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Research article

Finite-time flocking of particle models with nonlinear velocity coupling and inter-driving forces

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Abstract: In this paper, we propose a finite-time flocking particle model with nonlinear velocity coupling and inter-driving forces. Initially, we demonstrate that under specific initial conditions, the system achieves flocking within a finite timeframe, with all particles' velocities converging to the average of their initial velocities. Our results include some results in the literature. Furthermore, a special case of mutual driving is given. Finally, we validate the obtained results through numerical simulations to confirm their accuracy.

Keywords: finite-time flocking; nonlinear velocity coupling; inter-driving forces

1. Introduction

Flocking, as a prevalent multi-agent interaction phenomenon, holds significant implications across diverse fields such as biology, ecology, robotics and control theory, sensor networks, sociology, and economics (see [1–5]). Scholars have devoted considerable efforts to studying the collective behavior of multi-agent systems and have developed various models to understand and simulate these phenomena. Similar to physics, the study of idealized models can shed light on observed real-world phenomena, provided that the fundamental principles of these models are understood. Therefore, the analysis of these mechanisms and phenomena in mathematics is of paramount importance for comprehending and addressing complex multi-agent interactions and their practical applications.

For the investigation of flocking phenomena, numerous models have been put forward. In 1995, Vicsek et al. introduced a self-propelled particle model that has served as an inspiration for subsequent research [6]. In this model, each particle adjusts its speed based on the velocities of its neighboring particles to achieve flocking behavior. This influential model has significantly contributed to the understanding and exploration of collective motion and emergent behavior in multi-agent systems.

In their pursuit of understanding the flocking phenomenon in multi-agent systems, Cucker and

Smale introduced a mathematical model to elucidate the dynamic behavior of an N-particle system [7, 8]. The The Cucker-Smale model is a mathematical model for multi-agent systems, used to describe the interactions and collective behavior among individuals within a group. The model employs a set of ordinary differential equations (ODEs) to characterize the motion of each agent and simulates the relationships between agents based on the principles of Newtonian mechanics. In the Cucker-Smale model, the motion of each agent is influenced by the positions, velocities of its neighboring agents, as well as its own velocity and acceleration. The interaction forces between agents are described using Newton's second law of motion, where these forces are proportional to the differences in their velocities. Consequently, if agents are moving in the same direction, they tend to converge, while moving in opposite directions leads to divergence. The strength of these interaction forces can be adjusted by a parameter known as the "cohesion" term, representing the attraction between agents. The versatility of the Cucker-Smale model is demonstrated by its successful application across various domains, including biological scenarios such as bird flocking, fish schooling, and bacterial swarming. By adjusting the model's parameters, researchers have successfully replicated many observed behaviors in these systems. For example, increasing the cohesion term results in more tightly packed groups of agents, while reducing it leads to more dispersed groups. The Cucker-Smale model provides researchers with a powerful tool for studying the dynamic behavior and collective phenomena of multi-agent systems and plays a significant role in fields such as biology, engineering, and other scientific disciplines.

The Cucker-Smale particle model is represented as follows

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}x_i(t) = v_i(t), i = 1, 2, \cdots, N, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}v_i(t) = \dfrac{1}{N}\sum\limits_{j=1}^{N}\phi_{ij}(x)\left(v_j - v_i\right), \end{cases} \tag{1.1}$$

where $x_i(t) \in \mathbb{R}^d$ and $v_i(t) \in \mathbb{R}^d$ respectively represent the position and velocity of agent i at time t, $d \geq 1$ is a positive integer, and $\phi_{ij}(x)$ is the interaction weight between agents. The communication weight function $\phi(r) = \dfrac{\theta}{(1 + r^2)^\beta}$ with $\beta, \theta \geq 0$ is nonnegative and non-increasing.

Building on the pioneering work of Cucker and Smale, Ha and Liu [9] first proposed a method based on explicit Lyapunov functions to investigate the flockinging conditions of the Cucker-Smale model by establishing differential inequalities. They demonstrated that when the critical index $\beta \leq \frac{1}{2}$ is satisfied, the system exhibits unconditional flocking, thus improving the conditions for achieving flocking behavior in the Cucker-Smale model. Due to the versatility of the Cucker-Smale model, it has attracted the research interest of scholars across various fields. In recent years, numerous researchers have used the multi-particle flocking model established by Cucker, Smale, Ha, and others as a research platform and continuously improved the classical Cucker-Smale model to meet different practical application needs. For example, Shen [10] considered a Cucker-Smale model with a hierarchical structure; Li and Xue [11] studied Cucker-Smale models under root leadership with fixed and switching topologies; Li [12] examined emergence phenomena in particle flocking within the Cucker-Smale model under joint leadership; Ru et al. [13] considered a Cucker-Smale model with interactions affected by random failures; Ru et al. [14] studied the asymptotic behavior of multi-clustered Cucker-Smale models with hierarchical structures; Dong [15] explored Cucker-Smale models on general directed graphs; Cucker and Dong [16] addressed the collision-free problem in Cucker-Smale models; Ha and Kim [17] studied nonlinear velocity Cucker-Smale model finite-time flocking; and Liu [18] and Wang [19] investigated the impact of time delay on the formation of flocking behavior in Cucker-Smale models. For the emergence behavior of thermodynamic Cucker-Smale particles, please refer to the literatures such as [20–22].

Since the Cucker-Smale model is a Lipschitz system, only asymptotic flockinging can be achieved. Based on the significance of finite-time stability, Han [23] considered a new non-Lipschitz continuous system model and demonstrated that under certain conditions, the system forms flocking in a finite time. On one hand, Han [23] requires the system's coupling function to be a continuous function that is monotonically decreasing and has a positive lower bound. However, in the classic Cucker-Smale model (1.2), there is no requirement for the coupling function to have a positive lower bound, and the boundedness of the coupling function implies the boundedness of relative positions, which is precisely the key to proving the formation of clusters in the system. On the other hand, it is worthwhile to investigate whether the non-Lipschitz term or the Lipschitz term plays a decisive role in promoting the formation of flocking in the system.

Motivated by the above works, this paper aims to investigate finite-time flocking with inter-driving forces. We propose a modified Cucker-Smale model with a general inter-particle bonding force as follows:

$$\begin{cases} \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t} = v_i(t), \quad i = 1, \ldots, N, \quad t > 0, \\ \dfrac{\mathrm{d}v_i(t)}{\mathrm{d}t} = \displaystyle\sum_{j=1}^{N} \psi\left(\| x_j(t) - x_i(t) \|\right) \Gamma\left(v_j(t) - v_i(t)\right) + \gamma F_i(v_1 - \bar{v}, v_2 - \bar{v}, \ldots, v_N - \bar{v}), \end{cases} \tag{1.2}$$

and

$$\bar{v}(t) = \frac{1}{N} \sum_{i=1}^{N} v_i(t),$$

where $\psi(r) : [0, \infty) \to [0, \infty)$ is a communication weight function between two individuals, $\gamma \geq 0$, and $F_i(v_1, v_2, \ldots, v_N) : \mathbb{R}^{dN} \to \mathbb{R}^d$ is the inter-particle bonding force function. In (1.2), the Γ is a velocity nonlinear coupling term, and its introduction can be found in Assumption 2.

We also impose some restrictions on the initial states, which play important roles in our main results.

We give an important lemma (see Lemma 3.7) in this paper, which proves that finite-time flocking play a very important role. We propose Lemma 3.7, which states that the velocity and displacement satisfy a dissipative differential inequality. The right-hand side of this inequality includes finite-time and asymptotic-time flocking terms, respectively. We prove that this inequality is dominated by the finite-time flocking term. We show that system (1.2) satisfies the dissipative differential inequality in Lemma 3.7, demonstrate that the system achieves flocking within finite time, and provide an estimate of the upper bound of the finite time. In this paper, we present a more general finite-time flocking model with inter-driving forces. When the velocity coupling term takes the form of $\Gamma(v) = \mathrm{sig}\,(v)^{2\theta-1}$, $\frac{1}{2} < \theta < 1$ and the inter-driving forces vanishes. Our results are consistent with those in [23]. We also removed the lower bound of the communication function in [23]. We derive sufficient conditions for the finite-time flocking of the system and provide an estimate of the upper bound of the finite time.

This work innovatively replaces traditional linear velocity interactions with a nonlinear coupling mechanism in flocking control, enabling accelerated consensus and robust disturbance handling while guaranteeing finite-time flocking, an improvement over prior asymptotic models requiring infinite convergence. By introducing a generalized finite-time flocking model with internal forces, the study captures diverse real-world interaction dynamics in biological aggregations and robotic swarms. The research further establishes explicit sufficient conditions for finite-time flocking, offering a transparent theoretical framework, and derives an explicit upper bound for flocking time to quantify convergence speed, facilitating model optimization and performance prediction in multi-agent systems. Integrating nonlinear dynamics and internal forces, this work enhances the realism of interaction modeling and bridges theoretical analysis with practical applications in complex system coordination.

The rest of the paper is organized as follows: in Section 2, we provide assumptions on the communication weight function, velocity coupling term, and inter-driving forces, and demonstrate that our proposed model encompasses certain models in the literature. In Section 3, we first present the equivalent system (3.10) of the original system. Second, we prove an important Lemma 3.7 in this paper. Finally, we state the main Theorem 3.9 of this paper and obtain sufficient conditions for the system to achieve finite-time flocking. In Section 4, we provide a special case of the inter-driving force and demonstrate our theoretical results by numerical simulations.

2. Assumptions on key components and model generalization

In this section, we provide assumptions on the communication weight function, velocity coupling term, and inter-driving forces, and demonstrate that our proposed model encompasses certain models in the literature.

Assumption 1. ***(Communication weight function)*** *The communication weight function* $\psi : [0,\infty) \to [0,\infty)$ *is nonnegative and non-increasing:*

$$\psi(s) \geq 0 \ \text{ and } \ \psi(s_1) \geq \psi(s_2) \ \text{ for } s_1 \leq s_2. \tag{2.1}$$

Assumption 2. ***(Nonlinear velocity coupling)*** *The continuous function* $\Gamma : \mathbb{R}^d \to \mathbb{R}^d$ *satisfies the following properties: For a given* $v \in \mathbb{R}^d$*,*

(1) $\Gamma(v) = -\Gamma(-v), \langle \Gamma(v), v \rangle \geq 0.$

(2) There exist positive constants C_* *and* $\theta \in (1/2, 1]$ *such that*

$$\langle \Gamma(v), v \rangle \geq C_* \|v\|^{2\theta},$$

where C_* *depends only on the function* Γ.

Assumption 3. ***(Inter-driving force)*** *The inter-particle bonding force* F_i *satisfies the following conditions:*

(1) $\sum_{i=1}^{N} F_i = \mathbf{0}$*, where* $\mathbf{0} \in \mathbb{R}^d$ *is a zero vector.*

(2) There exists a constant $p > 0$ *such that*

$$\sum_{i=1}^{N} \langle F_i(v_1, v_2, \cdots, v_N), v_i \rangle \leq -p\|v\|^2, \tag{2.2}$$

where $\|v\|^2 = \sum_{i=1}^{N} \| v_i \|^2$.

Below we introduce several special models that are included in our model.

Remark 2.1. ***(Cucker-Smale model)*** *Note that the Cucker-Smale model in [7,8] with the following* ψ, γ, *and* Γ*:*

$$\Gamma(v_j - v_i) = v_j - v_i, \quad \gamma = 0 \ \textit{for} \ \ i = 1, 2, \cdots, N$$

and

$$\psi(\|x_j - x_i\|) = \frac{1}{(1 + \|x_j - x_i\|^2)^\beta}$$

satisfies the Assumptions 1 and 2.

Remark 2.2. ***(Finite-time model)*** *Note that the finite-time model in [23] with the following* ψ, γ, *and* Γ*:*

$$\Gamma(v) = \operatorname{sig}(v)^{2\theta - 1}, \gamma = 0, \ \ \frac{1}{2} < \theta < 1,$$

and $\psi \geq \psi*$ *is positive, where*

$$\operatorname{sig}\left(v_j - v_i\right)^\theta = (\operatorname{sign}(v_{j1} - v_{i1})|v_{j1} - v_{i1}|^\theta, \cdots, \operatorname{sign}(v_{jd} - v_{id})|v_{jd} - v_{id}|^\theta)^T$$

and sign A *is the sign function, and it is expressed as follows:*

$$\operatorname{sign}(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

Obviously, there is $\operatorname{sig}(v)^{2\theta-1} = -\operatorname{sig}(-v)^{2\theta-1}$. *For any* $v \in \mathbb{R}^d$, *by Lemma 3.3, we have that*

$$\sum_{k=1}^{d} |v_k|^{2\theta} = \sum_{k=1}^{d} \left(|v_k|^2\right)^\theta \geq \left(\sum_{k=1}^{d} |v_k|^2\right)^\theta = \|v\|^{2\theta}.$$

Therefore, we obtain that

$$\left\langle \operatorname{sign}(v)^{2\theta-1}, v \right\rangle = \sum_{k=1}^{d} |v_k|^{2\theta} \geq \|v\|^{2\theta}.$$

So when $\Gamma(v) = \operatorname{sig}(v)^{2\theta-1}, \theta \in \left(\frac{1}{2}, 1\right)$, *the Assumption 2 is satisfied.*

3. Finite-time flocking

In this section, we first introduce the definition of finite-time flocking, and then introduce several important lemmas, inspired by these lemmas and prove an important lemma (see Lemma 3.7), which plays a very important role in this paper. The main theorem of this article is presented at the end, which proves the sufficient conditions for the system to implement finite-time flocking under certain conditions.

We provide the definitions of asymptotic flocking and finite-time flocking, which can be referred to in [9,23].

Definition 3.1. *The system* $\{x_i(t), v_i(t)\}_{i=1}^N$ *represents the solutions of system (1.2) with given initial conditions.* $d_X(t)$ *and* $d_V(t)$ *denote the maximum differences in position and velocity magnitude between individual entities in the system at time t, given by*

$$\begin{aligned} d_X(t) &= \max_{i,j} \|x_j(t) - x_i(t)\|, \\ d_V(t) &= \max_{i,j} \|v_j(t) - v_i(t)\|. \end{aligned} \tag{3.1}$$

The system is said to form asymptotic flocking if all solutions of the system satisfy

$$\sup_{t\geq 0} d_X(t) < +\infty \quad \text{and} \quad \lim_{t\to+\infty} d_V(t) = 0. \tag{3.2}$$

The system is said to form finite-time flocking if there exists a positive number $T_1 > 0$*, such that for* $t \geq T_1$*, it holds that*

$$d_V(t) = 0, \quad \sup_{t\geq 0} d_X(t) < +\infty, \tag{3.3}$$

where $T_1 = \inf\{T : d_V(t) = 0, \forall t \geq T_1\}$ *is called the convergence time.*

The displacement and velocity of the system (1.2) are centralized below, and the corresponding equivalent system is provided. From the assumptions of the functions ψ and Γ, we have

$$\psi\left(\| x_j(t) - x_i(t) \|\right)\Gamma\left(v_j(t) - v_i(t)\right) = -\psi\left(\| x_j(t) - x_i(t) \|\right)\Gamma\left(v_i(t) - v_j(t)\right). \tag{3.4}$$

Thus

$$\sum_{i=1}^N \sum_{j=1}^N \psi\left(\| x_j(t) - x_i(t) \|\right)\Gamma\left(v_j(t) - v_i(t)\right) = 0. \tag{3.5}$$

By (3.5) and Assumption 3, we have that

$$\sum_{i=1}^N \frac{\mathrm{d}v_i(t)}{\mathrm{d}t} = \sum_{i=1}^N \sum_{j=1}^N \psi\left(\| x_j(t) - x_i(t) \|\right)\Gamma\left(v_j(t) - v_i(t)\right) + \gamma \sum_{i=1}^N F_i(v_1 - \bar{v}, v_2 - \bar{v}, \ldots, v_N - \bar{v}) = 0. \tag{3.6}$$

We consider averaged quantities

$$\bar{v}(t) = \frac{1}{N}\sum_{i=1}^N v_i(t), \quad x(t) = \frac{1}{N}\sum_{i=1}^N x_i(t).$$

By (3.6), we have $\bar{v}(t) = \bar{v}(0)$ and $\frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t} = \bar{v}(t) = \bar{v}(0)$. Therefore, we obtain $\bar{x}(t) = \bar{x}(0) + \bar{v}(0)t$.

Below, we centralize the velocity and displacement of the particles, and define

$$\hat{v}_i(t) = v_i(t) - \bar{v}(t), \hat{x}_i(t) = x_i(t) - \bar{x}(t),$$

and system (1.2) can be rewritten as

$$\begin{cases} \dfrac{\mathrm{d}\hat{x}_i(t)}{\mathrm{d}t} = \hat{v}_i(t), \\ \dfrac{\mathrm{d}\hat{v}_i(t)}{\mathrm{d}t} = \sum\limits_{j=1}^N \psi\left(\| \hat{x}_j(t) - \hat{x}_i(t) \|\right)\Gamma\left(\hat{v}_j(t) - \hat{v}_i(t)\right) + \gamma F_i(\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_N), \end{cases} \tag{3.7}$$

where the initial condition

$$(\hat{x}_i(0), \hat{v}_i(0)) = (\hat{x}_{i0}, \hat{v}_{i0}),$$

and zero sum constraints

$$\sum_{i=1}^{N} \hat{x}_i(t) = \sum_{i=1}^{N} \hat{v}_i(t) = 0.$$

The flocking property of system (3.7) implies the clustering property of system (1.2).

Remark 3.2. *We prove the finite-time flocking of system (3.7) by considering only the finite-time flocking of system (3.10). The reasons are as follows:*

$$\begin{aligned} d_V(t) &= \max_{i,j} \|v_j(t) - v_i(t)\| \\ &\le \max_i \|v_i(t) - \bar{v}(t)\| + \max_j \|v_j(t) - \bar{v}(t)\| \\ &\le 2\sqrt{\sum_{i=1}^{N} \|\hat{v}_i(t)\|^2} = 2V(t). \end{aligned} \tag{3.8}$$

So we only need to prove $V(t) = 0$ to deduce $d_V(t) = 0$ for $t \ge t_1$. Furthermore, we have derived the convergence of the system velocity to the initial average velocity, denoted as

$$v_i = \bar{v}(t) = \bar{v}(0), \quad for\ t \ge t_1.$$

This means that the velocities of all particles reach a consensus in finite time.

By the same calculation, we have that

$$\begin{aligned} d_X(t) &= \max_{i,j} \|x_j(t) - x_i(t)\| \\ &\le \max_i \|x_i(t) - \bar{x}(t)\| + \max_j \|x_j(t) - \bar{x}(t)\| \\ &\le 2\sqrt{\sum_{i=1}^{N} \|\hat{x}_i(t)\|^2} = 2X(t). \end{aligned} \tag{3.9}$$

So we only need to prove $X(t) < \infty$ to deduce $d_V(t) < \infty$ for $t \ge t_1$.

$X(t)$ and $V(t)$ above are defined in (3.12).

Based on Remark 3.2, we only need to discuss the finite-time flocking of system (3.7). For the sake of discussion, $(\hat{x}_i, \hat{v}_i)$ will be replaced by (x_i, v_i).

In this paper, we only need to discuss the following system:

$$\begin{cases} \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t} = v_i(t), \\ \dfrac{\mathrm{d}v_i(t)}{\mathrm{d}t} = \sum\limits_{j=1}^{N} \psi\left(\| x_j(t) - x_i(t) \|\right)\Gamma\left(v_j(t) - v_i(t)\right) + \gamma F_i(v_1, v_2, \ldots, v_N), \end{cases} \tag{3.10}$$

and

$$\sum_{i=1}^{N} x_i(t) = \sum_{i=1}^{N} v_i(t) = 0. \tag{3.11}$$

Considering the vectors $v(t) = (v_i(t), v_2(t), \dots, v_N(t)) \in \mathbb{R}^{dN}$ and $x(t) = (x_i(t), x_2(t), \dots, x_N(t)) \in \mathbb{R}^{dN}$, we denote

$$V^2(t) = \sum_{i=1}^{N} \| v_i(t) \|^2, X^2(t) = \sum_{i=1}^{N} \| x_i(t) \|^2 . \tag{3.12}$$

Through simple calculations we have that

$$\begin{aligned}\sum_{i,j=1}^{N} \|v_i - v_j\|^2 &= \sum_{i,j=1}^{N} \left\langle v_i - v_j, v_i - v_j \right\rangle \\ &= \sum_{i,j=1}^{N} \left(\|v_i\|^2 + \|v_j\|^2 - 2 \left\langle v_i, v_j \right\rangle \right) \\ &= 2NV^2(t) - \left\langle \sum_{i=1}^{N} v_i, \sum_{j=1}^{N} v_j \right\rangle \\ &= 2NV^2(t),\end{aligned}$$

where $\sum\limits_{i=1}^{N} v_i(t) = 0$ from (3.11).

By the same calculation, we have

$$\sum_{i,j=1}^{N} \|x_i - x_j\|^2 = 2NX^2(t).$$

Below, we present several important inequalities, and detailed proofs can be found in [24–26].

Lemma 3.3. *([24]) Let $a_1, a_2, \cdots, a_n > 0$ and $0 < r < p$, and then*

$$\left(\sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{n} a_i^r \right)^{\frac{1}{r}} .$$

Lemma 3.4. *([24]) If $a_1, a_2, \cdots, a_n \ge 0$ and $0 < p \le 1$, then*

$$\left(\sum_{i=1}^{n} a_i \right)^{p} \le \sum_{i=1}^{n} a_i^p .$$

By establishing differential inequalities and using the Lyapunov function method, we prove that flocking is an effective approach, which can be referred to in [9, 17, 27, 28].

Lemma 3.5. *([27, 28]) Suppose $\psi(x)$ is a monotone decreasing positive continuous function and negative differentiable functions $X(t), V(t)$ satisfy the following differential inequalities:*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} X(t) \le V(t), \\ \dfrac{\mathrm{d}}{\mathrm{d}t} V(t) \le -\alpha\psi\left(X(t)\right) V(t). \end{cases}$$

If

$$V(t_0) \leq \alpha \int_{X(t_0)}^{+\infty} \psi(r)\mathrm{d}r, \tag{3.13}$$

then

$$\sup_{t \geq t_0} X(t) < +\infty \quad \text{and} \quad \lim_{t \to +\infty} V(t) = 0.$$

In particular, if $\int_0^{+\infty} \psi(r)\mathrm{d}r = \infty$, *then (3.13) is unconditionally satisfied.*

Lemma 3.6. *([17]) Suppose* $\psi(x)$ *to monotone decreasing positive continuous function,* $0 < \theta < 2$, *and negative differentiable functions* $X(t), V(t)$ *satisfy the following differential inequalities:*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} X(t) \leq V(t), \\ \dfrac{\mathrm{d}}{\mathrm{d}t} V(t) \leq -\alpha\phi\left(X(t)\right) V^{\theta}(t). \end{cases}$$

If

$$\frac{V^{2-\theta}(t_0)}{2-\theta} \leq \alpha \int_{X(t_0)}^{+\infty} \phi(r)\mathrm{d}r, \tag{3.14}$$

then we have that

$$\frac{\mathrm{d}}{\mathrm{d}t} V(t) \leq -\alpha\phi_* V^{\theta}(t), \ \phi_* := \min_{0 \leq r \leq d_*} \phi(r) > 0,$$

where d_* *is given by the following formula:*

$$\frac{V^{2-\theta}(t_0)}{2-\theta} = \alpha \int_{X(t_0)}^{d_*} \phi(s)\mathrm{d}s.$$

In particular, if $\int_0^{+\infty} \phi(r)\mathrm{d}r = \infty$, *then (3.14) is unconditionally satisfied.*

Inspired by Lemmas 3.5 and 3.6, we prove the important lemma in this paper.

Lemma 3.7. *Suppose* $\phi(r), \psi(r)$ *to monotone decreasing positive continuous functions,* $0 < \theta < 1$, *and negative differentiable functions* $X(t), V(t)$ *satisfy the following differential inequalities:*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} X(t) \leq V(t), \\ \dfrac{\mathrm{d}}{\mathrm{d}t} V(t) \leq -\alpha\phi\left(X(t)\right) V^{\theta}(t) - \beta\psi\left(X(t)\right) V(t), \end{cases} \tag{3.15}$$

where α, β *are constant positive numbers,* $0 < \theta < 1$. *For any given* t_0, *if inequality*

$$\frac{V^{2-\theta}(t_0)}{2-\theta} \leq \alpha \int_{X(t_0)}^{+\infty} \phi(r)\mathrm{d}r \quad \text{or} \quad V(t_0) \leq \beta \int_{X(t_0)}^{+\infty} \psi(r)\mathrm{d}r \tag{3.16}$$

is satisfied, then we have that

$$\begin{aligned} V^{1-\theta}(t) \leq & \left(V^{1-\theta}(t_0) + \frac{\alpha\phi_*}{\beta\psi_*} \right) e^{-\beta\psi_*(1-\theta)(t-t_0)} - \frac{\alpha\phi_*}{\beta\psi_*}, \quad t_0 \leq t < t_1, \\ V(t) = & \ 0, \qquad\qquad\qquad\qquad\qquad\qquad \forall t \geq t_1, \end{aligned}$$

where $t_1 = t_0 + \frac{1}{\beta\psi_*(1-\theta)} \ln \frac{V^{1-\theta}(t_0) + \frac{\alpha\phi_*}{\beta\psi_*}}{\frac{\alpha\phi_*}{\beta\psi_*}}$ *,* $\phi_* := \min_{0\le r\le d_*} \phi(r), \psi_* := \min_{0\le r\le d_*} \psi(r)$*, and* $d*$ *is given by the following formula:*

$$\frac{V^{2-\theta}(t_0)}{2-\theta} = \alpha \int_{X(t_0)}^{d_*} \phi(r)\mathrm{d}r \quad or \quad V(t_0) = \beta \int_{X(t_0)}^{d_*} \psi(r)\mathrm{d}r.$$

In particular, if $\int_0^{+\infty} \phi(r)\mathrm{d}r = \infty$ *or* $\int_0^{+\infty} \psi(r)\mathrm{d}r = \infty$*, then (3.16) is unconditionally satisfied.*

Proof. We make the following two estimates of the second formula in (3.15).

The first method retains the $V(t)$ item on the right side of the above formula, and then there is

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \le -\beta\psi\left(X(t)\right)V(t).$$

By Lemma 3.5, when

$$V(t_0) \le \beta \int_{X(t_0)}^{+\infty} \psi(r)\mathrm{d}r \tag{3.17}$$

is satisfied, there is l_* such that for any $t > t_0$, we can obtain $X(t) \le l_*$.

The second method retains the $V^\theta(t)$ item on the right side of the above formula, then there is

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \le -\alpha\phi(X(t))(V(t))^\theta.$$

We know from Lemma 3.6 that when

$$\frac{V^{2-\theta}(t_0)}{2-\theta} \le \alpha \int_{X(t_0)}^{+\infty} \phi(r)\mathrm{d}r \tag{3.18}$$

is satisfied, there is s_* such that for any $t > t_0$, we have $X(t) \le s_*$.

In conclusion, when one of (3.16) is met, for any $t > 0$, we chooes $d_* = \max\{l_*, s_*\}$, and then $X(t) \le d_*$ is satisfied.

We denote $\phi_* := \min_{0\le r\le d_*} \phi(r), \psi_* := \min_{0\le r\le d_*} \psi(r)$, according to the monotonicity of $\phi(r), \psi(r)$ yields $\phi\left(X(t)\right) \ge \phi_*, \psi\left(X(t)\right) \ge \psi_*$. Therefore, $X(t), V(t)$ satisfy the following inequality:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}X(t) \le V(t), \\ \dfrac{\mathrm{d}}{\mathrm{d}t}V(t) \le -\alpha\phi_*V^\theta(t) - \beta\psi_*V(t). \end{cases}$$

We consider $U(t) = V^{1-\theta}(t) + \frac{\alpha\phi_*}{\beta\psi_*} > 0$, and then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}U(t) &= (1-\theta)V^{-\theta}(t)\frac{\mathrm{d}}{\mathrm{d}t}V(t) \\ &\le (1-\theta)V^{-\theta}(t)\left(-\alpha\phi_*V^\theta(t) - \beta\psi_*V(t)\right) \\ &= (1-\theta)\left(-\alpha\phi_* - \beta\psi_*V^{1-\theta}(t)\right) \\ &= -\beta\psi_*(1-\theta)\left(V^{1-\theta}(t) + \frac{\alpha\phi_*}{\beta\psi_*}\right) \\ &= -\beta\psi_*(1-\theta)U(t), \end{aligned}$$

that is,

$$\frac{U'(t)}{U(t)} \leq -\beta\psi_*(1-\theta).$$

We integrate the above formula from t_0 to t, it gets

$$U(t) \leq U(t_0)e^{-\beta\psi_*(1-\theta)(t-t_0)},$$

and by simplification we can see

$$V^{1-\theta}(t) \leq \left(V^{1-\theta}(t_0) + \frac{\alpha\phi_*}{\beta\psi_*}\right)e^{-\beta\psi_*(1-\theta)(t-t_0)} - \frac{\alpha\phi_*}{\beta\psi_*}. \tag{3.19}$$

Since $V^{1-\theta}(t) \geq 0$, and the right side of (3.19) is decreasing monotonically, when t_0 increased to t_1, have $V(t_1) = 0$. Therefore we have that

$$t_1 = t_0 + \frac{1}{\beta\psi_*(1-\theta)} \ln \frac{V^{1-\theta}(t_0) + \frac{\alpha\phi_*}{\beta\psi_*}}{\frac{\alpha\phi_*}{\beta\psi_*}}.$$

According to the continuity of $V(t)$, when $t \geq t_1$, we have $V(t) = 0$. □

Remark 3.8. *In Lemma 3.6, the parameter* $0 < \theta < 2$ *can be chosen, and in our Lemma 3.7, since we consider both* $V(t)$ *and* $V^\theta(t)$ *terms and need to neutralize both terms, we only consider the parameter range* $0 < \theta < 1$.

Based on Lemma 3.7, we present the main theorem of this paper.

Theorem 3.9. *Let* $(x_i(t), v_i(t))_{i=1}^N$ *be the solution of the system (3.10) given the initial conditions and it satisfies Assumptions 1, 2 and 3. When*

$$\frac{V^{2-\theta}(t_0)}{2-\theta} < \frac{C_*N}{2(2N)^{1-\theta}} \int_{\sqrt{2N}X(t_0)}^{\infty} \psi(r)\mathrm{d}r, \ \frac{1}{2} < \theta < 1. \tag{3.20}$$

is satisfied, then the system (3.10) achieves finite-time flocking. The convergence time is estimated by

$$t_1 = t_0 + \frac{1}{2\gamma p(1-\theta)} \ln\left(\frac{(\sqrt{2N}V(t_0))^{2-2\theta} + \frac{C_*N\psi_*}{2\gamma p}}{\frac{C_*N\psi_*}{2\gamma p}}\right),$$

where $\psi_* := \min_{0\leq r\leq d_*} \psi(r)$ *and* $d*$ *is given by the following formula:*

$$\frac{V^{2-\theta}(t_0)}{2-\theta} = \frac{C_*N}{2(2N)^{1-\theta}} \int_{\sqrt{2N}X(t_0)}^{d_*} \psi(r)\mathrm{d}r.$$

In particular, if $\int_0^{+\infty} \psi(r)\mathrm{d}r = \infty$, *then (3.20) is unconditionally satisfied.*

If $\theta = 1$ *and* $\gamma = 0$, *the system (3.10) achieves asymptotic flocking.*

If $\theta = 1, \gamma > 0$, *and*

$$V(t_0) \leq \frac{C_*N}{2\sqrt{2N}} \int_{\sqrt{2N}X(t_0)}^{+\infty} \psi(r)\mathrm{d}r$$

are satisfied, the system (3.10) achieves asymptotic flocking.

Proof. Considering the derivative of $V(t)$ along the second equation of system (3.10), we have

$$\begin{aligned}\frac{\mathrm{d}V^2(t)}{\mathrm{d}t} &= 2\sum_{i=1}^{N}\langle v_i(t), \dot{v}_i(t)\rangle \\ &= 2\sum_{i=1}^{N}\langle v_i(t), \sum_{j=1}^{N}\psi\left(\| x_j(t)-x_i(t) \|\right)\Gamma\left(v_j(t)-v_i(t)\right)+\gamma F_i(v_1,v_2,\dots,v_N)\rangle \\ &= 2\sum_{i=1}^{N}\sum_{j=1}^{N}\psi\left(\| x_j(t)-x_i(t) \|\right)\langle v_i(t),\Gamma\left(v_j(t)-v_i(t)\right)\rangle \\ &\quad + 2\gamma\sum_{i=1}^{N}\langle v_i(t), F_i(v_1,v_2,\dots,v_N)\rangle. \\ &:= I_1 + I_2.\end{aligned} \tag{3.21}$$

First, we calculate I_1. Notice that

$$2NX^2(t) = \sum_{i,j=1}^{N}\|x_i-x_j\|^2 \geq \|x_i-x_j\|^2,$$

that is,

$$\|x_i-x_j\| \leq \sqrt{2N}X(t), \forall i,j \in 1,2,\cdots,N.$$

By Lemma 3.4, we have that

$$\begin{aligned}\sum_{i=1}^{N}\sum_{j=1}^{N}\|v_i-v_j\|^{2\theta} &= \sum_{i=1}^{N}\sum_{j=1}^{N}(\|v_i-v_j\|^2)^{\theta} \\ &\geq \left(\sum_{i=1}^{N}\sum_{j=1}^{N}\|v_i-v_j\|^2\right)^{\theta} \\ &= \left(2NV^2(t)\right)^{\theta} \\ &= (2N)^{\theta}\,V^{2\theta}(t).\end{aligned} \tag{3.22}$$

Using Assumption 1, Assumption 2, and (3.22), we obtain that

$$\begin{aligned}I_1 &= \sum_{i=1}^{N}\sum_{j=1}^{N}\psi\left(\| x_j(t)-x_i(t) \|\right)\langle v_i(t),\Gamma\left(v_j(t)-v_i(t)\right)\rangle \\ &= -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\psi\left(\| x_j(t)-x_i(t) \|\right)\langle v_j(t)-v_i(t),\Gamma\left(v_j(t)-v_i(t)\right)\rangle \\ &\leq -\frac{1}{2}C_*\psi\left(\sqrt{2N}X(t)\right)\sum_{i=1}^{N}\sum_{j=1}^{N}\|v_i-v_j\|^{2\theta} \\ &\leq -\frac{1}{2}C_*\,(2N)^{\theta}\,\psi\left(\sqrt{2N}X(t)\right)V^{2\theta}(t).\end{aligned} \tag{3.23}$$

Next, we consider I_2. By Assumption 3, we have that

$$\begin{aligned} I_2 &= 2\gamma \sum_{i=1}^{N} \langle v_i(t), F_i(v_1, v_2, \ldots, v_N) \rangle \\ &\leq -2\gamma p \sum_{i=1}^{N} \| v_i(t) \|^2 \\ &= -2\gamma p V^2(t). \end{aligned} \tag{3.24}$$

Combining (3.21), (3.23), and (3.24), we obtain that

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \leq -\frac{1}{4} C_* (2N)^\theta \psi\left(\sqrt{2N} X(t)\right) V^{2\theta-1}(t) - \gamma p V(t).$$

We establish the differential inequality for X,

$$\begin{aligned} \frac{\mathrm{d}X^2(t)}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{N} \|x_i\|^2\right) = 2\sum_{i=1}^{N} \langle x_i, v_i \rangle \leq 2\sum_{i=1}^{N} \|x_i\| \|v_i\| \\ &\leq 2\left(\sum_{i=1}^{N} \|x_i\|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{N} \|v_i\|^2\right)^{\frac{1}{2}} \\ &= 2X(t)V(t), \end{aligned}$$

that is, $\frac{\mathrm{d}X(t)}{\mathrm{d}t} \leq V(t)$.

It follows that the system (3.10) satisfies the following differential inequality:

$$\begin{cases} \dfrac{\mathrm{d}X(t)}{\mathrm{d}t} \leq V(t), \\ \dfrac{\mathrm{d}V(t)}{\mathrm{d}t} \leq -\dfrac{1}{4} C_* (2N)^\theta \psi\left(\sqrt{2N} X(t)\right) V^{2\theta-1}(t) - \gamma p V(t), \end{cases} \tag{3.25}$$

where $\frac{1}{2} \leq \theta \leq 1$.

To apply Lemma 3.7, we need to parameterize, rewriting the differential inequality (3.25) as

$$\begin{cases} \dfrac{\mathrm{d}\sqrt{2N} X(t)}{\mathrm{d}t} \leq \sqrt{2N} V(t), \\ \dfrac{\mathrm{d}\sqrt{2N} V(t)}{\mathrm{d}t} \leq -\dfrac{1}{2} C_* N \psi\left(\sqrt{2N} X(t)\right)\left(\sqrt{2N} V(t)\right)^{2\theta-1} - \gamma p \sqrt{2N} V(t). \end{cases} \tag{3.26}$$

Case 1 : $\frac{1}{2} < \theta < 1$.

Applying Lemma 3.7 to

$$\psi = 1,\ \beta = \gamma p,\ \alpha = \frac{1}{2} C_* N,$$

we can obtain that

$$V(t) \equiv 0,\ t \geq t_1$$

and the convergence time is estimated by

$$t_1 = t_0 + \frac{1}{2\gamma p(1-\theta)} \ln\left(\frac{(\sqrt{2N}V(t_0))^{2-2\theta} + \frac{C_* N\psi_*}{2\gamma p}}{\frac{C_* N\psi_*}{2\gamma p}}\right),$$

where $\psi_* := \min_{0\le r\le d_*} \psi(r)$ and $d*$ are given by the following formula:

$$\frac{V^{2-\theta}(t_0)}{2-\theta} = \frac{C_* N}{2(2N)^{1-\theta}} \int_{\sqrt{2N}X(t_0)}^{d_*} \psi(r)\mathrm{d}r. \tag{3.27}$$

In particular, if $\int_0^{+\infty} \psi(r)\mathrm{d}r = \infty$, then (3.27) is unconditionally satisfied.

Notice $V(t) = 0$ for $t \ge t_1$, and we know $V(t) \le V(0)$ from (3.25), so we have

$$\begin{aligned} X(t) &\le X(0) + \int_0^t V(s)\mathrm{d}s \\ &\le X(0) + \int_0^\infty V(s)\mathrm{d}s \\ &= X(0) + \int_0^{t_1} V(s)\mathrm{d}s \\ &\le X(0) + \int_0^{t_1} V(0)\mathrm{d}s \\ &= X(0) + V(0)t_1. \end{aligned}$$

Thus, the system (3.10) achieves finite-time flocking.

Case 2 : $\theta = 1$**.** We have that

$$\begin{cases} \dfrac{\mathrm{d}\sqrt{2N}X(t)}{\mathrm{d}t} \le \sqrt{2N}V(t), \\ \dfrac{\mathrm{d}\sqrt{2N}V(t)}{\mathrm{d}t} \le -\left(\dfrac{1}{2}C_* N\psi\left(\sqrt{2N}X(t)\right) + \gamma p\right)\left(\sqrt{2N}V(t)\right). \end{cases} \tag{3.28}$$

(1) If $\gamma > 0$, it is obtained by inequality (3.28) that

$$V(t) \le V(t_0)e^{-\gamma p t} \to 0$$

as $t \to \infty$ and

$$X(t) \le X(0) + \int_0^\infty e^{-\gamma p s}\mathrm{d}s < \infty.$$

Therefore, the system (3.10) implements asymptotic flocking.

(2) If $\gamma = 0$, this is the classical inequality, see [9, 27]. It is obtained by inequality (3.28) and Lemma 3.5 that

$$V(t_0) \le \frac{C_* N}{2\sqrt{2N}} \int_{\sqrt{2N}X(t_0)}^{+\infty} \psi(r)\mathrm{d}r,$$

and then system (3.10) achieves asymptotic flocking.

□

Remark 3.10. *For any* $0 < \alpha < 1$, *when* $\gamma = 0$ *and* $\Gamma(v) = \mathrm{sig}(v)^{\alpha}$, *system (3.10) becomes the same model as that in [23]. By Remark 2.2, we have that* $\theta = \frac{1+\alpha}{2} \in \left(\frac{1}{2}, 1\right)$. *Moreover, regarding the convergence time shown in Theorem 3.9, we have*

$$\begin{aligned}
\lim_{\gamma\to 0} t_1 &= t_0 + \lim_{\gamma\to 0} \frac{1}{2\gamma p(1-\theta)} \ln\left(\frac{(\sqrt{2N}V(t_0))^{2-2\theta} + \frac{CN\psi_*}{2\gamma p}}{\frac{CN\psi_*}{2\gamma p}} \right) \\
&= t_0 + \lim_{\gamma\to 0} \frac{1}{\gamma p(1-\alpha)} \ln\left(\frac{(\sqrt{2N}V(t_0))^{1-\alpha} + \frac{CN\psi_*}{2\gamma p}}{\frac{CN\psi_*}{2\gamma p}} \right) \\
&= t_0 + \lim_{\gamma\to 0} \frac{1}{p(1-\alpha)} \frac{2p(\sqrt{2N}V(t_0))^{1-\alpha}}{CN\psi_*} \frac{1}{1 + \frac{2p\gamma(\sqrt{2N}V(t_0))^{1-\alpha}}{CN\psi_*}} \\
&= t_0 + \frac{2^{\frac{5-\alpha}{2}}(V(t_0))^{1-\alpha}}{C\psi^*(1-\alpha)} N^{-\frac{1+\alpha}{2}},
\end{aligned} \tag{3.29}$$

which is consistent with the result in [23].

4. A special case and numerical simulation

In this section, we present a specific form of inter-driving force $F_i(v_1 - \bar{v}, v_2 - \bar{v}, \ldots, v_N - \bar{v})$ to explore the convergence time of finite-time flocking, which is described by

$$F_i(v_1 - \bar{v}, v_2 - \bar{v}, \ldots, v_N - \bar{v}) = -p(v_i(t) - \bar{v}(t)), \tag{4.1}$$

where $\bar{v}(t) = \frac{1}{N}\sum_{i=1}^{N} v_i(t)$ and p is a positive constant. Then, we have that

$$\sum_{i=1}^{N} F_i(v_1 - \bar{v}, v_2 - \bar{v}, \ldots, v_N - \bar{v}) = 0 \tag{4.2}$$

and

$$\sum_{i=1}^{N} \langle F_i(v_1 - \bar{v}, v_2 - \bar{v}, \cdots, v_N - \bar{v}), v_i - \bar{v} \rangle = -p \sum_{i=1}^{N} \|v_i - \bar{v}\|^2.$$

So we have verified that F_i satisfies Assumption 3.

In this section, numerical simulations were carried out using MATLAB, the communication weight function ψ, and the target motion mode inter-driving forces function F_i. For system (1.2), we select the classical Cucker-Smale communication weight function:

$$\psi(r) = \frac{H}{(1+r^2)^{\beta}}, H, \beta > 0,$$

inter-driving forces $F_i = -p(v_i(t) - \bar{v}(t))$, and the nonlinear velocity coupling function $\Gamma(v) = \mathrm{sig}(v)^{\theta}, 0 < \theta < 1$.

The number of agents is $N = 20$; the spatial dimension is $d = 2$; and the parameters in the system are set as $\gamma = 2$, $H = 1$, $p = 4$, $\beta = \frac{1}{3}$, and $\theta = 0.8$. The initial displacement and velocity are random numbers in intervals $[-10, 0] \times [-8, 12]$ and $[-13, 17] \times [-12, 2]$, respectively.

We denote

$$\Gamma_{v_i}(t) = \|v_i(t) - \overline{v}(t)\|, \quad i = 1, 2, \cdots, 20,$$

and

$$\Gamma_{x_i}(t) = \|x_i(t) - x_1(t)\|, \quad i = 2, 3, \cdots, 20,$$

where $x_i = (x_{i1}, x_{i2})$, $v_i = (v_{i1}, v_{i2})$ and $\|x_i\| = \sqrt{x_{i1}^2 + x_{i2}^2}$, $\|v_i\| = \sqrt{v_{i1}^2 + v_{i2}^2}$.

Noticing that

$$d_X(t) = \max_{i,j} \|x_j(t) - x_i(t)\| \leq \max_i \|x_i(t) - x_1(t)\| + \max_j \|x_j(t) - x_1(t)\| = 2 \max_i \Gamma_{x_i}(t)$$

and

$$d_V(t) = \max_{i,j} \|v_j(t) - v_i(t)\| \leq \max_i \|v_i(t) - \bar{v}(t)\| + \max_j \|v_j(t) - \bar{v}(t)\| = 2 \max_i \Gamma_{v_i}(t).$$

Next, we will conduct simulations for $\Gamma_{v_i}(t)$ and $\Gamma_{x_i}(t)$.

Figures 1 and 2 show the evolution of velocity component $v_{i1}, i = 1, 2, \cdots, 20$, and velocity component $v_{i2}, i = 1, 2, \cdots, 20$, with respect to time t within 2 seconds, respectively. The magnitude of the velocity changes chaotically. As time evolves, it can be seen from the figures that the velocities of the system tend to be consistent at approximately 1.3 seconds . Through calculation, we can obtain that the average value of the initial velocity component $v_{i1}, i = 1, 2, \cdots, 20$, is -4.76, and the average value of the initial velocity component $v_{i2}, i = 1, 2, \cdots, 20$, is -4.53, which is consistent with the convergent velocity components obtained from the numerical simulation.

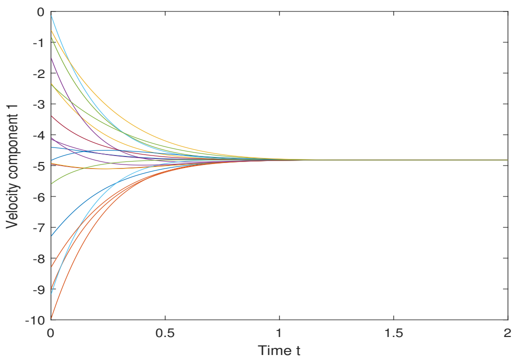

Figure 1. Velocity component 1.

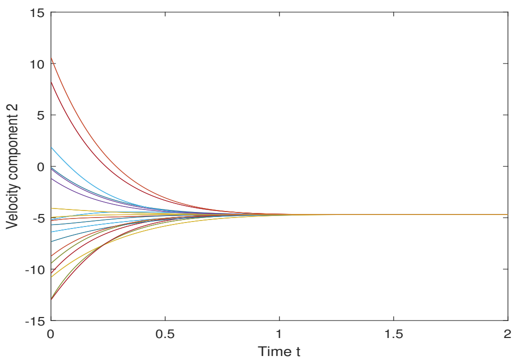

Figure 2. Velocity component 2.

Figure 3 shows the evolution of the Euclidean norm of the difference between each particle's velocity and the average velocity with time t within 2 seconds. When the time is 1.3 seconds, all particles tend to the average velocity. It suggests that the designed control or communication mechanism effectively promotes information exchange, enabling each particle to adapt its velocity to align with the group.

Figure 4 displays the evolution over 2 seconds of the Euclidean norm of the displacement difference between the first particle and the remaining 19 particles, implying that the displacement differences of all particles remain bounded within a finite time. It shows that despite the particles being in motion, the relative distances between them do not grow indefinitely, which is essential for applications requiring coordinated behavior. For instance, in a robotic formation, this ensures that the robots maintain a safe and functional distance from each other. The bounded evolution also suggests that the underlying interaction rules among particles are well-tuned to prevent divergence.

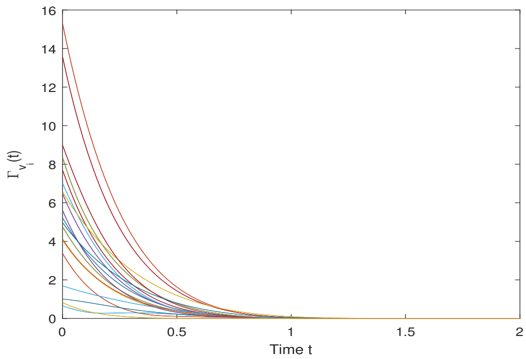

Figure 3. $\Gamma_{v_i}(t) = \|v_i(t) - \overline{v}(t)\|$.

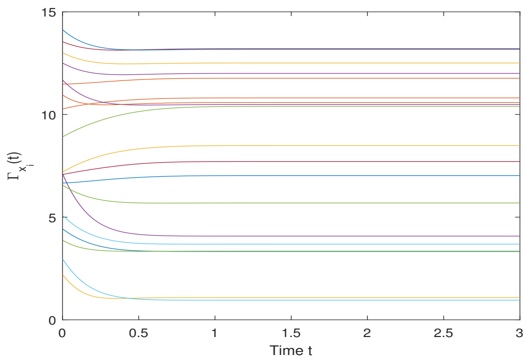

Figure 4. $\Gamma_{x_i}(t) = \|x_i(t) - x_1(t)\|$.

The following presents a comparison between the numerical simulation of the system's finite-time flocking and the finite time calculated by us.

System (3.10) achieves finite-time flocking. The convergence time is estimated by

$$t_1 = t_0 + \frac{1}{2\gamma p(1-\theta)} \ln\left(\frac{(\sqrt{2N}V(t_0))^{2-2\theta} + \frac{C_* N\psi_*}{2\gamma p}}{\frac{C_* N\psi_*}{2\gamma p}} \right).$$

The numerical simulation yields a finite-time flocking duration of $T_1 = 1.32$ s , while theoretical calculation (with $t_0 = 0$) gives $t_1 = 1.53$ s. According to the definition of finite-time stability, the relationship $t_1 \geq T_1$ is consistent with our theoretical framework. This discrepancy can be attributed to the nature of the stability analysis conducted in this study. Our theoretical derivation provides a sufficient condition for finite-time flocking, which inherently offers an upper bound estimate for the convergence time. In other words, the calculated t_1 represents a conservative estimate that ensures flocking will occur within this time frame under the specified assumptions. Mathematically, the sufficient condition approach often involves constructing Lyapunov functions with certain properties. These functions are designed to guarantee stability but may not necessarily capture the exact minimum time required for flocking to occur. By contrast, numerical simulations account for specific initial conditions and model nuances that might lead to faster convergence in practice. This gap between t_1 and T_1 actually validates the robustness of our theoretical results: even if the estimated t_1 is longer than the simulated T_1, it confirms that the system will achieve flocking within the predicted (albeit conservative) time bound.

The following numerical simulations are conducted for different values of θ to study the impact of θ on the finite-time convergence time of the system. We denote

$$V_{max} = \max_{i,j} \|v_i - v_j\|$$

for all $i, j = 1, 2, \cdots, 20$.

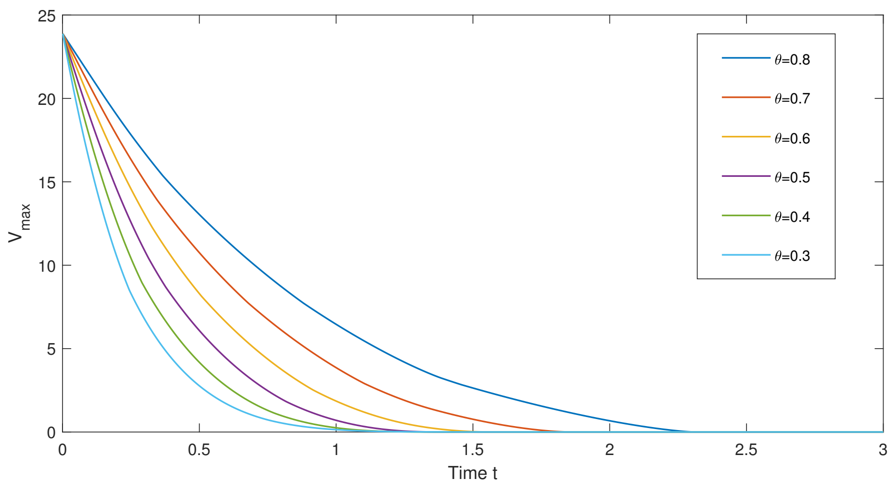

Figure 5. The evolution of V_{max} over time within 3 s when the parameter θ takes different values.

With the same initial data and parameters as above, different values of parameter $\theta = 0.3, 0.4, 0.5, 0.6, 0.7, 0.8$ were selected. The simulation results are shown in Figure 5, indicating that the smaller θ is, the faster the convergence speed. This relationship between the parameter θ and convergence speed provides valuable insights into the system's tunability. A faster convergence with smaller θ values suggests that this parameter acts as a control knob for the rate at which the system reaches its desired state. For practical applications, such as optimizing the performance of multi-agent systems or accelerating simulations, fine-tuning θ to lower values could significantly reduce the time required for the system to stabilize.

5. Conclusions

This paper addresses the finite-time flocking problem for particle models with nonlinear velocity coupling and inter-driving forces. By constructing a suitable Lyapunov function and centralizing the system, we derive dissipative differential inequalities for velocity and displacement differences. Rigorous mathematical proofs show that, under specific initial conditions, the system achieves finite-time flocking, and we provide an explicit upper bound estimation for the flocking time. When certain parameter values are assigned, our results subsume and generalize existing results in the literature, demonstrating the generality of our theoretical framework.

These findings have practical implications and research value. In the field of robotics, the finite-time convergence property can serve as a theoretical reference for the coordinated control of multi-robot systems. For example, when designing the formation control algorithms of small-scale robot swarms, the proposed model can help engineers better understand how to adjust the interaction parameters to achieve faster velocity synchronization among robots. In biological research, the model provides a new perspective for studying the collective motion of biological populations at the micro and macro levels.

In terms of future research directions, several avenues are worth exploring. One direction is to study the sensitivity of the flocking time bound with respect to different initial conditions and parameter settings. Rigorous mathematical analysis can reveal how small perturbations in these factors affect the convergence rate, providing insights into the robustness of the system. Another direction is to generalize the model to higher-dimensional spaces and more complex network topologies. This requires the development of new mathematical techniques for analyzing high-dimensional nonlinear systems and graph-based dynamics. In addition, incorporating stochastic elements into the model and studying the probabilistic properties of finite-time flocking can bridge the gap between deterministic theory and real-world stochastic environments, leading to more realistic and applicable mathematical models.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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