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Research article

Robust-observer design for nonlinear systems with delayed measurements using time-averaged Lyapunov stability criterion

Krishna Vijayaraghavan*

School of Mechatronic Systems Engineering, Simon Fraser University, 13450 102 Ave, Surrey, BC, V3B7J7 Canada

* **Correspondence:** Email: krishna@sfu.ca; Tel: +17788500320; Fax: +17787827514.

Abstract: This paper developed an observer design for a matrix-Lipchitz nonlinear system with measurement delay that can achieve a desired $\mathcal{L}_2$ performance in the presence of modeling uncertainties, input disturbance, and measurement noise. The observer was shown to be stable in the absence of disturbances and modeling uncertainties. The equations for the observer design were shown to be both necessary and sufficient. Furthermore, the observer design was formulated as linear matrix inequality (LMI) that can be solved offline using commercial solvers. Compared to previous literature, the proposed observer does not require the underlying system to be stable. The observer design procedure is demonstrated through two illustrative examples.

Keywords: time-averaged Lyapunov; nonlinear observer; LMI; measurement delay; robust observer

Notation

n, n_u, n_w, n_y are the number of states, inputs, input noise, and outputs, respectively.
q is the dimension of the nonlinearity ($q \leq n$).
$u \in \mathbb{R}^{n_u}$ is the vector of known inputs.
$\tilde{u}_{eq}$ is the equivalent output of the nonlinearity function.
$v \in \mathbb{R}^{n_y}$ is a vector of unknown measurement noise.
$w \in \mathbb{R}^{n_w}$ is a vector of unknown input disturbance.

$w_{ext} = col\big(w(t), x(t), \phi(t,u,x), u(t)\big)$ are the extended errors.

$x \in \mathbb{R}^n$ is the state vector.
$\hat{x} \in \mathbb{R}^n$ is the state estimate vector.
$\tilde{x} = x - \hat{x}$ is the state estimation error vector of the system.
x_d is the state of delayed measurement.
$\hat{x}_a$ is the approximation of $\hat{x}$ (when delay is approximated).
$\tilde{x}_a$ is the approximation of $\tilde{x}$ (when delay is approximated).
$y \in \mathbb{R}^{n_y}$ is the vector of measurement.
$\tilde{y}_{eq}$ is the equivalent input to the nonlinearity.
$y_{\bar{\delta}}$ is the effect of error in modeling the delay.
z_{eq} is the state of the augmented equivalent system.
$\mathcal{x}, \mathcal{y}, \mathcal{u}$ are the states, output, and input of system used in the proof of bounded real lemma.
$A, \Delta A \in \mathbb{R}^{n\times n}$ are the nominal system matrix and its deviation, respectively, with $A_{sys} = A + \Delta A$.
A_{Γ} is a delay system matrix.
A_d, B_d, and C_d are the system matrices of delay.
$\bar{A}, \bar{C}, \bar{L}, \bar{E}, \bar{F}_{ext}, \bar{G}$ are system matrices in extended state space.
$E, \Delta E \in \mathbb{R}^{n\times q}$ are the nominal nonlinearity gain matrix and its deviation, respectively, with $E_{sys} = E + \Delta E$.
$F \in \mathbb{R}^{n\times n_w}$ is the gain matrix for input disturbance.
$F_{ext} = [F\ \Delta A\ \Delta E\ \Delta H]$ is the extended error-gain matrix.
G is the matrix gain used in matrix Lipschitz condition.
$H, \Delta H \in \mathbb{R}^{n\times n_u}$ are the nominal input-gain matrix and its deviation, respectively, with $H_{sys} = H + \Delta H$.
L is the observer gain.
K is the degree of rational approximation of delay.
$\bar{P}$ is a Lyapunov matrix in the extended state-space.
$Q \in \mathbb{R}^{n_y\times n_y}, R_{ext} \in \mathbb{R}^{n_{wext}\times n_{wext}}$, and $W \in \mathbb{R}^{n\times n}$ are weighting matrices.
T is the time period of the moving time-averaged Lyapunov function.
$V_1: \mathbb{R}^{n_1} \to \mathbb{R}$ and $V_2: \mathbb{R}^{n_2} \to \mathbb{R}$, $V: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are arbitrary functions.
$V(t, x)$ is an average energy function.
$V_I(\tau, x)$ is an energy function (being averaged).
$\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are arbitrary matrices.
$\mathcal{G}(s)$ is the transfer function from the output to the input of the nonlinearity.
$\mathcal{G}_d(s)$ is the transfer function of delay.
$\mathcal{G}_{d'}(s)$ is the approximate rational transfer function of delay.
$\mathcal{U}, \mathcal{V}, \mathcal{W}, \mathcal{Z}$ are arbitrary matrices used in the matrix inverse lemma.
$\mathcal{V}$ is an arbitrary energy function used in the proof of bounded real lemma.
$\mathcal{V}_t \coloneqq \partial \mathcal{V}(t, x)/\partial t, \mathcal{V}_x \coloneqq \partial \mathcal{V}(t, x)/2\partial x$ are partial derivatives of $\mathcal{V}$.
$\mathcal{T}$ is the inverse-projection matrix with $\mathcal{T}^T = \left[\bar{0}^T_{Kn_y\times n}\ \ I_{n\times n}\right]^T$.
γ is H_∞ norm of the system used in the proof of bounded real lemma.
γ_C is the effective bound on the error dynamics.

$\varepsilon, \epsilon_\phi$ is an arbitrary constant.

δ_{Gd} is the approximation error transfer function.
$\bar{\delta}_{Gd}$ is an equivalent approximation error transfer function.

$\varsigma_k, \bar{\varsigma}_k$ are Taylor coefficients.
$\phi = \phi(t, u, x): \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^q$ is the vector nonlinearity.
$\tilde{\phi} = \tilde{\phi}(t, u, \hat{x}, \tilde{x}) = \phi(t, u, x) - \phi(t, u, \hat{x})$ is the error in the nonlinearity.
ω is the frequency.
ω_{max} is the frequency where gain of $\mathcal{G}_{NL}$ is maximum.
ω_δ is the frequency at which the gain of δ_{Gd} is maximum.
Γ is the time delay.
$\Theta(\omega)$ is the matrix frequency gain function of the observer error dynamics.
$\bar{\Theta}(\omega)$ is the matrix frequency gain function of the observer error dynamics with equivalent delay.
Ξ, Ω are arbitrary matrices in extended state space.

1. Introduction

All physical systems exhibit varying levels of nonlinearities. While some nonlinearities may be negligible, it is often necessary to implement a nonlinear controller to ensure stability and system performance [1]. Many advanced control techniques, such as backstepping and sliding mode control, use all system states [1]. Since not all system states are directly measured, it is often necessary to utilize a nonlinear observer to estimate unmeasured states. Physical systems also experience at least a small amount of delay between sensing and activation due to physical limits on processing and communication speed. Non-trivial amounts of delay are known to destabilize systems [2] and observers [3–5]. Most observer designs tend to neglect modeling uncertainties, unknown inputs, and measurement noises [6–10]. Hence, this paper focuses on developing observers for systems with measurement delays.

Lipschitz [11–15] and matrix Lipschitz [16,17] nonlinearities are the most common classes of nonlinearities considered in the literature. Here, the observer design problem is formulated as a solution to a linear matrix inequality (LMI). LMI-based formulation is particularly attractive owing to the availability of fast commercial solvers. Such LMI-based designs have been extended to delay-free Lipschitz and matrix Lipschitz nonlinear systems with input disturbances and measurement noise [6–8], as well as to perform sensor fault detection [18], actuator fault detection [19], and unknown parameter estimation [9,10]. Other literature has utilized high-gain sliding mode observers [20–22] to improve the robustness of systems without measurement noises or measurement delays. More recent literature has used a new time-averaged Lyapunov function to design sliding mode nonlinear observers for systems with sensor noise [23,24]. High-gain observers have also been used for extended state observers for uncertain nonlinear systems without sensor noise or measurement delay [25]. Fault reconstruction algorithms have also been extended to noise-free one-sided Lipchitz nonlinear descriptor systems [26]. However, these results require systems with special structures and cannot be extended to general systems.

The most common method for analyzing the stability of a system with delay is through the use of Lyapunov–Krasowskii (LK) or Lyapunov–Razumikhin (LR) functionals. LK and LR are constructed by adding a quadratic integral term to the traditional quadratic Lyapunov function. While there is ample literature on control [27–30] and parameter estimation [31–33] in systems with non-measurement delay (i.e., delay only in the state dynamics with, e.g., no measurement delay), this review instead focuses on the literature on measurement delay. Zhou et al. [34] proposed an observer for a stochastic linear system to implement non-fragile observer-based H_∞ control. Kazantzis and Wright [35] used a linearizing transform (potentially through feedback linearization) on a noise-free nonlinear system.

These works do not, however, provide an explicit method for calculating the observer gain. Cacace et al. [36] presented a state observer for drift observable nonlinear systems where the output measurements are affected by a known and bounded time-varying delay. Majeed et al. [3] developed a nonlinear observer-based control of a noise-free Lipschitz nonlinear system whose stability is demonstrated using an LK functional. He and Liu [4,5] designed a noise-free nonlinear observer for a system in feedback linearization. However, these works do not provide an explicit design procedure. Vafaei and Yazdanpanah [37] proposed a "chain observer" for a Lipschitz nonlinear system; again, the overall stability was demonstrated using the LF functional. Chakrabarty et al. [38] presented an LMI-based sufficient condition for the design of state and unknown input observers, which as in earlier results, requires the underlying system dynamics to be stable. Huong [39] proposed an observer design for a noise-free linear system with sensor delay without utilizing LK/LF functionals. However, the observer design relies on the existence of an esoteric state transformation that would be difficult to extend to nonlinear systems with noise.

More recently, Targui et al. [40] designed an observer for a noise-free nonlinear system adding a state-integral term. The state-integral term aims to project the sensor measurement to the current time. However, the need for an additional integration term would make the observer design impractical, and the conditions for the observer design are conservative. Furthermore, satisfying the provided condition does not appear to guarantee observer stability (a more detailed explanation is provided in the Appendix). Guarro et al. [41] proposed a hybrid observer for a linear system, which uses a similar state-integral term to project sensor measurements to current time. There have been other variations of LK/LF functionals for specific systems such as wind turbines [42] or using neural networks [43] for noise-free systems that also require inherent system stability.

It should be noted that all LMI-based formulations require us to find an observer gain that would make the LMI negative. The LK- and LR-based formulation used in previous literature typically result in a positive definite term being added to the LMI. In the case of observers, this implies that previous literature can only design stable observers when the underlying system is stable. Further, LK- and LR-based formulation would require the delay to be smaller than the time scale of the error dynamics [44]. Additionally, most previous results have focused on linear or nonlinear systems without any noise. Hence, this paper aims to develop an observer design that can be applied to any magnitude of delay. The observer design would make the nominal observer stable or would guarantee a $\mathcal{L}_2$ performance in the presence of modeling uncertainties, sensor noise, and input disturbance. The key contributions of this paper lie in developing a new observer design procedure that

- can be implemented on a very wide class of nonlinear systems in the presence of measurement noise and disturbances,
- works for large measurement delays,
- does not require stability of the underlying system,
- is robust to modeling uncertainties,
- can guarantee stability in the absence of noise and $\mathcal{L}_2$ performance in the presence of modeling uncertainties, sensor noise, and input disturbance,
- provides both necessary and sufficient conditions for the existence of the observer.

Additionally, the implementation of the algorithm has been demonstrated using multiple illustrative examples. The methodology for calculating the explicit solution can be extended to intermittent measurements. A previous observer design by Targui et al. [40] was shown to produce an unstable observer for certain systems.

2. Materials and methods

2.1. System model

Consider a nonlinear system with measurement delay given by

$$\dot{x}(t) = A_{sys}x(t) + E_{sys}\phi(t,u,x) + H_{sys}u(t) + Fw(t), \tag{2.1}$$

$$y(t) = Cx(t-\Gamma) + v(t), \tag{2.2}$$

$$A_{sys} = A + \Delta A, E_{sys} = E + \Delta E, H_{sys} = H + \Delta H, \tag{2.3}$$

v, w can be zero-mean white noises. The C-matrix is assumed to not deviate from its nominal value as any error in the C-matrix can be incorporated in $\Delta A, \Delta E,$ and ΔH. The nonlinearity $\phi = \phi(t,u,x)$: $\mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^q$ is assumed to satisfy a matrix Lipschitz condition

$$|\phi(t,u,x) - \phi(t,u,\hat{x})| \leq \left|G\left(x(t) - \hat{x}(t)\right)\right|, G \in \mathbb{R}^{q\times n}. \tag{2.4}$$

The matrix Lipschitz condition is a fairly general condition as all nonlinear systems would become matrix Lipschitz in a small neighborhood around their normal operating point. (Note: we may assume $\sigma_{max}(G) = 1$ or $\sigma_{max}(E) = 1$). We can construct an observer using the nominal system as follows:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + E\phi(t,u,\hat{x}) + L\left(y(t) - C\hat{x}(t-\Gamma)\right) + Hu(t), \tag{2.5}$$

The estimation error dynamics can be written as

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + F_{ext}w_{ext}(t) - LC\tilde{x}(t-\Gamma) - Lv(t) + E\tilde{\phi}(t,u,\hat{x},\tilde{x}). \tag{2.6}$$

We will now determine observer gains that can drive $\tilde{x}$ to an invariant subspace when $w \neq 0, v \neq 0$ and $[\Delta A\ \Delta E\ \Delta H] \neq 0$, or drive $\tilde{x} \to 0$ when $w = 0, v = 0$ and $[\Delta A\ \Delta E\ \Delta H] = 0$. To this end, we will define preliminary lemmas in section 2.2 that will help us formulate LMIs. Next, we will create a state-space approximation of the delay and define a bound on the approximation error in section 2.3.

2.2. Preliminary results

Lemma 1: S-Procedure Lemma [45]: If V_1: $\mathbb{R}^{n_1} \to \mathbb{R}$ and V_2: $\mathbb{R}^{n_2} \to \mathbb{R}$ be such that $V_2 \leq 0$, then $V_1 < 0$ iff $\exists \varepsilon > 0$ such that

$$V_1 - \varepsilon V_2 < 0. \tag{2.7}$$

Proof: If $V_1 < 0$, then we can choose $\varepsilon \leq \frac{\min(-V_1)}{\max(-V_2)}$ to obtain Eq (2.7). Now, if Eq (2.7) is valid, since $\varepsilon > 0$, and $V_2 \leq 0$, $-\varepsilon V_2 \geq 0$. Hence, we find that $0 > V_1 - \varepsilon V_2 \geq V_1$.

Lemma 2: Time-averaged Lyapunov function [23,24] or, equivalently, the moving average function: If there exists a moving average function

$$V(t,x) = \frac{1}{T}\int_{t-T}^{t} V_I(\tau,x)d\tau, \tag{2.8}$$

s.t., $V_I(\tau, x) > 0 \quad \forall x \neq \bar{0}$, $\frac{1}{\mathrm{T}}\int_{\mathrm{t-T}}^{t} \dot{V}_I(\tau, x) d\tau < 0$, when $\frac{1}{\mathrm{T}}\int_{\mathrm{t-T}}^{t} x^T W x d\tau > D$ then the system achieves a $\mathcal{L}_2$ performance $\frac{1}{\mathrm{T}}\int_{\mathrm{t-T}}^{t} x^T W x d\tau \leq D$.

Proof: Note that

$$V(t + dt) = \frac{1}{T}\int_{t+dt-T}^{t+dt} V_I(\tau) dt = \int_{t-T}^{t} V_I\big(\tau + dt, x(\tau + dt)\big) d\tau, \tag{2.9}$$

hence

$$\dot{V}(t) = \frac{1}{T}\int_{t-T}^{t} \dot{V}_I(\tau) d\tau. \tag{2.10}$$

Using LaSalle's invariance principle [46], we can show that the system converges to a subspace where $\dot{V} = 0$, which corresponds to $\frac{1}{\mathrm{T}}\int_{\mathrm{t-T}}^{t} x^T W x d\tau \leq D$.

Lemma 3: Matrix inverse lemma or Woodbury lemma [47]:

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathcal{A}^{-1} + \mathcal{A}^{-1}\mathcal{B}(\mathcal{D} - \mathcal{C}\mathcal{A}^{-1}\mathcal{B})^{-1}\mathcal{C}\mathcal{A}^{-1} & -\mathcal{A}^{-1}\mathcal{B}(\mathcal{D} - \mathcal{C}\mathcal{A}^{-1}\mathcal{B})^{-1} \\ (\mathcal{D} - \mathcal{C}\mathcal{A}^{-1}\mathcal{B})^{-1}\mathcal{C}\mathcal{A}^{-1} & (\mathcal{D} - \mathcal{C}\mathcal{A}^{-1}\mathcal{B})^{-1} \end{bmatrix}. \tag{2.11}$$

Lemma 4: Bounded real lemma extended to a system with delay: Consider a system with a transfer function

$$\mathcal{G}(s) = G\big[sI - \big(A + A_\Gamma \mathcal{G}_d(s)\big)\big]^{-1} E, \tag{2.12}$$

and an equivalent state space representation

$$\dot{x}(t) = Ax(t) + A_\Gamma x(t - \Gamma) + E u(t), y(t) = Gx(t). \tag{2.13}$$

Let $\exists \mathcal{V} = \mathcal{V}(t, x): \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be an energy function satisfying

$$\mathcal{V}(t, x) > 0 \ \forall x \neq \bar{0}, \mathcal{V}(t, x) = 0 \ if \ x = \bar{0}, \tag{2.14}$$

$$\mathcal{V}_t := \partial \mathcal{V}(t, x)/\partial t, \mathcal{V}_x := \partial \mathcal{V}(t, x)/2\partial x. \tag{2.15}$$

For this system, the following statements are equivalent:

i. The H_∞ norm of the transfer function is bounded by γ

$$\|\mathcal{G}(s)\|_\infty < \gamma. \tag{2.16}$$

ii. $\exists \mathcal{V}$ satisfying Eq (2.14) and $\epsilon_\phi \in \mathbb{R} \geq 0$ s.t., $\forall x$,

$$\begin{aligned} &\mathcal{V}_t + \mathcal{V}_x[Ax(t) + A_\Gamma x(t - \Gamma)] + [Ax(t) + A_\Gamma x(t - \Gamma)]^T \mathcal{V}_x^T \\ &\quad + \epsilon_\phi \gamma^2 x^T(t) G^T G x(t) + \epsilon_\phi^{-1} \mathcal{V}_x E E^T \mathcal{V}_x^T < 0 \end{aligned} \tag{2.17}$$

iii. $\exists \mathcal{V}$ satisfying Eq (2.14) and $\epsilon_\phi \in \mathbb{R} \geq 0$ s.t. $\forall x, u$

$$\mathcal{V}_t + \mathcal{V}_x[Ax(t) + A_\Gamma x(t-\Gamma) + Eu(t)] + [Ax(t) + A_\Gamma x(t-\Gamma) + Eu(t)]^T \mathcal{V}_x^T \\ +\epsilon_\phi x(t) G^T G x(t) - \epsilon_\phi \gamma^2 u^T(t) u(t) < 0 \tag{2.18}$$

iv. The system is stable with an input $|u| \le \gamma^{-1}|Gx|$.

Proof: Since $\left(E^T \mathcal{V}_x^T - \epsilon_\phi u(t)\right)^T \epsilon_\phi^{-1} \left(E^T \mathcal{V}_x^T - \epsilon_\phi u(t)\right) \ge 0$

$$-\epsilon_\phi^{-1} \mathcal{V}_x E E^T \mathcal{V}_x^T + u^T(t) E^T \mathcal{V}_x^T + \mathcal{V}_x E u(t) - \epsilon_\phi u^T(t) u(t) \le 0. \tag{2.19}$$

Adding Eq (2.19) to Eq (2.17) yields Eq (2.18); hence, ii ⇒ iii. If iii is valid, substituting $u(t) = E^T \mathcal{V}_x^T$ into Eq (2.18) yields Eq (2.17) ($u(t) = E^T \mathcal{V}_x^T$ corresponds to the value of u that optimizes Eq (2.18)). Hence, ii ⇔ iii. Now, from Eq (2.13), $Ax(t) + A_\Gamma x(t-\Gamma) + Eu(t) = \dot{x}(t)$; substituting $Ax(t) + A_\Gamma x(t-\Gamma) + Eu(t)$ with $\dot{x}(t)$ into Eq (2.18) and integrating the result yields

$$\mathcal{V}(t,x) + \epsilon_\phi \int_0^t \left(y^T(t) y(t) - \gamma^2 u^T(t) u(t)\right) d\tau < \mathcal{V}\left(0, x(0)\right). \tag{2.20}$$

Thus, when $|u| \le \gamma |y|$, we can choose $\mathcal{V}(t,x) + \epsilon_\phi \int_0^t \left(y^T(t) y(t) - \gamma^2 u^T(t) u(t)\right) d\tau > 0$ as our Lyapunov candidate to demonstrate stability. Hence, iii ⇔ iv.

When $x(0) = \bar{0}$, $\mathcal{V}\left(0, x(0)\right) = 0$. Since $\mathcal{V}(t,x) \ge 0$, we find that Eq (2.20) yields

$$\int_0^t \left(y^T(t) y(t) - \gamma^2 u^T(t) u(t)\right) dt < 0, \tag{2.21}$$

or ii-iv ⇒ i. To show i⇒iv, let us consider $|u| \le \gamma^{-1}|y|$. This is equivalent to interconnecting $\mathcal{G}(s)$ with another system $\|\mathcal{G}'\|_\infty \le \gamma^{-1}$. Since $\|\mathcal{G}\|_\infty \|\mathcal{G}'\|_\infty < 1$, it follows from the small gain theorem [46] that the interconnected system is stable. Thus i ⇒ iv.

2.3. Modeling feedback delay

2.3.1. State-space representation of the delay

A time delay can be approximated in the Laplace domain using K^{th}-degree numerator and denominator polynomials as follows [48]

$$\mathcal{G}_{d'}(s) = \{\textstyle\sum_{k=0}^K \varsigma_k (-s)^k\} / \{\sum_{k=0}^K \varsigma_k s^k\} = (-1)^K - \{\sum_{k=0}^K \bar{\varsigma}_k s^k\} / \{\sum_{k=0}^K \varsigma_k s^k\}, \tag{2.22}$$

where $\varsigma_k = (2K-k)!/[\Gamma^{K-k} k!\,(K-k)!]$, $\bar{\varsigma}_k = 2\varsigma_k \ if\ k\ is\ even, else\ 0$.

Notice that owing to symmetry, $|\mathcal{G}_{d'}(j\omega)| = 1$. The state-space representation of the delay can now be constructed such that

$$\mathcal{G}_{d'} = (-1)^K + C_d[sI - A_d]^{-1} B_d. \tag{2.23}$$

In the controller canonical form, A_d, B_d, and C_d are given by

$$A_d = \begin{bmatrix} \bar{0} & I_{K-1} \\ \varsigma_0 & \varsigma_1 \cdots \varsigma_{K-1} \end{bmatrix}, B_d = \begin{bmatrix} \bar{0} \\ 1 \end{bmatrix}, C_d = [-\bar{\varsigma}_0 \cdots - \bar{\varsigma}_{K-1}]. \tag{2.24}$$

If $\tilde{x}$ is known, then the approximation of the delayed measurement $x_d \approx C\tilde{x}(t-\Gamma)$ is written as

$$\dot{x}_d(t) = A_d x_d(t) + B_d C\tilde{x}(t). \tag{2.25}$$

In order to construct the overall approximate dynamics, we also need to replace $\tilde{x}$ with $\tilde{x}_a$. Hence, the approximate observer dynamics can be written as

$$\begin{aligned} &\dot{x}_d(t) = A_d x_d(t) + B_d C\tilde{x}_a(t) \\ &\dot{\tilde{x}}_a(t) = A\tilde{x}_a(t) + Fw(t) - L[(-1)^K C\tilde{x}_a(t) + C_d x_d(t)] - Lv(t) + E\tilde{\phi}(t,u,\hat{x}_a,\tilde{x}_a)' \end{aligned} \tag{2.26}$$

where $\tilde{x}_a$ is the approximation of $\tilde{x}$, and $\hat{x}_a$ is the approximation of $\hat{x}$. In the extended space,

$$\dot{z}(t) = (\bar{A} - \bar{L}\bar{C})z(t) + \bar{E}\tilde{\phi}(t,u,\hat{x},\tilde{x}) + \bar{F}w(t) - \bar{L}v(t), \tag{2.27}$$

where

$$z = \begin{bmatrix} x_d \\ \tilde{x}_a \end{bmatrix}, \bar{A} = \begin{bmatrix} A_d & B_d C \\ 0 & A \end{bmatrix}, \bar{C} = [C_d (-1)^K C], \tag{2.28}$$

$$\bar{L} = \mathcal{T}L, \bar{E} = \mathcal{T}E, \bar{F}_{ext} = \mathcal{T}F_{ext}, \bar{G} = G\mathcal{T}^T, \tag{2.29}$$

$$\mathcal{T} = \begin{bmatrix} \bar{0}_{Kn_y \times n} \\ I_{n\times n} \end{bmatrix}. \tag{2.30}$$

This can be easily extended to multiple sensors by stacking the state-space representation of the delay for the individual sensors.

2.3.2. Approximation error

Let us define the approximation error as

$$\delta_{Gd}(s) := \big(\mathcal{G}_d(s) - \mathcal{G}_{d'}(s)\big)\mathcal{G}_{d'}^{-1}(s) = \mathcal{G}_d(s)\mathcal{G}_{d'}^{-1}(s) - 1. \tag{2.31}$$

Figure 1 plots $|\delta_{Gd}(j\omega)|$ vs. $\Gamma\omega$ for various values of K.

Lemma 5: For $\delta_{Gd}(s)$ defined by Eq (2.31), $\mathcal{T}$ defined by Eq (2.30), $\bar{A}$, $\bar{L}$, and $\bar{C}$ defined by Eqs (2.28) and (2.29), and $\forall \Omega, \Xi$,

$$\Omega[sI - A + L\mathcal{G}_d(s)C]^{-1}\Xi = \Omega\mathcal{T}^T\left[sI - \big(\bar{A} - \bar{L}\big(1 + \delta_{Gd}(s)\big)\bar{C}\big)\right]^{-1}\mathcal{T}\Xi. \tag{2.32}$$

Proof: From Eqs (2.31) and (2.23), we can deduce that

$$\mathcal{G}_d(s) = \big(1 + \delta_{Gd}(s)\big)\mathcal{G}_{d'}(s) = \big(1 + \delta_{Gd}(s)\big)[(-1)^K + C_d(sI - A_d)^{-1}B_d], \tag{2.33}$$

Hence,

$$sI - A + L\mathcal{G}_d(s)C = (sI - A) + L\big(1 + \delta_{Gd}(s)\big)[(-1)^K + C_d(sI - A_d)^{-1}B_d]C \\ = \big[sI - A + (-1)^K L\big(1 + \delta_{Gd}(s)\big)C\big] - \big[L\big(1 + \delta_{Gd}(s)\big)C_d\big](sI - A_d)^{-1}[-B_dC] \tag{2.34}$$

Using Lemma 3 we find that

$$\left\{\big[sI - A + (-1)^K L\big(1 + \delta_{Gd}(s)\big)C\big] - \big[L\big(1 + \delta_{Gd}(s)\big)C_d\big](sI - A_d)^{-1}[-B_dC]\right\}^{-1} \\ = \mathcal{T}^T \begin{bmatrix} sI - A_d & -B_dC \\ L\big(1 + \delta_{Gd}(s)\big)C_d & sI - A + (-1)^K L\big(1 + \delta_{Gd}(s)\big)C \end{bmatrix}^{-1} \mathcal{T} \quad . \tag{2.35}$$

Substituting Eq (2.34) into Eq (2.35) and substituting for $\bar{A}, \bar{C}$, and $\bar{L}$ from Eq (2.28) yields Eq (2.32).

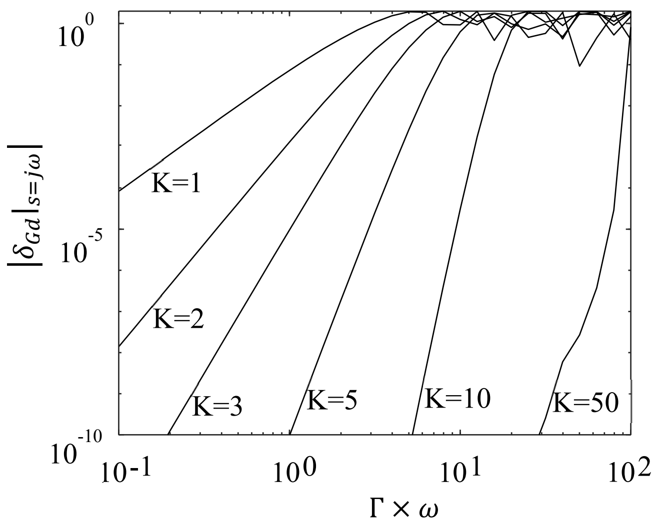

Figure 1. Plot of the magnitude of $\delta_{Gd}(s = j\omega)$ for different values of K.

2.3.3. Estimating the bound on the approximation error

We will now determine a bound on the approximation error that can then be used in the observer design. Since $|\mathcal{G}_{d'}(j\omega)| = |\mathcal{G}_d(j\omega)| = 1$, Eq (2.31) yields $\|\delta(s)\|_\infty = 2$. However, from Figure 1, it is clear that if we are only interested in frequencies below a particular threshold, it may be possible to define a lower effective bound for $\delta_{Gd}(j\omega)$.

Definition 1: The effective bound on the error of K^{th} Padé approximation is defined as $\gamma_C > 0$, s.t., $\forall \Omega, \Xi$ and $\bar{A}, \bar{C}, and\ \bar{L}$ as defined in Eq (2.28), $\exists \bar{\delta}_{Gd}$: $\left\|\bar{\delta}_{Gd}(s)\right\|_\infty \le \gamma_C$ such that,

$$\|\Omega[sI - A + L\mathcal{G}_d(s)C]^{-1}\Xi\|_\infty = \left\|\Omega\mathcal{T}^T\left[sI - \left(\bar{A} - \bar{L}\left(1 + \bar{\delta}_{Gd}(s)\right)\bar{C}\right)\right]^{-1}\mathcal{T}\Xi\right\|_\infty. \tag{2.36}$$

Lemma 6: Suppose $\mathcal{G}_{NL}(s) := \Omega[sI - A + L\mathcal{G}_d(s)C]^{-1}\Xi$, is strictly proper, then

$$\gamma_C \approx max_{\omega \le \omega_\delta} |\delta_{Gd}(j\omega)|, \tag{2.37}$$

where

$$\exists\omega_\delta \gtrsim \omega_{max}\ s.t., \forall\omega \geq \omega_\delta, |\mathcal{G}_{NL}(j\omega)| \leq \frac{\|\mathcal{G}_{NL}\|_\infty}{1+4|\bar{L}||\bar{C}|}. \quad (2.38)$$

Proof: Let ω_{max} be the frequency at which the H_∞ norm is reached or $|\mathcal{G}_{NL}(j\omega_{max})| = \|\mathcal{G}_{NL}\|_\infty$. Since $\mathcal{G}_{NL}(s)$ is strictly proper, $|\mathcal{G}_{NL}(j\omega)| \to 0$ as $\omega \to 0$. Hence, $\exists\omega_\delta \gtrsim \omega_{max}$ s.t. $\forall\omega \geq \omega_\delta$, $|\mathcal{G}_{NL}(j\omega)| \leq \|\mathcal{G}_{NL}\|_\infty/(1 + 4|\bar{L}||\bar{C}|)$. Now, consider

$$\Theta(\omega) \coloneqq \left[(j\omega I - \bar{A}) + \bar{L}\big(1 + \delta_{Gd}(j\omega)\big)\bar{C}\right]^{-1}, \quad (2.39)$$

$$\bar{\delta}_{Gd}(j\omega) = \begin{cases} \delta_{Gd}(j\omega) & \omega \leq \omega_\delta \\ \delta_{Gd}(j\omega_\delta) & \omega > \omega_\delta \end{cases}, \quad (2.40)$$

$$\bar{\Theta}(\omega) \coloneqq \left[sI - \left(\bar{A} - \bar{L}\left(1 + \bar{\delta}_{Gd}(s)\right)\bar{C}\right)\right]^{-1}. \quad (2.41)$$

Let $\Delta\delta_{\mathcal{G}d} = \bar{\delta}_{Gd}(j\omega) - \delta_{Gd}$. Hence

$$\bar{\Theta}(\omega) \coloneqq \left[(j\omega I - \bar{A}) + \bar{L}\big(1 + \delta_{Gd}(j\omega) + \Delta\delta_{\mathcal{G}d}\big)\bar{C}\right]^{-1}. \quad (2.42)$$

Now

$$(j\omega I - \bar{A}) + \bar{L}\big(1 + \delta_{Gd}(j\omega)\big)\bar{C} = (j\omega I - \bar{A}) + \bar{L}\big(1 + \delta_{Gd}(j\omega) + \Delta\delta_{\mathcal{G}d}\big)\bar{C} - \bar{L}\Delta\delta_{\mathcal{G}d}\bar{C}. \quad (2.43)$$

Substituting Eqs (2.39) and (2.42) into Eq (2.43) and simplifying yields,

$$\bar{\Theta}(\omega) = \left[I - \bar{L}\Delta\delta_{\mathcal{G}d}\bar{C}\right]^{-1}\Theta(\omega) \approx \left[I + \bar{L}\Delta\delta_{\mathcal{G}d}\bar{C}\right]\Theta(\omega). \quad (2.44)$$

Since $\|\delta(s)\|_\infty = 2$, for $\forall\omega \geq \omega_\delta$, $\left|\Delta\delta_{\mathcal{G}d}(\omega)\right| \leq 4$. From Eq (2.38), $\bar{\Theta}(\omega) < \|\mathcal{G}_{NL}\|_\infty \forall\omega \geq \omega_\delta$.

Hence, we are only interested in $\left|\bar{\delta}_{Gd}(j\omega)\right|$ for $\omega \leq \omega_\delta$. Since for $\omega \leq \omega_\delta$, $\bar{\delta}_{Gd}(j\omega) = \delta_{Gd}(j\omega)$, it follows that $\gamma_C \approx |\delta_{Gd}(j\omega_\delta)|$.

To illustrate the utility of Lemma 6, suppose it is known that $\omega_\delta = 0.1/\Gamma$, then for $K = 2$, the effective bound on $|\delta_{Gd}(j\omega)|$ is 2×10^{-10}.

Note: We can choose $\omega_\delta \approx -Re\left(\lambda_{\bar{A}-\bar{L}\bar{C}}^{min}\right)$.

Note: In principle, γ_C can be made arbitrarily small by increasing *K*. However, notice that $\varsigma_0 = (2K)!/[\Gamma^K K!]$, and $\varsigma_{K-1} = (K+1)K/\Gamma$; as we increase *K*, the variation between the smallest and the largest values of ς_k will increase and $\bar{A}$ and $\bar{C}$ will become less well conditioned. While there is no strict upper limit for *K*, a very large *K* may cause computation problems.

2.4. Observer stability

Theorem 7: The noise-free, disturbance-free, and uncertainty-free nominal error dynamics, Eq (2.6) with $w = 0, v = 0$ and $[\Delta A\ \Delta E\ \Delta H] = 0$, is stable $\forall \phi$ satisfying Eq (2.4) if, for the K^{th} Padé approximation of the delay, $\exists \bar{P} \in \mathbb{R}^{(n+Kn_y)\times(n+Kn_y)} > 0, \epsilon_\phi > 0$, and $\epsilon_C > 0$ s.t.,

$$\begin{bmatrix} N & \bar{P}\bar{E} & \bar{P}\bar{L} \\ \bar{E}^T\bar{P} & -\epsilon_\phi I & \bar{0} \\ \bar{L}^T P & \bar{0} & -\epsilon_C I \end{bmatrix} < 0, \tag{2.45}$$

where

$$N = \bar{A}^T\bar{P} + \bar{P}\bar{A} - \bar{P}\bar{L}\bar{C} - \bar{C}^T\bar{L}^T P + \epsilon_\phi \bar{G}^T\bar{G} + \epsilon_C \gamma_C^2 \bar{C}^T\bar{C}, \tag{2.46}$$

and $\bar{0}$-s are zero matrices of appropriate dimensions, $\bar{A}, \bar{C}, \bar{L}, \bar{E}$ *and* $\bar{G}$ are defined in Eq (2.28), and $\gamma_C > 0$ is the effective bound on the error dynamics as defined in Definition 1.

Further, conditions Eq (2.45) become necessary and sufficient as $\gamma_C \to 0$.

Proof: For $\Xi = \mathrm{E}$ and $\Omega = \mathrm{G}$ in Definition 1, $\exists \left\|\bar{\delta}_{Gd}(s)\right\|_\infty \leq \gamma_C$

$$\|G[sI - A + L\mathcal{G}_d(s)C]^{-1}E\|_\infty = \left\|\bar{G}\left[sI - \left(\bar{A} - \bar{L}\left(1 + \bar{\delta}_{Gd}(s)\right)\bar{C}\right)\right]^{-1}\bar{E}\right\|_\infty. \tag{2.47}$$

For such a $\bar{\delta}_{Gd}$, the state-space form of $\bar{G}\left[sI - \left(\bar{A} - \bar{L}\left(1 + \bar{\delta}_{Gd}(s)\right)\bar{C}\right)\right]^{-1}\bar{E}$ can be written as

$$\dot{z}_{eq}(t) = (\bar{A} - \bar{L}\bar{C})z_{eq}(t) - \bar{L}y_{\bar{\delta}}(t) + \bar{E}\tilde{u}_{eq}(t), \tag{2.48}$$

$$y_{\bar{\delta}}(t) = \mathcal{L}^{-1}\left[\bar{\delta}_{Gd}(s)\mathcal{L}\left(\bar{C}z_{eq}(t)\right)\right], \left\|\bar{\delta}_{Gd}(s)\right\|_\infty \leq \gamma_C, \tag{2.49}$$

$$\tilde{y}_{eq}(t) = \bar{G}z_{eq}(t). \tag{2.50}$$

Multiplying Eq (2.45) by $\left[z_{eq}^T\ \tilde{u}_{eq}^T\ y_{\bar{\delta}}^T\right]$ on the left and $col\left(z_{eq}\ \tilde{u}_{eq}\ y_{\bar{\delta}}\right)$ on the right,

$$\dot{V} \coloneqq \begin{bmatrix} z_{eq}(t) \\ \tilde{u}_{eq}(t) \\ y_{\bar{\delta}}(t) \end{bmatrix}^T \begin{bmatrix} N & \bar{P}\bar{E} & \bar{P}\bar{L} \\ \bar{E}^T\bar{P} & -\epsilon_\phi I & 0 \\ \bar{L}^T P & 0 & -\epsilon_C I \end{bmatrix} \begin{bmatrix} z_{eq}(t) \\ \tilde{u}_{eq}(t) \\ y_{\bar{\delta}}(t) \end{bmatrix} < 0. \tag{2.51}$$

Noting that $\bar{A}z_{eq}(t) - \bar{L}\bar{C}z_{eq}(t) - \bar{L}y_{\bar{\delta}}(t) + \bar{E}\tilde{u}_{eq}(t) = \dot{z}_{eq}(t)$ (from Eq (2.48)), we find

$$\begin{aligned} z_{eq}^T(t)\bar{P}\dot{z}_{eq}(t) + \dot{z}_{eq}^T(t)\bar{P}z_{eq}(t) + \epsilon_\phi\left[\tilde{y}_{eq}^T(t)\tilde{y}_{eq}(t) - \tilde{u}_{eq}^T(t)\tilde{u}_{eq}(t)\right] \\ + \epsilon_C\left[\gamma_C^2 z_{eq}^T(t)\bar{C}^T\bar{C}z_{eq}(t) - y_{\bar{\delta}}^T(t)y_{\bar{\delta}}(t)\right] < 0^{\cdot} \end{aligned} \tag{2.52}$$

Integrating the above equation yields

$$\begin{aligned} &z_{eq}^T(t)\bar{P}z_{eq}(t) + \epsilon_\phi \int_0^t \left[\tilde{y}_{eq}^T(\tau)\tilde{y}_{eq}(\tau) - \tilde{u}_{eq}^T(\tau)\tilde{u}_{eq}(\tau)\right]d\tau \\ &+\epsilon_C \int_0^t \left[\gamma_C^2 z_{eq}^T(t)\bar{C}^T\bar{C}z_{eq}(t) - y_{\bar{\delta}}^T(\tau)y_{\bar{\delta}}(\tau)\right]d\tau + z_{eq}^T(0)\bar{P}z_{eq}(0) < 0 \end{aligned}. \tag{2.53}$$

Since $\left\|\bar{\delta}_{Gd}(s)\right\|_\infty \leq \gamma_C$,

$$\int_0^t \left[y_{\bar{\delta}}^T(\tau)y_{\bar{\delta}}(\tau) - \gamma_C^2 z_{eq}^T(t)\bar{C}^T\bar{C}z_{eq}(t)\right]d\tau \leq 0. \tag{2.54}$$

Applying Lemma 1 to Eqs (2.53) and (2.54), we find

$$z_{eq}^T(t)\bar{P}z_{eq}(t) + \epsilon_\phi \int_0^t \left[\tilde{y}_{eq}^T(\tau)\tilde{y}_{eq}(\tau) - \tilde{u}_{eq}^T(\tau)\tilde{u}_{eq}(\tau)\right]d\tau + z_{eq}^T(0)\bar{P}z_{eq}(0) < 0. \tag{2.55}$$

Thus when $z_{eq}(0) = 0$, $\int_0^t \tilde{y}_{eq}^T(\tau)\tilde{y}_{eq}(\tau)d\tau < \int_0^t \tilde{u}_{eq}^T(\tau)\tilde{u}_{eq}(\tau)d\tau$ or,

$$\left\|\bar{G}\left[sI - \left(\bar{A} - \bar{L}\left((-1)^K + \bar{\delta}_{Gd}(s)\right)\bar{C}\right)\right]^{-1}\bar{E}\right\|_\infty < 1. \tag{2.56}$$

From Eqs (2.47) and (2.57), we can deduce that

$$\|G[sI - A + L\mathcal{G}_d(s)C]^{-1}E\|_\infty < 1. \tag{2.57}$$

Replacing u with $\tilde{\phi}$ and A_Γ with $-LC$, in Lemma 4, it is clear that Eq (2.57) $\Leftrightarrow$ $\exists\mathcal{V} = \mathcal{V}(t,\tilde{x})\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ satisfying $\mathcal{V}(t,\tilde{x}) > 0\ \forall\tilde{x} \neq \bar{0}, \mathcal{V}(t,\bar{0}) = 0$, and $\epsilon_\phi \in \mathbb{R} \geq 0$ s.t $\forall\ \tilde{x}, \tilde{\phi}$

$$\frac{d}{dt}\left\{\mathcal{V}(t,\tilde{x}) + \epsilon_\phi \int_0^t \left[\tilde{x}^T(t)G^TG\tilde{x}(t) - \tilde{\phi}^T(t,u,\hat{x},\tilde{x})\tilde{\phi}(t,u,\hat{x},\tilde{x})\right]d\tau\right\} < 0. \tag{2.58}$$

Since $\phi(t,u,\hat{x},\tilde{x})$ satisfies Eq (2.4),

$$\int_0^t \left[\tilde{x}^T(t)G^TG\tilde{x}(t) - \tilde{\phi}^T(t,u,\hat{x},\tilde{x})\tilde{\phi}(t,u,\hat{x},\tilde{x})\right]d\tau \geq 0. \tag{2.59}$$

Let us define a candidate Lyapunov function for the system Eq (2.6) as

$$V \coloneqq \mathcal{V}(t,\tilde{x}) + \epsilon_\phi \int_0^t \left[\tilde{x}^T(t)G^TG\tilde{x}(t) - \tilde{\phi}^T(t,u,\hat{x},\tilde{x})\tilde{\phi}(t,u,\hat{x},\tilde{x})\right]d\tau > 0. \tag{2.60}$$

Since $V > 0\ \forall\tilde{x} \neq \bar{0}$, and $\dot{V} < 0$, the system Eq (2.6) is stable.

On the limit $\gamma_C \to 0$,

$$\|G[sI - A + L\mathcal{G}_d(s)C]^{-1}E\|_\infty = \|\bar{G}[sI - (\bar{A} - \bar{L}\bar{C})]^{-1}\bar{E}\|_\infty. \tag{2.61}$$

If on the limit $\gamma_C \to 0$, $\nexists\bar{P}$ satisfying Eq (2.45), then $\|G[sI - A + L\mathcal{G}_dC]^{-1}E\|_\infty > 1$. Subsequently, $\exists\phi$, for instance $\phi = \sin(Gx)$ that would make the observer unstable. Hence the condition becomes necessary when $\gamma_C \to 0$.

Note: It is clear from the above theorem that we need to make γ_C as small as possible. Without the concept of effective bound, however, we would need to use $\gamma_C = 2$ (since $\bar{\delta} = \delta$ and $\|\delta(s)\|_\infty = 2$).

Lemma 8: Given weighting matrices $W \in \mathbb{R}^{n\times n}$, $R_{ext} \in \mathbb{R}^{n_{wext}\times n_{wext}}$, and $Q \in \mathbb{R}^{n_y\times n_y}$, the observer Eq (2.5) can eventually guarantee the following performance

$$\int_{t-T}^{t} \tilde{x}^T(\tau)W\tilde{x}(\tau)d\tau \le \int_{t-T}^{t}\left(v^T(\tau)Qv(\tau) + w_{ext}^T(\tau)R_{ext}w_{ext}(\tau)\right)d\tau, \tag{2.62}$$

for all nonlinearities ϕ satisfying Eq (2.4), if for the K^{th} Padé approximation of the delay $\exists\bar{P} > 0$, $\epsilon_\phi > 0, \epsilon_C > 0$, and $\epsilon_W > 0$ s.t.,

$$\begin{bmatrix} N & \bar{P}\bar{E} & \bar{P}\bar{F}_{ext} & \bar{P}\bar{L} & \bar{P}\bar{L} \\ \bar{E}^T\bar{P} & -\epsilon_\phi I & \bar{0} & \bar{0} & \bar{0} \\ \bar{F}_{ext}^T\bar{P} & 0 & -\epsilon_W R & \bar{0} & \bar{0} \\ \bar{L}^T P & \bar{0} & \bar{0} & -\epsilon_W Q & \bar{0} \\ \bar{L}^T P & \bar{0} & \bar{0} & \bar{0} & -\epsilon_C I \end{bmatrix} < 0, \tag{2.63}$$

where $N = \epsilon_W \mathcal{J}W^TW\mathcal{J}^T + \bar{A}^T\bar{P} + \bar{P}\bar{A} - \bar{P}\bar{L}\bar{C} - \bar{C}^T\bar{L}^T P + \epsilon_\phi\gamma_\phi^2\bar{G}^T\bar{G} + \epsilon_C\gamma_C^2\bar{C}^T\bar{C}$, $\bar{0}$ is a zero matrix, $\bar{A}, \bar{C}, \bar{L}, \bar{E}, and\ \bar{G}$ are the systems matrices for the error dynamics defined in Eq (2.28), and $\gamma_C > 0$ is the effective bound on the approximation error as defined in Definition 1.

Further, Eq (2.63) becomes necessary and sufficient on the limit $\gamma_C \to 0$.

Proof: Let $\mathbb{E} := \left[E \;\; F_{ext}(\epsilon_W/\epsilon_\phi R)^{-1/2} \;\; L(\epsilon_W/\epsilon_\phi Q)^{-1/2}\right]$ and $\mathbb{G} := col\left(G, (\epsilon_W/\epsilon_\phi)^{1/2}W\right)$. By multiplying Eq (2.63) by $diag\left(I, I, \left(\frac{\epsilon_W}{\epsilon_\phi R}\right)^{-1/2}, \left(\frac{\epsilon_W}{\epsilon_\phi}Q\right)^{-1/2}, I\right)$ on both sides, we find

$$\begin{bmatrix} N & \bar{P}\mathcal{J}\mathbb{E} & \bar{P}\bar{L} \\ \mathbb{E}^T\mathcal{J}^T\bar{P} & -\epsilon_\phi\begin{bmatrix} I & \bar{0} & \bar{0} \\ \bar{0} & I & \bar{0} \\ \bar{0} & \bar{0} & I \end{bmatrix} & \begin{bmatrix} \bar{0} \\ \bar{0} \\ \bar{0} \end{bmatrix} \\ \bar{L}^T P & [\bar{0} \;\; \bar{0} \;\; \bar{0}] & -\epsilon_C I \end{bmatrix} < 0. \tag{2.64}$$

Following the proof of Theorem 7, we can show that

$$\|\mathbb{G}[sI - A + L\mathcal{G}_d(s)C]^{-1}\mathbb{E}\|_\infty < 1. \tag{2.65}$$

Replacing u with $col\left(\tilde{\phi}, \left(\frac{\epsilon_W}{\epsilon_\phi}Q\right)^{1/2} v, (\epsilon_W/\epsilon_\phi R_{ext})^{1/2} w_{ext}\right)$, G with $\mathbb{G}$ and E with $\mathbb{E}$ in Lemma 4, we find $\exists\mathcal{V}$ satisfying Eq (2.14), s.t.,

$$\dot{\mathcal{V}} + \epsilon_\phi\left[\tilde{x}^TG^TG\tilde{x} - \tilde{\phi}^T\tilde{\phi}\right] + \epsilon_W[\tilde{x}^TW\tilde{x} - v^TQv - w_{ext}^TR_{ext}w_{ext}] < 0. \tag{2.66}$$

Hence, if we define

$$V_I(t,\tilde{x}) := \mathcal{V}(t,\tilde{x}) > 0, V = \int_{t-T}^{t} V_I(\tau,\tilde{x})d\tau. \tag{2.67}$$

Notice that $\dot{V} < 0$ when

$$\int_{t-T}^{t} \tilde{x}^T(\tau)W\tilde{x}(\tau)d\tau > \int_{t-T}^{t} [v^T Q v + w_{ext}^T R_{ext} w_{ext}]d\tau. \tag{2.68}$$

Hence, from Lemma 2, the system will converge to

$$\int_{t-T}^{t} \tilde{x}^T(\tau)W\tilde{x}(\tau)d\tau \le \int_{t-T}^{t} (v^T Q v + w_{ext}^T R_{ext} w_{ext})d\tau. \tag{2.69}$$

It also follows from the proof of Theorem 7 that the condition becomes necessary when $\gamma_C \to 0$.

Note: While not strictly necessary, if w and v are zero-mean white Gaussian noises, then we can choose $(R)^{-1}$ equal to the covariances of v and $(Q)^{-1}$ equal to the covariances of w.

Notice that Eq (2.63) ceases to be a linear matrix inequality if L is unknown. Hence, we will require an iterative procedure for determining L. Such an iterative method would require a method of evaluating the fitness of a chosen L.

Lemma 9: For the system

$$\begin{aligned} &\dot{z}_{eq}(t) = (\bar{A} - \bar{L}\bar{C})z_{eq}(t) + \bar{F}_{ext}w_{ext}(t) - \bar{L}y_{\bar{\delta}}(t) - \bar{L}v(t) + \bar{E}\tilde{u}_{eq}(t) \\ &y_{\bar{\delta}}(t) = \mathcal{L}^{-1}\left[\delta_{Gd}(s)\mathcal{L}\left(\bar{C}Z_{eq}(t)\right)\right],\ \left\|\bar{\delta}_{Gd}(s)\right\|_\infty \le \gamma_C \\ &\tilde{y}_{eq}(t) = \bar{G}z_{eq}(t) \end{aligned}. \tag{2.70}$$

If $\exists \bar{P}: I \ge \bar{P} > 0,\ \epsilon_\phi > 0, \epsilon_C > 0\ and\ \epsilon_W > 0$, s.t.,

$$\begin{bmatrix} N & \bar{P}\bar{E} & \bar{P}\bar{L} & \bar{P}\bar{L} & \bar{P}\bar{F}_{ext} \\ \bar{E}^T\bar{P} & -I & 0 & 0 & 0 \\ \bar{L}^T P & 0 & -\epsilon_C I & 0 & 0 \\ \bar{L}^T P & 0 & 0 & -\epsilon_W Q^{-1} & 0 \\ \bar{F}_{ext}^T\bar{P} & 0 & 0 & 0 & -\epsilon_W R^{-1} \end{bmatrix} < 0, \tag{2.71}$$

where $N = \epsilon_W \mathcal{T} W \mathcal{T}^T + \bar{A}^T\bar{P} + \bar{P}\bar{A} - \bar{P}\bar{L}\bar{C} - \bar{C}^T\bar{L}^T P + \bar{G}^T\bar{G} + \epsilon_C\gamma_C^2\bar{C}^T\bar{C} + \gamma_S I$, then if $\gamma_S > 0$, system Eq (2.70) has the performance

$$\int_{t-T}^{t} z_{eq}^T(\tau)\mathcal{T} W \mathcal{T}^T z_{eq}(\tau)d\tau \le \int_{t-T}^{t} \left(v^T(\tau)Rv(\tau) + w^T(\tau)Qw(\tau)\right)d\tau, \tag{2.72}$$

with an exponential rate $\ge \gamma_S$.

Proof: Defining $\dot{V}$ by multiplying Eq (2.71) by $\left[z_{eq}^T\ \tilde{\phi}_{eq}^T\ y_{\bar{\delta}}^T\ v^T\ w^T\right]$ on the left and $col\left(z_{eq}, \tilde{\phi}_{eq},\right.$

$\left.y_{\bar{\delta}}, v, w\right)$ on the right, and following the proof of Theorem 7, we can show that

$$\begin{aligned} \frac{d}{dt}[\tilde{x}^T(t)\bar{P}\tilde{x}(t)] + \epsilon_W\left[z_{eq}^T(t)\mathcal{T} W \mathcal{T}^T z_{eq}(t) - \left(v^T(t)Rv(t) + w^T(t)Qw(t)\right)\right] \\ < -\gamma_S\tilde{x}^T(t)\tilde{x}(t) \end{aligned}. \tag{2.73}$$

Since $I \geq \bar{P}$, it is evident that Eq (2.73) ensures that for $V(\mathrm{t}) = \frac{1}{\mathrm{T}}\int_{\mathrm{t-T}}^{t} \tilde{x}^T(\tau)\bar{P}\tilde{x}(\tau)d\tau$, when Eq (2.72) is not satisfied

$$\dot{V} < -\gamma_S V, \tag{2.74}$$

Q.E.D.

For numerical stability, we may need to impose the constraint $\bar{P} \geq 10^{-5}I$ and iteratively solve Eq (2.71) and move in the direction of increasing γ_S until $\gamma_S > 0$.

3. Results

3.1. Simple 2-D problem

3.1.1. Observer design

We will use a simple example to show that the proposed observer design works better than other algorithms from the literature. Consider a second-order unstable system with a dead-zone nonlinearity.

$$\begin{gathered}\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \phi(x_1) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t - 0.5)\end{gathered} \tag{3.1}$$

$$\phi(x_1) = \begin{cases} 10(x_1 + x_{dead})\ if\ x_1 < -x_{dead} \\ 0\ if\ -x_{dead} \leq x_1 \leq x_{dead} \\ 10(x_1 - x_{dead})\ if\ x_1 > x_{dead} \end{cases}. \tag{3.2}$$

Notice $|\phi(x_{1a}) - \phi(x_{1b}) - 5(x_{1a} - x_{1b})| \leq 5|x_{1a} - x_{1b}|$. Hence,

$$A = \begin{bmatrix} 0 & 1 \\ 4 & -3 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, E = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, F = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, G = \begin{bmatrix} 5 & 0 \end{bmatrix}, \Gamma = 0.5, R = 0, Q = 0. \tag{3.3}$$

The parameters of this system were chosen arbitrarily such that the eigenvalues of A are stable while those of $A + EG$ are unstable. Assuming that our observer eigenvalues would be of the same order of magnitude as our open-loop system, $\Gamma \times |Re(\lambda_{max})| \sim 5$. From Figure 1, let us choose *K = 6* with $\gamma = 10^{-5}$. We can solve the equation to get $L = [2.2\ 2.6]^T$, with

$$\bar{P} = 10^{-2} \times \begin{bmatrix} 7.28 & -1.24 & 3.34 & 6.03 & 2.06 & 3.16 & -1.60 & 0.10 \\ -1.24 & 16.14 & -17.54 & 4.01 & -14.06 & 4.09 & -3.77 & -0.77 \\ 3.34 & -17.54 & 41.43 & 1.41 & 37.92 & 1.81 & 2.25 & 2.74 \\ 6.03 & 4.01 & 1.41 & 17.40 & 0.27 & 11.09 & -3.57 & -1.80 \\ 2.06 & -14.06 & 37.92 & 0.27 & 36.22 & 1.90 & 3.70 & 2.08 \\ 3.16 & 4.09 & 1.81 & 11.09 & 1.90 & 9.03 & -3.63 & 0.41 \\ -1.60 & -3.77 & 2.25 & -3.57 & 3.70 & -3.63 & 15.04 & -9.59 \\ 0.10 & -0.77 & 2.74 & -1.80 & 2.08 & 0.41 & -9.59 & 8.90 \end{bmatrix}. \tag{3.4}$$

3.1.2. Comparison with previous literature

Since much of the literature on delay observers has focused on linear systems, let us consider an equivalent linear system with $x_{dead} = 0$, wherein $A_{eq} = A + EG$. The eigenvalues of A_{eq} are 1.8541 and -4.8541, and the system is unstable without feedback. We know that for a given L, the linear observer would be stable if all of the eigenvalues of the characteristic equations $\det(sI - A + Le^{-\Gamma s}C) = 0$ have negative real parts. Since L only has two elements, we have solved for the roots of the characteristic equations for different values of L. Figure 2 plots the maximum of the real part of the eigenvalues (either the least negative or the most positive eigenvalue), as well as the stability boundary wherein this maximum of the real part of the eigenvalues is zero.

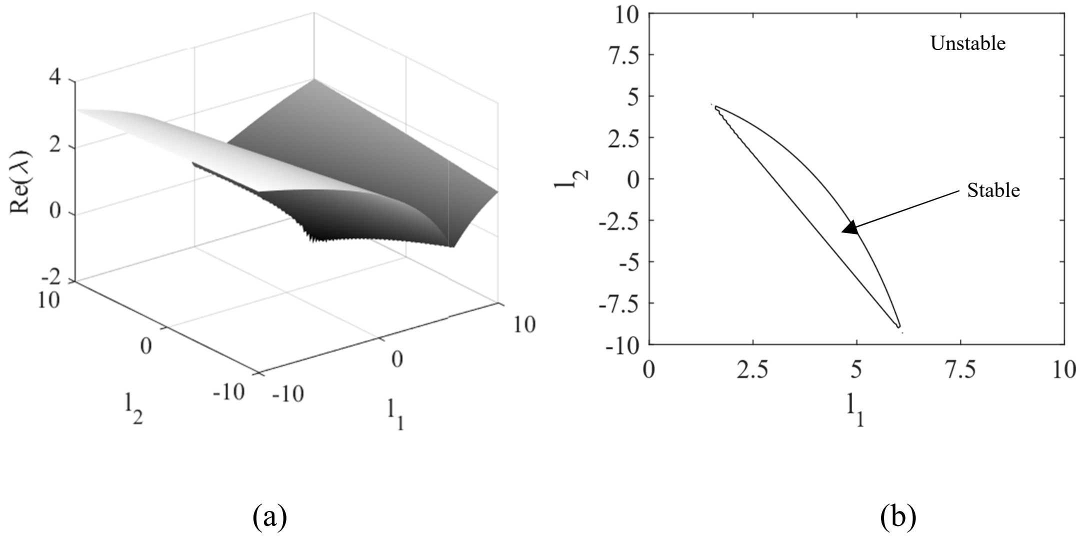

Figure 2. (a) 3D plot of $Re(\lambda)$. (b) Observer stability boundary.

Figure 3 plots the stability boundary obtained by applying the observer design proposed in this paper. This boundary matches the theoretical stability boundary in Figure 2.

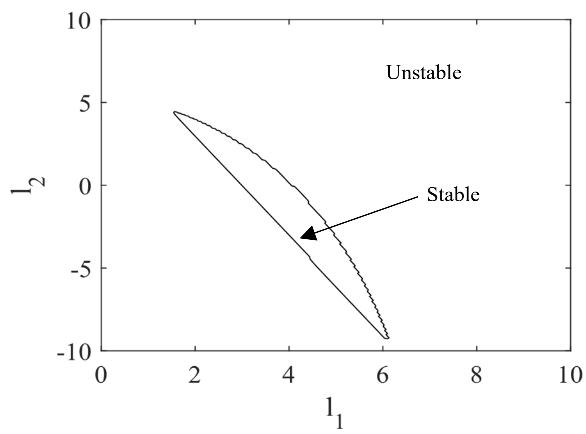

Figure 3. Observer gain stability boundary of the proposed algorithm.

We are unable to make a direct comparison to literature results as there are no equivalent explicit procedures for observer design for systems with measurement delay. Hence, we select possible observer gains over a dense grid ($L = [l_1\ l_2]^T$, $-100 \le l_1 \le 100$, and $-100 \le l_2 \le 100$). For each L, we examined the feasibility of the LMI that was proposed in Fridman and Shaked [44,49], by attempting to find $P_1 > 0$, P_2, P_3, P_4, s.t.,

$$\mathcal{M} = \begin{bmatrix} \left(A_{eq} - LC\right)^T P_2 + P_2\left(A_{eq} - LC\right) & P_1 - P_2^T + \left(A_{eq} - LC\right)^T P_3 & \Gamma P_2^T LC \\ \left(P_1 - P_2^T + \left(A_{eq} - LC\right)^T P_3\right)^T & -P_3 - P_3^T + \Gamma P_4 & \Gamma P_3^T LC \\ (\Gamma P_2^T LC)^T & (\Gamma P_3^T LC)^T & -\Gamma P_4 \end{bmatrix} < 0. \qquad (3.5)$$

As we were unable to find any L, we aimed to get a better understanding of feasibility by making $\mathcal{M} < \alpha$ under the constraint $P_1 \ge 10^{-5} I$ (without this constraint, it is possible to make P_1, P_2, P_3, P_4 arbitrarily small, which will lead to numerical problems). Figure 4 plots the minimum α for $L = [l_1\ l_2]^T$, $-100 \le l_1 \le 100$, and $-100 \le l_2 \le 100$ (the plot was generated by minimizing α using LMI solver for every combination of l_1 and l_2 in the given range). It is seen that $\alpha > 0$, indicating that an observer design is not possible.

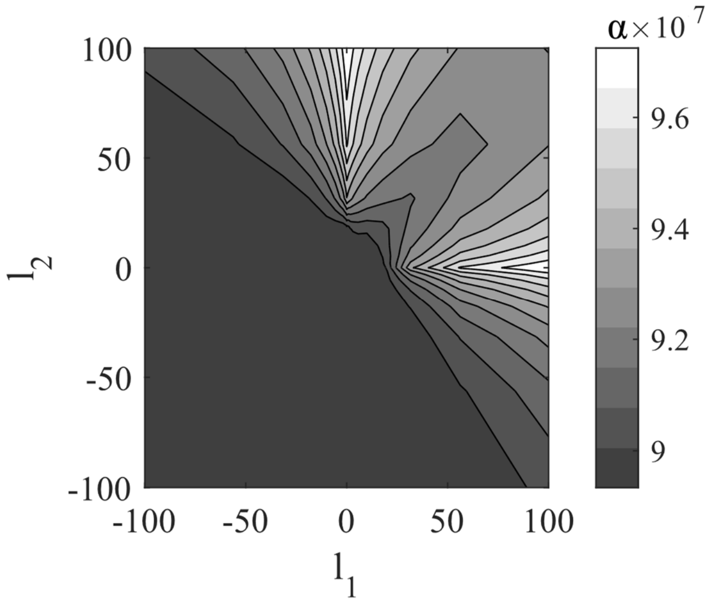

Figure 4. Observer design based on Fridman and Shaked [44,49].

We similarly attempted to solve the following LMI that was proposed by Park et al. [50] (presented in Fridman [44]), $P_1 > 0$, $P_2 > 0, P_3 > 0$, s.t.,

$$\begin{bmatrix} A_{eq}^T P_1 + P_1 A_{eq} + P_3 - P_2 & -P_1 LC + P_2 & \Gamma A_{eq}^T P_2 \\ -(LC)^T P_1 + P_2 & -P_2 - P_3 & -\Gamma (LC)^T P_2 \\ \Gamma P_2 A_{eq} & -\Gamma P_2 LC & -P_2 \end{bmatrix} < 0. \qquad (3.6)$$

The LMI was similarly found to be infeasible. This is expected since the condition based on earlier literature requires the eigenvalues of ΓLC to lie within a unit circle [44], while Figure 2 indicates a different stability boundary. We were similarly unable to design a nonlinear observer for the system based on the methodology proposed in Targui et al. [40], as for every L, we were unable to find $P > 0$ such that

$$\begin{bmatrix} (A_{eq} - LC)^T P + P(A_{eq} - LC) + \epsilon_2 \Gamma I & \Gamma PLCA_{eq} \\ (\Gamma PLCA_{eq})^T & -\epsilon_2 \Gamma I \end{bmatrix} < 0. \tag{3.7}$$

3.2. Robotic link with an elastic joint

In this example, we consider an unstable nonlinear system with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 20 & 0 & -19.5 & -1.25 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 21.6 \\ 0 \\ 0 \end{bmatrix}, E = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.333 \end{bmatrix}, F = B, . \tag{3.8}$$
$$C = [1 \quad 0 \quad 0 \quad 0], \phi(x,u) = sin(x_3)$$

This system is an unstable variation of the elastic joint robotic arm studied in the literature [6,24]. Notice that the eigenvalues of A are $0.2385, -0.6250 \pm 8.2501j, -1.4885$. The real system is assumed to have a mass that is 1% lower than the nominal system. This would result in

$$\Delta A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -0.486 & -0.0125 & 0.486 & 0 \\ 0 & 0 & 0 & 1 \\ 0.2 & 0 & -0.195 & -0.0125 \end{bmatrix}, \Delta B = \begin{bmatrix} 0 \\ 0.0216 \\ 0 \\ 0 \end{bmatrix}, \Delta E = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.00333 \end{bmatrix}. \tag{3.9}$$

We will further assume that the input disturbance is given by $w \sim (0, 10^{-2})$, the noise in the sensor is given by $v \sim (0, 10^{-2})$, and the sensor measurements are delayed by $\Gamma = 0.1\ s$. If we solve for L by ignoring the delay, we can obtain

$$L_{ignore\ delay} = [15.59 \quad 121.56 \quad 10.50 \quad -22.06]^T. \tag{3.10}$$

Notice that $\left\| E(sI - A + L_{ignore\ delay} C) G \right\|_\infty < 1$, and the observer is stable in the absence of delay. However, Figure 5 shows that the observer is unstable (when by $u = \sin(t)$).

We will use $K = 3$ and $\gamma_C = 10^{-5}$, and set $W = I$. Applying the proposed algorithm for a delay time of 0.1 s on outputs, we obtain:

$$L = [7.6708 \quad 9.8257 \quad 0.2664 \quad 1.4747]^T. \tag{3.11}$$

Figure 6 shows the observer states, and Figure 7 shows that the performance ratio $\int_{t-T}^{t} \tilde{x}^T(\tau) W \tilde{x}(\tau) d\tau / \int_{t-T}^{t} \left(v^T(\tau) Q v(\tau) + w_{ext}^T(\tau) R_{ext} w_{ext}(\tau) \right) d\tau$ is less than 1.

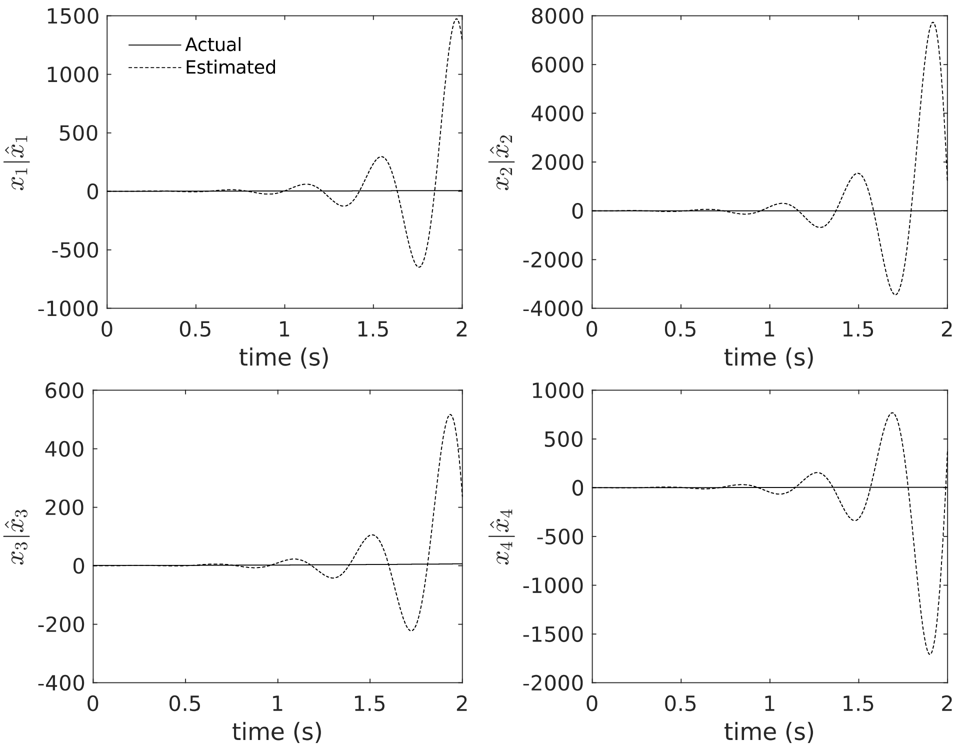

Figure 5. Observer design ignoring delay.

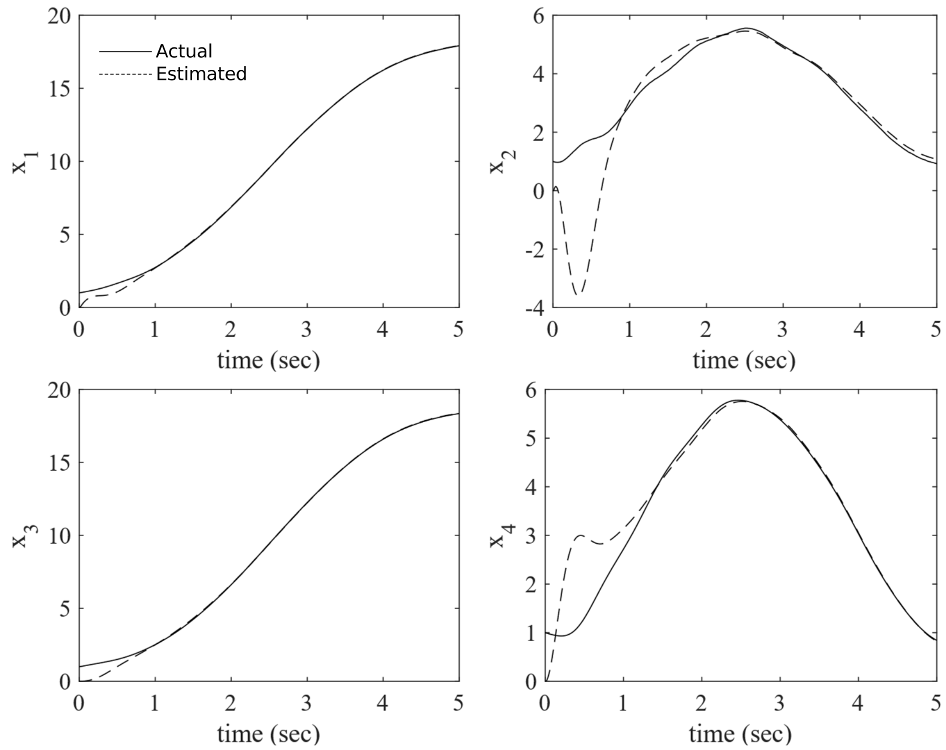

Figure 6. Proposed observer design: States.

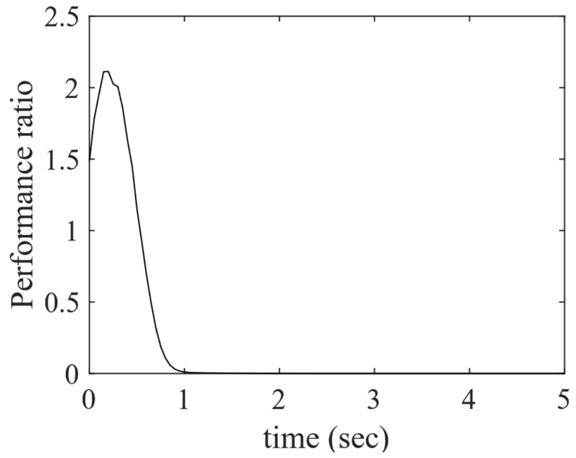

Figure 7. Proposed observer design: Performance ratio.

4. Discussion

This section discusses potential stability issues with the nonlinear delay observer that was proposed in Targui et al. [40]. Targui et al. [40] considered a noise-free system and an observer of the form

$$\dot{x}(t) = Ax(t) + \phi(t, u, x) + Hu(t), \tag{4.1}$$

$$y(t) = Cx(t - \Gamma), \tag{4.2}$$

$$\begin{aligned} \dot{\hat{x}}(t) = A\hat{x}(t) + L\left[y(t) + C\int_{t-\Gamma}^{t}\big(A\hat{x}(\tau) + \phi(\tau, u, x) + Hu(\tau)\big)d\tau - C\hat{x}\right] \\ +\phi(t, u, \hat{x}) + Hu(t) \end{aligned}. \tag{4.3}$$

The error dynamics of $\tilde{x} = x - \hat{x}$ is given by Eqs (4.1)–(4.3)

$$\begin{aligned} \dot{\tilde{x}}(t) = (A - LC)\tilde{x}(t) + L\left[Cx(t - \Gamma) - Cx + C\int_{t-\Gamma}^{t}\big(A\hat{x}(\tau) + \phi(\tau, u, x) + Hu(\tau)\big)d\tau\right] \\ +\phi(t, u, x) - \phi(t, u, \hat{x}) \end{aligned}. \tag{4.4}$$

Using

$$C[x - x(t - \Gamma)] = C\int_{t-\Gamma}^{t}\big(Ax(\tau) + \phi(\tau, u, x) + Hu(\tau)\big)d\tau. \tag{4.5}$$

They obtain

$$\begin{aligned} \dot{\tilde{x}}(t) = (A - LC)\tilde{x} + \phi(t, u, x) - \phi(t, u, \hat{x}) \\ +LC\int_{t-\Gamma}^{t}\big(A\tilde{x}(\tau) + \phi(\tau, u, x) - \phi(\tau, u, \hat{x})\big)d\tau \end{aligned}. \tag{4.6}$$

Using the following LK functional

$$V = \tilde{x}^T P\tilde{x} + \int_{-\Gamma}^{t}\int_{\theta}^{t}\tilde{x}^T(t + \psi)[\epsilon_2 I + \epsilon_3 G^T G]\tilde{x}(t + \psi)d\theta d\psi. \tag{4.7}$$

The authors show that the observer design would be stable if

$$\begin{bmatrix} \mathcal{N} & P & \Gamma PLCA & \Gamma PLC \\ P & -\epsilon_1 I & 0 & 0 \\ (\Gamma PLCA)^T & 0 & -\Gamma\epsilon_2 I & 0 \\ (\Gamma PLC)^T & 0 & 0 & -\Gamma\epsilon_3 I \end{bmatrix} < 0 \quad . \tag{4.8}$$
$$\mathcal{N} = P(A - LC) + (A - LC)^T P + \epsilon_1 G^T G + \Gamma\epsilon_2 I + \Gamma\epsilon_3 G^T G$$

We will now provide an example where the error dynamics could be unstable despite satisfying Eq (4.8). Consider the unstable linear system

$$\begin{aligned} \dot{x} &= x + u \\ y &= x(t - 0.75) \end{aligned} . \tag{4.9}$$

Here, $A = 1,\ C = 1,\ G = 0$ and $\Gamma = 0.75$. We can construct an observer of the form Eq (4.3)

$$\dot{\hat{x}}(t) = \hat{x}(t) + L\left[y(t) + \int_{t-\Gamma}^{t}\big(A\hat{x}(\tau) + u(\tau)\big)d\tau - \hat{x}\right]. \tag{4.10}$$

Since $G = 0$, we can set $\epsilon_1, \epsilon_3 \to \infty$. Notice Eq (4.14) is satisfied for $l = 5$, and $P = 1, \epsilon_2 = 5$, $\epsilon_1, \epsilon_3 \to \infty$, since

$$\begin{bmatrix} -4.25 & 3.75 \\ 3.75 & -3.75 \end{bmatrix} < 0. \tag{4.11}$$

Hence, the observer should be stable. Figure 8 shows the error dynamics for the above system for $u = \sin(100t)$. It is evident from the figure that the observer is unstable, and the observer design proposed by Targui et al. [40] may not be viable.

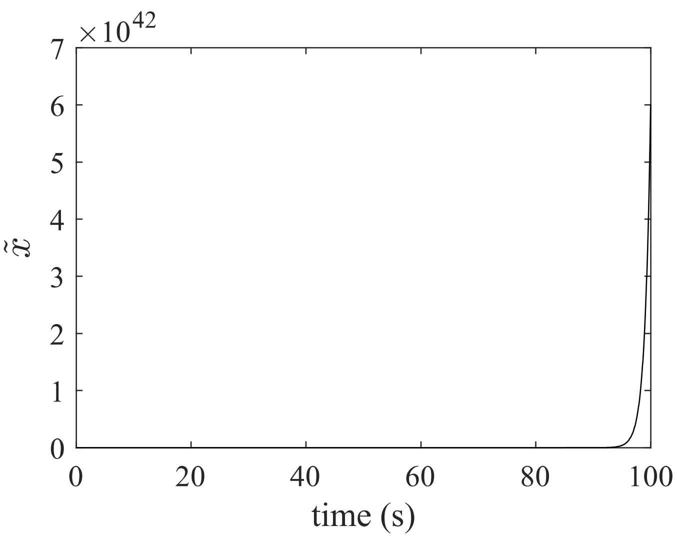

Figure 8. Observer design based on Targui et al. [40].

We will provide further analysis to get to the root of the problem. Let us begin by considering a more general form of the above system using $A = a,\ L = l,\ G = 0$.

$$\begin{aligned} \dot{x} &= ax + u \\ y &= x(t - 0.75) \end{aligned} . \tag{4.12}$$

$$\dot{\hat{x}}(t) = \hat{x}(t) + l\left[y(t) + \int_{t-\Gamma}^{t}\left(A\hat{x}(\tau) + u(\tau)\right)d\tau - \hat{x}\right]. \tag{4.13}$$

As before, since $G = 0$, we can set $\epsilon_1, \epsilon_3 \to \infty$. Hence, Eq (4.8) can be written as

$$\begin{bmatrix} 2Pa - 2Pl + \epsilon_2\Gamma & \Gamma Pla \\ \Gamma Pla & -\epsilon_2\Gamma \end{bmatrix} < 0. \tag{4.14}$$

Taking Schur's complement, we find that we need

$$\mathcal{M} \coloneqq 2Pa - 2Pl + \epsilon_2\Gamma + \Gamma\epsilon_2^{-1}(Pla)^2 < 0. \tag{4.15}$$

Setting $d\,\mathcal{M}/\mathrm{d}\epsilon_2 = 0$ yields

$$\Gamma - \Gamma\epsilon_2^{-2}(Pla)^2 = 0 \text{ or } \epsilon_2 = P|la|. \tag{4.16}$$

Notice that $\epsilon_2 = P|la|$ is a minimum as $d^2\,\mathcal{M}/\mathrm{d}\epsilon_2^2 > 0$. Hence, substituting in Eq (4.15) we find

$$\mathcal{M}_{min-\epsilon 2} = 2Pa - 2Pl + 2P|la|\Gamma < 0. \tag{4.17}$$

Since $P > 0$, we need

$$a - l + (|la|)\Gamma < 0. \tag{4.18}$$

Since l would usually be positive, we need

$$l > \frac{a}{1-|a|\Gamma}. \tag{4.19}$$

There does not seem to be an upper bound for stability (which contradicts our understanding of system delay). It should be noted that even if the observer would work under most normal circumstances, it would be extremely difficult to implement.

5. Conclusions

This paper has developed an observer design procedure for a nonlinear system in the presence of unknown input disturbance, sensor delay, and sensor noise. Necessary and sufficient conditions have been presented for the observer stability in the form of linear matrix inequalities. The observer design procedure was demonstrated for a simple 2D system and an elastic joint robotic arm with delay. Additionally, the paper provided a means of calculating the H_∞ gain of the state-space representation of a delay system, which can be extended to a robust control of the system with delay.

The proposed design procedure is less conservative than results that use Lyapunov–Krasowskii (LK) or Lyapunov–Razumikhin (LR) functionals. In the future, the proposed observer design may be extended to systems with variable delay and to hybrid systems by treating the discrete-time digital signal as a delayed measurement. Additionally, controller and observer designs are often dual problems, and it will be possible to adapt these results to several controller designs.

Use of AI tools declaration

The author declares that they have not used any Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that there is no conflict of interest.

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