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Theory article

Gorenstein invariants under right Quasi-Frobenius extensions

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Abstract: The focus of this work was to establish relations between Gorenstein projective modules linked by right quasi-Frobenius extensions of rings. As applications, the right quasi-Frobenius extensions of (weakly) Gorenstein algebras, Cohen-Macaulay finite algebras and the Cohen-Macaulay free algebra were studied. Suppose that Γ was a right quasi-Frobenius extension of an Artin algebra Λ with Γ a completely faithful left Λ-module. We demonstrated that if Γ exhibited (weakly) Gorenstein properties, then Λ did so. Additionally, under the condition of Γ being a separable extension of Λ, the converse became valid.

Keywords: right quasi-Frobenius extension; Gorenstein projective module; CM-finite algebra; CM-free algebra; Gorenstein algebra

1. Introduction

Right (resp., left, null) quasi-Frobenius extensions of rings were first introduced by Müller [1] as a generalization of quasi-Frobenius rings and Frobenius extensions. They are closely related with separable extensions and group rings [1, 2]. For instance, K. Hirata [2] established that a central projective separable extension equivalently characterized as a quasi-Frobenius extension. Extensive research has focused on the invariant properties of rings under right (resp., null) quasi-Frobenius extensions. Notably, key homological invariants such as the dominant dimension, injectivity, and related properties are preserved under quasi-Frobenius extensions, as demonstrated in [1, 3].

In the 1960s, Auslander and Bridge [4] initially introduced the concept of finitely generated modules having Gorenstein dimension zero over a Noetherian ring; these modules are now widely termed Gorenstein projective (see [4, 5] for further details). A number of studies, including [6–10], have explored the invariance of Gorenstein projective modules under various ring extensions such as excellent extensions, Frobenius extensions, trivial ring extensions, and separable equivalences.

It is known that the Gorenstein algebra, the Cohen-Macaulay finite algebra (CM-finite algebra, for short), and the Cohen-Macaulay finite algebra (CM-free algebra, for short) are characterized in terms of Gorenstein projective modules [3, 11]. Recall that a ring Λ is called a *Gorenstein ring*, if the injective dimension of ${}_\Lambda\Lambda$ and that of Λ_Λ are finite. Let $\Gamma \geq \Lambda$ be a quasi-Frobenius extension of Λ with Γ_Λ a generator for Λ-modules. It follows from [3, Proposition 7] that Λ is a Gorenstein ring when Γ is so. Note that a generator for Λ-modules is completely faithful, but a completely faithful module is not a generator in general (see [12, P234], for detail). In this study, we shall develop their arguments and apply obtained results to right quasi-Frobenius extensions ultimately establishing the following theorem.

Theorem A. *Let Γ be a right quasi-Frobenius extension of a two-sided Noetherian ring Λ.*

(1) *Suppose that Γ is a completely faithful left Λ-module. Then, Λ is a Gorenstein ring, when Γ is so.*

(2) *Suppose that Γ is a separable extension of Λ. Then, Γ is a Gorenstein ring, when Λ is so.*

In [10], Zhao proved that CM-free (resp., CM-finite) properties are invariant under separable Frobenius extensions of commutative Artin rings. It is shown by Huang et al. [7] that CM-free (resp., CM-finite) properties are invariant under the strongly separable quasi-Frobenius extensions. In this paper, we shall improve them and prove the same results for right quasi-Frobenius extensions of Artin algebras.

Theorem B. *Let Γ be a separable right quasi-Frobenius extension of an Artin algebra Λ.*

(1) *If Λ is CM-free, then so is Γ. Furthermore, if Γ is a generator for Λ-modules, the converse also holds.*

(2) *If Λ is CM-finite, then so is Γ. Furthermore, if M is a relative generator for $\mathcal{FG}(\Gamma)$, where $\mathcal{FG}(\Gamma)$ is the category formed by all finitely generated Gorenstein projective left Λ-modules, then* $\mathrm{End}_\Lambda M$ *is a right quasi-Frobenius extension of* $\mathrm{End}_\Gamma M$.

Essential definitions and results frequently employed in this work are given in Section 2. Theorems A and B are proved in Section 3.

2. Preliminaries

In this study, unless otherwise specified, all rings are assumed to be two-sided Noetherian rings, and all modules are finitely generated left modules. Given a ring Λ, the category composed of all finitely generated left Λ-modules is denoted by $\operatorname{mod}\Lambda$.

Definition 2.1. ([1, Definition 1.1]) Let Λ be a subring of Γ containing the identity of Γ. Then Γ is said to be a ring extension of Λ, represented as $\Gamma \geq \Lambda$. Such an extension $\Gamma \geq \Lambda$ is said to be a right quasi-Frobenius extension (right QF-extension, for short), if

(1) Γ_Λ is projective;

(2) ${}_\Lambda\Gamma_\Gamma \in \mathrm{add}_\Lambda(\mathrm{Hom}_\Lambda({}_\Gamma\Gamma_\Lambda, {}_\Lambda\Lambda_\Lambda)_\Gamma$, where $\mathrm{add}_\Lambda(\mathrm{Hom}_\Lambda({}_\Gamma\Gamma_\Lambda, {}_\Lambda\Lambda_\Lambda)_\Gamma$ denotes the category formed by direct summands of finite copies of ${}_\Lambda\,\mathrm{Hom}_\Lambda({}_\Gamma\Gamma_\Lambda, {}_\Lambda\Lambda_\Lambda)_\Gamma$.

The definition of a left quasi-Frobenius extension follows analogously. When a right QF-extension $\Gamma \geq \Lambda$ additionally satisfies the criteria for a left QF-extension, it is termed a *quasi-Frobenius extension* (QF-extension, for short). Moreover, a QF-extension $\Gamma \geq \Lambda$ is called a *Frobenius extension* if there is a bimodule isomorphism ${}_\Lambda\Gamma_\Gamma \cong_\Lambda \mathrm{Hom}_\Lambda({}_\Gamma\Gamma_\Lambda, {}_\Lambda\Lambda_\Lambda)_\Gamma$; see [13, Theorem 1.2] for detail.

The following remark is easy, which is important for our arguments.

Remark 2.2. *Let* $\Gamma \geq \Lambda$ *be a right QF-extension, then*
(1) ${}_\Gamma \operatorname{Hom}_\Lambda({}_\Lambda\Gamma_\Gamma, {}_\Lambda \Lambda)_\Lambda \in \operatorname{add}_\Gamma \Gamma_\Lambda$.
(2) ${}_\Lambda\Gamma$ *is projective.*

Recall from [13, Section 2.4] that a ring extension $\Gamma \geq \Lambda$ is said to be a *separable extension*, if the map

$$\pi : \Gamma \otimes_\Lambda \Gamma \to \Gamma \qquad via \qquad \pi(a \otimes b) = ab$$

is a split epimorphism of the Γ-bimodule. By the definition of separable extensions, we have the following lemma.

Lemma 2.3. *Let* $\Gamma \geq \Lambda$ *be a separable extension and* $N \in \operatorname{mod}\Gamma$. *Then, we have* ${}_\Gamma N \in \operatorname{add}_\Gamma(\Gamma \otimes_\Lambda N)$.

Proof. By the definition of separable extensions, there exist a Γ-bimodule Y and a Γ-bimodule isomorphism ${}_\Gamma\Gamma_\Gamma \oplus {}_\Gamma Y_\Gamma \cong {}_\Gamma(\Gamma \otimes_\Lambda \Gamma)_\Gamma$. Then, for a Γ-module N, one gets Γ-module isomorphisms

$${}_\Gamma\Gamma \otimes_\Lambda N \cong {}_\Gamma(\Gamma \otimes_\Lambda \Gamma) \otimes_\Gamma N \cong {}_\Gamma(\Gamma \oplus Y) \otimes_\Gamma N \cong {}_\Gamma N \oplus {}_\Gamma(Y \otimes_\Gamma N).$$

Thus, we have ${}_\Gamma N \in \operatorname{add}_\Gamma(\Gamma \otimes_\Lambda N)$.

For a module $M \in \operatorname{mod}\Lambda$ with the projective presentation,

$$P_1 \xrightarrow{f} P_0 \longrightarrow M \longrightarrow 0.$$

Following [4], the transpose of M, denoted by $\operatorname{Tr} M$, is defined as the cokernel of the dualized map of f^*, where $(-)^* = \operatorname{Hom}_\Lambda(-, \Lambda)$.

Lemma 2.4. *([6, Lemma 3.1]) Suppose that* $\Gamma \geq \Lambda$ *is a ring extension.*
(1) *For any* $M \in \operatorname{mod}\Lambda$*, there exist projective right* Γ*-modules* Q_1, Q_2 *and a right* Γ*-module isomorphism* $\operatorname{Tr}(\Gamma \otimes_\Lambda M)_\Gamma \oplus Q_1 \cong (\operatorname{Tr} M \otimes_\Lambda \Gamma)_\Gamma \oplus Q_2$.
(2) *If* Γ_Λ *is projective and* $N \in \operatorname{mod}\Gamma$*, then there exist projective right* Λ*-modules* P_1, P_2 *and a right* Λ*-module isomorphism* $(\operatorname{Tr} N)_\Lambda \oplus P_1 \cong \operatorname{Tr} N \otimes_\Gamma \operatorname{Hom}_\Lambda({}_\Lambda\Gamma_\Gamma, {}_\Lambda \Lambda)_\Lambda \oplus P_2$.

Recall from [4, Proposition 3.8] that M is said to have *Gorenstein dimension zero*, denoted by $\operatorname{Gpd}{}_\Lambda M = 0$, if $\operatorname{Ext}^{\geq 1}_\Lambda({}_\Lambda M, \Lambda) = 0 = \operatorname{Ext}^{\geq 1}_{\Lambda^o}((\operatorname{Tr} M)_\Lambda, \Lambda)$. These modules are now often termed *Gorenstein projective* (see [5, Definition 10.2.1] for detail). The full subcategory of $\operatorname{mod}\Lambda$ comprising all finitely generated Gorenstein projective modules is denoted by $\mathcal{FG}(\Lambda)$. The *Gorenstein projective dimension* (or *Gorenstein dimension*) of M, denoted by $\operatorname{Gpd}{}_\Lambda M$, is defined as the minimal integer $n \geq 1$ for which there exists an exact sequence in $\operatorname{mod}\Lambda$: $0 \to D_n \to \cdots \to D_1 \to D_0 \to M \to 0$ with each $D_i \in \mathcal{FG}(\Lambda)$ (see [4, 14] for detail). If no such finite sequence exists, $\operatorname{Gpd}{}_\Lambda M$ is defined to be infinite. The *finitistic dimension* of Λ, denoted $\operatorname{fin.dim}\Lambda$, is defined as the supremum of projective dimensions across all modules M in $\operatorname{mod}\Lambda$ having finite projective dimensions.

Lemma 2.5. *[6, Lemma 2.5] For a two-sided Noetherian ring* Λ*,* $\operatorname{fin.dim}\Lambda = \sup\{\operatorname{Gpd} M \mid M \in \operatorname{mod}\Lambda$ *and* $\operatorname{Gpd} M < \infty\}$.

Let B be a Λ-module. From [15, P87], we know that *its character module* B^+ is the right Λ-module $\mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})$. Recall from [12, P233] that B is called *completely faithful*, provided that for every homomorphism f, $f \otimes_\Lambda B = 0$ implies $f = 0$. Recall that B is a generator (resp., cogenerator) for Λ-modules if each Λ-module is a quotient (resp., submodule) of direct sums (resp., products) of B. Clearly, every generator is completely faithful, but the converse is not true in general.

Lemma 2.6. *([12, P234]) The equivalence of the following statements holds for a projective Λ-module Q.*

(1) *Q is a completely faithful module.*

(2) *Let V be a right Λ-module. Then, $V \otimes_\Lambda Q = 0$ implies $V = 0$,*

(3) *The character module Q^+ is a cogenerator in* $\mathrm{mod}\,\Lambda^{op}$.

Lemma 2.7. *Let Λ and Γ be Noetherian rings.*

(1) *[15, Corollary 10.65] In the situation $({}_\Lambda A, {}_\Gamma B_\Lambda, {}_\Gamma C)$, assume that B_Λ is projective. Then, for any $n \geq 1$, there is an isomorphism*

$$\mathrm{Ext}^n_\Gamma(B \otimes_\Lambda A, C) \cong \mathrm{Ext}^n_\Lambda(A, \mathrm{Hom}_\Gamma(B, C)).$$

(2) *[5, Theorem 3.2.15] In the situation $({}_\Lambda A, {}_\Gamma B_\Lambda, {}_\Gamma C)$, assume that ${}_\Lambda A$ is projective. Then, for any $n \geq 1$, there is an isomorphism*

$$\mathrm{Ext}^n_\Gamma(C, B \otimes_\Lambda A) \cong \mathrm{Ext}^n_\Gamma(C, B) \otimes_\Lambda A.$$

3. Main results

Lemma 3.1. *Let $\Gamma \geq \Lambda$ be a right QF-extension, and $X \in \mathrm{mod}\,\Lambda$. Then, for any $i \geq 1$, we have the following isomorphisms*

(1) $\mathrm{Ext}^i_\Gamma(\Gamma \otimes_\Lambda X, \Gamma) \cong \mathrm{Ext}^i_\Lambda(X, \Lambda) \otimes_\Lambda \Gamma$*;*

(2) $\mathrm{Ext}^i_\Gamma(\Gamma \otimes_\Lambda X, \Gamma) \cong \mathrm{Ext}^i_\Lambda({}_\Lambda X, {}_\Lambda \Gamma)$.

Proof. Noting that ${}_\Lambda\Gamma$ and Γ_Λ are projective by assumption and by Remark 2.2(2), the assertion follows directly from Lemma 2.7.

Lemma 3.2. *Suppose that $\Gamma \geq \Lambda$ is a right QF-extension with ${}_\Lambda\Gamma$ completely faithful. Then,*

(1) $\mathrm{Hom}_\Lambda({}_\Lambda\Gamma, {}_\Lambda\Lambda_\Lambda)$ *is a completely faithful right Λ-module.*

(2) Γ_Λ *is completely faithful.*

Proof. (1) It suffices to prove that the character module $\mathrm{Hom}_\Lambda({}_\Lambda\Gamma, {}_\Lambda\Lambda_\Lambda)^+_\Lambda$ is a cogenerator by Lemma 2.6(3), because $\mathrm{Hom}_\Lambda({}_\Lambda\Gamma, {}_\Lambda\Lambda_\Lambda)_\Lambda$ is projective by Remark 2.2(1).

Let X be a finitely generated right Λ-module satisfying $\mathrm{Hom}_\Lambda(X, \mathrm{Hom}_\Lambda({}_\Lambda\Gamma_\Gamma, {}_\Lambda\Lambda_\Lambda)^+) = 0$. Since ${}_\Lambda\Gamma$ is projective, by [14, P78, Theorem 3.2.11], there are isomorphisms

$$0 = \mathrm{Hom}_\Lambda(X_\Lambda, \mathrm{Hom}_\Lambda({}_\Lambda\Gamma_\Gamma, \Lambda)^+_\Lambda) \cong \mathrm{Hom}_\Lambda(X_\Lambda, \Lambda^+ \otimes_\Lambda \Gamma) \cong \mathrm{Hom}_\Lambda(X_\Lambda, \Lambda^+) \otimes_\Lambda \Gamma,$$

So, $\mathrm{Hom}_\Lambda({}_\Lambda X, \Lambda^+) = 0$ by the completely faithful property of ${}_\Lambda\Gamma$. Since Λ^+ is also a cogenerator, $X = 0$, which implies that $\mathrm{Hom}_\Lambda({}_\Lambda\Gamma, {}_\Lambda \Lambda)^+$ is a cogenerator. Thus, we obtain our claim by Lemma 2.6(3).

(2) Since $\mathrm{Hom}_\Lambda({}_\Lambda\Gamma_\Gamma, \Lambda)_\Lambda \in \mathrm{add}\,\Gamma_\Lambda$ by Remark 2.2(1), Γ is a completely faithful right Λ-module as desired.

Remark 3.3. (1) *Let* $\Gamma \geq \Lambda$ *be a right QF-extension such that* Γ *is a completely faithful right* Λ*-module. It is not known that* Γ *is completely faithful as a left* Λ*-module.*

(2) *Let* $\Gamma \geq \Lambda$ *be a right QF-extension with* ${}_{\Lambda}\Gamma$ *a generator for* Λ*-modules, then* Γ_{Λ} *is completely faithful by Lemma 3.2. For example, if* Γ *is an excellent extension of* Λ*. It follows from [6, Proposition 4.3] that* Γ *is a right QF-extension of* Λ*. By the definition of excellent extensions,* Γ *is a free* Λ*-module. It follows from Lemma 3.2 that* Γ *is a completely faithful right* Λ*-module.*

Proposition 3.4. *Let* $\Gamma \geq \Lambda$ *be a right QF-extension, and let* $M \in \operatorname{mod}\Lambda$*. If* ${}_{\Lambda}M$ *is Gorenstein projective, so is* ${}_{\Gamma}(\Gamma \otimes_{\Lambda} M)$*. Furthermore, if* ${}_{\Lambda}\Gamma$ *is a completely faithful* Λ*-module, then the converse holds.*

Proof. By assumption, for any $i \geq 1$, we have $\operatorname{Ext}^i_{\Lambda}(M, \Lambda) = 0 = \operatorname{Ext}^i_{\Lambda}((\operatorname{Tr} M)_{\Lambda}, \Lambda)$. Since $\Gamma \geq \Lambda$ is a right QF-extension, one obtains that ${}_{\Lambda}\Gamma$ is projective by Remark 2.2(2). By Lemma 3.1(1), one has $\operatorname{Ext}^i_{\Gamma}(\Gamma \otimes_{\Lambda} M, \Gamma) \cong \operatorname{Ext}^i_{\Lambda}(M, \Lambda) \otimes_{\Lambda} \Gamma = 0$ for any $i \geq 1$. On the other hand, since both ${}_{\Lambda}\Gamma$ and Γ_{Λ} are projective, by Lemma 2.4 and Lemma 3.1(2), for any $i \geq 1$, there are isomorphisms

$$\operatorname{Ext}^i_{\Gamma}(\operatorname{Tr}(\Gamma \otimes_{\Lambda} M)_{\Gamma}, \Gamma) \cong \operatorname{Ext}^i_{\Gamma}((\operatorname{Tr} M) \otimes_{\Lambda} \Gamma, \Gamma) \cong \operatorname{Ext}^i_{\Lambda}(\operatorname{Tr} M, \Gamma) = 0.$$

This implies that $\Gamma \otimes_{\Lambda} M \in \mathcal{FG}(\Gamma)$.

Conversely, assume that ${}_{\Lambda}\Gamma$ is completely faithful and $\Gamma \otimes_{\Lambda} M \in \mathcal{FG}(\Gamma)$. Then, $\operatorname{Ext}^i_{\Gamma}(\Gamma \otimes_{\Lambda} M, \Gamma) = 0 = \operatorname{Ext}^i_{\Gamma}(\operatorname{Tr}(\Gamma \otimes_{\Lambda} M), \Gamma)$ for any $i \geq 1$.

By Lemma 3.1(1), one gets $0 = \operatorname{Ext}^i_{\Gamma}(\Gamma \otimes_{\Lambda} M, \Gamma) \cong \operatorname{Ext}^i_{\Lambda}(M, \Lambda) \otimes_{\Lambda} \Gamma$ for any $i \geq 1$. Since ${}_{\Lambda}\Gamma$ is completely faithful, by Lemma 2.6(2), $\operatorname{Ext}^i_{\Lambda}(M, \Lambda) = 0$ for each $i \geq 1$.

It remains to show $\operatorname{Ext}^{\geq 1}_{\Lambda}(\operatorname{Tr} M, \Lambda) = 0$. Since ${}_{\Lambda}\Gamma$ is projective by Remark 2.2(2), for $i \geq 1$, we have

$$\begin{aligned} 0 &= \operatorname{Ext}^i_{\Gamma}(\operatorname{Tr}(\Gamma \otimes_{\Lambda} M), \Gamma_{\Gamma}) \\ &\cong \operatorname{Ext}^i_{\Gamma}(\operatorname{Tr} M \otimes_{\Lambda} \Gamma, \Gamma_{\Gamma}) \text{ (by Lemma 2.4)} \\ &\cong \operatorname{Ext}^i_{\Lambda}((\operatorname{Tr} M)_{\Lambda}, \Gamma_{\Lambda}) \\ &\cong \operatorname{Ext}^i_{\Lambda}((\operatorname{Tr} M)_{\Lambda}, \Gamma \otimes_{\Lambda} \Lambda) \\ &\cong \Gamma \otimes_{\Lambda} \operatorname{Ext}^i_{\Lambda}(\operatorname{Tr} M, \Lambda). \text{ (by Lemma 2.7(2))} \end{aligned}$$

Note that Γ is a completely faithful right Λ-module by Lemma 3.2(2), then $\operatorname{Ext}^{\geq 1}_{\Lambda}((\operatorname{Tr} M)_{\Lambda}, \Lambda_{\Lambda})$=0.

Corollary 3.5. *Let* $\Gamma \geq \Lambda$ *be a right QF-extension with* ${}_{\Lambda}\Gamma$ *completely faithful, and let* $M \in \operatorname{mod}\Lambda$*. Then*

$$\operatorname{Gpd}_{\Gamma}(\Gamma \otimes_{\Lambda} M) = \operatorname{Gpd}{}_{\Lambda}M.$$

Proof. Since Γ_{Λ} is projective, by [6, Proposition 3.2], one obtains $\operatorname{Gpd}_{\Gamma}(\Gamma \otimes_{\Lambda} M) \leq \operatorname{Gpd}{}_{\Lambda}M$.

On the other hand, without loss of generality, let $\operatorname{Gpd}_{\Gamma}(\Gamma \otimes_{\Lambda} M) = n < \infty$. Taking an exact sequence in $\operatorname{mod}\Lambda$,

$$0 \to K_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$$

with each $G_i \in \mathcal{FG}(\Lambda)$. Applying the exact functor $\Gamma \otimes_{\Lambda} -$ to the above sequence yields an exact sequence in $\operatorname{mod}\Gamma$

$$0 \to \Gamma \otimes_{\Lambda} K_n \to \Gamma \otimes_{\Lambda} G_{n-1} \to \cdots \to \Gamma \otimes_{\Lambda} G_1 \to \Gamma \otimes_{\Lambda} G_0 \to \Gamma \otimes_{\Lambda} M \to 0.$$

By the first part of Proposition 3.4, one gets $\Gamma \otimes_\Lambda G_i \in \mathcal{FG}(\Gamma)$ for all $0 \le i \le n-1$. So, $\Gamma \otimes_\Lambda K_n \in \mathcal{FG}(\Gamma)$ by assumption and by [16, Theorem 2.20]. It follows from the second part of Proposition 3.4 that $K_n \in \mathcal{FG}(\Lambda)$. This implies $\operatorname{Gpd}{}_\Lambda M \le n$ by [16, Corollary 2.21].

Corollary 3.6. *Suppose that $\Gamma \ge \Lambda$ is a right QF-extension of Λ and $M \in \operatorname{mod}\Lambda$. If M is Gorenstein projective as a Λ-module, then so is $\operatorname{Hom}_\Lambda({}_\Lambda\Gamma_\Gamma, M)$ as a Γ-module.*

Proof. Since $\Gamma \ge \Lambda$ is a right QF-extension, from Remark 2.2 one has $\operatorname{Hom}_\Lambda({}_\Lambda\Gamma_\Gamma, M) \cong \operatorname{Hom}_\Lambda({}_\Lambda\Gamma_\Gamma, \Lambda) \otimes_\Lambda M \in \operatorname{add}_\Gamma(\Gamma \otimes_\Lambda M)$. By Proposition 3.4, $\Gamma \otimes_\Lambda M$ is Gorenstein projective. The assertion follows from [16, Theorem 2.5].

Proposition 3.7. *Let Γ be a right QF-extension of Λ and $N \in \operatorname{mod}\Gamma$. Suppose that N is Gorenstein projective as a Γ-module, then so is the underlying module N as a Λ-module. Furthermore, the converse holds when $\Gamma \ge \Lambda$ is separable.*

Proof. By assumption, for any $i \ge 1$, one has $\operatorname{Ext}^i_\Gamma(N, \Gamma) = 0 = \operatorname{Ext}^i_\Gamma((\operatorname{Tr} N)_\Gamma, \Gamma)$. Hence, for each $i \ge 1$, $\operatorname{Ext}^i_\Gamma({}_\Gamma N, \operatorname{Hom}_\Lambda({}_\Lambda\Gamma_\Gamma, \Lambda)) = 0$, because ${}_\Gamma\operatorname{Hom}_\Lambda({}_\Lambda\Gamma_\Gamma, \Lambda)$ is projective by Remark 2.2(1). By Lemma 2.7(1), for any positive integer i, we have

$$\operatorname{Ext}^i_\Lambda({}_\Lambda N, \Lambda) \cong \operatorname{Ext}^i_\Lambda({}_\Lambda\Gamma \otimes_\Gamma N, \Lambda) \cong \operatorname{Ext}^i_\Gamma({}_\Gamma N, \operatorname{Hom}_\Lambda({}_\Lambda\Gamma_{\Gamma,\Lambda} \Lambda)) = 0.$$

On the other hand, since ${}_\Lambda\Gamma$ is projective by Remark 2.2(2), for any positive integer i, we have

$$\begin{aligned}
&\operatorname{Ext}^i_\Lambda((\operatorname{Tr} N)_\Lambda, \Lambda)\\
\cong\ &\operatorname{Ext}^i_\Lambda(\operatorname{Tr}_\Gamma N \otimes_\Gamma \operatorname{Hom}_\Lambda({}_\Lambda\Gamma_{\Gamma,\Lambda} \Lambda), \Lambda) \text{ (by Lemma 2.4)}\\
\cong\ &\operatorname{Ext}^i_\Gamma(\operatorname{Tr}_\Gamma N, \operatorname{Hom}_\Lambda(\operatorname{Hom}_\Lambda({}_\Lambda\Gamma_{\Gamma,\Lambda} \Lambda), \Lambda)) \text{ (by Lemma 2.7(1))}\\
\cong\ &\operatorname{Ext}^i_\Gamma((\operatorname{Tr} N)_\Gamma, \Gamma) = 0.
\end{aligned}$$

This means ${}_\Lambda N \in \mathcal{FG}(\Lambda)$ as desired.

Conversely, assume that $N \in \mathcal{FG}(\Lambda)$. By the first part of Proposition 3.4, $\Gamma \otimes_\Lambda N \in \mathcal{FG}(\Gamma)$. Noting that $\Gamma \ge \Lambda$ is separable, then, by Lemma 2.3, we obtain ${}_\Gamma N \in \operatorname{add}_\Gamma(\Gamma \otimes_\Lambda N)$. Thus, by [16, Theorem 2.5], one gets $N \in \mathcal{FG}(\Gamma)$.

Proposition 3.8. *Let $\Gamma \ge \Lambda$ be a right QF-extension, and let $N \in \operatorname{mod}\Gamma$. Then,* $\operatorname{Gpd}{}_\Lambda N \le \operatorname{Gpd}{}_\Gamma N$. *Moreover, the equality holds true when $\Gamma \ge \Lambda$ is separable.*

Proof. We may assume, without loss of generality, that $\operatorname{Gpd}{}_\Gamma N = m$, then we have the following exact sequence in $\operatorname{mod}\Gamma$:

$$0 \to D_m \to D_{m-1} \to \cdots \to D_0 \to N \to 0$$

in $\operatorname{mod}\Gamma$, where each ${}_\Gamma D_i$ lies in $\mathcal{FG}(\Gamma)$. Clearly, the above sequence remains exact in $\operatorname{mod}\Lambda$. Also, by the first part of Proposition 3.7, we have ${}_\Lambda D_i \in \mathcal{FG}(\Lambda)$, for each $0 \le i \le m$. This implies $\operatorname{Gpd}{}_\Lambda N \le m$.

Let $\Gamma \ge \Lambda$ be a separable right QF-extension, and we have ${}_\Gamma N \in \operatorname{add}_\Gamma(\Gamma \otimes_\Lambda N)$, by Lemma 2.3. So, one has $\operatorname{Gpd}{}_\Gamma N \le \operatorname{Gpd}{}_\Gamma(\Gamma \otimes_\Lambda N) \le \operatorname{Gpd}{}_\Lambda N$ by ([6, Proposition 3.2].

Theorem 3.9. *If $\Gamma \ge \Lambda$ is a separable right QF-extension with ${}_\Lambda\Gamma$ completely faithful, then*

$$\operatorname{fin.dim}\Gamma = \operatorname{fin.dim}\Lambda.$$

Proof. This assertion follows immediately from Lemma 2.5, Corollary 3.5, and the second part of Proposition 3.8.

Theorem 3.10. *Let* Γ *and* Λ *be two Artin algebras.*

(1) *Let* $\Gamma \geq \Lambda$ *be a right QF-extension with* ${}_\Lambda\Gamma$ *completely faithful. If* Γ *is Gorenstein, then so is* Λ*.*

(2) *Let* $\Gamma \geq \Lambda$ *be a separable right QF-extension. If* Λ *is Gorenstein, then so is* Γ*.*

Proof. According to [17, Theorem 1.1], an Artin algebra is Gorenstein precisely when every finitely generated left module has finite Gorenstein projective dimension. (1) follows directly from Corollary 3.5, while (2) is an immediate consequence of the first part of Proposition 3.8.

Let Λ be an Artin algebra over a commutative Artin ring R. Recall from [18, Section 1] that a Λ-module M is termed *semi-Gorenstein projective*, provided $\mathrm{Ext}_\Lambda^{\geq 1}(M, \Lambda) = 0$. We write ${}^\perp\Lambda = \{X \in \mathrm{mod}\,\Lambda \mid \mathrm{Ext}_\Lambda^{\geq 1}(X, \Lambda) = 0\}$. Λ is termed *left weakly Gorenstein*, if $\mathcal{GP}(\Lambda) = {}^\perp\Lambda$. The notion of left weakly Gorenstein algebras is a generalization of that of Gorenstein algebras. Hence, Λ is left weakly Gorenstein if, and only if, every semi-Gorenstein projective Λ-module is Gorenstein projective.

Theorem 3.11. *Let* Γ *and* Λ *be two Artin* R*-algebras.*

(1) *Let* $\Gamma \geq \Lambda$ *be a right QF-extension with* ${}_\Lambda\Gamma$ *completely faithful. If* Γ *is left weakly Gorenstein, then so is* Λ*.*

(2) *Let* $\Gamma \geq \Lambda$ *be a separable right QF-extension. If* Λ *is left weakly Gorenstein, then so is* Γ*.*

Proof. (1) Let M be a semi-Gorenstein projective Λ-module. Then, one has $\mathrm{Ext}_\Lambda^{\geq 1}(M, \Lambda) = 0$. By assumption and by Remark 2.2(2), both ${}_\Lambda\Gamma$ and Γ_Λ are projective. Then, by Lemma 2.7(1), for any $i \geq 1$, we have $\mathrm{Ext}^i_\Gamma(\Gamma \otimes_\Lambda M, \Gamma) \cong \mathrm{Ext}^i_\Lambda(M, \Gamma) = 0$, which implies that $\Gamma \otimes_\Lambda M$ is semi-Gorenstein projective. So, $\Gamma \otimes_\Lambda M \in \mathcal{FG}(\Gamma)$ by assumption. It follows from the second part of Proposition 3.4 that $M \in \mathcal{FG}(\Lambda)$. Hence, Λ is a left weakly Gorenstein algebra.

(2) suppose that Λ is left weakly Gorenstein. Given a semi-Gorenstein Γ-module N, by Lemma 2.7(1) and by [6, Proposition 3.2], there are isomorphisms $\mathrm{Ext}^i_\Lambda(N, \Lambda) \cong \mathrm{Ext}^i_\Lambda(\Gamma \otimes_\Gamma N, \Lambda) \cong \mathrm{Ext}^i_\Gamma({}_\Gamma N, \mathrm{Hom}_\Lambda({}_\Lambda\Gamma_\Gamma, \Lambda)) = 0$ for any $i \geq 1$, because ${}_\Gamma\,\mathrm{Hom}_\Lambda({}_\Lambda\Gamma_\Gamma, \Lambda)$ is projective by Remark 2.2(1). It implies that N is also a semi-Gorenstein Λ-module, and, hence, one obtains $N \in \mathcal{FG}(\Lambda)$ by assumption. Therefore, $N \in \mathcal{FG}(\Gamma)$ by the second part of Proposition 3.7. Thus, we complete this proof.

The following lemma is due to Y. Kitamura in [19].

Lemma 3.12. *[19, Theorem 1.2] Let* $\Gamma \geq \Lambda$ *be a right QF-extension and* $N \in \mathrm{mod}\,\Gamma$*. If* ${}_\Gamma\Gamma \otimes_\Lambda N \in \mathrm{add}\,{}_\Gamma N$*, then* $\mathrm{End}_\Lambda N \geq \mathrm{End}_\Gamma N$ *also is a right QF-extension.*

Let Λ be an Artin R-algebra. By $\mathbf{D}(-)$, we denote the standard duality $\mathrm{Hom}_R(-, E(R/rad(R)))$, where $E(R/rad(R))$ is the injective envelope of $R/rad(R)$. Following [11, Section 1], Λ is termed *Cohen-Macaulay free*, or simply, CM-free, provided $\mathcal{FG}(\Lambda) = \mathcal{P}(\Lambda)$ (where $\mathcal{P}(\Lambda)$ is the category of finitely generated projective Λ-modules). Additionally, Λ is called *Cohen-Macaulay finite* [20, Section 8], or simply, CM-finite, if there exists a Gorenstein projective Λ-module G such that $\mathcal{FG}(\Lambda) = \mathrm{add}\,{}_\Lambda G$, and G is called *a relative generator for* $\mathcal{FG}(\Lambda)$. Clearly, a Cohen-Macaulay free algebra is Cohen-Macaulay finite.

Theorem 3.13. *Suppose Γ is a separable right QF-extension of an Artin R-algebra Λ. Then,*

$$\begin{array}{ccc} \Lambda\ is\ CM-free & \xleftrightarrow{\text{generator}} & \Gamma\ is\ CM-free \\ \downarrow & & \downarrow \\ \Lambda\ is\ CM-finite & \longrightarrow & \Gamma\ is\ CM-finite. \end{array}$$

That is,

(1) *If Λ is CM-free, then so is Γ. Moreover, the converse holds when Γ is a generator for* mod Λ.

(2) *If Λ is CM-finite, then so is Γ. Moreover, if M is a relative generator for $\mathcal{FG}(\Gamma)$, then the ring homomorphism $\rho : \mathrm{End}_\Gamma M \to \mathrm{End}_\Lambda M$ is a right QF-extension.*

Proof. (1) For $N \in \mathcal{FG}(\Gamma)$, one has $N \in \mathcal{FG}(\Lambda)$ by the first part of Proposition 3.7. Hence, by assumption, one obtains ${}_\Lambda N \in \mathcal{P}(\Lambda)$. So, $\Gamma \otimes_\Lambda N \in \mathcal{P}(\Gamma)$. Noting that $\Gamma \geq \Lambda$ is separable, ${}_\Gamma N \in \mathrm{add}_\Gamma(\Gamma \otimes_\Lambda N)$ by Lemma 2.3. Then, ${}_\Gamma N$ is projective. Thus, Γ is a CM-free algebra.

Conversely, assume that ${}_\Lambda\Gamma$ is a generator, and Γ is CM-free. Let $M \in \mathcal{FG}(\Lambda)$. By Corollary 3.6, we have $\mathrm{Hom}_\Lambda({}_\Lambda\Gamma_\Gamma, M) \in \mathcal{FG}(\Gamma)$, which is projective by assumption. Hence, $\mathbf{D}(\mathrm{Hom}_\Lambda({}_\Lambda\Gamma_\Gamma, M))$ is an injective right Γ-module. By [15, Lemma 3.60], there is an isomorphism $\mathbf{D}(\mathrm{Hom}_\Lambda({}_\Lambda\Gamma_\Gamma, M)) \cong \mathbf{D}M \otimes_\Lambda \Gamma$. Hence, $\mathbf{D}M$ is injective by [3, Proposition 7] , which implies that M is projective. Therefore, Λ is CM-free.

(2) Assume that Λ is Cohen-Macaulay finite, with G being a relative generator for $\mathcal{FG}(\Lambda)$. By Proposition 3.4, we have $\Gamma \otimes_\Lambda G \in \mathcal{FG}(\Gamma)$. For any $X \in \mathcal{FG}(\Gamma)$, one gets ${}_\Lambda X \in \mathcal{FG}(\Lambda)$ by Proposition 3.7. So, ${}_\Lambda X \in \mathrm{add}_\Lambda G$ and, hence, one gets ${}_\Gamma(\Gamma \otimes_\Lambda X) \in \mathrm{add}_\Gamma(\Gamma \otimes_\Lambda G)$. On the other hand, noting that the ring extension $\Gamma \geq \Lambda$ is separable, we have ${}_\Gamma X \in \mathrm{add}_\Gamma(\Gamma \otimes_\Lambda X)$. It follows that ${}_\Gamma X \in \mathrm{add}_\Gamma(\Gamma \otimes_\Lambda G)$, which means that Γ is CM-finite.

By assumption and by the above discussion, one obtains that Γ is Cohen-Maulay finite. If M is a relative generator for $\mathcal{FG}(\Gamma)$, then we have that $M \in \mathcal{FG}(\Lambda)$ by Proposition 3.7. It follows from the first part of Proposition 3.4 that ${}_\Gamma(\Gamma \otimes_\Lambda M) \in \mathcal{FG}(\Gamma)$. So, ${}_\Gamma(\Gamma \otimes_\Lambda M) \in \mathrm{add}_\Gamma M$ by assumption and by Proposition 3.4, which satisfies the condition of Lemma 3.12. Therefore, $\mathrm{End}_\Lambda M \geq \mathrm{End}_\Gamma M$ is a right QF-extension as a direct consequence of Lemma 3.12.

Recall that a separable extension $\Gamma \geq \Lambda$ is said to be *strongly separable*, if $M \in \mathrm{add}_\Lambda(\Gamma \otimes_\Lambda M)$ for any Λ-module M. Under this condition, ${}_\Lambda\Gamma$ is a generator for Λ-modules. Due to Theorems 3.10 and 3.13, we re-obtain results in [7].

Corollary 3.14. *Let Γ be a strongly separable quasi-Frobenius extension of Λ. Then*

(1) *([7, Corollary 3.9]) Λ is Gorenstein if, and only if, Γ is Gorenstein.*

(2) *([7, Corollary 3.10]) Λ is CM-free if, and only if, Γ is CM-free.*

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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