Electronic Research Archive

https://www.aimspress.com/journal/era

ERA, 33(6): 3794–3810.
DOI: 10.3934/era.2025168
Received: 04 April 2025
Revised: 19 May 2025
Accepted: 26 May 2025
Published: 16 June 2025

Research article

Semi-local convergence of a sixth-order iterative method for solving nonlinear systems

Li Yu and Xiaofeng Wang*

School of Mathematical Sciences, Bohai University, Jinzhou 121000, China

* **Correspondence:** Email: w200888w@163.com.

Abstract: This paper verifies the semi-local convergence of a sixth-order iterative method for solving nonlinear systems. In the proof, the recursive relation method is used to prove the semi-local convergence of the iterative method in Banach spaces. The proof process outlined above does not necessitate the higher-order continuous differentiability of the function $L(u)$. The advantage of the iterative method is that by employing auxiliary sequences and functions, convergence can be established using only first-order Fréchet derivatives. The proposed sixth-order method reduces computational costs compared to existing sixth-order methods by requiring only one LU decomposition per iteration instead of two. Finally, this sixth-order iterative approach is utilized for solving Hammerstein equations along with real gas equation of state. The experimental results prove the effectiveness of the iterative method.

Keywords: nonlinear equations; semi-local convergence; recurrence relation; domain of existence and uniqueness

1. Introduction

The primary purpose of this paper is to determine the zeros of the function, that is, by solving the equation presented below:

$$L(u) = 0, \tag{1.1}$$

here, $L : \Gamma \subseteq \mathbb{X} \to \mathbb{Y}$ is a continuously nonlinear Fréchet differentiable operator defined on an upper convex subset of a Banach space $\mathbb{X}$, where $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces. For (1.1), the solutions can be in the form of numbers (single unknown equation), vectors (linear or nonlinear equation systems), or functions (differential and integral equations). In the majority of real-world problems, it is very difficult to solve analytically, and our main concern is with the approximate values of the equations in the problem, rather than imposing any particular demands on the exact solution. Thus, we should find an approximate solution that meets a certain precision, and we can use iteration methods to find this solution. One of the well-known iterative methods for solving nonlinear systems is Newton's method [1], and its format can be defined as follows:

$$\mu^{(k+1)} = \mu^{(k)} + \varsigma^{(k)} L(\mu^{(k)}), \tag{1.2}$$

with $k = 0, 1, 2, \cdots$ and $\varsigma^{(k)} = L'(\mu^{(k)})^{-1}$. Here, $L'(\mu^{(k)})$ represents the Jacobian matrix corresponding to L at the k-th iteration. If L is nonsingular and Lipschitz continuous on Γ, Newton's method converges quadratically. The method can be expressed as follows:

$$\begin{cases} L'(\mu^{(k)})\varsigma^{(k)} = L(\mu^{(k)}), \\ \mu^{(k+1)} = \mu^{(k)} - \varsigma^{(k)}. \end{cases} \tag{1.3}$$

To enhance the computational accuracy and iterative efficiency of Newton's method, many effective methods for finding solutions to nonlinear problems have been proposed. In recent years, breakthroughs have been achieved in the local convergence analysis of iterative methods for solving nonlinear systems. Ren and Wu [2] analyzed the convergence ball of the secant method under Hölder continuous divided differences. Wang et al. [3, 4] analyzed the local convergence and stability of some high-order iterative methods under weak conditions. To enhance error accuracy, Zabrejko and Nguen [5] proposed a correlation method for selecting approximate values and errors. Ruan et al. [6] analyzed the local convergence of the seventh-order iterative method under weak conditions. Argyros and Nguen [7] analyzed the complexity of extending the convergence domain of Newton's method under the weak majorant condition. Singh et al. [8] proposed a simple and efficient two-step fifth-order weighted Newton method for nonlinear models, which required two matrix inversions per iteration. Cordero et al. [9] proposed maximally efficient damped composed Newton-type methods to solve nonlinear systems of equations. Liu et al. [10, 11] applied the ADI compact difference scheme for the three-dimensional nonlocal evolution problem with a weakly singular kernel. Yuan and Zhu [12, 13] analyzed the well-posedness and stabilities of mean-field stochastic differential equations driven by G-Brownian motion and event-triggered sampling problems. Cordero et al. [14] put forward a fourth-order method, which constitutes an improvement and refinement of Newton's method. The iterative expression of this method is presented as follows:

$$\begin{cases} \upsilon^{(k)} = \mu^{(k)} - L'(\mu^{(k)})^{-1} L(\mu^{(k)}), \\ \mu^{(k+1)} = \upsilon^{(k)} - [(2I - L'(\mu^{(k)})^{-1} L'(\upsilon^{(k)})] L'(\mu^{(k)})^{-1} L(\upsilon^{(k)}), \end{cases} \tag{1.4}$$

where I represents the identity matrix. The method (1.4) requires only a single Lower-Upper Decomposition (also called LU decomposition) of the Jacobian matrix per iteration. The computational cost of the iterative method is defined by [15] $C(m) = a(m) + p(m)$, where $a(m)$ represents the number of evaluations of the scalar functions used in the evaluations of $L, L', [v, u, L]$, $p(m)$ represents the total number of multiplication and division operations used in the iterative method. Each iteration of the iterative method (1.4) requires $2m^2 + 2m$ scalar functions and $\frac{m^3-m}{3} + 4m^2$ multiplications and divisions. The computational cost of the iterative method (1.4) is $\frac{1}{3}m^3 + 6m^2 + \frac{5}{3}m$.

Cordero et al. [16] also suggested a sixth-order method , which is represented as follows:

$$\begin{cases} \upsilon^{(k)} = \mu^{(k)} - 1/2L'(\mu^{(k)})^{-1} L(\mu^{(k)}), \\ \omega^{(k)} = \mu^{(k)} + [L'(\mu^{(k)}) - 2L'(\upsilon^{(k)})]^{-1}[3L(\mu^{(k)}) - 4L(\upsilon^{(k)})], \\ \mu^{(k+1)} = \omega^{(k)} + [L'(\mu^{(k)}) - 2L'(\upsilon^{(k)})]^{-1} L(\omega^{(k)}), \end{cases} \tag{1.5}$$

method (1.5) entails the performance of two LU decompositions of the Jacobian matrix, with one applied to $L'(\mu^{(k)})$ and the other to $[L'(\mu^{(k)}) - 2L'(\upsilon^{(k)})]$. Each interation of the interative method (1.5) requires $2m^2 + 3m$ scalar functions and $2\left(\frac{m^3-m}{3} + m^2\right) + m^2$ multiplications and divisions. The computational cost of the iterative method (1.5) is $\frac{2}{3}m^3 + 5m^2 + \frac{7}{3}m$.

Although the above method has a relatively high convergence order, in the process of proving convergence, high-order derivatives are required to ensure local convergence. When the nonlinear function does not have higher-order derivatives or the higher-order derivatives are unbounded, it cannot guarantee local convergence. Compared to local convergence, semi-local convergence does not require precise neighborhood information, relying solely on the Kantorovich condition [17] of the initial point. This makes the method more practical in engineering problems, as it can still ensure convergence even when the initial estimate is far from the true solution. Therefore, by studying the semi-local convergence of the iterative publication, we avoid the problem of higher-order derivatives in the proof process. To prove the semi-local convergence of the iterative method in Banach spaces, many scholars have proposed higher-order iterative methods and semi-local proofs, as referenced in [18, 19]. Despite these advancements, the semi-local convergence analysis of high-order systems remains challenging. To address these issues, this paper rigorously verifies the semi-local convergence of a sixth-order iterative method for nonlinear systems and examines its practical applications through numerical examples.

In light of the methodology described in (1.5), we devise an iterative scheme possessing sixth-order convergence, which takes on the following form:

$$\begin{cases} \upsilon^{(k)} = \mu^{(k)} - \varsigma^{(k)} L(\mu^{(k)}), \\ \omega^{(k)} = \upsilon^{(k)} - (2I - \varsigma^{(k)} L'(\upsilon^{(k)}))\varsigma^{(k)} L(\upsilon^{(k)}), \\ \mu^{(k+1)} = \omega^{(k)} - (2I - \varsigma^{(k)} L'(\upsilon^{(k)}))\varsigma^{(k)} L(\omega^{(k)}). \end{cases} \tag{1.6}$$

Each iteration of the interative method (1.6) requires $2m^2 + 3m$ scalar functions and $\frac{m^3-m}{3} + 7m^2$ multiplications and divisions. The computational cost of the iterative method (1.6) is $\frac{1}{3}m^3 + 9m^2 + \frac{8}{3}m$. Our method (1.6) needs fewer LU decompositions than method (1.5) with the same order. Therefore, the computational cost of our method is lower. The main purpose of this paper is to analyze the semi-local convergence of (1.6) based on the weak Lipschitz condition.

There are six parts in this article. In the second section, the three scalar functions and auxiliary sequences crucial for proving semi-local convergence are thoroughly discussed. It also introduces the characteristics of the auxiliary sequences and conducts an analysis of these scalar functions. In the third section, a recursive relationship is proposed with the aim of demonstrating the semi-local convergence property of the iterative method (1.6). In the fourth section, we demonstrate the uniqueness of the semi-local convergence solution of method (1.6). The fifth section focuses on applying the iterative method (1.6) proposed in this paper to solve problems related to Hammerstein-type nonlinear integral equations and equations of state for real gases.

2. Preparatory knowledge

Lemma 2.1.(Banach lemma [20]) *Let A be a bounded linear operator on a Banach space $\mathbb{X}$. If*

$$\|I - A\| < 1, \tag{2.1}$$

then A is invertible, and

$$\left\|A^{-1}\right\| \leq \frac{1}{1-\|I-A\|}. \tag{2.2}$$

This lemma is crucial for establishing the existence of inverse operators in the convergence analysis.

We define $\mathbb{X}$ and $\mathbb{Y}$ as two Banach spaces. For a continuous nonlinear completely differentiable operator L, if L operates on an upper convex subset of $\mathbb{X}$ that is a Banach space, we can represent it as $L : \Gamma \subseteq \mathbb{X} \to \mathbb{Y}$. The Jacobian matrix for the initial iteration in the iterative method (1.6) satisfies $\tau_0 \subseteq \mathbb{X} \to \mathbb{Y}$ and u_0 satisfies $u_0 \subseteq \Gamma$, where $f(\mathbb{X}, \mathbb{Y})$ represents the set of operators that satisfy the linear relationship from $\mathbb{X}$ to $\mathbb{Y}$.

Furthermore, the Kantorovich condition [17] will be employed to derive the semi-local convergence outcome of the approach (1.6).
$(T_1)\ \|\varsigma_0\| \leq \beta$,
$(T_2)\ \|\varsigma_0 L(\mu_0)\| \leq \eta$,
$(T_3)\ \|L'(\mu) - L'(\upsilon)\| \leq K\,\|\mu - \upsilon\|$,
here, K,β and η are real numbers, all of which are greater than or equal to 0. For the purpose of simplicity, we define their product as $\theta_0 = K\beta\eta$, and define $a_0 = g(\theta_0)f(\theta_0)$, $\eta_0 = \eta$. If ξ is the positive root obtained when $uh(u) - 1 = 0$, and ξ is the smallest of all positive roots, then $\theta_0 < \xi$ and $\xi \approx 0.60 < 1$. For $n \geq 0$, we define:

$$\eta_{n+1} = a_n \eta_n, \tag{2.3}$$

$$\theta_{n+1} = g(\theta_n)^2 f(\theta_n), \tag{2.4}$$

$$a_{n+1} = g(\theta_{n+1}) f(\theta_{n+1}), \tag{2.5}$$

Then the scalar functions can be formulated as

$$h(\mu) = 1 + \frac{\mu}{2} + \frac{\mu^3}{2} + \frac{\mu^3}{8}(1+\mu)^3, \tag{2.6}$$

$$f(\mu) = \frac{1}{1 - \mu h(\mu)}, \tag{2.7}$$

$$g(\mu) = \frac{1}{2}\mu(h(\mu))^2 + h(\mu) - 1. \tag{2.8}$$

This is crucial for studying the semi-local convergence of iterative methods.

3. Recursive relation

We define the required recursive relations and auxiliary functions in the previous section, and then in the third part, we begin by analyzing the iterative method (1.6). We define $B(u,r) = \{v \in \mathbb{X}, \|v-u\| < r\}$, $\overline{B(u,r)} = \{v \in \mathbb{X}, \|v-u\| \leq r\}$. Given the assumptions (T_1)–(T_3) presented in Section 2, the recursive relation for the iterative method in (1.6) has the following definition.

We conduct the expansion of the Taylor series of v_0 with respect to L, which is estimated in the vicinity of u_0 to

$$L(\upsilon_0) = L(\mu_0) + L'(\mu_0)(\upsilon_0 - \mu_0) + \int_{\mu_0}^{\upsilon_0} (L'(\mu) - L'(\mu_0))d\mu, \tag{3.1}$$

we obtain the value of $L(\mu_0) + L'(\mu_0)(\upsilon_0 - \mu_0) = 0$ since it stems from the Newton's step. When $n = 0$, through variable substitution $\mu = \mu_0 + s(\upsilon_0 - \mu_0)$, where $s \in [0, 1]$, we can get

$$L(\upsilon_0) = \int_0^1 (L'(\mu_0 + s(\upsilon_0 - \mu_0)) - L'(\mu_0))(\upsilon_0 - \mu_0)ds. \tag{3.2}$$

From the assumption (T_1)–(T_3), ς_0 exists, which means υ_0 also exists, so there is

$$||\upsilon_0 - \mu_0|| \leq ||\varsigma_0 L(\mu_0)|| \leq \eta_0 \leq \eta\varsigma^{(k)}. \tag{3.3}$$

This indicates that $\upsilon_0 \in B(u_0, R\eta)$

$$\begin{aligned} \omega_0 - \mu_0 =&\upsilon_0 - \mu_0 - (2I - L'(\mu_0)^{-1}L'(\upsilon_0))L'(\mu)^{-1}L(\upsilon_0), \\ =&\upsilon_0 - \mu_0 - (I + \varsigma_0 L'(\mu_0) - L'(\upsilon_0))\varsigma_0 L(\upsilon_0), \\ \leq&\upsilon_0 - \mu - \int_0^1 L'(\mu_0 + s(\upsilon_0 - \mu_0) - L'(\mu_0))(\upsilon_0 - \mu_0)\varsigma_0 ds, \\ &+\varsigma_0(L'(\upsilon_0) - L'(\mu_0))\varsigma_0 \int_0^1 L'(\mu_0 + s(\upsilon_0 - \mu_0) - L'(\mu_0))(\upsilon_0 - \mu_0)ds. \end{aligned} \tag{3.4}$$

Apply the Lipschitz condition [21] from (3.2) and take the norm (3.4). We get

$$\begin{aligned} ||\omega_0 - \mu_0|| \leq& ||\upsilon_0 - \mu_0|| + \frac{K}{2} ||\varsigma_0|| \, ||\upsilon_0 - \mu_0||^2 + ||\varsigma_0||^2 K \, ||\upsilon_0 - \mu_0|| \, \frac{K}{2} ||\varsigma_0|| \, ||\upsilon_0 - \mu_0||^2, \\ \leq& ||\upsilon_0 - \mu_0|| + \frac{K}{2} ||\varsigma_0|| \, ||\upsilon_0 - \mu_0||^2 + \frac{K}{2} ||\varsigma_0||^2 ||\upsilon_0 - \mu_0||^3, \\ \leq&\eta + \frac{K}{2}\beta\eta^3 + \frac{K}{2}\beta^2\eta^3 = \eta(1 + \frac{\theta_0}{2} + \frac{{\theta_0}^2}{2}), \end{aligned} \tag{3.5}$$

so that

$$||\omega_0 - \mu_0|| \leq \eta(1 + \frac{\theta_0}{2} + \frac{{\theta_0}^2}{2}). \tag{3.6}$$

From (3.2), we can obtain

$$\begin{aligned} \omega_0 - \upsilon_0 =& - [2I - L'(\mu_0)^{-1}L'(\upsilon_0)]L'(\mu_0)^{-1}L(\upsilon_0), \\ =& - [I + L'(\mu_0)^{-1}L'(\mu_0) - L'(\mu_0)^{-1}L'(\upsilon_0)]L'(\mu_0)^{-1}L(\upsilon_0), \\ \leq& - L'(\mu_0)^{-1}L(\upsilon_0) - L'(\mu_0)^{-1}(L'(\mu_0) - L'(\upsilon_0))L'(\mu_0)^{-1}L(\upsilon_0), \\ =& - L'(\mu_0)^{-1} \int_0^1 (L'(\mu_0 + s(\upsilon_0 - \mu_0)) - L'(\mu_0))(\upsilon_0 - \mu_0)ds, \\ &-L'(\mu_0)^{-1}(L'(\mu_0) - L'(\upsilon_0))L'(\mu_0)^{-1} \int_0^1 (L'(\mu_0 + s(\upsilon_0 - \mu_0)) - L'(\mu_0))(\upsilon_0 - \mu_0)ds. \end{aligned} \tag{3.7}$$

Then, we have

$$
\begin{aligned}
\|\omega_0 - \upsilon_0\| &= \|\varsigma_0\| \frac{K}{2} \|\upsilon_0 - \mu_0\|^2 + \|\varsigma_0\|^2 K \|\upsilon_0 - \mu_0\| \frac{K}{2} \|\varsigma_0\| \|\upsilon_0 - \mu_0\|^2, \\
&\leq \frac{K}{2} \beta \eta^2 + \frac{K^2}{2} \beta^2 \eta^3, \\
&= \frac{\eta}{2}(K\beta\eta + K^2\beta^2\eta^2) + \frac{K}{2}, \\
&= \frac{\eta}{2}(\theta_0 + {\theta_0}^2).
\end{aligned}
\tag{3.8}
$$

By applying Banach's lemma [20], we obtain

$$
\|I - \varsigma_0 L(\upsilon_0)\| \leq \|\varsigma_0\| \|L'(\mu_0) - L'(\upsilon_0)\| \leq K\beta \|\upsilon_0 - \mu_0\| \leq K\beta\eta = \theta_0. \tag{3.9}
$$

Thus, $L'(\upsilon_0)^{-1}$ exists and

$$
\|L'(\upsilon_0)\|^{-1} = \frac{\beta}{1 - \theta_0}. \tag{3.10}
$$

The Taylor series expansion of L around v_0, when evaluated at ω_0, is

$$
L(\omega_0) = \int_0^1 (L'(\upsilon_0 + s(\omega_0 - \upsilon_0)) - L'(\upsilon_0))(\omega_0 - \upsilon_0) ds. \tag{3.11}
$$

Applying the Lipschitz condition [21] to the norms, we obtain

$$
\begin{aligned}
\|L(\omega_0)\| &\leq \int_0^1 K\|(s(\omega_0 - \upsilon_0) + \upsilon_0 - \upsilon_0)(\omega_0 - \upsilon_0)\| ds, \\
&\leq K\|\omega_0 - \upsilon_0\|^2 \int_0^1 s ds, \\
&\leq \frac{K}{2} \|\omega_0 - \upsilon_0\|^2.
\end{aligned}
\tag{3.12}
$$

Thus, from (3.6), (3.11), (3.12), and Kantorovich condition, we can obtain

$$
\begin{aligned}
\|\mu_1 - \mu_0\| &\leq \|\omega_0 - \mu_0\| - \left\| L'(\mu_0)^{-1} L(\omega_0) + (L'(\mu_0)^{-1})^2 L'(\upsilon_0) - L'(\mu_0)(\upsilon_0 - \mu_0) L(\omega_0) \right\|, \\
&\leq \eta(1 + \frac{\theta_0}{2} + \frac{{\theta_0}^2}{2}) - L'(\mu_0)^{-1} L(\omega_0) + (L'(\mu_0)^{-1})^2 L'(\upsilon_0) - L'(\mu_0)(\upsilon_0 - \mu_0) L(\omega_0), \\
&\leq \eta(1 + \frac{\theta_0}{2} + \frac{{\theta_0}^2}{2}) + \frac{K}{2} \beta \|\omega_0 - \upsilon_0\|^2 + \beta^2 K \|\upsilon_0 - \mu_0\| \frac{K}{2} \|\omega_0 - \upsilon_0\|^2, \\
&\leq \eta(1 + \frac{\theta_0}{2} + \frac{{\theta_0}^2}{2}) - \frac{K}{2} \beta \eta^2 (\frac{\theta_0}{2} + \frac{{\theta_0}^2}{2})^2 + K\beta^2 \eta \frac{K}{2} \eta^2 (\frac{\theta_0}{2} + \frac{{\theta_0}^2}{2})^2, \\
&\leq \eta(1 + \frac{\theta_0}{2} + \frac{{\theta_0}^2}{2}) - \frac{K}{8} \beta \eta^2 (\theta_0 + {\theta_0}^2)^2 + \frac{K^2}{8} \beta^2 \eta^2 (\theta_0 + {\theta_0}^2)^2, \\
&\leq \eta(1 + \frac{\theta_0}{2} + \frac{{\theta_0}^2}{2} - \frac{\theta_0}{8} (\theta_0 + {\theta_0}^2)^2 + \frac{{\theta_0}^2}{8} (\theta_0 + {\theta_0}^2)^2).
\end{aligned}
\tag{3.13}
$$

Substitute the θ definition and simplify it to obtain

$$
\|\mu_1 - \mu_0\| \leq \eta h(\theta_0), \tag{3.14}
$$

where $\theta_0 = K\beta\eta$ and $h(u) = 1 + \frac{u}{2} + \frac{u^2}{2} - \frac{u^3}{8}(1+u)^3$.

Reapplying the Banach Lemma, it follows that:

$$\begin{aligned} \|I - \varsigma_0 L(\mu_0)\| &\leq \|\varsigma_0 L'(\mu_0) - L'(\mu_1)\| \leq \|\varsigma_0\| \, \|L'(\mu_0) - L'(\mu_1)\| \leq K\beta \, \|\mu_1 - \mu_0\| \, , \\ &\leq K\beta\eta(1 + \frac{\theta_0}{2} + \frac{{\theta_0}^2}{2} - \frac{{\theta_0}^3}{8}(1+\theta_0)^3) = \theta_0 h(\theta_0) < 1. \end{aligned} \tag{3.15}$$

Then, Banach's lemma guarantees that $(\varsigma_0 L'(\mu_0))^{-1} = \varsigma_0 {\varsigma_0}^{-1}$exists when $\theta_0 h(\theta_0) < 1$ (by taking $\theta_0 < \xi$) and

$$\|\varsigma_1\| \leq \frac{1}{1 - \theta_0 h(\theta_0)} \|\varsigma_0\| = f(\theta_0) \, \|\varsigma_0\| \, , \tag{3.16}$$

where $f(\mu) = \frac{1}{1-\mu h(\mu)}$.

By repeating the above process, we are going to derive the recurrence relationship from the following lemma.

Lemma 3.1. *When $n \geq 1$, the following propositions are verified using the principle of mathematical induction:*
(I_n) $\|\varsigma_0\| \leq f(\theta_{n-1}) \, \|\varsigma_{n-1}\| \, ,$
(II_n) $\|\upsilon_n - \mu_n\| \leq \|\varsigma_n L(\mu_n)\| \leq \eta_n,$
(III_n) $K \, \|\varsigma_n\| \, \|\upsilon_n - \mu_n\| \leq \theta,$
(IV_n) $\|\mu_n - \mu_{n-1}\| \leq h(\theta_{n-1}) \, \|\upsilon_{n-1} - \mu_{n-1}\| \leq h(\theta_{n-1})\eta_{n-1}.$

Proof: Commencing with $n = 1$, the property (I_1) is demonstrated in (3.16). Regarding (II_1), by performing the Taylor expansion of $L(\mu_1)$ in the neighborhood of υ_0, we obtain

$$\begin{aligned} L(\mu_1) =& L(\upsilon_0) + L'(\upsilon_0)(\mu_1 - \upsilon_0) + \int_{\upsilon_0}^{\mu_1} (L'(\mu_0) - L'(\upsilon_0)) d\mu, \\ =& L(\upsilon_0) + L'(\upsilon_0) - L'(\mu_0)(\mu_1 - \upsilon_0) + L'(\mu_0)(\mu_1 - \upsilon_0) \\ &+ \int_0^1 L'(\upsilon_0 + s(\mu_1 - \mu_0) - L'(\upsilon_0))(\mu_1 - \upsilon_0) ds. \end{aligned} \tag{3.17}$$

Taking the norm of $L(\mu_1)$, we obtain

$$\begin{aligned} \|L(\mu_1)\| &= \|L(\upsilon_0)\| + K \, \|\upsilon_0 - \mu_0\| \, \|\mu_1 - \upsilon_0\| + \|L'(\mu_0)\| \, \|\mu_1 - \upsilon_0\| + \frac{K}{2} \|\mu_1 - \upsilon_0\|^2, \\ &\leq \frac{K}{2} \|\upsilon_0 - \mu_0\|^2 + K \, \|\upsilon_0 - \mu_0\| \, \|\mu_1 - \upsilon_0\| + \|L'(\mu_0)\| \, \|\mu_1 - \upsilon_0\| + \frac{K}{2} \|\mu_1 - \upsilon_0\|^2. \end{aligned} \tag{3.18}$$

From (2.6), (3.8), (3.10), and (3.12), we have

$$\begin{aligned}\|\mu_1 - \upsilon_0\| &= \left\|\omega_0 - \upsilon_0 - L'(\upsilon_0)^{-1}L(\omega_0)\right\|,\\ &\leq \|\omega_0 - \upsilon_0\| + \left\|L'(\upsilon_0)^{-1}L(\omega_0)\right\|,\\ &\leq \frac{\eta}{2}(\theta_0 + {\theta_0}^2) + \left\|L'(\upsilon_0)^{-1}\frac{K}{2}(\omega_0 - \upsilon_0)^2\right\|,\\ &\leq \frac{\eta}{2}(\theta_0 + {\theta_0}^2) + \frac{\beta}{1-\theta_0}\frac{K}{2}\frac{\eta^2}{4}(\theta_0 + {\theta_0}^2)^2,\\ &= \eta(\frac{\theta_0}{2} + \frac{{\theta_0}^2}{2} + \frac{{\theta_0}^3}{8(1-\theta_0)}(1+\theta_0)^2),\\ &= \eta h(\theta_0 - 1),\end{aligned} \tag{3.19}$$

then

$$L(\mu_1)^2 \leq \frac{K}{2}\eta^2 + K\eta^2(h(\theta_0) - 1) + \frac{1}{\beta}\eta(h(\theta_0) - 1) + \frac{K}{2}\eta^2(h(\theta_0) - 1)^2. \tag{3.20}$$

By applying (I_1), we can obtain

$$\begin{aligned}\|\upsilon_1 - \mu_1\| &\leq \|\varsigma_1 L(\mu_1)\| \leq f(\theta_0)\,\|\varsigma_0\|\,\|L(\mu_1)\|,\\ &\leq f(\theta_0)\beta\eta\left(\frac{K}{2}\eta + K\eta h(\theta_0) - 1\right) + \frac{1}{\beta}(h(\theta_0) - 1)\frac{K}{2}\eta(h(\theta_0) - 1)^2,\\ &\leq f(\theta_0)\eta\left(\frac{\theta_0}{2} + \theta_0\,(h(\theta_0) - 1) + +h(\theta_0) - 1 + \frac{\theta_0}{2}(h(\theta_0) - 1)^2\right),\\ &= \left(\frac{\theta_0}{2}h(\theta_0)^2 + h(\theta_0) - 1\right)f(\theta_0)\eta,\\ &= \theta_0\eta = \eta_1.\end{aligned} \tag{3.21}$$

Let

$$\|\upsilon_1 - \mu_1\| \leq f(\theta_0)g(\theta_0)\eta, \tag{3.22}$$

where $a_0 = g(\theta_0)f(\theta_0)$ and

$$g(u) = \frac{1}{2}\mu(h(\mu))^2 + h(\mu) - 1. \tag{3.23}$$

(III_1): For the case of $n = 1$, the outcome derived from the corresponding formula (I_1) and (II_1) is utilized to conduct the proof.

$$\begin{aligned}K\,\|\varsigma_1\|\,\|\upsilon_1 - \mu_1\| &\leq K f(\theta_0)\,\|\varsigma_0\|\,f(\theta_0)g(\theta_0)\eta,\\ &\leq \theta_0(f(\theta_0))^2 g(\theta_0),\\ &= \theta_1.\end{aligned} \tag{3.24}$$

(IV_1): For the case of $n = 1$, this has been proven in (3.13).

4. Semi-local convergence analysis

This part has provided a theorem regarding the sixth-order convergent iterative approach (1.6) and proved the semi-local convergence of (1.6). In order to guarantee the convergence of the sequence

$\{u_n\}$, it is essential to demonstrate that $\{u_n\}$ constitutes a Cauchy sequence. With the aim of attaining this objective, we have carried out an analysis of the characteristics of the aforementioned recursive sequences $\{\theta_n\}$, $\{a_n\}$ as well as $h(u), f(u), g(u)$, and acquired the following preliminary results:

Lemma 4.1. *The functions $h(u), f(u)$, and $g(u)$ are determined by (2.6)–(2.8), and several of the properties of these functions are presented in the following:*
(i) $h(u), f(u)$, and $g(u)$ are increasing ,where $h(u) > 1$ and $f(u) > 1$ for $0 < u < \xi$,
(ii) $f(\theta_0)g(\theta_0) < 1$ for $\theta_0 < 0.3814$,
(iii) $f(\theta_0)^2 g(\theta_0) < 1$ for $\theta_0 < 0.3127$.

Proof: (i) For $h(u)$, compute the derivative to get: $h'(u) = \frac{1}{2} + \frac{3u^2}{2} + \frac{3u^2(1+u)^3}{8} + \frac{3u^3(1+u)^2}{8}$. Since all terms are positive for $u > 0$, $h'(u) > 0$. Thus, $h(u)$ is monotonically increasing in $(0, \xi)$. Similarly, $f(u)$ and $g(u)$ are increasing. For the (ii), (iii). When $f(\theta_0)^2 g(\theta_0) < 1$, we design θ_n such that θ_n is monotonically decreasing. From (2.4)–(2.6), the minimum roots 0.3814 and 0.3127 of $f(\theta_0)g(\theta_0) = 1$ and $f(\theta_0)^2 g(\theta_0) = 1$ can be calculated according to the definition of $f(\theta_0)$ and $g(\theta_0)$.As a result, when $n \geq 1$, it holds that $\theta_n < \theta_0 \leq 0.3127$.

Lemma 4.2. *For the auxiliary functions $f(u)$,$h(u)$, and $g(u)$, and the smallest positive root ξ of $uh(u) - 1$, if*

$$\theta_0 < \xi, f(\theta_0) a_0 < 1, \tag{4.1}$$

then
(i) $f(\theta_0) > 1, a_n < 1 (n \geq 0)$.
(ii) When $n \geq 0$, if $\theta_n < 0.3127$, then $\{\theta_n\}$, $\{a_n\}$, and $\{\eta_n\}$ are decreasing.
(iii) $h(\theta_n)\theta_n < 1, f(\theta_n) a_n < 1 \, (n \geq 0)$.

Proof: (i) Based on Lemma 4.1 and (2.7), it is evident to us that $f(\theta_0) > 1$ is true and $a_0 < 1$, so it holds for all cases when $n = 0$. When $n = 1$, given that $\mu_1 < 1$ holds based on the same principle, we can demonstrate that, by the same token as $a_1 < 1$, it can be inferred through mathematical induction.
(ii) We can obtain $a_n < 1$ from the definition of the sequence (2.1)–(2.3) and (i), thus, $\eta_{n+1} < \eta_n$, and η_n is decreasing. According to Lemma 4.1, when $n = 0$, $f(\theta_0)^2 g(\theta_0) < 1$, so $\theta_1 < \theta_0$. $\{\lambda_n\}$ is a decreasing sequence by mathematical induction. By the same token, $a_1 < a_0$ and $\{a_n\}$ is also a decreasing sequence.
(iii) According to the above results and lemma 4.1, we can obtain that $h(\theta_1)^2\theta_1 < h(\theta_0)^2\theta_0 < 1$ and $f(\theta_1)^2\theta_1 < f(\theta_0)^2\theta_0 < 1$ are true. At the same time, (iii) is also true by induction.

Theorem 4.1. *Suppose that $L : \Gamma \subseteq \mathbb{X} \to \mathbb{Y}$ represents a twice continuously differentiable Fréchet nonlinear operator defined on the open set Γ. Assume that $u_0 \in \tau$ and $\varsigma^{(k)} = L'(\mu^{(k)})^{-1}$ exist, and the conditions (T_1)–(T_3) are fulfilled. Let $\theta_0 = k\beta\eta$ and $\theta_0 < \xi$ and define $\eta_{n+1} = a_n\eta_n$, $a_{n+1} = g(\theta_{n+1})f(\theta_{n+1})$, $\theta_0 < \xi$, and $f(\theta_0)a_0 < 1$. If $B_e(s_0 - R\eta) = \{s \in X : \|u - u_0\| < R\eta\} \subset \tau$, $R = \frac{h(\theta_0)}{g(\theta_0)f(\theta_0)}$, then the iterated sequence u_0 specified in (1.5) converges, starting from the initial point u_0, to the solution u^* of $L(u) = 0$. In this case, the iterated sequences $\{u_n\}$ and $\{v_n\}$ are contained within $u^* \in B(u_0 - R\eta)$, where u^* is the unique solution of the equation $L(u) = 0$ in the domain $B_n(u_0 - \frac{2}{k\beta}R\eta) \cap \tau$.*

Proof: Based on Lemma 4.1, the following result can be obtained by us.

$$\eta_n = g(\theta_{n-1})f(\theta_{n-1})\eta_{n-1} = \prod_{i=1}^{n}(g(\theta_i)f(\theta_i))\,\eta \leq (g(\theta_0)f(\theta_0))^n\eta. \tag{4.2}$$

Thus,

$$\sum_{i=0}^{n}\eta_i \leq \sum_{i=1}^{n}(g(\theta_0)f(\theta_0))^i\eta = \frac{1-(g(\theta_0)f(\theta_0))^{n+1}}{g(\theta_0)f(\theta_0)}\eta. \tag{4.3}$$

In accordance with Lemmas 4.1 and 2.2, the functions $f(u)$ and $g(u)$ exhibit an increasing trend. Hence, we represent $u_{n+1} - u_0$ by means of the partial sums of a geometric series,

$$\begin{aligned} \|u_{n+1} - u_0\| &\leq \sum_{i=0}^{n}\|u_{i+1} - u_i\| \leq \sum_{i=0}^{n} h(\theta_i)\eta_i \leq h(\theta_0)\sum_{i=0}^{n}\eta_i, \\ &\leq h(\theta_0)\eta\frac{1-(g(\theta_0)f(\theta_0))^{n+1}}{g(\theta_0)f(\theta_0)} < R\eta. \end{aligned} \tag{4.4}$$

Therefore, $f(\theta_0)g(\theta_0) < 1$ in Lemma 4.1 holds, leading to the conclusion that $\{u_n\}$ that all belong to $\overline{B_e(u_0 - R\eta)}$. With lemmas 4.1 and 4.2, we can see that $f(u)$, $g(u)$, and $h(u)$, while $\{\theta_n\}$ decreases, thus proving that $\{u_n\}$ is a Cauchy sequence.

$$\begin{aligned} \|u_{n+m} - u_m\| &\leq \sum_{i=0}^{n+m-1}\|u_{i+1} - u_i\|, \\ &\leq \sum_{i=0}^{n+m-1} h(\theta_i)\eta_i \leq h(\theta_0)\sum_{i=0}^{n+m-1}\eta_i, \\ &\leq h(\theta_0)\eta\frac{1-(g(\theta_0)f(\theta_0))^{n+m}}{g(\theta_0)f(\theta_0)}. \end{aligned} \tag{4.5}$$

So, the sequence $\{u_n\}$ is a convergent Cauchy sequence. Consequently, there exists u^*, such that $\lim_{n\to\infty} u_n = u^*$. In (4.3), by setting $n = 0, m \to \infty$, we obtain that $\| u^* - u_0 \| \leq R\eta$, which implies that $\overline{B_e(u^*, R\eta)}$.

Ultimately, we have demonstrated the uniqueness of u^* within $B_n(s_0 - \frac{2}{k\beta}R\eta) \cap \tau$, as presented below:

$$\frac{2}{k\beta} - R\eta = \left(\frac{2}{\theta_0} - R\right)\eta > \frac{1}{\theta_0}\eta > R\eta, \tag{4.6}$$

so $\overline{B_e(u_0 - R\eta)} \subset B_n(u_0 - \frac{2}{k\beta}R\eta) \cap \tau$. Subsequently, we postulate that v^* constitutes another solution of $L(u) = 0$ in $B_n(u_0 - \frac{2}{k\beta}R\eta) \cap \tau$ and demonstrate that $u^* - v^* = 0$. To begin with, we perform the Taylor expansion of L in the vicinity of u^* so as to obtain

$$L(v^*) = L(u^*) + \int_0^1 L'(u^* + s(v^* - u^*))(v^* - u^*)ds, \tag{4.7}$$

so that

$$0 = L(v^*) - L(u^*) = (v^* - u^*)\int_0^1 L'(u^* + s(v^* - u^*))ds. \tag{4.8}$$

It is necessary for us to demonstrate the invertibility of the operator $\int_0^1 (F'(u^* + t(v^* - u^*))$, thereby ensuring that $v^* - u^* = 0$. Subsequently, by applying the hypothesis (T_3), we get

$$\begin{aligned} \|\varsigma_0\| \left\| \int_0^1 L'(u^* + s(v^* - u^*)) - L'(u_0) \right\| ds \leq & K\beta \int_0^1 \|u^* + s(v^* - u^*) - u_0\| ds, \\ \leq & K\beta \int_0^1 ((1-s)\|u - u_0\| + s\|v^* - u_0\|) ds, \\ \leq & K\beta (R\eta + \frac{2}{K\beta} - R\eta) = 1. \end{aligned} \tag{4.9}$$

In accordance with Banach's lemma, it can be deduced that the operator $\int_0^1 (F'(u^* + t(v^* - u^*))$ is possesses invertibility, and $\int_0^1 (F'(u^* + t(v^* - u^*)) \in L(\mathbb{X}, \mathbb{Y})$. The proof is accomplished by conducting an estimation on $0 = L(v^*) = L(u^*) = (v^* - u^*) \int_0^1 (F'(u^* + t(v^* - u^*) ds$ to obtain $u^* = v^*$.

5. Numerical experiments

In the current section, with the aim of validating the practicability of the theoretical proof and demonstrating the rationality of the recursive relationship we have deduced, we will employ the iterative method (1.6) to address nonlinear systems. Moreover, we will apply the iterative method (1.6) to resolve practical chemistry problems. The practical examples here are the Hammerstein equations and the real gas equation of state (RGES). The Hammerstein equations lead to large-scale nonlinear systems during the discretization process. By solving them with the iterative method presented in this paper, the computational effort can be greatly reduced, as LU decomposition is performed only once per step, significantly lowering computation time and cost. In the case of RGES, solving the Van der Waals equation requires extremely high precision. The iterative method described in this paper ensures the accurate determination of gas volume while guaranteeing both iteration speed and stability.

The numerical results can be found in Tables 1–5, which present the parameters β, η, K, and θ_0 of the Kantorovich condition for different initial iteration values, as well as the radius R_e and R_n, the error $\| u_k - u_{k-1} \|$, and the function values $\| L(u_k) \|$.

Problem 1. Tens of thousands of nonlinear equation-solving problems, such as integral equation problems, ordinary (partial) differential equation problems, nonlinear electrical problems, nonlinear programming problems, etc., and a large number of basic work research can be converted into solving nonlinear equations or nonlinear equations. Nonlinear integral equations In mathematical physics and chemistry, many important phenomena can be solved by nonlinear integral equations, such as electrodynamics, electrostatics, fluid mechanics, etc. Therefore, it is important to solve the mathematical theory and numerical solutions of linear or nonlinear integral equations. In the present paper, the focus is placed on the nonlinear integral equations of Hammerstein type [22]. The obtained results are utilized to tackle the integral equations of this type, thereby verifying the applicability of the theoretical findings. The Hammerstein equation is presented in the following form:

$$u(t) = 1 + \frac{1}{5} \int_0^1 G(t,s) u(s)^3 ds, \tag{5.1}$$

where $s \in \mathbb{C}(0,1)$, $t \in [0,1]$, $s \in [0,1]$, with the kernel G as

$$G(t,s) = \begin{cases} (1-t)s & \text{if } s \leq t, \\ t(1-s) & \text{if } t < s. \end{cases} \tag{5.2}$$

We discretized (5.1) to transform it into a nonlinear equation. Then, we use the Gauss–Legendre quadrature approximation from the integral in (5.1):

$$\int_0^1 L(y)dy \;= \frac{1}{2}\int_{-1}^1 L(\frac{1}{2}+\frac{1}{2}x)dx \approx \frac{1}{2}\sum_{j=1}^{n} \delta_j L(y_i), \tag{5.3}$$

where $y_j = \frac{1}{2}+\frac{1}{2}x_j$, y_j and δ_j are the nodes and weights of the Gauss–Legendre polynomial, respectively.

Sampling the Hammerstein equation at the Gauss–Legendre nodes $\{s_i\}_{i=1}^n$ (typically taken as the same points as the quadrature nodes, i.e., $s_i = t_i$), we obtain:

$$u(t_i) = 1 + \frac{1}{10}\sum_{j=1}^{n} \delta_j G(s_j, t_i) u(s_j)^3 \quad (i = 1, 2, \ldots, n). \tag{5.4}$$

Denoting $u_i = u(t_i)$, and $G_{ij} = G(t_i, s_j)$, the equation can be simplified as:

$$u_i = 1 + \frac{1}{10}\sum_{j=1}^{n} \delta_j G_{ij} {u_j}^3 \quad (i = 1, 2, \ldots, n). \tag{5.5}$$

This constitutes a system of nonlinear algebraic equations for $u_1, u_2, \ldots, u_n$.

When $n = 7$,we obtain

$$u_i = 1 + \frac{1}{10}\sum_{j=1}^{7} \alpha_{ij} u_j^3, \tag{5.6}$$

where

$$\alpha_{ij} = \begin{cases} \delta_j s_j (1-s_i) & \text{if } j \leq i, \\ \delta_j s_i (1-s_j) & \text{if } j > i. \end{cases} \tag{5.7}$$

where $s_1 = 0.0254460438, s_2 = 0.1292344072, s_3 = 0.2970774243, s_4 = 0.5, s_5 = 0.7029225757, s_6 = 0.8707655928, s_7 = 0.9745539561$. $\delta_1 = \delta_7 = 0.1294849662, \delta_2 = \delta_6 = 0.2797053915, \delta_3 = \delta_5 = 0.3818303915, \delta_4 = 0.4179591837$.

Rewrite (5.5), we obtain

$$\begin{aligned} L(u) &= u - 1 - \frac{1}{10}A\varsigma_u, \quad \varsigma_u = (u_1^3, u_2^3, ..., u_7^3)^T, \\ L'(s) &= I - \frac{3}{10}AD(u), \quad D(u) = diag(u_1^2, u_2^2, ..., u_7^2), \end{aligned} \tag{5.8}$$

here $A = (\alpha_{ij})_{7\times 7}$, L' is the Fréchet derivative of L. If $\mathcal{F}(\mathbb{R}^n, \mathbb{R}^n)$ denotes the set of a nonlinear operator on the Banach space, then L belongs to $\mathbb{R}^n$. Based on this, the nonlinear problem can be solved using (1.6).

Using the infinite norm while taking $u_0 = (1.8, 1.8, 1.8, 1.8, 1.8, 1.8, 1.8)^T$, we can obtain

$$\begin{aligned} \| \varsigma_0 \| &\leq \beta, & \beta &\approx 1.1135, \\ \|\varsigma_0 L(\mu_0)\| &\leq \eta, & \eta &\approx 2.1563, \\ \| \ L'(u) - \ L'(v) \| &\leq K \| u - v \|, & K &\approx 0.0335. \end{aligned} \tag{5.9}$$

The aforementioned results satisfy the semi-local convergence condition, thus enabling its application to the system. Additionally, the existence of the solution u_0 in $B_e(u_0, 2.4701)$ is considered, and uniqueness in $B_n(u_0, 51.1994)$ are guaranteed by Theorem 4.1. In Table 1, we provide the existence radius R_e and uniqueness radius R_n for varying values of the initial estimator u_0 with equal components. In the meantime, we observe that when $u_{0i} > 1.7, i = 1, 2, ..., 7$, the iterative method fails to satisfy the convergence condition; there is no guarantee of its convergence.

When applying the iterative method (1.6) to solve Eq (5.1), the exact solution obtained is

$$u^* = \{1.003, 1.016, 1.029, 1.035, 1.029, 1.016, 1.003\}^T . \tag{5.10}$$

In Table 1, the existence radius R_e and the uniqueness radius R_n corresponding to diverse initial values are acquired by assigning the values of the pertinent parameters at those initial values. Table 2 presents the errors and function values associated with different initial values, which verifies the sixth-order convergence of the iterative method (1.6).

Table 1. Parameter values with different initial values for Problem 1.

u_{0i}	β	η	K	θ_0	R_e	R_n
0.40	1.0051	1.5939	0.0168	0.0269	1.6619	116.9498
0.60	1.0115	1.0703	0.0193	0.0209	1.1054	101.3732
0.80	1.0205	0.5460	0.0220	0.0123	0.5563	88.5296
1.00	1.0325	0.0223	0.0338	0.0008	0.0223	57.2273
1.20	1.0474	0.5127	0.0258	0.0138	0.5237	73.5120
1.40	1.0657	1.0507	0.0285	0.0319	1.1045	64.6856
1.60	1.0876	1.5976	0.0310	0.0539	1.7433	57.5042

Table 2. Experimental results of method (1.6) for Problem 1.

u_{0i}	*iter*	$\| u_k - u_{k-1} \|$	$\| \ L(u_k) \|$	ρ
0.6	4	5.733×10^{-248}	1.065×10^{-1478}	6
0.7	4	6.463×10^{-270}	7.279×10^{-1620}	6
0.8	4	8.415×10^{-294}	1.064×10^{-1762}	6
0.9	4	2.898×10^{-342}	5.919×10^{-2054}	6
1.0	4	4.669×10^{-449}	3.106×10^{-2694}	6
1.1	4	1.415×10^{-410}	8.025×10^{-2464}	6
1.2	4	6.387×10^{-311}	2.036×10^{-1865}	6
1.3	4	1.181×10^{-263}	2.709×10^{-1582}	6
1.4	4	1.157×10^{-2284}	7.204×10^{-1348}	6

Table 3. Experimental results of method (1.5) for Problem 1.

u_{0i}	*iter*	$\| u_k - u_{k-1} \|$	$\| L(u_k) \|$	ρ
0.6	4	2.292×10^{-220}	4.257×10^{-1322}	6
0.7	4	9.717×10^{-242}	2.474×10^{-1450}	6
0.8	4	1.466×10^{-271}	2.912×10^{-1629}	6
0.9	4	4.564×10^{-319}	2.655×10^{-1914}	6
1.0	4	1.516×10^{-430}	3.557×10^{-2583}	6
1.1	4	2.772×10^{-393}	1.338×10^{-2359}	6
1.2	4	5.203×10^{-299}	5.838×10^{-1794}	6
1.3	4	1.263×10^{-250}	1.191×10^{-1503}	6
1.4	4	3.180×10^{-217}	3.041×10^{-1303}	6

where, ρ represents the convergence order obtained from numerical experiments. We can find that the results of Tables 1 and 2 are similar. By selecting different initial values under the Kantorovich condition [18], we can converge to the unique solution, and the more proximal the initial value is to the root, the lower the estimated error. Under specific assumptions, the semi-local convergence proof guarantees the existence and uniqueness of solutions. This is of great significance when the existence of the solution remains uncertain. From the comparative evaluation of Tables 2 and 3, the empirical results indicate that the iterative approach introduced in this work outperforms method (1.5) in terms of error reduction, achieving enhanced numerical accuracy at a higher order of magnitude.

Problem 2. The problem of the gas equation of state is crucial in addressing practical chemical issues. We utilize the iterative method (1.6) to deal with this problem. (real gas equation of state – RGES): The container holds 2 moles of nitrogen, and its volume is calculated using the van der Waals equation, which is written as follows:

$$L(V) = (p + \frac{an^2}{V^2})(V - nb) - nRT. \tag{5.11}$$

The van der Waals constants a ($4.17atm * L^2/mol^2$) and $b(0.0371L/mol)$ correct for gas pressure and volume, respectively, for nitrogen and are standard parameters from physical chemistry literature, ensuring the RGES is well-posed and the initial approximation lies within the convergence domain specified by Theorem 4.1. Given 2 moles of nitrogen at 9.33 atm and 300.2 K, calculate the container volume using Eq (5.11):

$$L(V) = 9.33V^3 - 46.9611V^2 + 16.68V - 1.23766. \tag{5.12}$$

Taking $u_0 = 0.6$ and the infinity norm, we get

$$\theta_0 = K\beta\eta, \theta \approx 0.3404, \tag{5.13}$$

$$a_0 = g(\theta_0)f(\theta_0), a_0 \approx 0.7576. \tag{5.14}$$

Consequently, the method meets the convergence condition, the solution lies in $B_e(u_0, 0.1906)$, and the domain of uniqueness is $B_n(u_0, 0.0353)$. Provided that the initial value meets the Kantorovich condition, one can choose an initial value from this range to address the nonlinear system. Employing the iterative method (1.6) to solve system (5.11) leads to the acquisition of the root $u^* = 0.10911$.

Table 4. Experimental results of method (1.6) for Problem 2.

u_{0i}	*iter*	$\| u_k - u_{k-1} \|$	$\| L(u_k) \|$	ρ
0.06	4	2.6428×10^{-472}	1.6134×10^{-471}	6
0.07	4	6.0167×10^{-551}	3.6732×10^{-550}	6
0.08	4	2.0568×10^{-662}	1.2557×10^{-661}	6
0.09	4	6.6023×10^{-838}	4.0307×10^{-837}	6
0.1	4	3.7793×10^{-1184}	2.3073×10^{-1183}	6
0.11	4	4.3161×10^{-2409}	2.3019×10^{-2408}	6

Table 5. Experimental results of method (1.5) for Problem 2.

u_{0i}	*iter*	$\| u_k - u_{k-1} \|$	$\| L(u_k) \|$	ρ
0.06	4	4.7843×10^{-464}	2.9208×10^{-463}	6
0.07	4	9.2335×10^{-548}	5.637×10^{-547}	6
0.08	4	1.2906×10^{-657}	7.8791×10^{-659}	6
0.09	4	5.4014×10^{-816}	3.2975×10^{-815}	6
0.1	4	2.7032×10^{-1061}	1.6503×10^{-1060}	6
0.11	4	5.2314×10^{-2397}	2.3019×10^{-2395}	6

where, ρ represents the convergence order obtained from numerical experiments. Tables 4 and 5 present the errors as well as the function values corresponding to various initial values. Numerical result demonstrate that method (1.6) exhibits significantly smaller errors compared to method (1.5), achieving higher-order significant digit accuracy. Under the Kantorovich condition, the method ensures convergence to the unique solution by selecting different initial values. Moreover, the closer the initial value is to the root, the smaller the error estimate becomes. This serves as compelling numerical evidence that method (1.6) converges at a sixth-order rate, further validating its efficiency and stability in the iterative process.

6. Conclusions

This paper focuses on the study of the semi-local convergence of the iteration method (1.6). By applying the Lipschitz condition to the first derivative, we prove iteration method (1.6) using the recursion method. Initially, it is crucial to explore the properties of the auxiliary sequences η_n, θ_n, a_n as well as the scalar functions $h(u)$, $g(u)$, and $f(u)$. Subsequently, we have also managed to obtain the radius of convergence R. Then, we demonstrate that the iterated sequence converges to $u^* \in \overline{B(u_0, R)}$, where u^* satisfies $L(u^*) = 0$, which demonstrates the existence of the roots. Eventually, the Banach lemma is utilized to prove the uniqueness of the solution. Numerical experiments carried out on Hammerstein - type nonlinear integral equations as well as real gas state equations have validated the efficacy of the iteration method. In the future, efforts will be made to explore whether it is possible to further reduce the computational cost of each iteration based on the methods presented in this paper while improving the convergence order of the algorithm.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This research was supported by Educational Commission Foundation of Liaoning Province of China (No. LJ212410167008) and the Key Project of Bohai University (No. 0522xn078).

Conflict of interest

The authors declare there is no conflicts of interest.

References

1. J. M. Ortega, W. C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, 1970. https://doi.org/10.1016/C2013-0-11263-9
2. H. Ren, Q. Wu, The convergence ball of the secant method under Hölder continuous divided differences, *J. Comput. Appl. Math.*, **194** (2006), 284–293. http://doi.org/10.1016/j.cam.2005.07.008
3. X. Wang, N. Shang, Local convergence analysis of a novel derivative-free method with and without memory for solving nonlinear systems, *Int. J. Comput. Math.*, **102** (2025), 846–863. https://doi.org/10.1080/00207160.2025.2464701
4. W. Li, X. Wang, Fractal behavior of King's optimal eighth-order iterative method and its numerical application, *Math. Commun.*, **29** (2024), 217–236.
5. P. P. Zabrejko, D. F. Nguen, The majorant method in the theory of newton-kantorovich approximations and the pták error estimates, *Numer. Funct. Anal. Optim.*, **9** (1987), 671–684. http://doi.org/10.1080/01630568708816254
6. D. Ruan, X. Wang, Y. Wang, Local convergence of seventh-order iterative method under weak conditions and its applications, *Eng. Comput.*, **42** (2025), 1599–1615. https://doi.org/10.1108/EC-08-2024-0775
7. I. K. Argyros, S. George, On the complexity of extending the convergence domain of Newton's method under the weak majorant condition, *Can. Math. Bull.*, **67** (2024), 781–795. https://doi.org/10.4153/S000843952400016X
8. H. Singh, J. R. Sharma, S. Kumar, A simple yet efficient two-step fifth-order weighted-Newton method for nonlinear models, *Numer. Algor.*, **93** (2023), 203–225. https://doi.org/10.1007/s11075-022-01412-w
9. A. Cordero, R. V. Rojas-Hiciano, J. R. Torregrosa, M. P. Vassileva, Maximally efficient damped composed Newton-type methods to solve nonlinear systems of equations, *Appl. Math. Comput.*, **492** (2025), 129231. https://doi.org/10.1016/j.amc.2024.129231

10. T. Liu, H. Zhang, X. Yang, The ADI compact difference scheme for three-dimensional integro-partial differenqtial equation with three weakly singular kernels, *J. Appl. Math. Comput.*, **71** (2025), 3861–3889. https://doi.org/10.1007/s12190-025-02386-3

11. K. Liu, Z. He, H. Zhang, X. Yang, A Crank-Nicolson ADI compact difference scheme for the three-dimensional nonlocal evolution problem with a weakly singular kernel, *Comput. Appl. Math.*, **44** (2025), 164. https://doi.org/10.1007/s40314-025-03125-x

12. H. Yuan, Q. Zhu, The well-posedness and stabilities of mean-field stochastic differential equations driven by G-Brownian motion, *SIAM J. Control Optim.* **63** (2025), 596–624. https://doi.org/10.1137/23M1593681

13. Q. Zhu, Event-triggered sampling problem for exponential stability of stochastic nonlinear delay systems driven by Lévy processes. *IEEE Trans. Autom. Control*, **70** (2025), 1176–1183. https://doi.org/10.1109/TAC.2024.3448128

14. A. Cordero, E. Martínez, J. R. Torregrosa, Iterative methods of order four and five for systems of nonlinear equations, *J. Comput. Appl. Math.*, **231** (2009), 541–551. https://doi.org/10.1016/j.cam.2009.04.015

15. A. Cordero, J. L. Hueso, E. Martínez, J. R. Torregrosa, A modified Newton-Jarratt's composition, *Numer. Algor.*, **55** (2010), 88–99. https://doi.org/10.1007/s11075-009-9359-z

16. A. Cordero, J. R. Torregrosa, M. P. Vassileva, Pseudocomposition: A technique to design predictor-corrector methods for systems of nonlinear equations, *Appl. Math. Comput.*, **218** (2012), 11496–11504. https://doi.org/10.1016/j.amc.2012.04.081

17. G. E. Alefeld, F. A. Potra, Z. Shen, On the existence theorems of Kantorovich, Moore and Miranda, in *Topics in Numerical Analysis* , **15** (2001), 21–28. http://doi.org/10.1007/978-3-7091-6217-0_3

18. A. Cordero, J. G. Maimó, E. Martinez, J. R. Torregrosa, M. P. Vassileva, Semilocal convergence of the extension of Chun's method, *Axioms*, **10** (2021), 161. http://doi.org/10.3390/axioms10030161

19. S. Yadav, S. Singh, R. P. Badoni, A. Kumar, M. Singh, Semilocal convergence of Chebyshev Kurchatov type methods for non-differentiable operators, *Comput. Math. Appl.*, **170** (2024), 275–281. https://doi.org/10.1016/j.camwa.2024.07.003

20. I. K. Argyros, A new convergence theorem for the Steffenssen method in Banach space and applications, *Rev. Anal. Numér. Théor. Approx.*, **29** (2000), 119–127. https://doi.org/10.33993/jnaat292-661

21. J. M. Ortega, W. C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, Society for Industrial and Applied Mathematics, 2000.

22. J. A. Ezquerro, M. A. Hernández, Halley's method for operators with unbounded second derivative, *Appl. Numer. Math.*, **57** (2007), 354–360. https://doi.org/10.1016/j.apnum.2006.05.001