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Research article

Module algebra structures of nonstandard quantum group $X_q(A_1)$ on the quantum plane

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Abstract: In this paper, for $n \geq 2$ and $n \neq 3$, the module algebra structures of $X_q(A_1)$ on the quantum n-space were discussed, where the quantum n-space is denoted by $A_q(n)$. In particular, a complete list of $X_q(A_1)$-module algebra structures on the quantum plane $A_q(2)$ was produced and the isomorphism classes of these structures were described.

Keywords: nonstandard quantum group; quantum n-space; Hopf action; module algebra; weight

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1. Introduction

The nonstandard quantum groups were studied in [1], where Ge et al. [1] obtained new solutions of the Yang-Baxter equations. For these new solutions, they followed the Faddeev-Reshetikhin-Takhtajan [2] method to establish the related quantum group structure, which, in general, may not be the same as the standard ones. In [3] one class of nonstandard quantum deformation corresponding to simple Lie algebra sl_n was given, which is denoted by $X_q(A_{n-1})$. For each vertex i ($i = 1, \cdots, n-1$) of the Dynkin diagram, the parameter q_i is equal to q or $-q^{-1}$, and if $q_i = q$ for all i, then $X_q(A_{n-1})$ is just $U_q(sl_n)$. However, if $q_i \neq q_{i+1}$ for some $1 \leq i \leq n-1$, it has the relation $E_i^2 = F_i^2 = 0$ in $X_q(A_{n-1})$, such that $X_q(A_{n-1})$ is different from $U_q(sl_n)$. For more results for nonstandard quantum groups, one can refer to [4–6].

The notion of Hopf algebra actions on algebras was introduced by Sweedler [7] in 1969. The Brauer groups of H-module and H-dimodule algebras were researched by Beattie [8]. A duality theorem for Hopf module algebras was studied by Blattner and Montgomery [9] in 1985. Moreover, the actions of Hopf algebras and their generalizations [10, 11] play an important role in quantum group theory [12, 13], and the actions of Hopf algebras have various applications in physics [14]. Duplij and Sinel'shchikov [15, 16] used a general form of the automorphism of the quantum plane to render the notion of weight for $U_q(sl_2)$-actions, and they completely classified $U_q(sl_2)$-module algebra structures on the quantum plane, which consist of 6 non-isomorphic cases. Moreover, in [17] the authors used the method of weights [15, 16] to study the module algebra structures of $U_q(sl_{m+1})$ on the coordinate algebra of quantum vector spaces. More relevant research can be found at [18, 19]. However, the module algebras of nonstandard quantum groups have not yet achieved research results. Consequently, based on the above research results, we consider here the actions of the nonstandard quantum group $X_q(A_1)$ on the quantum n-space $A_q(n)$. In particular, a complete list of $X_q(A_1)$-module algebra structures on the quantum plane $A_q(2)$ is produced and the isomorphism classes of these structures are described.

This paper is organized as follows. In Section 1, we introduce some necessary notations and concepts, as well as prove a lemma about actions on generators and any elements of $A_q(n)$. In Section 2, using the method of weights [15–17], the 0-th homogeneous component and 1-st homogeneous component of the action matrix are given. We have $2n + 1$ cases for the 0-th homogeneous component $(M_{EF})_0$, and $2n(n - 1) + 1$ cases for the 1-st homogeneous component $(M_{EF})_1$. In Section 3, we study the actions of $X_q(A_1)$ on $A_q(2)$, and characterize all module algebra structures of $X_q(A_1)$ on the quantum plane $A_q(2)$, which rely upon considering the 0-th and 1-st homogeneous components of an action.

2. Preliminaries

Throughout, we work over the complex field $\mathbb{C}$ unless otherwise stated. All algebras, Hopf algebras, and modules are defined over $\mathbb{C}$; all maps are $\mathbb{C}$-linear.

Let $(H, m, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra, where Δ, ε, and S are the comultiplication, counit, and antipode of H, respectively. Let A be a unital algebra with unit $\mathbf{1}$. Sweedler's notations [7] are used in the sequel. For example, for $h \in H$, we denote

$$\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}.$$

Definition 2.1. *By a structure of an H-module algebra on A, we mean a homomorphism $\pi : H \to \mathrm{End}_{\mathbb{C}}A$ such that:*

1) for all $h \in H, a, b \in A$, $\pi(h)(ab) = \sum_{(h)} \pi(h_{(1)})(a)\pi(h_{(2)})(b)$;

2) for all $h \in H, \pi(h)(1) = \varepsilon(h)1$.

Let π_1 and π_2 be two H-module algebras on A, and the structures π_1, π_2 are said to be isomorphic, if there exists an automorphism Ψ of the algebra A, such that $\Psi\pi_1(h)\Psi^{-1} = \pi_2(h)$ for all $h \in H$.

Throughout the paper we assume that $q \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is not a root of the unit ($q^n \neq 1$ for all non-zero integers n). A class of the nonstandard quantum group $X_q(A_1)$ was studied by the authors of [3, 4]. Now, we recall the definition of $X_q(A_1)$.

Definition 2.2. *The nonstandard quantum group $X_q(A_1)$ is a unital associative $\mathbb{C}$-algebra generated by $E, F, K_1, K_2, K_1^{-1}, K_2^{-1}$ subject to the relations:*

$$K_1K_1^{-1} = K_1^{-1}K_1 = 1, \quad K_2K_2^{-1} = K_2^{-1}K_2 = 1, \quad K_1K_2 = K_2K_1, \tag{2.1}$$

$$K_1E = q^{-1}EK_1, \tag{2.2}$$
$$K_1F = qFK_1, \tag{2.3}$$
$$K_2E = -q^{-1}EK_2, \tag{2.4}$$
$$K_2F = -qFK_2, \tag{2.5}$$
$$EF - FE = \frac{K_2K_1^{-1} - K_2^{-1}K_1}{q - q^{-1}}, \tag{2.6}$$
$$E^2 = F^2 = 0. \tag{2.7}$$

The algebra $X_q(A_1)$ is also a Hopf algebra, and the comultiplication Δ, counit ε, and antipode S are given as the following:

$$\Delta(K_1) = K_1 \otimes K_1, \quad \Delta(K_2) = K_2 \otimes K_2, \tag{2.8}$$
$$\Delta(E) = E \otimes 1 + K_2K_1^{-1} \otimes E, \tag{2.9}$$
$$\Delta(F) = 1 \otimes F + F \otimes K_2^{-1}K_1, \tag{2.10}$$
$$\varepsilon(K_1) = 1,\ \varepsilon(K_2) = 1,\ \varepsilon(E) = 0,\ \varepsilon(F) = 0, \tag{2.11}$$
$$S(K_1) = K_1^{-1},\ S(K_2) = K_2^{-1},\ S(E) = -K_1K_2^{-1}E,\ S(F) = -FK_2K_1^{-1}. \tag{2.12}$$

Let us review the definition of the quantum n-space (see [20, 21]).

Definition 2.3. *The quantum n-space $A_q(n)$ is a unital algebra, generated by n generators x_i for $i \in \{1, 2, \cdots, n\}$, and for any $i > j$ it satisfies the relation:*

$$x_ix_j = qx_jx_i. \tag{2.13}$$

The quantum n-space $A_q(n)$ is also called a coordinate algebra of quantum n-dimensional vector space. If $n = 2$, $A_q(2)$ is called a quantum plane.

For all $n \geq 2$ and $n \neq 3$, by [22–24], one has a description of automorphisms of the algebra $A_q(n)$, as follows. Let Ψ be an automorphism of $A_q(n)$, and then there exist nonzero constants α_i for $i \in \{1, 2, 4, \cdots, n\}$, such that

$$\Psi : x_i \to \alpha_i x_i.$$

All such automorphisms form the automorphism group of $A_q(n)$, which we denote by $\mathrm{Aut}(A_q(n))$, and in addition, one can get

$$\mathrm{Aut}(A_q(n)) \cong (\mathbb{C}^*)^n.$$

It should be pointed out that there are more automorphisms of $A_q(3)$. Let σ be an automorphism of $A_q(3)$, and then there exist nonzero constants $\alpha, \beta, \gamma \in \mathbb{C}^*$ and $t \in \mathbb{C}$, such that

$$\sigma : x_1 \to \alpha x_1, \quad x_2 \to \beta x_2 + tx_1x_3, \quad x_3 \to \gamma x_3,$$

and $\mathrm{Aut}(A_q(3)) \cong \mathbb{C} \rtimes (\mathbb{C}^*)^3$. Obviously, the automorphism group of $A_q(3)$ is more complex, and therefore, we separately discussed the module algebra structures of nonstandard quantum group $X_q(A_1)$ on $A_q(3)$, as detailed in [25].

Unless otherwise specified, in the following text, we fix the integers $n \geq 2$ and $n \neq 3$.

Next, we give a lemma which will be useful for checking the module algebra structures of $X_q(A_1)$ on $A_q(n)$.

Lemma 2.4. *Given the module algebra actions of the generators E, F, K_1, K_2 of $X_q(A_1)$ on $A_q(n)$, if an element in the ideal generated by the relations (2.1)–(2.7) of $X_q(A_1)$, which acting on the generators x_i of $A_q(n)$ produces zero for all $i = 1, 2, 4, \cdots, n$, then this element acting on any $v \in A_q(n)$ produces zero.*

Proof. Here, we only prove that, if

$$\left[(EF - FE) - \frac{K_2K_1^{-1} - K_2^{-1}K_1}{q - q^{-1}}\right](x_i) = 0,$$
$$\left[(EF - FE) - \frac{K_2K_1^{-1} - K_2^{-1}K_1}{q - q^{-1}}\right](x_j) = 0,$$

where x_i, x_j are arbitrary generators of $A_q(n)$, then

$$\left[(EF - FE) - \frac{K_2K_1^{-1} - K_2^{-1}K_1}{q - q^{-1}}\right](x_ix_j) = 0.$$

The other relationships can be proven similarly. Indeed, by (2.9) and (2.10), we have

$$\begin{aligned}\Delta(E)\Delta(F) - \Delta(F)\Delta(E) &= (E \otimes 1 + K_2K_1^{-1} \otimes E)(1 \otimes F + F \otimes K_2^{-1}K_1)\\ &= (EF - FE) \otimes K_2^{-1}K_1 + K_2K_1^{-1} \otimes (EF - FE),\end{aligned}$$

and by Definition (2.1), then,

$$\begin{aligned}(EF - FE)(x_ix_j) &= \pi(EF - FE)(x_i)\pi(K_2^{-1}K_1))(x_j) + \pi((K_2K_1^{-1})(x_i)\pi((EF - FE))(x_j)\\ &= \frac{K_2K_1^{-1} - K_2^{-1}K_1}{q - q^{-1}}(x_i)K_2K_1^{-1}(x_j) + K_2K_1^{-1}(x_i)\frac{K_2K_1^{-1} - K_2^{-1}K_1}{q - q^{-1}}(x_j)\\ &= \frac{K_2K_1^{-1} - K_2^{-1}K_1}{q - q^{-1}}(x_ix_j).\end{aligned}$$

Thus, $\left[(EF - FE) - \frac{K_2K_1^{-1} - K_2^{-1}K_1}{q - q^{-1}}\right](x_ix_j) = 0$, and the lemma holds. □

Therefore, by Lemma 2.4, in checking whether the relations of $X_q(A_1)$, acting on any $v \in A_q(n)$, produces zero, we only need to check whether they produce zero when they act on the generators $x_1, x_2, \cdots, x_n$.

3. Properties of $X_q(A_1)$-module algebras on $A_q(n)$

In this section, we will study the module algebra structures of $X_q(A_1)$ on $A_q(n)$, where $K_1, K_2 \in \mathrm{Aut}(A_q(n))$, $n \geq 2$, and $n \neq 3$.

The s-th homogeneous component of $A_q(n)$ is denoted by $A_q(n)_s$, which is linear spanned by the monomials $x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n}$ with $m_1 + m_2 + \cdots + m_n = s$. Also, given a polynomial $p \in A_q(n)$, the s-th homogeneous component of p is denote by $(p)_s$, which is the projection of p onto $A_q(n)_s$ parallel to the direct sum of all other homogeneous components of $A_q(n)$.

By the definition of module algebra, it is easy to see that any action of $X_q(A_1)$ on $A_q(n)$ is determined by the following $4 \times n$ matrix with entries from $A_q(n)$:

$$M \overset{\text{definition}}{=} \begin{pmatrix} K_1(x_1) & K_1(x_2) & \cdots & K_1(x_n) \\ K_2(x_1) & K_2(x_2) & \cdots & K_2(x_n) \\ E(x_1) & E(x_2) & \cdots & E(x_n) \\ F(x_1) & F(x_2) & \cdots & F(x_n) \end{pmatrix}, \tag{3.1}$$

which is called the full action matrix, see [22]. Given a $X_q(A_1)$-module algebra structure on $A_q(n)$, obviously, the action of K_1 or K_2 determines an automorphism of $A_q(n)$. Therefore, by the assumption $K_1, K_2 \in \mathrm{Aut}(A_q(n)$, we can set

$$\begin{aligned} M_{K_1K_2} &\overset{\text{definition}}{=} \begin{pmatrix} K_1(x_1) & K_1(x_2) & \cdots & K_1(x_n) \\ K_2(x_1) & K_2(x_2) & \cdots & K_2(x_n) \end{pmatrix} \\ &= \begin{pmatrix} \alpha_1 x_1 & \alpha_2 x_2 & \cdots & \alpha_n x_n \\ \beta_1 x_1 & \beta_2 x_2 & \cdots & \beta_n x_n \end{pmatrix}, \end{aligned} \tag{3.2}$$

where $\alpha_i, \beta_i \in \mathbb{C}^*$ for $i \in \{1, 2, \cdots, n\}$.

It is easy to see that every monomial $x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \in A_q(n)$ is an eigenvector of K_1 and K_2, and the associated eigenvalues $\alpha_1^{m_1}\alpha_2^{m_2}\cdots\alpha_n^{m_n}$ and $\beta_1^{m_1}\beta_2^{m_2}\cdots\beta_n^{m_n}$ are called the K_1-weight and K_2-weight of this monomial, respectively, which will be written as

$$\mathrm{wt}_{K_1}(x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}) = \alpha_1^{m_1}\alpha_2^{m_2}\cdots\alpha_n^{m_n},$$
$$\mathrm{wt}_{K_2}(x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}) = \beta_1^{m_1}\beta_2^{m_2}\cdots\beta_n^{m_n}.$$

We will also need another matrix M_{EF} as follows:

$$M_{EF} \overset{\text{definition}}{=} \begin{pmatrix} E(x_1) & E(x_2) & \cdots & E(x_n) \\ F(x_1) & F(x_2) & \cdots & F(x_n) \end{pmatrix}, \tag{3.3}$$

and we call $M_{K_1K_2}$ and M_{EF} the action K_1K_2-matrix and EF-matrix, respectively. It follows from relations (2.2)–(2.5) that all entries of M are weight vectors for K_1 and K_2, and we have

$$\begin{aligned} \mathrm{wt}_{K_1}(M) &\overset{\text{definition}}{=} \begin{pmatrix} \mathrm{wt}_{K_1}(K_1(x_1)) & \mathrm{wt}_{K_1}(K_1(x_2)) & \cdots & \mathrm{wt}_{K_1}(K_1(x_n)) \\ \mathrm{wt}_{K_1}(K_2(x_1)) & \mathrm{wt}_{K_1}(K_2(x_2)) & \cdots & \mathrm{wt}_{K_1}(K_2(x_n)) \\ \mathrm{wt}_{K_1}(E(x_1)) & \mathrm{wt}_{K_1}(E(x_2)) & \cdots & \mathrm{wt}_{K_1}(E(x_n)) \\ \mathrm{wt}_{K_1}(F(x_1)) & \mathrm{wt}_{K_1}(F(x_2)) & \cdots & \mathrm{wt}_{K_1}(F(x_n)) \end{pmatrix} \\ &\bowtie \begin{pmatrix} \mathrm{wt}_{K_1}(x_1) & \mathrm{wt}_{K_1}(x_2) & \cdots & \mathrm{wt}_{K_1}(x_n) \\ \mathrm{wt}_{K_1}(x_1) & \mathrm{wt}_{K_1}(x_2) & \cdots & \mathrm{wt}_{K_1}(x_n) \\ q^{-1}\mathrm{wt}_{K_1}(x_1) & q^{-1}\mathrm{wt}_{K_1}(x_2) & \cdots & q^{-1}\mathrm{wt}_{K_1}(x_n) \\ q\mathrm{wt}_{K_1}(x_1) & q\mathrm{wt}_{K_1}(x_2) & \cdots & q\mathrm{wt}_{K_1}(x_n) \end{pmatrix} \\ &= \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ q^{-1}\alpha_1 & q^{-1}\alpha_2 & \cdots & q^{-1}\alpha_n \\ q\alpha_1 & q\alpha_2 & \cdots & q\alpha_n \end{pmatrix}, \end{aligned} \tag{3.4}$$

$$\mathrm{wt}_{K_2}(M) \overset{\text{definition}}{=} \begin{pmatrix} \mathrm{wt}_{K_2}(K_1(x_1)) & \mathrm{wt}_{K_2}(K_1(x_2)) & \cdots & \mathrm{wt}_{K_2}(K_1(x_n)) \\ \mathrm{wt}_{K_2}(K_2(x_1)) & \mathrm{wt}_{K_2}(K_2(x_2)) & \cdots & \mathrm{wt}_{K_2}(K_2(x_n)) \\ \mathrm{wt}_{K_2}(E(x_1)) & \mathrm{wt}_{K_2}(E(x_2)) & \cdots & \mathrm{wt}_{K_2}(E(x_n)) \\ \mathrm{wt}_{K_2}(F(x_1)) & \mathrm{wt}_{K_2}(F(x_2)) & \cdots & \mathrm{wt}_{K_2}(F(x_n)) \end{pmatrix}$$
$$\bowtie \begin{pmatrix} \mathrm{wt}_{K_2}(x_1) & \mathrm{wt}_{K_2}(x_2) & \cdots & \mathrm{wt}_{K_2}(x_n) \\ \mathrm{wt}_{K_2}(x_1) & \mathrm{wt}_{K_2}(x_2) & \cdots & \mathrm{wt}_{K_2}(x_n) \\ -q^{-1}\mathrm{wt}_{K_2}(x_1) & -q^{-1}\mathrm{wt}_{K_2}(x_2) & \cdots & -q^{-1}\mathrm{wt}_{K_2}(x_n) \\ -q\mathrm{wt}_{K_2}(x_1) & -q\mathrm{wt}_{K_2}(x_2) & \cdots & -q\mathrm{wt}_{K_2}(x_n) \end{pmatrix} \tag{3.5}$$
$$= \begin{pmatrix} \beta_1 & \beta_2 & \cdots & \beta_n \\ \beta_1 & \beta_2 & \cdots & \beta_n \\ -q^{-1}\beta_1 & -q^{-1}\beta_2 & \cdots & -q^{-1}\beta_n \\ -q\beta_1 & -q\beta_2 & \cdots & -q\beta_n \end{pmatrix},$$

where the relation $(a_{st}) \bowtie (b_{st})$ means that for every pair of indices s, t such that both a_{st} and b_{st} are nonzero, one has $a_{st} = b_{st}$.

In the following, we denote the j-th homogeneous component of M, whose elements are just the j-th homogeneous components of the corresponding entries of M, by $(M)_j$. Set

$$(M)_0 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix}_0,$$

where $a_i, b_i \in \mathbb{C}$ for all $i \in \{1, 2, \cdots, n\}$. Then, we obtain

$$\mathrm{wt}_{K_1}((M_{EF})_0) \bowtie \begin{pmatrix} q^{-1}\alpha_1 & q^{-1}\alpha_2 & \cdots & q^{-1}\alpha_n \\ q\alpha_1 & q\alpha_2 & \cdots & q\alpha_n \end{pmatrix}$$
$$\bowtie \begin{pmatrix} \varepsilon(K_1) & \varepsilon(K_1) & \cdots & \varepsilon(K_1) \\ \varepsilon(K_1) & \varepsilon(K_1) & \cdots & \varepsilon(K_1) \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \end{pmatrix}, \tag{3.6}$$

$$\mathrm{wt}_{K_2}((M_{EF})_0) \bowtie \begin{pmatrix} -q^{-1}\beta_1 & -q^{-1}\beta_2 & \cdots & -q^{-1}\beta_n \\ -q\beta_1 & -q\beta_2 & \cdots & -q\beta_n \end{pmatrix}$$
$$\bowtie \begin{pmatrix} \varepsilon(K_2) & \varepsilon(K_2) & \cdots & \varepsilon(K_2) \\ \varepsilon(K_2) & \varepsilon(K_2) & \cdots & \varepsilon(K_2) \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \end{pmatrix}. \tag{3.7}$$

Therefore, the relations (3.6) and (3.7) imply that a_i and b_i are at most one nonzero for any $i \in \{1, 2, \cdots, n\}$, and

$$a_i \neq 0 \Rightarrow \alpha_i = q, \beta_i = -q, \tag{3.8}$$
$$b_i \neq 0 \Rightarrow \alpha_i = q^{-1}, \beta_i = -q^{-1}. \tag{3.9}$$

An application of E and F to the relation (2.13) and by Eq (3.2), one has the following equalities:

$$E(x_i)x_j + \alpha_i^{-1}\beta_i x_i E(x_j) = qE(x_j)x_i + q\alpha_j^{-1}\beta_j x_j E(x_i), \quad \text{for} \quad i > j, \tag{3.10}$$
$$x_i F(x_j) + \beta_j^{-1}\alpha_j F(x_i)x_j = qx_j F(x_i) + q\beta_i^{-1}\alpha_i F(x_j)x_i, \quad \text{for} \quad i > j. \tag{3.11}$$

After projecting (3.10) and (3.11) to $A_q(n)_1$, we obtain

$$\begin{aligned} a_i\left(1 - q\alpha_j^{-1}\beta_j\right)x_j + a_j\left(\alpha_i^{-1}\beta_i - q\right)x_i = 0, \text{ for } i > j, \\ b_j\left(1 - q\beta_i^{-1}\alpha_i\right)x_i + b_i\left(\beta_j^{-1}\alpha_j - q\right)x_j = 0, \text{ for } i > j, \end{aligned}$$

which certainly implies

$$a_i\left(1 - q\alpha_j^{-1}\beta_j\right) = a_j\left(\alpha_i^{-1}\beta_i - q\right) = b_j\left(1 - q\beta_i^{-1}\alpha_i\right) = b_i\left(\beta_j^{-1}\alpha_j - q\right) = 0.$$

For any $i, j \in \{1, 2, \cdots, n\}$ and $i > j$, we will determine the weight constants α_i and β_i as follows:

$$a_i \neq 0 \Rightarrow \beta_j\alpha_j^{-1} = q^{-1}, \tag{3.12}$$
$$a_j \neq 0 \Rightarrow \beta_i\alpha_i^{-1} = q, \tag{3.13}$$
$$b_i \neq 0 \Rightarrow \alpha_j\beta_j^{-1} = q, \tag{3.14}$$
$$b_j \neq 0 \Rightarrow \alpha_i\beta_i^{-1} = q^{-1}. \tag{3.15}$$

Lemma 3.1. *For any $i, j, s, t \in \{1, 2, \cdots, n\}$, a_i, a_j, b_s, and b_t are at most one nonzero.*

Proof. For any $i, j \in \{1, 2, \cdots, n\}$, and $i > j$, we only prove that a_i and a_j are at most one nonzero. Assume $a_i \neq 0$ and $a_j \neq 0$, and then

$$\begin{aligned} a_i \neq 0 \quad &\Rightarrow \quad \alpha_i = q, \quad \beta_i = -q, \quad \beta_j\alpha_j^{-1} = q^{-1}, \\ a_j \neq 0 \quad &\Rightarrow \quad \alpha_j = q, \quad \beta_j = -q, \quad \beta_i\alpha_i^{-1} = q, \end{aligned}$$

by Eqs (3.8), (3.12), and (3.13). However

$$\beta_j\alpha_j^{-1} = -qq^{-1} = -1 = q^{-1} \text{ and } \beta_i\alpha_i^{-1} = -qq^{-1} = -1 = q,$$

which are impossible, since it is contradictory to q not being a root of the unit. Therefore at least one of a_i and a_j is zero for $i, j \in 1, 2, \cdots, n$.

The remaining statements can be proven in a similar way. □

In summary, we have obtained the following results for the 0-th homogeneous component $(M_{EF})_0$ of M_{EF}.

Theorem 3.2. *There are $2n+1$ cases for the 0-th homogeneous component $(M_{EF})_0$ of M_{EF}, as follows:*

1) $a_i \neq 0, a_j = 0$ for $i \neq j$ and all $b_s = 0$ for any $i, j, s \in \{1, 2, \cdots, n\}$, i.e.,

$$\begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}_0, \begin{pmatrix} 0 & a_2 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}_0, \cdots, \begin{pmatrix} 0 & 0 & \cdots & a_n \\ 0 & 0 & \cdots & 0 \end{pmatrix}_0,$$

and we have

$$\begin{aligned} a_i \neq 0 \quad \Rightarrow \quad &\alpha_i = q, \beta_i = -q, \\ &\beta_1\alpha_1^{-1} = \beta_2\alpha_2^{-1} = \cdots = \beta_{i-1}\alpha_{i-1}^{-1} = q^{-1}, \\ &\beta_{i+1}\alpha_{i+1}^{-1} = \beta_{i+2}\alpha_{i+2}^{-1} = \cdots = \beta_n\alpha_n^{-1} = q; \end{aligned} \tag{3.16}$$

2) $b_i \neq 0, b_j = 0$ for $i \neq j$ and all $a_s = 0$ for any $i, j, s \in \{1, 2, \cdots, n\}$, i.e.,

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ b_1 & 0 & \cdots & 0 \end{pmatrix}_0, \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \end{pmatrix}_0, \cdots, \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & b_n \end{pmatrix}_0,$$

and we have

$$\begin{aligned} b_i \neq 0 \quad \Rightarrow \quad & \alpha_i = q^{-1}, \beta_i = -q^{-1}, \\ & \beta_1^{-1}\alpha_1 = \beta_2^{-1}\alpha_2 = \cdots = \beta_{i-1}^{-1}\alpha_{i-1} = q, \\ & \beta_{i+1}^{-1}\alpha_{i+1} = \beta_{i+2}^{-1}\alpha_{i+2} = \cdots = \beta_n^{-1}\alpha_n = q^{-1}; \end{aligned} \tag{3.17}$$

3) all $a_i = b_i = 0$ for any $i \in \{1, 2, \cdots, n\}$, i.e.,

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}_0.$$

Therefore, it does not determine the weight constants at all.

Next, for the 1-st homogeneous component $(M_{EF})_1$, due to q not being a root of the unit, one has

$$\begin{aligned} \mathrm{wt}_{K_1}(E(x_i)) &= q^{-1}\alpha_i = q^{-1}\mathrm{wt}_{K_1}(x_i) \neq \mathrm{wt}_{K_1}(x_i), \\ \mathrm{wt}_{K_2}(E(x_i)) &= -q^{-1}\beta_i = -q^{-1}\mathrm{wt}_{K_2}(x_i) \neq \mathrm{wt}_{K_2}(x_i), \end{aligned}$$

which implies

$$(E(x_i))_1 = \sum_{s=1}^{i-1} c_{is}x_s + \sum_{s=i+1}^{n} c_{is}x_s,$$

for some $c_{is} \in \mathbb{C}$. In a similar way, we have

$$(F(x_i))_1 = \sum_{s=1}^{i-1} d_{is}x_s + \sum_{s=i+1}^{n} d_{is}x_s,$$

for some $d_{is} \in \mathbb{C}$. Hence

$$(M_{EF})_1 = \begin{pmatrix} \sum\limits_{s=2}^{n} c_{1s}x_s & c_{21}x_1 + \sum\limits_{s=3}^{n} c_{2s}x_s & \cdots & \sum\limits_{s=1}^{i-1} c_{is}x_s + \sum\limits_{s=i+1}^{n} c_{is}x_s & \cdots & \sum\limits_{s=1}^{n-1} c_{ns}x_s \\ \sum\limits_{s=2}^{n} d_{is}x_s & d_{21}x_1 + \sum\limits_{s=3}^{n} d_{2s}x_s & \cdots & \sum\limits_{s=1}^{i-1} d_{is}x_s + \sum\limits_{s=i+1}^{n} d_{is}x_s & \cdots & \sum\limits_{s=1}^{n-1} d_{ns}x_s \end{pmatrix}_1$$

where $c_{is}, d_{is} \in \mathbb{C}$.

Now project (3.10) and (3.11) to $A_q(n)_2$, and we can obtain

$$\begin{aligned}
c_{ij}(1-q\beta_j\alpha_j^{-1})x_j^2 \quad & + c_{ji}(\beta_i\alpha_i^{-1}-q)x_i^2 + \sum_{s=1}^{j-1} c_{is}(1-q^2\beta_j\alpha_j^{-1})x_sx_j \\
& + \sum_{s=j+1}^{i-1} c_{is}q(1-\beta_j\alpha_j^{-1})x_jx_s + \sum_{s=i+1}^{n} c_{is}q(1-\beta_j\alpha_j^{-1})x_jx_s \\
& + \sum_{t=1}^{j-1} c_{jt}q(\beta_i\alpha_i^{-1}-1)x_tx_i + \sum_{t=j+1}^{i-1} c_{jt}q(\beta_i\alpha_i^{-1}-1)x_tx_i \\
& + \sum_{t=i+1}^{n} c_{jt}(\beta_i\alpha_i^{-1}-q^2)x_ix_t = 0, \\
d_{ji}(1-q\beta_i^{-1}\alpha_i)x_i^2 \quad & + d_{ij}(\beta_j^{-1}\alpha_j-q)x_j^2 + \sum_{t=1}^{j-1} d_{jt}q(1-\beta_i^{-1}\alpha_i)x_tx_i \\
& + \sum_{t=j+1}^{i-1} d_{jt}q(1-\beta_i^{-1}\alpha_i)x_tx_i + \sum_{t=i+1}^{n} d_{jt}(1-q^2\beta_i^{-1}\alpha_i)x_ix_t \\
& + \sum_{s=1}^{j-1} d_{is}(\beta_j^{-1}\alpha_j-q^2)x_sx_j + \sum_{s=j+1}^{i-1} d_{is}q(\beta_j^{-1}\alpha_j-1)x_jx_s \\
& + \sum_{s=i+1}^{n} d_{is}q(\beta_j^{-1}\alpha_j-1)x_jx_s = 0.
\end{aligned}$$

for any $i, j \in \{1, 2, \cdots, n\}$ and $i > j$, Where

$$\begin{aligned}
& c_{ij}(1-q\beta_j\alpha_j^{-1}) = 0, && \text{for } i > j, \\
& c_{ji}(\beta_i\alpha_i^{-1}-q) = 0, && \text{for } i > j, \\
& c_{is}(1-q^2\beta_j\alpha_j^{-1}) = 0, && \text{for } 1 \le s \le j-1, \\
& c_{is}q(1-\beta_j\alpha_j^{-1}) = 0, && \text{for } j+1 \le s \le i-1, \\
& c_{is}q(1-\beta_j\alpha_j^{-1}) = 0, && \text{for } i+1 \le s \le n, \\
& c_{jt}q(\beta_i\alpha_i^{-1}-1) = 0, && \text{for } 1 \le t \le j-1, \\
& c_{jt}q(\beta_i\alpha_i^{-1}-1) = 0, && \text{for } j+1 \le t \le i-1, \\
& c_{jt}(\beta_i\alpha_i^{-1}-q^2) = 0, && \text{for } i+1 \le t \le n.
\end{aligned}$$

$$\begin{aligned}
& d_{ji}(1-q\beta_i^{-1}\alpha_i) = 0, && \text{for } i > j, \\
& d_{ij}(\beta_j^{-1}\alpha_j-q) = 0, && \text{for } i > j, \\
& d_{jt}q(1-\beta_i^{-1}\alpha_i) = 0, && \text{for } 1 \le t \le j-1, \\
& d_{jt}q(1-\beta_i^{-1}\alpha_i) = 0, && \text{for } j+1 \le t \le i-1, \\
& d_{jt}(1-q^2\beta_i^{-1}\alpha_i) = 0, && \text{for } i+1 \le t \le n, \\
& d_{is}(\beta_j^{-1}\alpha_j-q^2) = 0, && \text{for } 1 \le s \le j-1, \\
& d_{is}q(\beta_j^{-1}\alpha_j-1) = 0, && \text{for } j+1 \le s \le i-1, \\
& d_{is}q(\beta_j^{-1}\alpha_j-1) = 0, && \text{for } i+1 \le s \le n.
\end{aligned}$$

As a consequence, for any $i, j \in \{1, 2, \cdots, n\}$ and $i > j$, we have

$$\begin{aligned}
c_{ij} \neq 0 \quad &\Rightarrow \quad \beta_j\alpha_j^{-1} = q^{-1}, \\
c_{ji} \neq 0 \quad &\Rightarrow \quad \beta_i\alpha_i^{-1} = q,
\end{aligned}$$

$$\begin{array}{llll} c_{is} \neq 0 & \Rightarrow & \beta_j \alpha_j^{-1} = q^{-2}, & \text{for } 1 \leq s \leq j-1, \\ c_{is} \neq 0 & \Rightarrow & \beta_j \alpha_j^{-1} = 1, & \text{for } j+1 \leq s \leq i-1, \\ c_{is} \neq 0 & \Rightarrow & \beta_j \alpha_j^{-1} = 1, & \text{for } i+1 \leq s \leq n, \\ c_{jt} \neq 0 & \Rightarrow & \beta_i \alpha_i^{-1} = 1, & \text{for } 1 \leq t \leq j-1, \\ c_{jt} \neq 0 & \Rightarrow & \beta_i \alpha_i^{-1} = 1, & \text{for } j+1 \leq t \leq i-1, \\ c_{jt} \neq 0 & \Rightarrow & \beta_i \alpha_i^{-1} = q^{2}, & \text{for } i+1 \leq t \leq n. \end{array} \tag{3.18}$$

$$\begin{array}{llll} d_{ji} \neq 0 & \Rightarrow & \beta_i^{-1} \alpha_i = q^{-1}, & \\ d_{ij} \neq 0 & \Rightarrow & \beta_j^{-1} \alpha_j = q, & \\ d_{jt} \neq 0 & \Rightarrow & \beta_i^{-1} \alpha_i = 1, & \text{for } 1 \leq t \leq j-1, \\ d_{jt} \neq 0 & \Rightarrow & \beta_i^{-1} \alpha_i = 1, & \text{for } j+1 \leq t \leq i-1, \\ d_{jt} \neq 0 & \Rightarrow & \beta_i^{-1} \alpha_i = q^{-2}, & \text{for } i+1 \leq t \leq n, \\ d_{is} \neq 0 & \Rightarrow & \beta_j^{-1} \alpha_j = q^{2}, & \text{for } 1 \leq s \leq j-1, \\ d_{is} \neq 0 & \Rightarrow & \beta_j^{-1} \alpha_j = 1, & \text{for } j+1 \leq s \leq i-1, \\ d_{is} \neq 0 & \Rightarrow & \beta_j^{-1} \alpha_j = 1, & \text{for } i+1 \leq s \leq n. \end{array} \tag{3.19}$$

Lemma 3.3. *For any $i \in \{1, 2, \cdots, n\}$, every 1-st homogeneous component $(E(x_i))_1$ and $(F(x_i))_1$, if nonzero, reduces to a monomial.*

Proof. We assume that

$$E(x_i)_1 = \sum_{s=1}^{i-1} c_{is} x_s + \sum_{s=i+1}^{n} c_{is} x_s,$$

and $c_{is} \neq 0$, $c_{is'} \neq 0$ $(s \neq s')$ for some $s, s' \in \{1, 2, \cdots, i-1, i+1, \cdots, n\}$. Without loss of generality, we stipulate that $s < s'$.

If $s, s' < i$, then

$$\begin{array}{lll} c_{is} \neq 0 & \Rightarrow & \beta_s \alpha_s^{-1} = q^{-1}, \\ c_{is'} \neq 0 & \Rightarrow & \beta_1 \alpha_1^{-1} = \beta_2 \alpha_2^{-1} = \cdots = \beta_{s'-1} \alpha_{s'-1}^{-1} = 1. \end{array}$$

However, s must be one of the $\{1, 2, \cdots, s'-1\}$, and one gets $q^{-1} = 1$, which is impossible. Hence, c_{is} and $c_{is'}$ are at most one nonzero, and $(E(x_i))_1$ is equal to zero or a monomial. The remaining situations can be proven in a similar way.

Similarly, $(F(x_i))_1$ is equal to zero or a monomial. □

Additionally, since

$$\mathrm{wt}_{K_1}((M_{EF})_1) \bowtie \begin{pmatrix} q^{-1}\alpha_1 & q^{-1}\alpha_2 & \cdots & q^{-1}\alpha_n \\ q\alpha_1 & q\alpha_2 & \cdots & q\alpha_n \end{pmatrix}, \tag{3.20}$$

$$\mathrm{wt}_{K_2}((M_{EF})_1) \bowtie \begin{pmatrix} -q^{-1}\beta_1 & -q^{-1}\beta_2 & \cdots & -q^{-1}\beta_n \\ -q\beta_1 & -q\beta_2 & \cdots & -q\beta_n \end{pmatrix}. \tag{3.21}$$

We obtain the following result.

Lemma 3.4. *For any $i, j, s, t \in \{1, 2, \cdots, n\}$, $(E(x_i))_1$, $(E(x_j))_1$, $(F(x_s))_1$, $(F(x_t))_1$ are at most one nonzero.*

Proof. Here, we only prove that $(E(x_i))_1$ and $(E(x_j))_1$ are at most one nonzero. The other statements can be proven similarly.

By Lemma 3.3, we get that if $(E(x_i))_1$ and $(E(x_j))_1$ are nonzero, then they are a monomial for any $i, j \in \{1, 2, \cdots, n\}$. Assume

$$E(x_i)_1 = c_{is}x_s \neq 0 \text{ and } E(x_j)_1 = c_{js'}x_{s'} \neq 0.$$

Without loss of generality, we stipulate that $i > j$. According to the Eqs (3.20) and (3.21), we have

$$\mathrm{wt}_{K_1}(E(x_i)_1) = q^{-1}\alpha_i, \quad \mathrm{wt}_{K_2}(E(x_i)_1) = -q^{-1}\beta_i,$$

$$\mathrm{wt}_{K_1}(E(x_j)_1) = q^{-1}\alpha_j, \quad \mathrm{wt}_{K_2}(E(x_j)_1) = -q^{-1}\beta_j.$$

In addition,

$$\mathrm{wt}_{K_1}(E(x_i)_1) = \alpha_s, \quad \mathrm{wt}_{K_2}(E(x_i)_1) = \beta_s,$$

$$\mathrm{wt}_{K_1}(E(x_j)_1) = \alpha_{s'}, \quad \mathrm{wt}_{K_2}(E(x_j)_1) = \beta_{s'}.$$

So, $\alpha_i = q\alpha_s$, $\beta_i = -q\beta_s$, $\alpha_j = q\alpha_{s'}$, $\beta_j = -q\beta_{s'}$.

On the other hand, since $c_{is} \neq 0$ and $c_{js'} \neq 0$, it follows that

$$\beta_j\alpha_j^{-1} = \begin{cases} q^{-1} & s = j, \\ q^{-2} & 1 \leq s \leq j-1, \\ 1 & j+1 \leq s \leq n, \end{cases}$$

$$\beta_i\alpha_i^{-1} = \begin{cases} q & s' = i, \\ 1 & 1 \leq s' \leq j-1, \\ q^2 & j+1 \leq s' \leq n, \end{cases}$$

by (3.18). Then $q^{-1} = -q^{-2}$ or $q^{-1} = -1$, and $q = -q^2$ or $q = -1$, which are impossible. Hence, $(E(x_i))_1$ and $(E(x_j))_1$ are at most one nonzero. □

From the above discussion, we have the following result for the 1-st homogeneous component $(M_{EF})_1$ of M_{EF}.

Theorem 3.5. *There are $2n(n-1)+1$ cases for the 1-st homogeneous component $(M_{EF})_1$ of M_{EF}, as follows:*

1) $c_{is} \neq 0$ $(i \neq s)$, and otherwise $c_{i's'} = 0$ and all $d_{jt} = 0$ for any $i, s, j, t, i', s' \in \{1, 2, \cdots, n\}$, i.e.,

$$\begin{pmatrix} 0 & \cdots & 0 & c_{is}x_s & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_1,$$

and we have $\alpha_i = q\alpha_s, \beta_i = -q\beta_s$, and

$$\begin{aligned} \text{if } i > s, \text{then } \beta_s\alpha_s^{-1} = q^{-1}, \quad & \beta_{i+1}\alpha_{i+1}^{-1} = \beta_{i+2}\alpha_{i+2}^{-1} = \cdots = \beta_n\alpha_n^{-1} = 1, \\ & \beta_{i-1}\alpha_{i-1}^{-1} = \beta_{i-2}\alpha_{i-2}^{-1} = \cdots = \beta_{s+1}\alpha_{s+1}^{-1} = q^{-2}, \\ & \beta_{s-1}\alpha_{s-1}^{-1} = \beta_{s-2}\alpha_{s-2}^{-1} = \cdots = \beta_1\alpha_1^{-1} = 1; \end{aligned} \tag{3.22}$$

$$\text{if } i < s, \text{ then } \beta_s\alpha_s^{-1} = q, \quad \begin{aligned} &\beta_{i-1}\alpha_{i-1}^{-1} = \beta_{i-2}\alpha_{i-2}^{-1} = \cdots = \beta_1\alpha_1^{-1} = 1, \\ &\beta_{i+1}\alpha_{i+1}^{-1} = \beta_{i+2}\alpha_{i+2}^{-1} = \cdots = \beta_{s-1}\alpha_{s-1}^{-1} = q^2, \\ &\beta_{s+1}\alpha_{s+1}^{-1} = \beta_{s+2}\alpha_{s+2}^{-1} = \cdots = \beta_n\alpha_n^{-1} = 1; \end{aligned} \tag{3.23}$$

2) $d_{is} \neq 0$ $(i \neq s)$, and otherwise $d_{i's'} = 0$ and all $c_{jt} = 0$ for any $i, s, j, t, i', s' \in \{1, 2, \cdots, n\}$, i.e.,

$$\begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & d_{is}x_s & 0 & \cdots & 0 \end{pmatrix}_1,$$

and we have $\alpha_i = q^{-1}\alpha_s, \beta_i = -q^{-1}\beta_s$, and

$$\text{if } i > s, \text{ then } \beta_s^{-1}\alpha_s = q, \quad \begin{aligned} &\beta_{i+1}^{-1}\alpha_{i+1} = \beta_{i+2}^{-1}\alpha_{i+2} = \cdots = \beta_n^{-1}\alpha_n = 1, \\ &\beta_{i-1}^{-1}\alpha_{i-1} = \beta_{i-2}^{-1}\alpha_{i-2} = \cdots = \beta_{s+1}^{-1}\alpha_{s+1} = q^2, \\ &\beta_{s-1}^{-1}\alpha_{s-1} = \beta_{s-2}^{-1}\alpha_{s-2} = \cdots = \beta_1^{-1}\alpha_1 = 1; \end{aligned} \tag{3.24}$$

$$\text{if } i < s, \text{ then } \beta_s^{-1}\alpha_s = q^{-1}, \quad \begin{aligned} &\beta_{i-1}^{-1}\alpha_{i-1} = \beta_{i-2}^{-1}\alpha_{i-2} = \cdots = \beta_1^{-1}\alpha_1 = 1, \\ &\beta_{i+1}^{-1}\alpha_{i+1} = \beta_{i+2}^{-1}\alpha_{i+2} = \cdots = \beta_{s-1}^{-1}\alpha_{s-1} = q^{-2}, \\ &\beta_{s+1}^{-1}\alpha_{s+1} = \beta_{s+2}^{-1}\alpha_{s+2} = \cdots = \beta_n^{-1}\alpha_n = 1; \end{aligned} \tag{3.25}$$

3) all $c_{is} = 0$ and $d_{i's'} = 0$, for any $i, s, i', s' \in \{1, 2, \cdots, n\}$, i.e.,

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}_1.$$

Therefore, it does not determine the weight constants at all.

4. The structures of $X_q(A_1)$-module algebra on $A_q(2)$

In this section, our aim is to describe the concrete $X_q(A_1)$-module algebra structures on the quantum plane $A_q(2)$, where $K_1, K_2 \in \mathrm{Aut}(A_q(2)) \cong (\mathbb{C}^*)^2$.

By Theorems 3.2 and 3.5, it follows that if both the 0-th homogeneous component and the 1-st homogeneous component of M_{EF} are nonzero, it is easy to see that these series are empty, so we only need to consider 9 possibilities.

$$\left[\begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix}_0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_1\right], \left[\begin{pmatrix} 0 & a_2 \\ 0 & 0 \end{pmatrix}_0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_1\right], \left[\begin{pmatrix} 0 & 0 \\ b_1 & 0 \end{pmatrix}_0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_1\right],$$

$$\left[\begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix}_0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_1\right], \left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_0, \begin{pmatrix} c_{12}x_2 & 0 \\ 0 & 0 \end{pmatrix}_1\right], \left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_0, \begin{pmatrix} 0 & c_{21}x_1 \\ 0 & 0 \end{pmatrix}_1\right],$$

$$\left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_0, \begin{pmatrix} 0 & 0 \\ d_{12}x_2 & 0 \end{pmatrix}_1\right], \left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_0, \begin{pmatrix} 0 & 0 \\ 0 & d_{21}x_1 \end{pmatrix}_1\right], \left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_1\right]$$

where $a_i \neq 0, b_i \neq 0$ for $i = 1, 2$ and $c_{12}, c_{21}, d_{12}, d_{21}$ are not zero.

Lemma 4.1. *If the 0-th homogeneous component of M_{EF} is zero and the 1-st homogeneous component of M_{EF} is nonzero, then these series are empty.*

Proof. Now we show that the $\left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_0, \begin{pmatrix} c_{12}x_2 & 0 \\ 0 & 0 \end{pmatrix}_1\right]$-series is empty. If we suppose the contrary, then it follows from

$$EF - FE = \frac{K_2K_1^{-1} - K_2^{-1}K_1}{q - q^{-1}}$$

that within this series, one can have

$$\frac{K_2K_1^{-1} - K_2^{-1}K_1}{q - q^{-1}}(x_1) = \frac{\beta_1\alpha_1^{-1} - \beta_1^{-1}\alpha_1}{q - q^{-1}}x_1.$$

By $c_{12} \neq 0$, one can get $\alpha_1 = q\alpha_2, \beta_1 = -q\beta_2$, and $\beta_2\alpha_2^{-1} = q$. Hence, $\beta_1\alpha_1^{-1} = -q$, and

$$\frac{K_2K_1^{-1} - K_2^{-1}K_1}{q - q^{-1}}(x_1) = -x_1.$$

On the other hand, projecting $(EF - FE)(x_1)$ to $A_q(2)_1$, we obtain

$$(EF - FE)(x_1) = E(F(x_1)) - F(E(x_1)) = E(0) - F(c_{12}x_2) = 0.$$

However, $0 \neq -x_i$. We get the contradiction, and prove our claim.

In a similar way, one can prove that all other series where the 0-st homogeneous component of M_{EF} is zero and the 1-st homogeneous component of M_{EF} is nonzero are empty. □

Lemma 4.2. *If the* 0*-th homogeneous component of* M_{EF} *is nonzero and the* 1*-st homogeneous component of* M_{EF} *is zero, then these series are empty.*

Proof. We only show that the $\left[\begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix}_0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_1\right]$-series is empty, and in a similar way, one can prove that all other series are empty.

Consider this series and we obtain that

$$a_1 \neq 0 \quad \Rightarrow \quad \alpha_1 = q, \beta_1 = -q, \beta_2\alpha_2^{-1} = q,$$

and suppose that it is not empty. We set

$$\begin{array}{lll}
K_1(x_1) = \alpha_1 x_1 = qx, \quad K_2(x_1) = \beta_1 x_1 = -qx_1, & & \\
K_2(x_2) = \alpha_2 x_2, \qquad\qquad\quad K_2(x_2) = \beta_2 x_2, & & \\
E(x_1) = a_1 + \sum\limits_{m_1+m_2\geq 2} \rho_{m_1m_2} x_1^{m_1} x_2^{m_2} & & \text{for } m_1, m_2 \in \mathbb{N}, \\
E(x_2) = \sum\limits_{l_1+l_2\geq 2} \theta_{l_1l_2} x_1^{l_1} x_2^{l_2} & & \text{for } l_1, l_2 \in \mathbb{N}, \\
F(x_1) = \sum\limits_{t_1+t_2\geq 2} \sigma_{t_1t_2} x_1^{t_1} x_2^{t_2} & & \text{for } t_1, t_2 \in \mathbb{N}, \\
F(x_2) = \sum\limits_{h_1+h_2\geq 2} \tau_{h_1h_2} x_1^{h_1} x_2^{h_2} & & \text{for } h_1, h_2 \in \mathbb{N},
\end{array}$$

where $\alpha_2, \beta_2 \in \mathbb{C}^*$, and $\rho_{m_1m_2},\ \theta_{l_1l_2},\ \sigma_{t_1t_2},\ \tau_{h_1h_2} \in \mathbb{C}$.

Then we apply the relations (2.1)–(2.7) to the generators of $A_q(2)$. It is easy to see that the application of relation (2.1) to the generators of $A_q(2)$ produces zero. So, we consider the residue, as follows.

$$(K_1E - q^{-1}EK_1)(x_1) = K_1(E(x_1)) - q^{-1}E(K_1(x_1))$$

$$
\begin{aligned}
&= K_1(a_1 + \sum_{m_1+m_2\geq 2} \rho_{m_1m_2} x_1^{m_1} x_2^{m_2}) - q^{-1}qE(x_1) \\
&= a_1 + \sum_{m_1+m_2\geq 2} \rho_{m_1m_2} \alpha_1^{m_1}\alpha_2^{m_2} x_1^{m_1} x_2^{m_2} - E(x_1) \\
&= \sum_{m_1+m_2\geq 2} \rho_{m_1m_2} (\alpha_1^{m_1}\alpha_2^{m_2} - 1) x_1^{m_1} x_2^{m_2} = 0,
\end{aligned}
$$

and then $\rho_{m_1m_2} = 0$ for all $m_1, m_2 \in \mathbb{N}$ with $m_1 + m_2 \geq 2$, or $\alpha_2^{m_2} = q^{-m_1}$ for some $m_1, m_2 \in \mathbb{N}$ with $m_1 + m_2 \geq 2$.

$$
\begin{aligned}
(K_2E + q^{-1}EK_2)(x_1) &= K_2(E(x_1)) + q^{-1}E(K_2(x_1)) \\
&= K_2(a_1 + \sum_{m_1+m_2\geq 2} \rho_{m_1m_2} x_1^{m_1} x_2^{m_2}) - q^{-1}qE(x_1) \\
&= a_1 + \sum_{m_1+m_2\geq 2} \rho_{m_1m_2} \beta_1^{m_1}\beta_2^{m_2} x_1^{m_1} x_2^{m_2} - E(x_1) \\
&= \sum_{m_1+m_2\geq 2} \rho_{m_1m_2} (\beta_1^{m_1}\beta_2^{m_2} - 1) x_1^{m_1} x_2^{m_2} = 0,
\end{aligned}
$$

and then $\rho_{m_1m_2} = 0$ for all $m_1, m_2 \in \mathbb{N}$ with $m_1 + m_2 \geq 2$, or $\beta_2^{m_2} = (-q)^{-m_1}$ for some $m_1, m_2 \in \mathbb{N}$ with $m_1 + m_2 \geq 2$.

If some $\rho_{m_1m_2} \neq 0$, and it meets the conditions, i.e.,

$$
\begin{cases}
\alpha_2^{m_2} = q^{-m_1}, \\
\beta_2^{m_2} = (-q)^{-m_1},
\end{cases}
$$

and $\beta_2\alpha_2^{-1} = q$, one can get $q^{m_2} = (-1)^{m_1}$, since q is not a unit root, which is impossible. Therefore, we have $E(x_1) = a_1$.

Similar to the discussion above, we can obtain that

$$E(x_2) = 0,$$

$$F(x_1) = 0 \ \text{ or } \ F(x_1) = \sigma_{20}x_1^2,$$

$$F(x_2) = 0 \ \text{ or } \ F(x_2) = \tau_{11}x_1x_2,$$

where $\sigma_{20}, \tau_{11} \in \mathbb{C}$.

From $EF - FE = \frac{K_2K_1^{-1} - K_2^{-1}K_1}{q - q^{-1}}$, we have

$$
\begin{aligned}
\frac{K_2K_1^{-1} - K_2^{-1}K_1}{q - q^{-1}}(x_1) &= \frac{\beta_1\alpha_1^{-1} - \beta_1^{-1}\alpha_1}{q - q^{-1}} = 0, \\
\frac{K_2K_1^{-1} - K_2^{-1}K_1}{q - q^{-1}}(x_2) &= \frac{\beta_2\alpha_2^{-1} - \beta_2^{-1}\alpha_2}{q - q^{-1}} = x_2.
\end{aligned}
$$

If $F(x_2) = 0$, then

$$(EF - FE)(x_2) = 0 \neq x_2;$$

if $F(x_2) = \tau_{11}x_1x_2 \neq 0$, then

$$(EF - FE)(x_2) = \tau_{11}a_1x_2 = x_2.$$

Hence, we have $\tau_{11} = \frac{1}{a_1}$ and $F(x_2) = \frac{1}{a_1}x_1x_2$.

By $F^2 = 0$, one has that

$$\begin{aligned} F^2(x_2) = \frac{1}{a_1}F(x_1x_2) &= \frac{1}{a_1}(x_1F(x_2) + F(x_1)K_2^{-1}K_1(x_2)) \\ &= \frac{1}{a_1}(\frac{1}{a_1}x_1^2x_2 + q^{-1}F(x_1)x_2). \end{aligned}$$

If $F(x_1) = 0$, then

$$F^2(x_2) = \frac{1}{a_1^2}x_1^2x_2 \neq 0;$$

if $F(x_1) = \sigma_{20}x_1^2$, then

$$F^2(x_2) = \frac{1}{a_1^2}x_1^2x_2 + q^{-1}\frac{1}{a_1}\sigma_{20}x_1^2x_2 = 0.$$

So $\sigma_{20} = -\frac{q}{a_1}$ and $F(x_1) = -\frac{q}{a_1}x_1^2$.

With an application of F to $x_2x_1 = qx_1x_2$, we have

$$\begin{aligned} F(x_2x_1 - qx_1x_2) &= x_2F(x_1) - F(x_2)x_1 - qx_1F(x_2) - F(x_1)x_2 \\ &= -\frac{q}{a_1}x_2x_1^2 - \frac{1}{a_1}x_1x_2x_1 - \frac{q}{a_1}x_1^2x_2 + \frac{q}{a_1}x_1^2x_2 \\ &= -\frac{q}{a_1}(1 + q^2)x_1^2x_2 \neq 0. \end{aligned}$$

In summary, this series is empty.

In a similar way, one can prove that all other series where the 0-th homogeneous component of M_{EF} is nonzero and the 1-st homogeneous component of M_{EF} is zero are empty. □

Theorem 4.3. *The* $\left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_1\right]$ *-series has $X_q(A_1)$-module algebra structures on the quantum plane $A_q(2)$ given by*

$$K_1(x_1) = \lambda_1x_1, \quad K_2(x_1) = \pm\lambda_1x_1, \tag{4.1}$$

$$K_1(x_2) = \lambda_2x_2, \quad K_2(x_2) = \pm\lambda_2x_2, \tag{4.2}$$

$$E(x_1) = F(x_1) = E(x_2) = F(x_2) = 0, \tag{4.3}$$

where $\lambda_1, \lambda_2 \in \mathbb{C}^*$, and therefore, they are pairwise nonisomorphic.

Proof. It is easy to check that (4.1)–(4.3) determine a well-defined $X_q(A_1)$-action consistent with the multiplication in $X_q(A_1)$ and in the quantum plane $A_q(2)$, as well as with comultiplication in $X_q(A_1)$. We prove that there are no other $X_q(A_1)$-actions here. Note that an application of (2.6) to x_1 or x_2 has zero projection to $A_q(2)_1$, i.e., $(EF - FE)(x_i) = 0, (i = 1, 2)$, because in this series E and F send any monomial to a sum of the monomials of higher degree. Therefore,

$$\frac{K_2K_1^{-1} - K_2^{-1}K_1}{q - q^{-1}}(x_1) = \frac{\beta_1\alpha_1^{-1} - \beta_1^{-1}\alpha_1}{q - q^{-1}}x_1 = 0,$$
$$\frac{K_2K_1^{-1} - K_2^{-1}K_1}{q - q^{-1}}(x_2) = \frac{\beta_2\alpha_2^{-1} - \beta_2^{-1}\alpha_2}{q - q^{-1}}x_2 = 0,$$

and we have

$$\beta_1\alpha_1^{-1} - \beta_1^{-1}\alpha_1 = \beta_2\alpha_2^{-1} - \beta_2^{-1}\alpha_2 = 0,$$

which leads to $\beta_1^2 = \alpha_1^2$ and $\beta_2^2 = \alpha_2^2$. Let $\alpha_1 = \lambda_1$ and $\alpha_2 = \lambda_2$, and we have $\beta_1 = \pm\lambda_1$ and $\beta_2 = \pm\lambda_2$. To prove (4.3), note that if $E(x_i) \neq 0$ or $F(x_i) \neq 0$, for $i = 1, 2$, then they are a sum of the monomials with degrees greater than 1. Similar to the proof of Lemma 4.2, we get that this is impossible, because they cannot satisfy the conditions of $X_q(A_1)$-module algebra on $A_q(2)$.

To see that the $X_q(A_1)$-module algebra structures are pairwise nonisomorphic, observe that all the automorphisms of the quantum plane commute with the actions of K_1 and K_2. □

Next, our immediate intention is to describe the composition series for these representations.

Proposition 4.4. *The representations corresponding to the* $\left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_1\right]$*-series described in (4.1)–(4.3) split into the direct sum* $A_q(2) = \oplus_{m=0}^{\infty} \oplus_{n=0}^{\infty} \mathbb{C}x_1^m x_2^n$ *of one-dimensional subrepresentations. These subrepresentations may belong to two isomorphism classes, depending on the weights of a specific monomial* $x_1^m x_2^n$ *which can be* $K_1(x_1^m x_2^n) = \lambda_1^m\lambda_2^n x_1^m x_2^n$ *and* $K_2(x_1^m x_2^n) = (\pm 1)^{m+n}\lambda_1^m\lambda_2^n x_1^m x_2^n$.

Proof. Since E and F are represented by zero operators and the monomials $x_1^m x_2^n$ are eigenvectors for K_1 and K_2, then every direct summand is $X_q(A_1)$-invariant. □

5. Conclusions

In this paper, we discuss the module algebra structures of $X_q(A_1)$ on the quantum n-space $A_q(n)$ for $n \geq 2$ and $n \neq 3$. However, we have presented only a complete list of $X_q(A_1)$-module algebra structures on the quantum plane $A_q(2)$, and described the isomorphism classes of these structures. For all $n \geq 4$, it is complicated to give the solutions of (3.7) and (3.8). We will continue to classify the module algebra structures of $X_q(A_1)$ on the quantum n-space $A_q(n)$ for $n \geq 4$ in the future.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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