globally asymptotically stable, which means the disease will eventually die out. The simulation of this phenomenon is presented in Figure* 1.

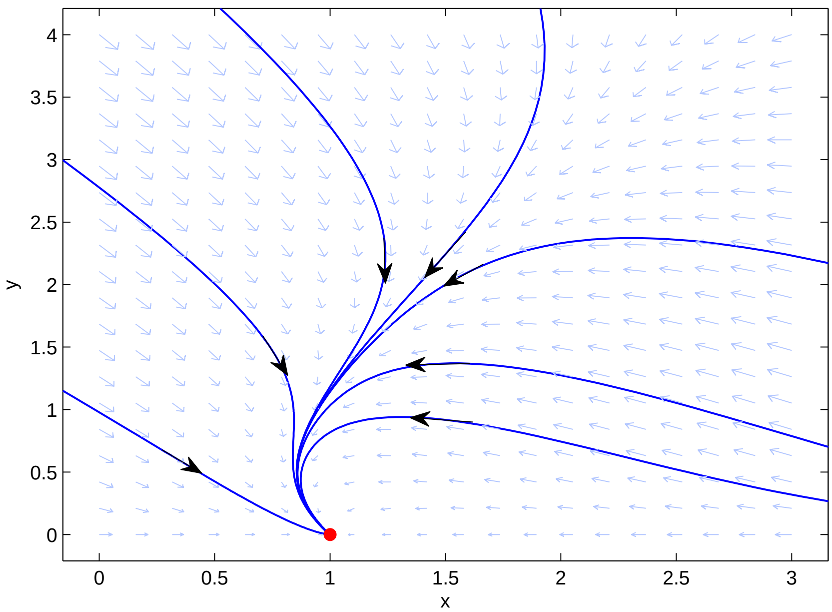


Figure 1. The globally asymptotical stable disease-free equilibrium.

Next, we proceed to investigate the topological type of the endemic equilibrium. As we can learn from (2.3), we are aware of the Jacobian matrix of system (2.1) at the equilibrium $E(x, y)$. Subsequently, the determinant of $J(E)$ is

$$det(J(E)) = \frac{(2mp + 2 - 2w)y^2 + (q - qw - \Lambda)y}{1 + py^2},$$

and its sign is determined by

$$S_D = (2mp + 2 - 2w)y^2 + (q - qw - \Lambda)y. \tag{2.7}$$

Similarly, we obtain the trace of $J(E)$,

$$tr(J(E)) = \frac{(-m^2p - m - 2pm + w - 1)y^2 + (\Lambda - mq)y - m^2}{m(1 + py^2)},$$

and its sign is determined by

$$S_T = (-m^2p - m - 2pm + w - 1)y^2 + (\Lambda - mq)y - m^2. \tag{2.8}$$

Theorem 9. *For system* (2.1)*, according to Lemma* 2*, if* $\Lambda q < m < \frac{(q - qw - \Lambda)^2}{4(1 - w)} + \Lambda q, \Lambda > q - qw$*, and* $p < p_*$ *holds, there exist two endemic equilibria* $E_1(x_1, y_1)$ *and* $E_2(x_2, y_2)$*. Furthermore,* $E_1(x_1, y_1)$ *must be a hyperbolic saddle, and then* $E_2(x_2, y_2)$ *will be*
1) a stable hyperbolic focus (or node) if $trJ(E_2) < 0$*; or*
2) a weak focus (or a center) if $trJ(E_2) = 0$*; or*
3) an unstable hyperbolic focus (or node) if $trJ(E_2) > 0$.

Proof. Upon computing the determinants of $J(E_1)$ and $J(E_2)$, it turns out that their signs are determined by S_D in (2.7). Through several straightforward calculations, we derive

$$S_D(y_1) = (2mp + 2 - 2w)y_1^2 + (q - qw - \Lambda)y_1 < 0,$$
$$S_D(y_2) = (2mp + 2 - 2w)y_2^2 + (q - qw - \Lambda)y_2 > 0.$$

Therefore, the endemic equilibrium E_1 is a hyperbolic saddle.
(i) If $trJ(E_2) < 0$, the eigenvalues of $J(E_2)$ have negative real parts, so E_2 is a stable hyperbolic focus (or node);
(ii) If $trJ(E_2) = 0$, the eigenvalues of $J(E_2)$ are a pair of pure imaginary roots, so E_2 is a weak focus (or a center);
(iii) If $trJ(E_2) > 0$, the eigenvalues of $J(E_2)$ have positive real parts, so E_2 is an unstable hyperbolic focus (or node). □

Remark 10. *When* $m = \Lambda q$ *and* $\Lambda > q - qw$*, or* $m < \Lambda q$*, system* (2.1) *has a unique positive equilibrium* $E_2(x_2, y_2)$ *along with a disease-free equilibrium* E_0*.* E_0 *is either a hyperbolic saddle or saddle-node. The nature and stability of* E_2 *accord with those in Theorem* 9*. Furthermore, by Theorem* 9*, if* $R_0 > 1$ *(i.e.,* $m < \Lambda q$*), signifying that an infected individual generates more than one new infection on average, or if* $R_0 = 1$ *and* $\frac{k_1}{k_2} < \frac{b}{d + \varepsilon}$*, implying that the proportion of linear to nonlinear infection hazards is less than* $\frac{b}{d + \varepsilon}$ *and an infected individual yields one new infection on average, the disease will endure as multiple periodic coexisting oscillations bifurcating from* E_2.

The topological types of the endemic equilibrium $E_*(x_*, y_*)$ will be discussed in the following section.

3. Bifurcation analysis

3.1. Transcritical bifurcation and saddle-node bifurcation

In this section, we will conduct a bifurcation analysis of system (2.1).

Theorem 11. *For the system* (2.1)*, we choose R_0 as the bifurcation parameter. When $R_0 = 1$,*
1) system (2.1) *undergoes forward bifurcation if $\Lambda < q - qw$, and undergoes backward bifurcation if $\Lambda > q - qw$;*
2) system (2.1) *undergoes pitchfork bifurcation if $\Lambda = q - qw$.*

Proof. Given that R_0 can be regarded as a function depending on the parameters Λ, q, and m, without sacrificing generality, we may select m as the bifurcation parameter. Set $m = \Lambda q + \delta$, where the case of $\delta = 0$ corresponds to $R_0 = 1$.

We linearize the system (2.1) at E_0. First, let $u' = x - \dfrac{\Lambda}{\Lambda q + \delta}, v' = y$ (for simplicity, we still use u, v for u', v'), and using Taylor expansion, the following system is obtained.

$$\begin{cases} \dfrac{du}{dt} = a_{11}u + a_{12}v + a_{13}uv + a_{14}v^2 + a_{15}uv^2 + a_{16}v^3 + o(|u, v|^4), \\ \dfrac{dv}{dt} = a_{21}v + a_{22}uv + a_{23}v^2 + a_{24}uv^2 + a_{25}v^3 + o(|u, v|^4), \end{cases} \tag{3.1}$$

where

$$\begin{aligned} &a_{11} = -(\Lambda q + \delta), \quad a_{12} = w - \frac{\Lambda q}{\Lambda q + \delta}, \quad a_{13} = -q, \\ &a_{14} = -\frac{\Lambda}{\Lambda q + \delta}, \quad a_{15} = -1, \quad a_{16} = \frac{\Lambda q p}{\Lambda q + \delta}, \quad a_{21} = -\frac{\delta}{\Lambda q + \delta}, \\ &a_{22} = q, \quad a_{23} = \frac{\Lambda}{\Lambda q + \delta}, \quad a_{24} = 1, \quad a_{25} = -\frac{\Lambda q p}{\Lambda q + \delta}. \end{aligned}$$

Next, we diagonalize the linear part of the system (3.1), let $X = v, Y = u - \dfrac{w(\Lambda q + \delta) - \Lambda q}{(\Lambda q + \delta)^2 - \delta}v$, $\tau = -(\Lambda q + \delta)t$ (for simplicity, we still use t for τ), and the system (3.1) is transformed into

$$\begin{cases} \dfrac{dX}{dt} = b_{11}X + b_{12}XY + b_{13}X^2 + b_{14}X^2Y + b_{15}X^3 + o(|X, Y|^4), \\ \dfrac{dY}{dt} = b_{21}Y + b_{22}XY + b_{23}X^2 + b_{24}X^2Y + b_{25}X^3 + o(|X, Y|^4), \end{cases}$$

where

$$\begin{aligned} &b_{11} = \frac{\delta}{(\Lambda q + \delta)^2}, \quad b_{12} = -\frac{q}{\Lambda q + \delta}, \quad b_{13} = -\frac{1}{\Lambda q + \delta}\Big(\frac{wq(\Lambda q + \delta) - \Lambda q^2}{(\Lambda q + \delta)^2 - \delta} + \frac{\Lambda}{\Lambda q + \delta}\Big), \\ &b_{14} = -\frac{1}{\Lambda q + \delta}, \quad b_{15} = -\frac{1}{\Lambda q + \delta}\Big(\frac{wq(\Lambda q + \delta) - \Lambda q^2}{(\Lambda q + \delta)^2 - \delta} - \frac{\Lambda q p}{\Lambda q + \delta}\Big), \quad b_{21} = 1, \\ &b_{22} = \frac{q}{\Lambda q + \delta}\Big(\frac{wq(\Lambda q + \delta) - \Lambda q^2}{(\Lambda q + \delta)^2 - \delta} + 1\Big), \\ &b_{23} = \frac{1}{\Lambda q + \delta}\Big(\frac{wq(\Lambda q + \delta) - \Lambda q^2}{(\Lambda q + \delta)^2 - \delta} + 1\Big)\Big(\frac{wq(\Lambda q + \delta) - \Lambda q^2}{(\Lambda q + \delta)^2 - \delta} + \frac{\Lambda}{\Lambda q + \delta}\Big), \\ &b_{24} = \frac{1}{\Lambda q + \delta}\Big(\frac{wq(\Lambda q + \delta) - \Lambda q^2}{(\Lambda q + \delta)^2 - \delta} + 1\Big), \\ &b_{25} = \frac{1}{\Lambda q + \delta}\Big(\frac{wq(\Lambda q + \delta) - \Lambda q^2}{(\Lambda q + \delta)^2 - \delta} + 1\Big)\Big(\frac{wq(\Lambda q + \delta) - \Lambda q^2}{(\Lambda q + \delta)^2 - \delta} - \frac{\Lambda q p}{\Lambda q + \delta}\Big). \end{aligned}$$

The following reduced model on the center manifold can be obtained by applying the center manifold theorem with the parameter δ,

$$\frac{dX}{dt} = \frac{\delta}{(\Lambda q+\delta)^2}X - \frac{1}{\Lambda q+\delta}\Big(\frac{wq(\Lambda q+\delta)-\Lambda q^2}{(\Lambda q+\delta)^2-\delta} + \frac{\Lambda}{\Lambda q+\delta}\Big)X^2 + o(X^3). \tag{3.2}$$

Denoting the righthand side of the model (3.2) as $F(X,\delta)$, we can derive

$$\begin{aligned} &F(0,0)=0, \quad \frac{\partial F}{\partial X}(0,0)=0, \quad \frac{\partial F}{\partial \delta}(0,0)=0,\\ &\frac{\partial^2 F}{\partial X\partial\delta}(0,0)=\frac{1}{(\Lambda q)^2}\neq 0, \quad \frac{\partial^2 F}{\partial^2 X}(0,0)=-2\frac{\Lambda+qw-q}{(\Lambda q)^2}. \end{aligned}$$

Therefore, model (3.2) undergoes a transcritical bifurcation([14]) if $\Lambda \neq q-qw$.

Since $\frac{\partial R_0}{\partial \varepsilon} = -\frac{1}{\Lambda q} < 0$, when R_0 crosses $R_0 = 1$, the system (2.1) undergoes a forward bifurcation if $\Lambda < q-qw$, and a backward bifurcation if $\Lambda > q-qw$, respectively.

If $\Lambda = q-qw$, model (3.2) in the center manifold becomes

$$\frac{dX}{dt} = b_{31}X + b_{32}X^2 + b_{33}X^3 + o(X^4), \tag{3.3}$$

where

$$\begin{aligned} b_{31} &= \frac{\delta}{(q^2-q^2w+\delta)^2}, \quad b_{32} = -\frac{q^3\delta-q^3w\delta+q\delta^2-q\delta-qw\delta}{((q^2-q^2w+\delta)^2-\delta)(q^2-q^2w+\delta)^2},\\ b_{33} &= -\frac{2wq^3-w^2q^3+wq\delta-q^3}{(q^2-q^2w+\delta)^3-\delta(q^2-q^2w+\delta)} - \frac{q^2p-q^2wp}{(q^2-q^2w+\delta)^2}\\ &\quad + \frac{q^4\delta-q^4w\delta+q^2\delta^2-q^2\delta-q^2w\delta}{((q^2-q^2w+\delta)^2-\delta)(q^2-q^2w+\delta)^3}\Big(\frac{2wq^3-w^2q^3+wq\delta-q^3}{(q^2-q^2w+\delta)^2-\delta}+1\Big). \end{aligned}$$

For simplicity, we still denote the right side of the model (3.3) as $F(X,\varepsilon)$, and derive

$$\begin{aligned} &F(0,0)=0, \quad \frac{\partial F}{\partial X}(0,0)=0, \quad \frac{\partial F}{\partial \delta}(0,0)=0,\\ &\frac{\partial^2 F}{\partial X\partial\delta}(0,0)=\frac{1}{(q^2-q^2w)^2}\neq 0, \quad \frac{\partial^2 F}{\partial^2 X}(0,0)=0,\\ &\frac{\partial^3 F}{\partial^3 X}(0,0)=\frac{qp+1}{q^3(1-w)}\neq 0. \end{aligned}$$

Therefore, model (3.3) undergoes pitchfork bifurcation([14]) if $\Lambda = q-qw$. □

Remark 12. *In epidemiological models, the backward bifurcation is an important phenomenon, and the value of $R_0 \leq 1$ is not sufficient to reflect the prevalence of the disease. Theorem* 11 *proves the conditions for the occurrence of a backward bifurcation. The existence of backward bifurcation indicates that even if some parameters control the basic reproduction number $R_0 < 1$, the disease will still spread and certain parameters need to be controlled to make $R_0 < R_a$ to eliminate the disease.*

Theorem 13. *For system* (2.1)*, according to Lemma* 2*, if $\Lambda q < m < \frac{(q-qw-\Lambda)^2}{4(1-w)} + \Lambda q, \Lambda > q-qw$, and $p = p_*$ hold, there is a unique endemic equilibrium $E_*(x_*, y_*)$. Furthermore,*
1) if $w \neq w_*$, then $E_*(x_*, y_*)$ is a saddle node, which is attracting (or repelling) if $w < w_2$, or $w > w_*$ (or $w_2 < w < w_*$);
2) if $w = w_*$, then $E_*(x_*, y_*)$ is a cusp of codimension two.

Proof. We plug $y = y_*, p = p_*$ in (2.7) and (2.8), and then obtain $S_D(y_*) = 0$ and

$$S_T(y_*) = \frac{C_1 w^2 + C_2 w + C_3}{m(\Lambda + qw - q)^2},$$

where

$$\begin{aligned} C_1 &= q^2(-2m^3 - 2m^2 + m^2\Lambda q + 2m\Lambda q), \\ C_2 &= 2q(\Lambda - q)(-2m^3 - 2m^2 + m^2\Lambda q + 2m\Lambda q) \\ &\quad -(4m + 4m^2)(m - \Lambda q)^2 + 2mq(\Lambda - qm)(m - \Lambda q), \\ C_3 &= (\Lambda - q)^2(-2m^3 - 2m^2 + m^2\Lambda q + 2m\Lambda q) + 4m(m - \Lambda q)^2 \\ &\quad +2m(\Lambda - q)(\Lambda - qm)(m - \Lambda q). \end{aligned}$$

Therefore, the symbol of $S_T(y_*)$ is determined by $C_1 w^2 + C_2 w + C_3$, and we denote

$$F(w) = C_1 w^2 + C_2 w + C_3.$$

By calculation,

$$\Delta_{F(w)} = 4m^2(m - \Lambda q)^2(2m(m + 1) - q(\Lambda + mq))^2 > 0.$$

It is easy to determine that $2m(m + 1) - q(\Lambda + mq) > 0$, then

$$\sqrt{\Delta_{F(w)}} = 2m(m - \Lambda q)(2m(m + 1) - q(\Lambda + mq)).$$

We assume that the two roots of $F(w)$ are w_1 and w_2, and we can get

$$\begin{aligned} w_1 &= \frac{-C_2 - \sqrt{\Delta_{F(w)}}}{2C_1} = \frac{2m - 2\Lambda q - m\Lambda^2 + m\Lambda q}{2m^2 + 2m - 2\Lambda q - m\Lambda q}, \\ w_2 &= \frac{-C_2 + \sqrt{\Delta_{F(w)}}}{2C_1} = \frac{q^2 + \Lambda q - 2m}{q^2}. \end{aligned}$$

Since $\Lambda q < m < \dfrac{(q - qw - \Lambda)^2}{4(1 - w)} + \Lambda q, \Lambda > q - qw$, we can get $0 < w_2 < w_1 < 1$; moreover, when $w = w_2$, it conflicts with the condition $m > \Lambda q$. Hence, we denote

$$w_* = \frac{2m - 2\Lambda q - m\Lambda^2 + m\Lambda q}{2m^2 + 2m - 2\Lambda q - m\Lambda q}.$$

Therefore, if $w = w_*, S_T(y_*) = 0$, $S_T(y_*) < 0$ if, and only if, $w < w_2$ or $w > w_*$. Conversely, if $w_2 < w < w_*, S_T(y_*) > 0$. Using the transformation of $u_1 = y - y_*, u_2 = x - x_*, p = p_*$, we rewrite system (2.1) as follows:

$$\begin{cases} \dfrac{du_1}{dt} = \dfrac{q(u_1 + y_*) + (u_1 + y_*)^2}{1 + p_*(u_1 + y_*)^2}(u_2 + x_*) - (u_1 + y_*), \\ \dfrac{du_2}{dt} = \Lambda - m(u_2 + x_*) - \dfrac{q(u_1 + y_*) + (u_1 + y_*)^2}{1 + p_*(u_1 + y_*)^2}(u_2 + x_*) + w(u_1 + y_*). \end{cases} \tag{3.4}$$

Letting $U_1 = u_1, U_2 = u_1 + u_2$ (U_1, U_2 are rewritten as u_1, u_2) and using Taylor expansion, then the following system is obtained.

$$\begin{cases} \dfrac{du_1}{dt} = c_{11}u_1 + c_{12}u_2 + c_{13}u_1^2 + c_{14}u_1u_2 + o(|u_1, u_2|^3), \\ \dfrac{du_2}{dt} = c_{21}u_1 + c_{22}u_2 + o(|u_1, u_2|^3), \end{cases} \tag{3.5}$$

where

$$\begin{aligned} c_{11} =& \frac{-4(m-\Lambda q)^2(m+w-1) - 2q(m-\Lambda q)(\Lambda+qw-q)(m+w-1)}{(2m-\Lambda q)(\Lambda+qw-q)^2 - 4(1-w)(m-\Lambda q)^2}, \\ c_{12} =& \frac{2mq(m-\Lambda q)(\Lambda+qw-q) + 4m(m-\Lambda q)^2}{(2m-\Lambda q)(\Lambda+qw-q)^2 - 4(1-w)(m-\Lambda q)^2}, \\ c_{13} =& \frac{-3(m-\Lambda q)^3 - 2(\Lambda - mq)(\Lambda+qw-q)^2 + 8(\Lambda+qw-q)(m-\Lambda q)(1-m-w)}{(2m-\Lambda q)(\Lambda+qw-q)^2 - 4(1-w)(m-\Lambda q)^2} \\ &+ \frac{-4m(\Lambda+qw-q)^5(m-\Lambda q) + 4m(mq-\Lambda)(\Lambda+qw-q)^4(m-\Lambda q)}{m((2m-\Lambda q)(\Lambda+qw-q)^2 - 4(1-w)(m-\Lambda q)^2)^2} \\ &+ \frac{(40m - 40mw + 8m^2)(\Lambda+qw-q)^3(m-\Lambda q)^2}{m((2m-\Lambda q)(\Lambda+qw-q)^2 - 4(1-w)(m-\Lambda q)^2)^2} \\ &- \frac{32m(1-w)(3-3w+m)(\Lambda+qw-q)(m-\Lambda q)^3}{m((2m-\Lambda q)(\Lambda+qw-q)^2 - 4(1-w)(m-\Lambda q)^2)^2}, \\ c_{14} =& \frac{m\Lambda q^2(\Lambda+qw-q)^4 + 4m^2(m-\Lambda q)(\Lambda+qw-q)^3}{((2m-\Lambda q)(\Lambda+qw-q)^2 - 4(1-w)(m-\Lambda q)^2)^2} \\ &+ \frac{4mq(1-w)(m-\Lambda q)^2(\Lambda+qw-q)^2}{((2m-\Lambda q)(\Lambda+qw-q)^2 - 4(1-w)(m-\Lambda q)^2)^2}, \\ c_{21} =& m+w-1, \quad c_{22} = -m. \end{aligned}$$

Defining

$$\kappa_1 = \frac{4(1-w)(m-\Lambda q)^2 - (2m-\Lambda q)(\Lambda+qw-q)^2}{4(m-\Lambda q)^2 + 2q(m-\Lambda q)(\Lambda+qw-q)}, \quad \kappa_2 = \frac{m+w-1}{m},$$

and letting

$$\rho_1 = \frac{\kappa_1 u_1 - u_2}{\kappa_1 - \kappa_2}, \quad \rho_2 = \frac{-\kappa_2 u_1 + u_2}{\kappa_1 - \kappa_2}.$$

We transform (3.5) into the following form:

$$\begin{cases} \dfrac{d\rho_1}{dt} = d_{11}\rho_1^2 + d_{12}\rho_1\rho_2 + d_{13}\rho_2^2 + o(|\rho_1, \rho_2|^3), \\ \dfrac{d\rho_2}{dt} = d_{20}\rho_2 + d_{21}\rho_1^2 + d_{22}\rho_1\rho_2 + d_{23}\rho_2^2 + o(|\rho_1, \rho_2|^3), \end{cases} \tag{3.6}$$

where

$$\begin{aligned} d_{11} &= \frac{\kappa_1(\kappa_2 c_{14} + c_{13})}{\kappa_1 - \kappa_2}, \quad d_{12} = \frac{\kappa_1((\kappa_1 + \kappa_2)c_{14} + 2c_{13})}{\kappa_1 - \kappa_2}, \\ d_{13} &= \frac{\kappa_1(\kappa_1 c_{14} + c_{13})}{\kappa_1 - \kappa_2}, \quad d_{20} = \frac{-\kappa_1\kappa_2 c_{12} + \kappa_1 c_{22} - \kappa_2 c_{11} + c_{21}}{\kappa_1 - \kappa_2}, \\ d_{21} &= \frac{-\kappa_2(\kappa_2 c_{14} + c_{13})}{\kappa_1 - \kappa_2}, \quad d_{22} = \frac{-\kappa_2((\kappa_1 + \kappa_2)c_{14} + 2c_{13})}{\kappa_1 - \kappa_2}, \quad d_{23} = \frac{-\kappa_2(\kappa_1 c_{14} + c_{13})}{\kappa_1 - \kappa_2}. \end{aligned}$$

We further compute d_{20} and obtain

$$d_{20} = \frac{(m+w-1)(4(1-w)(m-\Lambda q)^2 - (2m-\Lambda q)(\Lambda + qw - q)^2)}{4(m-\Lambda q)^2 + 2q(m-\Lambda q)(\Lambda + qw - q)} - m.$$

When $w \neq w_*$, $d_{20} \neq 0$. Introduce a new time variable τ through the relation $d\tau = d_{20}dt$, and then rewrite τ as t. From (3.6), we obtain

$$\begin{cases} \dfrac{d\rho_1}{dt} = e_{11}\rho_1^2 + e_{12}\rho_1\rho_2 + e_{13}\rho_2^2 + o(|\rho_1, \rho_2|^3), \\ \dfrac{d\rho_2}{dt} = \rho_2 + e_{21}\rho_1^2 + e_{22}\rho_1\rho_2 + e_{23}\rho_2^2 + o(|\rho_1, \rho_2|^3), \end{cases} \tag{3.7}$$

where $e_{ij} = \dfrac{d_{ij}}{d_{20}}$ $(i = 1, 2; j = 1, 2, 3)$. Letting the right side of the second equation system (3.7) be equal to zero, we can solve the implicit function $\rho_2 = \Phi(\rho_1)$ as follows:

$$\rho_2 = \Phi(\rho_1) = -e_{21}\rho_1^2 + \cdots. \tag{3.8}$$

Substituting (3.8) into the right side of the first equation of the system (3.7), we get

$$\frac{d\rho_1}{dt} = e_{11}\rho_1^2 + e_{12}\rho_1\Phi(\rho_1) + e_{13}\Phi^2(\rho_1) + \cdots,$$

where

$$e_{11} = \frac{\kappa_1(\kappa_2 c_{14} + c_{13})}{-\kappa_1\kappa_2 c_{12} + \kappa_1 c_{22} - \kappa_2 c_{11} + c_{21}}.$$

After calculation, it can be concluded that if $w \neq w_*$, $e_{11} \neq 0$. Therefore, according to Theorems 7.1–7.3 in Zhang [13] et al., E_* is a saddle node of codimension one. Then, we obtain the conclusion in (1).

In addition, the saddle-node bifurcation of system (2.1) will occur as the parameter p crosses the critical value p_* from the right to the left.

For the second conclusion in (2), substituting $w = w_*$ into (3.4) and applying the Taylor expansion, the system (3.4) is transformed into

$$\begin{cases} \dfrac{du_1}{dt} = a_1u_1 + a_2u_2 + a_3u_1^2 + a_4u_1u_2 + o(|u_1, u_2|^3), \\ \dfrac{du_2}{dt} = -\dfrac{a_1^2}{a_2}u_1 - a_1u_2 - a_3u_1^2 - a_4u_1u_2 + o(|u_1, u_2|^3), \end{cases} \tag{3.9}$$

where a_1, a_2, a_3, a_4 are listed in the Appendix.

Letting $v_1 = u_1, v_2 = a_1u_1 + a_2u_2$, the system (3.9) is transformed into

$$\begin{cases} \dfrac{dv_1}{dt} = v_2 + b_1v_1^2 + b_2v_1v_2 + o(|v_1, v_2|^3), \\ \dfrac{dv_2}{dt} = b_3v_1^2 + b_4v_1v_2 + o(|v_1, v_2|^3), \end{cases} \tag{3.10}$$

where

$$b_1 = \frac{a_2a_3 - a_1a_4}{a_2}, \quad b_2 = \frac{a_4}{a_2},$$
$$b_3 = \frac{(a_1 - a_2)(a_2a_3 - a_1a_4)}{a_2}, \quad b_4 = \frac{(a_1 - a_2)a_4}{a_2},$$

and the parameter a_1, a_2, a_3, a_4 is the same as in system (3.9). To eliminate the quadratic terms in the first equation of the system (3.10), we transform $\eta_1 = v_1 - \frac{b_2}{2}v_1^2, \eta_2 = v_2 + b_1 v_1^2$ and obtain

$$\begin{cases} \frac{d\eta_1}{dt} = \eta_2 + o(|\eta_1, \eta_2|^3), \\ \frac{d\eta_2}{dt} = b_3\eta_1^2 - (2b_1 + b_4)\eta_1\eta_2 + o(|\eta_1, \eta_2|^3). \end{cases}$$

From conditions $w = w_*, \Lambda q < m < \frac{(q - qw - \Lambda)^2}{4(1 - w)} + \Lambda q, \Lambda > q - qw$, we have

$$b_3 \neq 0, \quad 2b_1 + b_4 \neq 0.$$

Due to the complexity of the formula, the calculation process is omitted here. From the results in [15], $E_*(x_*, y_*)$ is a cusp of codimension two. □

Next, we select parameter values of Λ, m, q, w, and p to simulate the result of Theorem 13. First, letting $\Lambda = 0.2, m = 0.1, q = 0.45, p = 4.129$, and $w = 0.88$, we find that E_* is an attracting saddle node, and the phase portrait is shown in Figure 2. Second, letting $\Lambda = 0.2, m = 0.1, q = 0.4, p = 0.712$, and $w = 0.84$, the phase portrait of the repelling saddle-node is shown in Figure 3. Finally, letting $\Lambda = 0.4, m = 0.2, q = 0.4, p = \frac{5}{64}$, and $w = 0.625$, we can obtain E_* as a codimension two cusp, and the phase portrait is shown in Figure 4.

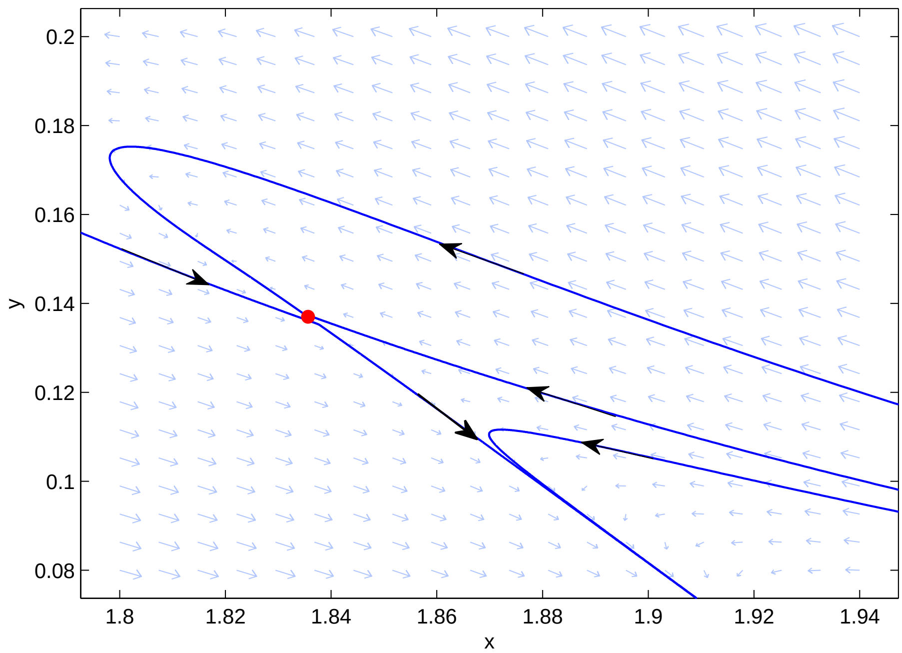


Figure 2. The attracting saddle-node.

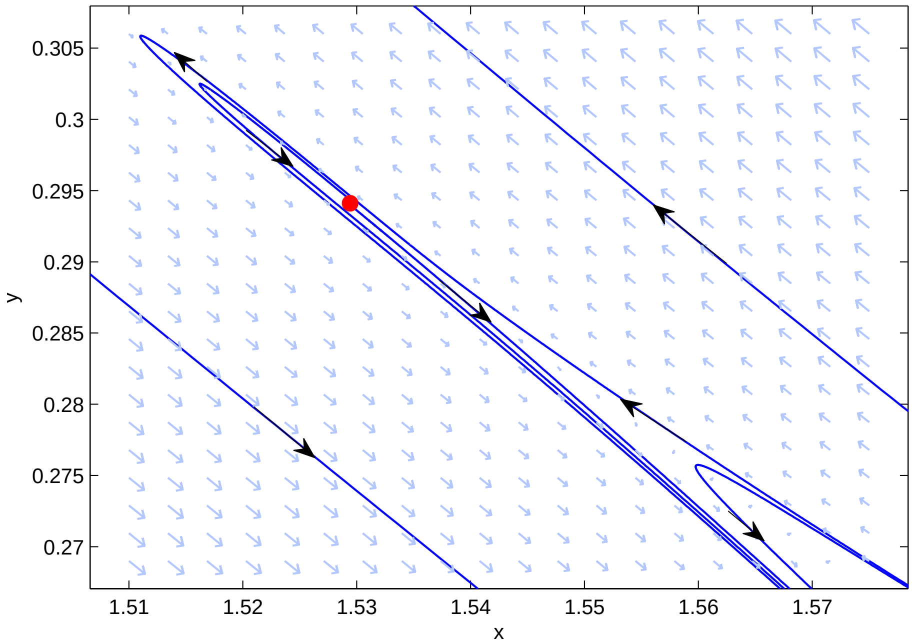


Figure 3. The repelling saddle-node.

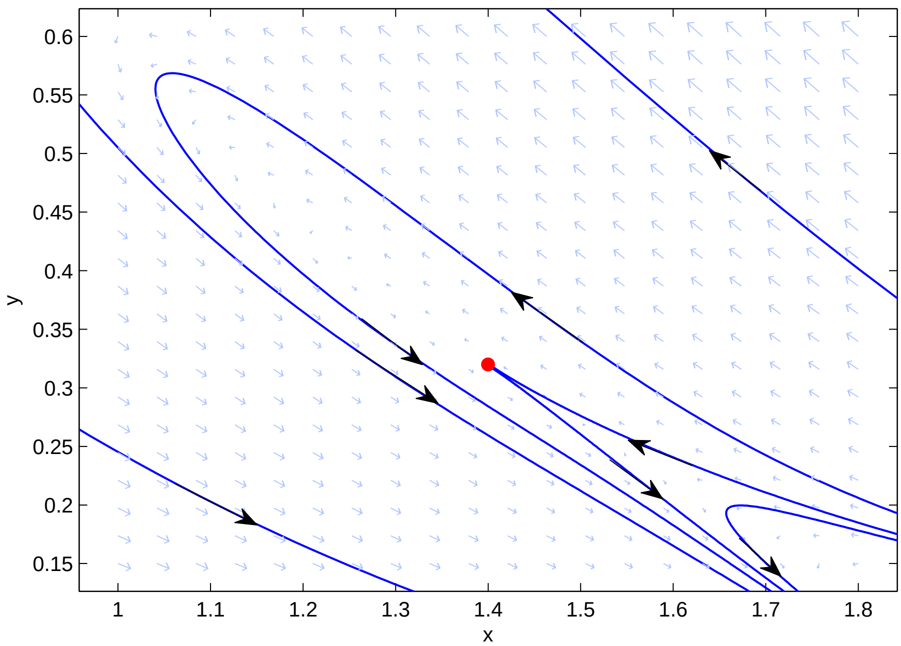


Figure 4. Phase portrait when $(\theta_1, \theta_2) = (0, 0)$. There is a codimension two-cusp point.

Remark 14. *When $q = 0$ (that is, $k_1 = 0$), system* (2.1) *is similar to model (1.3) of Tang et al. [16]. From Theorem* 13*, we prove that model* (2.1) *undergoes the Bogdanov-Takens bifurcation of codimension at most two near $E_*(x_*, y_*)$. From Theorems* 11 *and* 13*, when $k_1^* < k_1 < k_{10}$ (i.e., $\Lambda > q - qw$) for system* (2.1)*, the disease will be eliminated if $R_0 < R_a$ (i.e., $p > p_*$ and $k_1 > k_1^*$), and if $R_0 = R_a$ (i.e., $p = p_*$*

and $k_1 > k_1^*$*), system* (2.1) *will present complex dynamics; these conditions are not enough to determine dynamical behaviors and the disease will persist or die out, depending on the values of* k_1 *and* k_2*.*

3.2. Hopf bifurcation

In this section, we discuss Hopf bifurcation at endemic equilibrium $E_2(x_2, y_2)$. First, we introduce the new state variable $u = x + y - \frac{\Lambda}{m}$, $v = y$, time transformation $dt = (1 + py^2)d\tau$, and change the system (2.1) into the following form:

$$\begin{cases} \frac{du}{dt} = -mu + (m + w - 1)v - mpuv^2 + (m + w - 1)pv^3, \\ \frac{dv}{dt} = (\frac{\Lambda q}{m} - 1)v + (\frac{\Lambda}{m} - q)v^2 + quv + uv^2 - (1 + p)v^3. \end{cases} \tag{3.11}$$

Since $1 + py^2 > 0$, system (3.11) is topologically equivalent to the system (2.1) and has an endemic equilibrium $\widetilde{E}_2(u_2, v_2)$ corresponding to $E_2(x_2, y_2)$ of (2.1), where

$$u_2 = \frac{m + w - 1}{m} v_2, \quad v_2 = \frac{\Lambda + qw - q + \sqrt{(q - qw - \Lambda)^2 - 4(mp + 1 - w)(m - \Lambda q)}}{2(mp + 1 - w)}.$$

The Jacobian matrix of the system (3.11) at equilibrium $\widetilde{E}_2(u_2, v_2)$ is given by

$$J(\widetilde{E}_2) = \begin{bmatrix} -m - mpv_2^2 & (m + w - 1)(1 + pv_2^2) \\ qv_2 + v_2^2 & \frac{\Lambda q - m}{m} + \frac{2\Lambda + wq - mq - q}{m} v_2 + \frac{-3mp - m + 2w - 2}{m} v_2^2 \end{bmatrix}$$

We know that

$$\begin{aligned} sgn(detJ(\widetilde{E}_2)) &= sgn((1 + pv_2^2)((\Lambda + qw - q)v_2 - 2(m - \Lambda q)) \\ &= sgn((q - qw - \Lambda)^2 - 4(mp + 1 - w)(m - \Lambda q) \\ &\quad + \sqrt{(q - qw - \Lambda)^2 - 4(mp + 1 - w)(m - \Lambda q)}) = 1, \\ sgn(trJ(\widetilde{E}_2)) &= -sgn((mp + m + m^2 p)v_2^2 + (mq - q + qw)v_2 + \Lambda q - m + m^2), \end{aligned}$$

and $sgn(trJ(\widetilde{E}_2)) = 0$ is equivalent to

$$(mp + m + m^2 p)v_2^2 + (mq - q + qw)v_2 + \Lambda q - m + m^2 = 0. \tag{3.12}$$

Since $\Lambda q < m < \frac{(q - qw - \Lambda)^2}{4(1 - w)} + \Lambda q, \Lambda > q - qw$, we can obtain $\Lambda q - m + m^2 < 0$; thus, $(mq - q + qw)^2 - 4(mp + m + m^2 p)(\Lambda q - m + m^2) > 0$. Eq (3.12) has two real solutions

$$v_2 \pm = \frac{-(mq - q + qw) \pm \sqrt{(mq - q + qw)^2 - 4(mp + m + m^2 p)(\Lambda q - m + m^2)}}{2(mp + m + m^2 p)},$$

where $v_2^+ > 0, v_2^- < 0$. Since $v_2 > 0$, in determining the sign of $trJ(\widetilde{E}_2)$, we may consider the relationship between v_2 and v_2^+ as follows:

$$v_2 > v_2^+, v_2 = v_2^+, v_2 < v_2^+,$$

which are equivalent to $\vartheta > \vartheta^*, \vartheta = \vartheta^*, \vartheta < \vartheta^*$, respectively, where

$$\vartheta \triangleq (mp + m + m^2 p)(\Lambda + qw - q + \sqrt{(q - qw - \Lambda)^2 - 4(mp + 1 - w)(m - \Lambda q)}),$$
$$\vartheta^* \triangleq (mp + 1 - w)(-(mq - q + qw) + \sqrt{(mq - q + qw)^2 - 4(mp + m + m^2 p)(\Lambda q - m + m^2)}).$$

Therefore, we can come to the following conclusion: if $\vartheta > \vartheta^*$, then $sgn(trJ(\widetilde{E}_2)) < 0$; if $\vartheta = \vartheta^*$, then $sgn(trJ(\widetilde{E}_2)) = 0$; if $\vartheta < \vartheta^*$, then $sgn(trJ(\widetilde{E}_2)) > 0$. Then, we have the following theorem.

Theorem 15. *Equilibrium $\widetilde{E}_2(x_2, y_2)$ satisfies one of the following cases:*
1) when $\vartheta > \vartheta^*$, equilibrium $\widetilde{E}_2$ is a locally stable focus or node;
2) when $\vartheta = \vartheta^*$, equilibrium $\widetilde{E}_2$ is a weak focus or center;
3) when $\vartheta < \vartheta^*$, equilibrium $\widetilde{E}_2$ is a locally unstable focus or node.

Next, we discuss the Hopf bifurcation around the equilibrium $\widetilde{E}_2$. From expressions of $tr(\widetilde{E}_2)$ and ϑ, we can represent $tr(\widetilde{E}_2)$ as a function of ϑ,

$$trJ(\widetilde{E}_2) = -(mp + m + m^2 p)\left(\frac{\vartheta}{2(mp + m + m^2 p)(mp + 1 - w)}\right)^2 - (mq - q + qw)\left(\frac{\vartheta}{2(mp + m + m^2 p)(mp + 1 - w)}\right) - (\Lambda q - m + m^2).$$

We compute the derivative of $trJ(\widetilde{E}_2)$ with respect to ϑ at ϑ^* and obtain

$$\left.\frac{trJ(\widetilde{E}_2)}{d\vartheta}\right|_{\vartheta=\vartheta^*} = \frac{-\sqrt{(mq - q + qw)^2 - 4(mp + m + m^2 p)(\Lambda q - m + m^2)}}{2(mp + m + m^2 p)(mp + 1 - w)} < 0.$$

Using a transformation of $\varphi = u - u_2, \psi = v - v_2$ and Taylor expansion, system (3.11) can be changed into

$$\begin{cases} \dfrac{d\varphi}{dt} = a_{11}\varphi + a_{12}\psi + a_{13}\varphi\psi + a_{14}\psi^2 + a_{15}\varphi\psi^2 + a_{16}\psi^3, \\ \dfrac{d\psi}{dt} = a_{21}\varphi + a_{22}\psi + a_{23}\varphi\psi + a_{24}\psi^2 + a_{25}\varphi\psi^2 + a_{26}\psi^3, \end{cases} \tag{3.13}$$

where

$$a_{11} = -m - mpv_2^2, \quad a_{12} = (m + w - 1)(1 + pv_2^2), \quad a_{13} = -2mpv_2,$$
$$a_{14} = 2p(m + w - 1)v_2, \quad a_{15} = -mp, \quad a_{16} = p(m + w - 1),$$
$$a_{21} = qv_2 + v_2^2, \quad a_{22} = \frac{\Lambda q - m}{m} + \frac{2\Lambda + wq - mq - q}{m}v_2 + \frac{-3mp - m + 2w - 2}{m}v_2^2,$$
$$a_{23} = q + 2v_2, \quad a_{24} = \frac{\Lambda - mq}{m} + \frac{-3mp - 2m + w - 1}{m}v_2, \quad a_{25} = 1, \quad a_{26} = -(1 + p).$$

Defining $D = \sqrt{detJ(\widetilde{E}_2)}$ and introducing transformation of $\eta = -\varphi, \rho = \dfrac{a_{11}}{D}\varphi + \dfrac{a_{12}}{D}\psi$, we obtain, from system (3.13), that

$$\begin{cases} \dfrac{d\eta}{dt} = -D\rho + f(\eta, \rho), \\ \dfrac{d\rho}{dt} = D\eta + g(\eta, \rho), \end{cases}$$

where

$$f(\eta,\rho) = b_{11}\eta\rho + b_{12}\rho^2 + b_{13}\eta^2\rho + b_{14}\eta\rho^2 + b_{15}\rho^3,$$
$$g(\eta,\rho) = b_{21}\eta^2 + b_{22}\eta\rho + b_{23}\rho^2 + b_{24}\eta^3 + b_{25}\eta^2\rho + b_{26}\eta\rho^2 + b_{27}\rho^3,$$

and

$$\begin{aligned}
b_{11} &= \frac{2mpv_2D}{(m+w-1)(1+pv_2^2)}, \quad b_{12} = -\frac{2pv_2D^2}{(m+w-1)(1+pv_2^2)^2},\\
b_{13} &= -\frac{m^2pD}{(m+w-1)^2(1+pv_2^2)}, \quad b_{14} = \frac{2mpD^2}{(m+w-1)^2(1+pv_2^2)^2},\\
b_{15} &= -\frac{pD^3}{(m+w-1)^2(1+pv_2^2)^3},\\
b_{21} &= \frac{m(1+pv_2^2)((-3mp+3w+3)v_2+\Lambda+qw-q)}{(m+w-1)D},\\
b_{22} &= \frac{(2m^2p+6mp+2m-4w+4)v_2+mq-2\Lambda-qw+q}{m+w-1},\\
b_{23} &= \frac{((-2m^2p-3mp-2m+w-1)v_2+\Lambda-mq)D}{m(m+w-1)(1+pv_2^2)},\\
b_{24} &= \frac{m^2(1+pv_2^2)(mp-w+1)}{(m+w-1)^2D}, \quad b_{25} = -\frac{m(m^2p+3mp+m+2-2w)}{(m+w-1)^2},\\
b_{26} &= \frac{(2m^2p+3mp+2m-w+1)D}{(m+w-1)^2(1+pv_2^2)}, \quad b_{27} = -\frac{(mp+p+1)D}{(m+w-1)^2(1+pv_2^2)^2}.
\end{aligned}$$

Based on the results in [17, 18], the formula of the first Lyapunov number is given by

$$\begin{aligned}
l_1 =& \Big(\frac{f_{\eta\eta\eta}+f_{\eta\rho\rho}+f_{\eta\eta\rho}+f_{\rho\rho\rho}}{16}\\
&+\frac{f_{\eta\rho}(f_{\eta\eta}+f_{\rho\rho})-g_{\eta\rho}(g_{\eta\eta}+g_{\rho\rho})-f_{\eta\eta}g_{\eta\eta}+f_{\rho\rho}g_{\rho\rho}}{16D}\Big)\Big|_{\eta=0,\rho=0}.
\end{aligned}$$

With a simple computation, we have

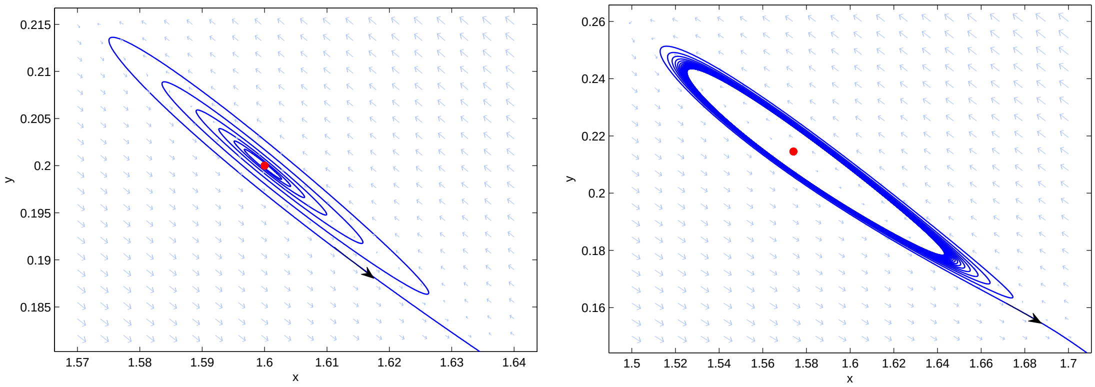
$$l_1 = \frac{\sigma}{8(m+w-1)^2},$$

where

$$\begin{aligned}
\sigma =& \frac{2mpD^2}{(1+pv_2^2)^2} - \frac{m^2pD}{1+pv_2^2} - \frac{3pD^3}{(1+pv_2^2)^3} - \frac{4mp^2v_2^2D^2}{(1+pv_2^2)^3} - \frac{1}{D^2}\Big((m(1+pv_2^2)((-3mp+3w\\
&+3)v_2+\Lambda+qw-q))((2m^2p+6mp+2m-4w+4)v_2+mq-2\Lambda-qw+q)\Big)\\
&-\Big(\frac{((2m^2p+6mp+2m-4w+4)v_2+mq-2\Lambda-qw+q)}{m(1+pv_2^2)}\\
&\times((-2m^2p-3mp-2m+w-1)v_2+\Lambda-mq)\Big)\\
&-\frac{4pv_2D((-2m^2p-3mp-2m+w-1)v_2+\Lambda-mq)}{m(1+pv_2^2)^3}.
\end{aligned}$$

Based on the above discussion, we have the following theorem.

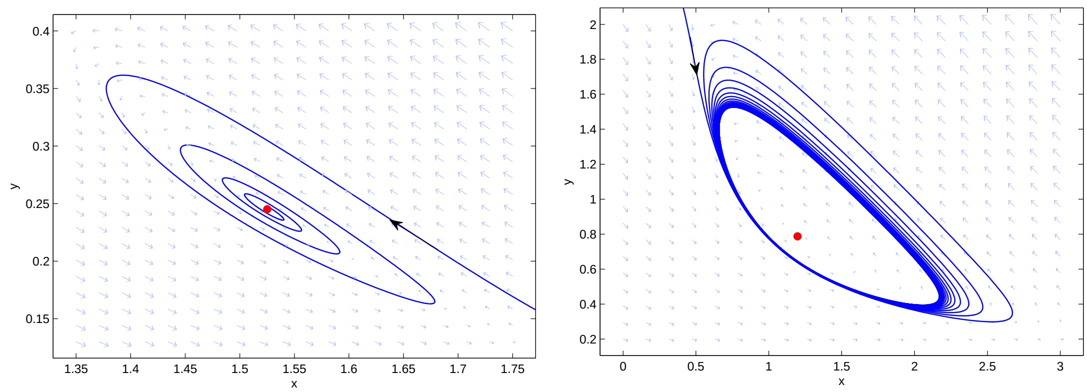
Theorem 16. *Assuming* $\Lambda q < m < \dfrac{(q - qw - \Lambda)^2}{4(1-w)} + \Lambda q, \Lambda > q - qw$, *and* $p < p_*$ *hold,*
1) when $\sigma > 0$ *and the parameter* ϑ *passes through* ϑ^* *from left to right, the system* (3.11) *undergoes a subcritical Hopf bifurcation around the equilibrium* $\widetilde{E}_2$ *in the sufficiently small neighborhood of* ϑ^**, and an unstable limit cycle appears.*
2) when $\sigma < 0$ *and the parameter* ϑ *passes through* ϑ^* *from right to left, the system* (3.11) *undergoes a supercritical Hopf bifurcation around the equilibrium* $\widetilde{E}_2$ *in the sufficiently small neighborhood of* ϑ^**, and a stable limit cycle appears.*



(a) the unstable focus. (b) the unstable limit cycle.

Figure 5. $\sigma > 0$, subcritical Hopf bifurcation.

We take parameters of $\Lambda = 0.2, m = 0.1, q = 0.45, p = 1, w = 0.8$, then $\sigma > 0$ and $\vartheta < \vartheta_*$. According to (1) of Theorem 16, the system (2.1) has an unstable focus (see Figure 5 (a)). When we change the parameter m from 0.1 to 0.0998, the relationship of $\vartheta < \vartheta_*$ becomes $\vartheta > \vartheta_*$, and the system (2.1) exists as an unstable limit cycle (see Figure 5 (b)).



(a) the stable focus. (b) the stable limit cycle.

Figure 6. $\sigma < 0$, supercritical Hopf bifurcation.

We take parameters of $\Lambda = 0.2, m = 0.099, q = 0.45, p = 1, w = 0.8$, then $\sigma < 0$ and $\vartheta > \vartheta_*$.

According to (2) of Theorem 16, (2.1) has a stable focus (see Figure 6 (a)). When we take parameters of $\Lambda = 0.5, m = 0.1, q = 0.1, p = 0.1, w = 0.517$, then $\sigma < 0$ and $\vartheta < \vartheta_*$, and system (2.1) exists as a stable limit cycle (see Figure 6 (b)).

3.3. Bogdanov-Takens bifurcation

From the conclusion (2) of Theorem 13 in the previous section, we know that the endemic equilibrium E_* is a codimension two-cusp point. It indicates that system (2.1) may exhibit the Bogdanov-Takens bifurcation under a small parameter perturbation. In this section, we study whether this bifurcation phenomenon can occur in the small neighborhood of E_* or not. Let Λ_0, m_0, p_0, q_0, and w_0 be the fixed values for the parameters Λ, m, p, q, and w, respectively, to ensure that conditions $p = p_*$ and $w = w_*$ hold, that is,

$$p_0 = \frac{(q_0 - q_0 w_0 - \Lambda_0)^2 - 4(1 - w_0)(m_0 - \Lambda_0 q_0)}{4m_0(m_0 - \Lambda_0 q_0)},$$
$$w_0 = \frac{2m_0 - 2\Lambda_0 q_0 - m_0\Lambda_0^2 + m_0\Lambda_0 q_0}{2m_0^2 + 2m_0 - 2\Lambda_0 q_0 - m_0\Lambda_0 q_0}.$$

Selecting Λ and m as bifurcation parameters, θ_1 and θ_2 as their perturbations at Λ_0 and m_0, respectively, and using the transformation of $u_1 = y - y_*, u_2 = x - x_*$, we have, from system (2.1), that

$$\begin{cases} \dfrac{du_1}{dt} = \dfrac{q_0(u_1 + y_*) + (u_1 + y_*)^2}{1 + p_0(u_1 + y_*)^2}(u_2 + x_*) - (u_1 + y_*), \\ \dfrac{du_2}{dt} = (\Lambda_0 + \theta_1) - (m_0 + \theta_2)(u_2 + x_*) \\ \qquad - \dfrac{q_0(u_1 + y_*) + (u_1 + y_*)^2}{1 + p_0(u_1 + y_*)^2}(u_2 + x_*) + w_0(u_1 + y_*). \end{cases} \tag{3.14}$$

By performing Taylor expansion, system (3.14) becomes

$$\begin{cases} \dfrac{du_1}{dt} = c_{11}u_1 + c_{12}u_2 + c_{13}u_1^2 + c_{14}u_1u_2 + R_1(\theta_1, \theta_2, u_1, u_2), \\ \dfrac{du_2}{dt} = c_{20} + c_{21}u_1 + c_{22}u_2 + R_2(\theta_1, \theta_2, u_1, u_2), \end{cases} \tag{3.15}$$

where $R_1(\theta_1, \theta_2, u_1, u_2)$ and $R_2(\theta_1, \theta_2, u_1, u_2)$ are C^∞ functions at least of the third order concerning (u_1, u_2), and their coefficients smoothly depend on θ_1, θ_2, and c_{ij} $(i = 1, 2; j = 0, 1, ..., 4)$, which are listed in the Appendix. Let

$$v_1 = u_1,$$
$$v_2 = c_{11}u_1 + c_{12}u_2 + c_{13}u_1^2 + c_{14}u_1u_2 + R_1(\theta_1, \theta_2, u_1, u_2),$$

and system (3.15) becomes

$$\begin{cases} \dfrac{dv_1}{dt} = v_2, \\ \dfrac{dv_2}{dt} = e_0 + e_1v_1 + e_2v_2 + e_3v_1^2 + e_4v_1v_2 + R_3(\theta_1, \theta_2, v_1, v_2), \end{cases} \tag{3.16}$$

where $R_3(\theta_1, \theta_2, v_1, v_2)$ are C^∞ functions at least of the third order concerning (v_1, v_2), and its coefficients smoothly depend on θ_1, θ_2, and

$$e_0 = c_{12}c_{20}, \quad e_1 = c_{14}c_{20} + c_{12}c_{21} - c_{11}c_{22}, \quad e_2 = c_{11} + c_{22},$$
$$e_3 = c_{14}c_{21} - c_{22}c_{13} - c_{13}c_{12} + c_{11}c_{14},$$
$$e_4 = -c_{14} + 2c_{13} - \frac{c_{14}c_{11}c_{12}}{c_{12}^2}, \quad e_5 = \frac{c_{14}}{c_{12}}.$$

Introducing a new time variable τ by $dt = (1 - \frac{c_{14}}{c_{12}}v_1)d\tau$ and rewriting τ as t, we change the system (3.16) into

$$\begin{cases} \dfrac{dv_1}{dt} = (1 - \dfrac{c_{14}}{c_{12}}v_1)v_2, \\ \dfrac{dv_2}{dt} = (1 - \dfrac{c_{14}}{c_{12}}v_1)(e_0 + e_1v_1 + e_2v_2 + e_3v_1^2 + e_4v_1v_2 + R_3(\theta_1, \theta_2, v_1, v_2)). \end{cases} \tag{3.17}$$

Let $\eta_1 = v_1, \eta_2 = (1 - \frac{c_{14}}{c_{12}}v_1)v_2$, and the system (3.17) becomes

$$\begin{cases} \dfrac{d\eta_1}{dt} = \eta_2, \\ \dfrac{d\eta_2}{dt} = e_0 + \gamma_1\eta_1 + e_2\eta_2 + \gamma_3\eta_1^2 + \gamma_4\eta_1\eta_2 + R_4(\theta_1, \theta_2, \eta_1, \eta_2)), \end{cases}$$

where $R_4(\theta_1, \theta_2, \eta_1, \eta_2)$ are C^∞ functions at least of the third order concerning (η_1, η_2), and its coefficients smoothly depend on θ_1, θ_2, and

$$\gamma_1 = e_1 - 2e_0e_5, \quad \gamma_3 = e_3 - 2e_1e_5 + e_0e_5^2, \quad \gamma_4 = e_4 - e_2e_5.$$

When $\theta_1 = 0, \theta_2 = 0$,

$$\gamma_3 = \frac{(c_{12} - c_{11})(c_{11}c_{14} - c_{12}c_{13})}{c_{12}}, \quad \gamma_4 = \frac{c_{11}^2c_{14} - c_{12}^2c_{14} + 2c_{12}^2c_{13}}{c_{12}^2},$$

where c_{11}, c_{12}, c_{13}, and c_{14} are the same as in the system (3.15). Substituting their values for the calculation yields $\gamma_1 = 0, \gamma_2 = 0, \gamma_3 \neq 0$, and $\gamma_4 \neq 0$. Due to the expression being too long, the calculation process and results are ignored here. Therefore, γ_3 and γ_4 are not equal to zero in the small enough neighborhood of $(\theta_1, \theta_2) = (0, 0)$. Letting $\zeta_1 = \eta_1 - \frac{\gamma_1}{2\gamma_3}$ and $\zeta_2 = \eta_2$, we have

$$\begin{cases} \dfrac{d\zeta_1}{dt} = \zeta_2, \\ \dfrac{d\zeta_2}{dt} = \gamma_0 + \gamma_2\zeta_2 + \gamma_3\zeta_1^2 + \gamma_4\zeta_1\zeta_2 + R_5(\theta_1, \theta_2, \zeta_1, \zeta_2)), \end{cases} \tag{3.18}$$

where $R_5(\theta_1, \theta_2, \zeta_1, \zeta_2)$ are C^∞ functions at least of the third order concerning (ζ_1, ζ_2) and its coefficients depend smoothly on θ_1, θ_2, and

$$\gamma_0 = e_0 - \frac{\gamma_1^2}{4\gamma_3}, \quad \gamma_2 = e_2 - \frac{\gamma_4\gamma_1}{2\gamma_3}.$$

Using the following transformations of state variables and time variables as

$$\rho_1 = \frac{\gamma_4^2}{\gamma_3}\zeta_1, \quad \rho_2 = \frac{\gamma_4^2}{\gamma_3^2}\zeta_2, \tau = \frac{\gamma_3}{\gamma_4}t,$$

and still denoting τ as t, system (3.18) becomes

$$\begin{cases} \dfrac{d\rho_1}{dt} = \rho_2, \\ \dfrac{d\rho_2}{dt} = \mu_1 + \mu_2\rho_2 + \rho_1^2 + \rho_1\rho_2 + R_6(\theta_1, \theta_2, \rho_1, \rho_2)), \end{cases} \tag{3.19}$$

where

$$\mu_1 = \frac{\gamma_0\gamma_4^4}{\gamma_3^3}, \quad \mu_2 = \frac{\gamma_2\gamma_4}{\gamma_3}.$$

From the above calculation of the coefficient parameters, μ_1 and μ_2 can be expressed in terms of θ_1 and θ_2 as follows:

$$\begin{cases} \mu_1 = \kappa_{11}\theta_1 + \kappa_{12}\theta_2 + \kappa_{13}\theta_1^2 + \kappa_{14}\theta_1\theta_2 + \kappa_{15}\theta_2^2 + K_1(\theta_1, \theta_2), \\ \mu_2 = \kappa_{21}\theta_1 + \kappa_{22}\theta_2 + \kappa_{23}\theta_1^2 + \kappa_{24}\theta_1\theta_2 + \kappa_{25}\theta_2^2 + K_2(\theta_1, \theta_2), \end{cases}$$

where

$$\begin{aligned}
\kappa_{11} &= \frac{m_0\gamma_4^4 c_{12}}{\gamma_3^3}, \quad \kappa_{12} = \frac{\gamma_4^4 c_{12}}{\gamma_3^3}\Big(\frac{2m_0 - \Lambda_0 q_0 - \Lambda_0^2 + \Lambda_0 q_0 w_0 - 2m_0 w_0}{\Lambda_0 + q_0 w_0 - q_0}\Big), \\
\kappa_{13} &= -\frac{m_0^2\gamma_4^4 c_{14}^2}{8\gamma_3^5}(24\gamma_3(c_{14} + c_{12}) + c_{12}), \\
\kappa_{14} &= \frac{m_0\gamma_4^4 c_{14}}{8\gamma_3^5}\Big(c_{12}(24\gamma_3 - 1) - c_{14}\frac{2m_0 - \Lambda_0 q_0 - \Lambda_0^2 + \Lambda_0 q_0 w_0 - 2m_0 w_0}{\Lambda_0 + q_0 w_0 - q_0}\Big), \\
\kappa_{15} &= -\frac{\gamma_4^4}{8\gamma_3^5}\Big(c_{12} + c_{14}\frac{2m_0 - \Lambda_0 q_0 - \Lambda_0^2 + \Lambda_0 q_0 w_0 - 2m_0 w_0}{\Lambda_0 + q_0 w_0 - q_0}\Big)^2, \\
\kappa_{21} &= \frac{m_0\gamma_4^2 c_{14}}{2\gamma_3^3}, \quad \kappa_{22} = \frac{\gamma_4^2}{2\gamma_3^3}\Big(c_{12} + c_{14}\frac{2m_0 - \Lambda_0 q_0 - \Lambda_0^2 + \Lambda_0 q_0 w_0 - 2m_0 w_0}{\Lambda_0 + q_0 w_0 - q_0}\Big), \\
\kappa_{23} &= \frac{m_0^2\gamma_4^4 c_{14}^3(2\gamma_3 - 1)}{4\gamma_3^4 c_{12}}, \\
\kappa_{24} &= \frac{m_0\gamma_4^2 c_{14}^2}{4\gamma_3^4 c_{12}}\Big(c_{12}(3 + 2\gamma_3) + c_{14}(2\gamma_3 - 1)\frac{2m_0 - \Lambda_0 q_0 - \Lambda_0^2 + \Lambda_0 q_0 w_0 - 2m_0 w_0}{\Lambda_0 + q_0 w_0 - q_0}\Big), \\
\kappa_{25} &= \frac{\gamma_4^2 c_{14}(2\gamma_3 - 1)}{4\gamma_3^4 c_{12}}\Big(c_{14}^2(\frac{2m_0 - \Lambda_0 q_0 - \Lambda_0^2 + \Lambda_0 q_0 w_0 - 2m_0 w_0}{\Lambda_0 + q_0 w_0 - q_0})^2 - c_{12}^2\Big).
\end{aligned}$$

By calculation, we get

$$\left|\frac{\partial(\mu_1, \mu_2)}{\partial(\theta_1, \theta_2)}\right|_{(\theta_1,\theta_2)=(0,0)} = \frac{m_0\gamma_4^6 c_{12}^2}{2\gamma_3^6} \neq 0.$$

It shows that parameters μ_1 and μ_2 are independent and reversible in a small neighborhood of the origin of (θ_1, θ_2). By the results in Bogdanov [19] and Takens [20], we can see that the system (3.19) (i.e., (2.1)) undergoes the Bogdanov-Takens bifurcation when (θ_1, θ_2) changes in a small neighborhood of $(0, 0)$. That is the following theorem.

Theorem 17. *When* $\Lambda q < m < \dfrac{(q - qw - \Lambda)^2}{4(1 - w)} + \Lambda q, \Lambda > q - qw, p = p_*$, *and* $w = w_*$ *hold, the system* (2.1) *has a cusp* $E_*(x_*, y_*)$ *of codimension two (i.e., Bogdanov-Takens singularity). Thus, system* (2.1) *undergoes the Bogdanov-Takens bifurcation of codimension two around the endemic equilibrium* E_**. Specifically, if* Λ *and* m *are selected as the bifurcation parameters, the system has an unstable limit cycle for some parameter values of* Λ *and* m*, and it has an unstable homoclinic loop for some other parameter values of* Λ *and* m*.*

Based on the study of Perko [21], the approximate representations up to the second order of the bifurcation curves are given by the following.
(i) The saddle-node bifurcation curve is:

$$SN = \{(\mu_1, \mu_2)|\mu_1 = 0, \mu_2 \neq 0\}$$
$$= \{(\theta_1, \theta_2)|\kappa_{11}\theta_1 + \kappa_{12}\theta_2 + \kappa_{13}\theta_1^2 + \kappa_{14}\theta_1\theta_2 + \kappa_{15}\theta_2^2 = 0, \mu_2 \neq 0\},$$

(ii) The Hopf bifurcation curve is:

$$H = \{(\mu_1, \mu_2)|\mu_2 = \sqrt{-\mu_1}, \mu_1 < 0\} = \{(\theta_1, \theta_2)|\kappa_{21}\theta_1 + \kappa_{22}\theta_2 + \kappa_{23}\theta_1^2 + \kappa_{24}\theta_1\theta_2$$
$$+\kappa_{25}\theta_2^2 - \sqrt{-\kappa_{11}\theta_1 - \kappa_{12}\theta_2 - \kappa_{13}\theta_1^2 - \kappa_{14}\theta_1\theta_2 - \kappa_{15}\theta_2^2} = 0, \mu_1 < 0\},$$

(iii) The homoclinic bifurcation curve is:

$$HL = \{(\mu_1, \mu_2)|\mu_2 = \frac{5}{7}\sqrt{-\mu_1}, \mu_1 < 0\} = \{(\theta_1, \theta_2)|\kappa_{21}\theta_1 + \kappa_{22}\theta_2 + \kappa_{23}\theta_1^2 + \kappa_{24}\theta_1\theta_2$$
$$+\kappa_{25}\theta_2^2 - \frac{5}{7}\sqrt{-\kappa_{11}\theta_1 - \kappa_{12}\theta_2 - \kappa_{13}\theta_1^2 - \kappa_{14}\theta_1\theta_2 - \kappa_{15}\theta_2^2} = 0, \mu_1 < 0\}.$$

The bifurcation diagram and the corresponding phase portraits of the system (2.1) are presented below.
(i) In Figure 7, the small neighborhood of origin in the parameter (θ_1, θ_2)-plane is divided into four regions by bifurcation curves of SN(Saddle-Node), H(Hopf), and HL(Homoclinic).

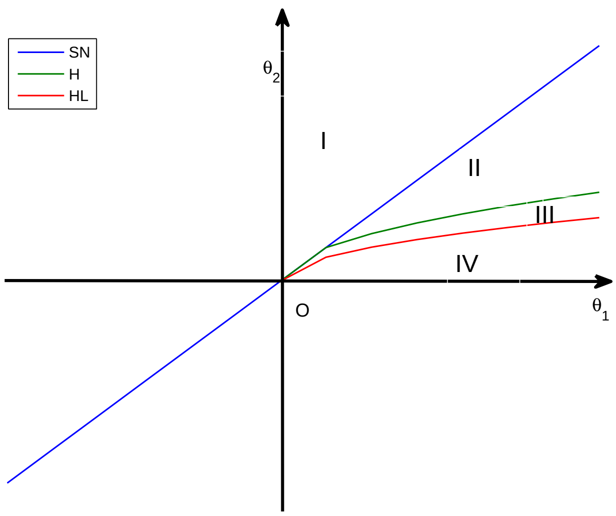


Figure 7. The bifurcation diagram in (θ_1, θ_2)-plane.

(ii) When $(\theta_1, \theta_2) = (0, 0)$, Figure 4 shows the system (2.1) having a unique endemic equilibrium point, which is a cusp of codimension two.

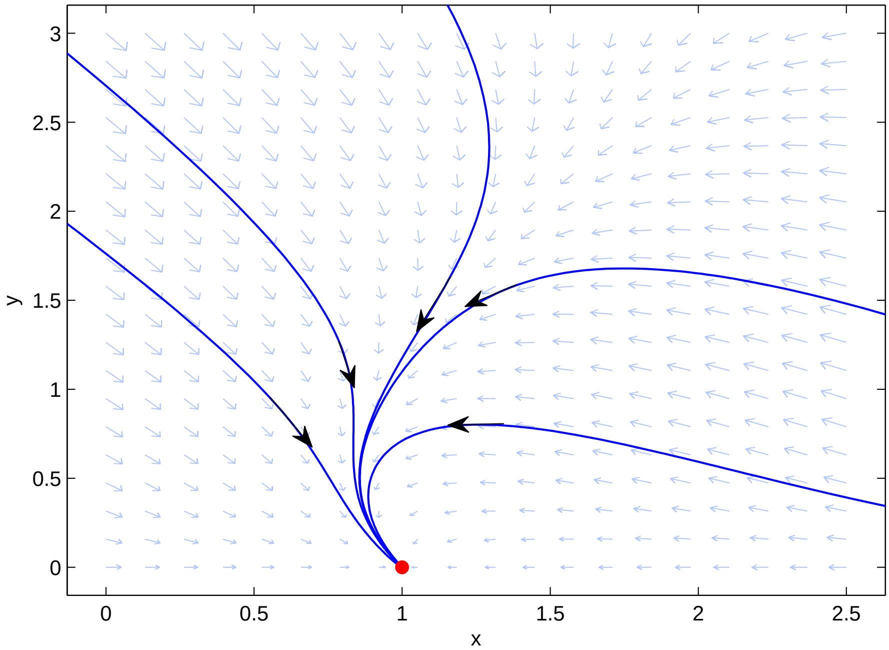


Figure 8. No endemic equilibrium.

(iii) If the perturbation parameters (θ_1, θ_2) fall in area I, system (2.1) does not exist in endemic equilibrium as shown in Figure 8, which means that the disease has been eliminated.

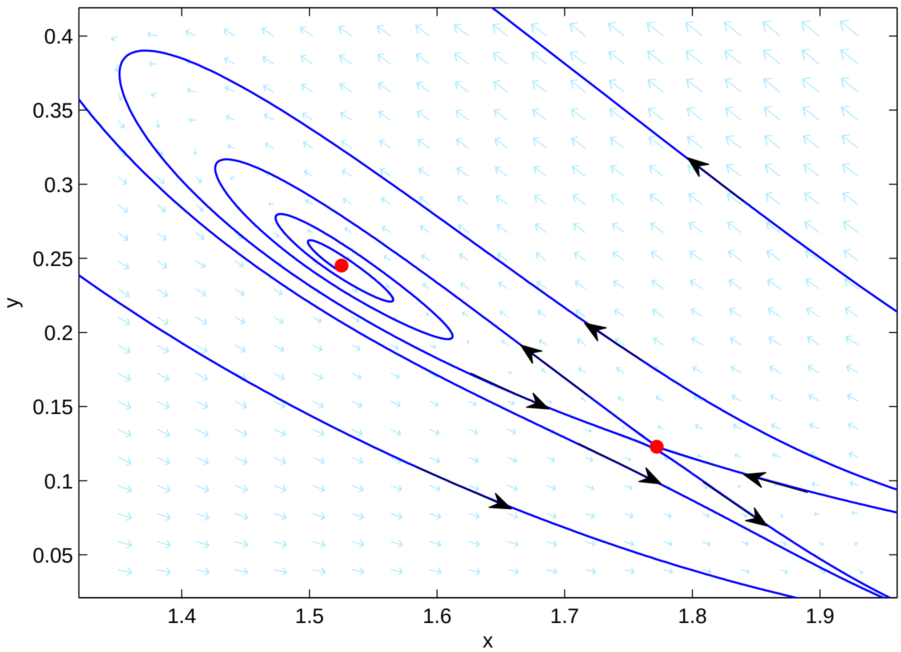


Figure 9. The stable focus and saddle.

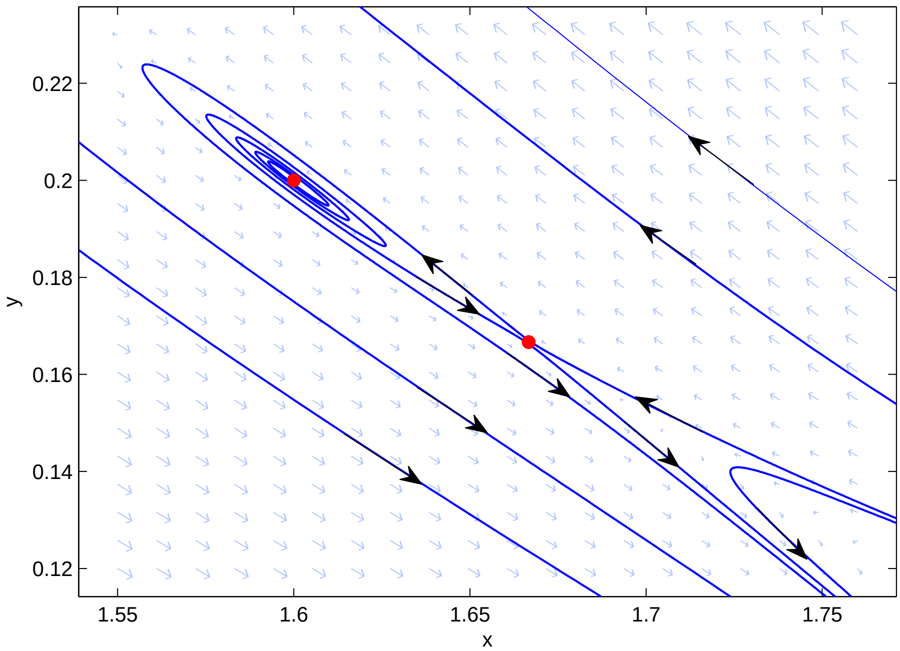


Figure 10. The unstable focus and saddle.

(iv) When perturbation parameters (θ_1, θ_2) fall on the curve SN, system (2.1) has a unique endemic equilibrium which is a saddle node. When the parameters cross through curve SN from region I to II or IV, the system undergoes a saddle-node bifurcation, and the number of endemic equilibrium changes from zero to two, which are shown in Figures 9 and 10, respectively.

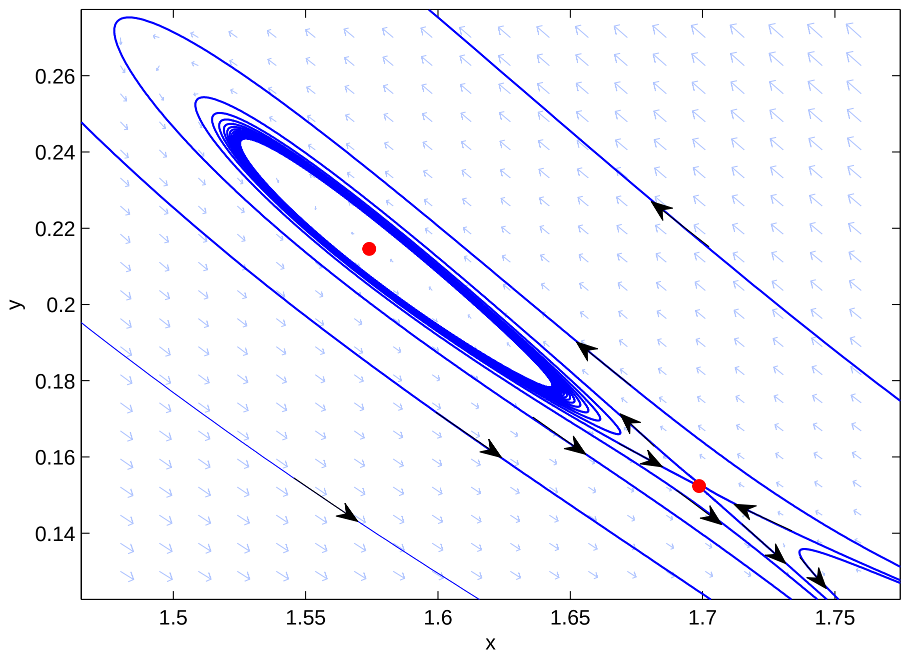


Figure 11. The unstable limit cycle.

(v) When perturbation parameters (θ_1, θ_2) cross curve H into area III, the system (2.1) undergoes a supercritical Hopf bifurcation of the endemic equilibrium point E_2 and produces an unstable limit cycle as displayed in Figure 11. If the initial value of (x, y) is within the limit cycle rather than in an endemic equilibrium E_2, the number of infectious individuals periodically varies.

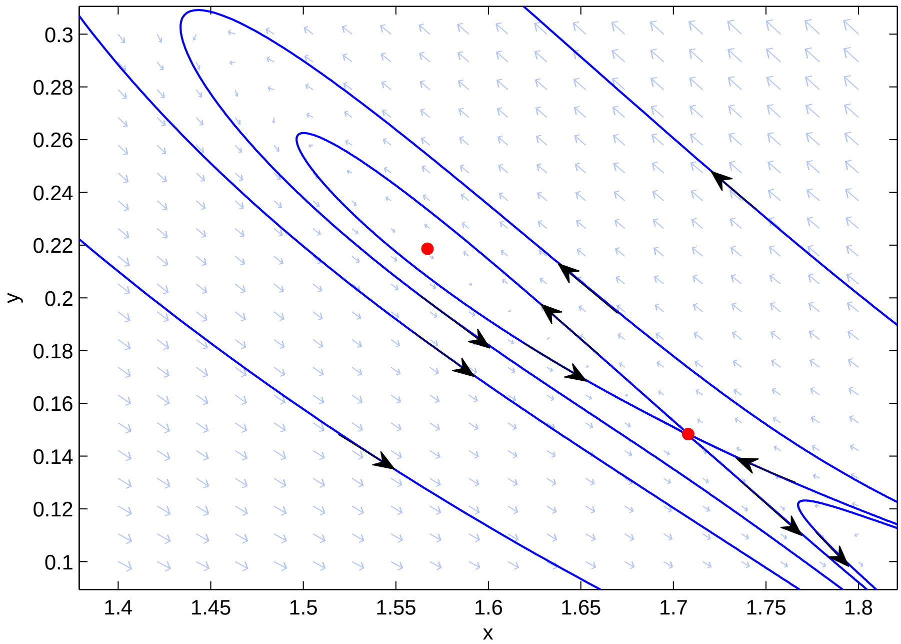


Figure 12. The homoclinic loop.

(vi) When the parameters (θ_1, θ_2) change from left to right in region III, the limit cycle becomes bigger and larger, then reaches equilibrium E_2 and becomes a homoclinic loop at the moment when (θ_1, θ_2) lies on the curve HL. After the parameter (θ_1, θ_2) is across the curve HL, the homoclinic loop breaks up. Figure 12 illustrates this phenomenon.

4. Conclusions

In this paper, the purpose of our research is to explore the influence of psychological reactions of people on the spread of disease. We choose the SIS model to describe the spread mechanism of some infectious diseases that do not give survivors immunity. In the model, the component that can reflect the influence of psychological behavior on the spread of disease is the incidence function. Many researchers have studied various incidence functions in SIS models.

Based on pioneering research work, in 2019, a generalized nonmonotone and saturated incidence function $\frac{kI^2}{1+\beta I+\alpha I^2}$ in the SIRS(Susceptible-Infectious-Recovered-Infectious) system was proposed by Lu et al. [7]; they considered that the incidence function, which describes the infection force, should not be just monotonic, nonmonotonic, or saturated, but a combination of monotonicity, nonmonotonicity, and saturation properties. Taking into account the psychological and crowding effect, the incidence function seemed reasonable to describe the infection force of some specific infectious diseases. However, we have found that the basic reproduction number of the SIRS model with the incidence rate $\frac{kI^2}{1+\beta I+\alpha I^2}$ is zero, but the disease can still be persistent; therefore, how to calculate the basic reproduction number of this model is still an open question. Therefore, the incidence rate is difficult to understand. Furthermore, the biological meaning of the parameter β is not defined, and the condition $\beta > -2\sqrt{\alpha}$ is mandatory.

So, we propose a more reasonable incidence rate $\frac{k_1 I + k_2 I^2}{1+aI^2}S$ with the combination of monotonicity, nonmonotonicity, and saturation properties. When $k_1 = 0$, the incidence function $f(I) = \frac{k_1 I + k_2 I^2}{1+aI^2}$ becomes the saturated incidence function in [22], which increases monotonously and then increases to $\frac{k_2}{a}$ as $I \to \infty$. When $k_1 > 0$, $f(I)$ increases first and then decreases to $\frac{k_2}{a}$ as $I \to \infty$, which describes the fact that the infection force of some infectious diseases grows rapidly to maximum as a new disease emerges or an old disease reemerges, and then trends to a value.

In this paper, we conducted a qualitative analysis. The basic reproduction number R_0 of the model (1.1) is $\frac{bk_1}{d(d+\varepsilon+\mu)}$, and we present that model (1.1) can undergo backward bifurcation if we take R_0 as the bifurcation parameter if $\Lambda > q - qw$. Backward bifurcation was proposed by Castillo-Chavez and Song [23] to illustrate that even if the basic reproduction number is $R_0 < 1$, disease outbreaks are still possible. The backward bifurcation has further epidemiological implications by providing a threshold R_a; when $\Lambda > q - qw$, model (1.1) shows bistable behavior (endemic equilibrium E_2 and disease-free equilibrium are stable) if $R_a < R_0 < 1$, and the model has a unique endemic equilibrium E_2 and the disease-free equilibrium becomes unstable if $R_0 > 1$. Moreover, a saddle-node bifurcation at the threshold R_a has been studied. In our paper, there is the basic reproduction number R_0, the threshold R_a, and the critical values k_1^* and k_{10} of k_1, which measure the linear infection risk to determine the dynamic of the system (2.1).

Briefly, when $R_0 \le 1$ and $\frac{k_1}{k_2}$, which measures the proportion of linear over nonlinear infection hazards, is greater than or equal to $\frac{b}{d+\varepsilon}$, the disease can be eliminated for all initial populations. When $R_0 > 1$ or $R_0 = 1$ and $\frac{k_1}{k_2} < \frac{b}{d+\varepsilon}$, the disease will persist in the form of multiple periodic coexistent oscillations bifurcated from the equilibrium E_2, or coexisting steady states for some initial populations. When $\frac{k_1}{k_2} < \frac{b}{d+\varepsilon}$, system (2.1) will present complex dynamics, including backward, saddle-node, Hopf bifurcation, etc. When $k_1^* < k_1 < k_{10}$, the disease will disappear if $R_0 < R_a$, but the disease will persist if $R_0 > R_a$, and for $R_0 = R_a$, these conditions are not enough to determine dynamical behaviors, which implies the disease will persist or die out, which depends on the values of independent parameters k_1 and k_2, and requires further evaluation.

Moreover, we have proved that the model undergoes Hopf bifurcation and undergoes Bogdanov-Takens bifurcation of, at most, codimension two, which indicates that the disease may exhibit multiple homeostasis and periodic outbreaks. However, our discussion of the nature of equilibrium E_2 is not exhaustive and needs further study. If the first Lyapunov coefficient l_1 is equal to zero, the equilibrium E_2 may be a weak focus with a multiplicity of at least two; meanwhile, if the parameters are suitable, there may be two limit cycles in system (2.1). Due to the complex computation, we will not discuss it here.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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