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A dissipative third-order boundary value problem with distributional potentials and eigenparameter-dependent boundary conditions

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Abstract: This paper investigates a class of dissipative boundary value problems arising from a third-order differential equation with distributional potentials and eigenparameter-dependent boundary conditions. Initially, we transform the boundary value problem into the corresponding operator problem. We then demonstrate that the operator is dissipative and examine certain eigenvalue properties of the operator. Furthermore, by applying Krein's theorem, we establish the completeness theorems for both the boundary value problem and the corresponding operator.

Keywords: third-order boundary value problem; dissipative differential operator; distributional potentials; completeness theorems

1. Introduction

Boundary value problems (BVPs) with distributional potentials hold significant theoretical and practical importance. They are not only utilized in the study of Schrödinger operators with distributional potentials in quantum mechanics but also encompass boundary value problems with transmission conditions. Within quantum mechanics, Schrödinger equations incorporating generalized potential functions are commonly employed to portray the interactions between individual particles [1, 2]. These potentials are frequently associated with issues described as point interactions. Beyond quantum mechanics, such potentials are present in disciplines like solid-state physics, atomic and nuclear physics, and electromagnetism [1, 3]. To provide a robust mathematical framework for addressing problems with distributional potentials, it is necessary to relax the integrability condition of the potential function in classical Sturm-Liouville theory. This approach not only extends classical Sturm-Liouville theory but also introduces new characteristics pertinent to physical problems. In recent years, scholars have explored this area from multiple perspectives, resulting in significant discoveries [1–6].

Recently, third-order boundary value problems with non-smooth coefficients and distributional potentials have garnered significant attention from scholars [7–10]. Third-order differential equations are pivotal in a variety of physical applications, such as in modeling thin membrane flow of viscous liquid and elastic beam vibrations [11–13].

In addition, one of the research topics of boundary value problems is boundary value problems with eigenparameter-dependent boundary conditions. A specific example is the equation of motion of a clamped-free elastic beam with a mass-spring system attached at its free end, which leads to a boundary eigenvalue problem [14]

$$y^{(4)} = \lambda(y^{(2)} - cy),$$

$$y(0) = 0,\ y'(0) = 0,\ y^{(2)}(1) = 0,\ \beta(\lambda)y^{(3)}(1) + \alpha(\lambda)y(1) = 0.$$

Here the coefficients $\alpha(\lambda)$ and $\beta(\lambda)$ are polynomials in λ of degree 3 and 2, respectively. The constant c in the differential equation is nonzero if, in addition, a fluid is flowing over the bar with constant velocity, which may be regarded as a model for pulling out glass or plastics on a solid foundation. Furthermore, many other problems in various engineering fields can also be transformed into boundary value problems with eigenparameter-dependent boundary conditions, such as heat conduction problems, and vibrating string problems and so on [14–16]. In recent years, scholars have had a strong interest in this kind of problem, and published a series of excellent research results [14–17].

There are many methods to study boundary value problems; one of the most important and effective methods is the spectral analysis method. That is, by defining an appropriate inner product space, the boundary value problem is transformed into a related operator problem. The operator has the same eigenvalues as the original boundary value problem, and their eigenfunctions are the same or have special correspondences.

Dissipative differential operators are a significant topic in the study of the spectral theory of differential operators and have a broad range of applications. For instance, they are widely used in various areas, including the analysis of Cauchy problems in partial differential equations, scattering theory, and infinite-dimensional dynamical systems [18, 19].

In the past period of time, dissipative differential operators with general separated or coupled boundary conditions have been investigated by many authors. For example, in [20, 21] the authors have investigated the general dissipative Sturm-Liouville operator and gave all the dissipative boundary conditions of order 2. Significantly, the determinant of perturbation connected with the dissipative operator L generated in $L^2(I)$ by the Sturm-Liouville differential expression has been studied by Bairamov and Uğurlu in [22], they used the Livšic theorem to prove the completeness of the system of eigenfunctions and associated functions of this operator. They also have studied the dissipative boundary value problems with transmission conditions and have shown the completeness of the root functions by using Krein's theorem [23, 24]. The study of fractional dissipative Sturm-Liouville operator can be found in recent work [25] and the studies of higher order dissipative operators can be found in [26–28], respectively.

However, there remains relatively little research on dissipative operators with special boundary conditions, such as those with eigenparameter-dependent boundary conditions [29, 30]. Particularly, no conclusions have been drawn yet for such problems of odd orders. Based on the aforementioned studies, the goal of this paper is to study a dissipative third-order boundary value problem with distributional potentials and eigenparameter-dependent boundary conditions, in further by using

Krein's theorem to prove the completeness theorems of the root vectors of the boundary value problem.

The paper is organized as follows: Following this introduction, in Section 2 we introduce the notation of the problems studied here and transform the original boundary value problem into a related isospectral operator problem. Section 3 shows the proof of the operator being dissipative and lists some properties of the eigenvalues of the operator. In Section 4, we review the characteristic function and the Green's function of the dissipative boundary value problem and prepare for the proof of the completeness theorems. Then we prove the completeness theorems of the boundary value problem and the operator by using Krein's theorem in Section 5. Finally, a brief conclusion is given in Section 6.

2. Notation

Consider the general third-order differential equation with distributional potentials

$$l(u) = \frac{1}{w}\{-i(q_0(q_0u')')' + i[q_1u' + (q_1u)'] - (p_0[u' + su])' + sp_0[u' + su] + p_1u\} = \lambda u \quad on\ \ J, \tag{2.1}$$

here

$$J = [a, b], \ -\infty < a < b < \infty, \tag{2.2}$$

and the coefficients satisfy:

$$\begin{aligned} &q_0,\ q_1,\ p_0,\ p_1,\ s,\ w : J \to \mathbb{R},\ q_0^{-1},\ q_1,\ p_0,\ p_1,\ s,\ sp_0,\ s^2p_0\ ,\ \frac{q_1}{q_0},\ \frac{sp_0}{q_0},\ \frac{p_0}{q_0^2},\ w \in L^1(J),\\ &q_0 > 0 \textit{ on } J \textit{ and } w > 0 \textit{ a.e. on } J. \end{aligned} \tag{2.3}$$

Similar to [9] and [28], we can introduce the following notations. Firstly, let us introduce the quasi-derivative $u^{[j]}$ (since the equation is third-order here, we chose $j = 0,1,2$) of a function u as follows:

$$u^{[0]} = u,\ u^{[1]} = q_0u',\ u^{[2]} = iq_0(q_0u')' - iq_1u + p_0(u' + su),$$

then (2.1) can be expressed as

$$l(u) = \frac{1}{w}[-(u^{[2]})' + \frac{iq_1 + sp_0}{q_0}u^{[1]} + (s^2p_0 + p_1)u] = \lambda u, \quad on\ \ J, \tag{2.4}$$

and further, it can be handled as the following Hamiltonian system

$$GY' = (\lambda W + P)Y,$$

where W and P are 3×3 matrices, Y is a 3×1 vector such that

$$G = \begin{pmatrix} 0 & 0 & -1 \\ 0 & i & 0 \\ 1 & 0 & 0 \end{pmatrix},\ W = \begin{pmatrix} w & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ Y = \begin{pmatrix} u \\ u^{[1]} \\ u^{[2]} \end{pmatrix},$$

and

$$P = \begin{pmatrix} -(s^2 p_0 + p_1) & -\frac{iq_1 + sq_0}{q_0} & 0 \\ \frac{iq_1 - sq_0}{q_0} & -\frac{p_0}{q_0^2} & \frac{1}{q_0} \\ 0 & \frac{1}{q_0} & 0 \end{pmatrix}.$$

For the next discussions, we first define a weighted space

$$H = L^2_w(J) = \left\{ u : \int_a^b |u(x)|^2 w(x) \mathrm{d}x < \infty \right\},$$

and the inner product in this space as $\langle f, g\rangle_H = \int_a^b f\bar{g}w(x)\mathrm{d}x$ for any $f, g \in H$.

For arbitrary $u,\ v \in H$, the Lagrange identity can be introduced as

$$[u, v] := u\overline{v}^{[2]} - u^{[2]}\overline{v} + iu^{[1]}\overline{v}^{[1]}. \tag{2.5}$$

Now consider the set

$$\Omega = \left\{ u \in H : u, u^{[1]}, u^{[2]} \in AC[a, b],\ l(u) \in H \right\},$$

for arbitrary two functions $u,\ v \in \Omega$, we have the Green's formula

$$\langle l(u), v\rangle_H - \langle u, l(v)\rangle_H = [u, v]_a^b, \tag{2.6}$$

where $[u, v]_{t_1}^{t_2} = [u, v](t_2) - [u, v](t_1)$.

Then we consider the BVP consisting of the following differential equation

$$-i(q_0(q_0u')')' + i[q_1u' + (q_1u)'] - (p_0[u' + su])' + sp_0[u' + su] + p_1u = \lambda wu \ on \ J, \tag{2.7}$$

and the boundary conditions (BCs):

$$l_1(u) = (\hat{\alpha}_1\lambda - \alpha_1)u(a) - (\hat{\alpha}_2\lambda - \alpha_2)u^{[2]}(a) = 0, \tag{2.8}$$

$$l_2(u) = u^{[1]}(a) - ru^{[1]}(b) - \beta_1 u^{[2]}(b) = 0, \tag{2.9}$$

$$l_3(u) = u(b) - ir\overline{\beta_1}u^{[1]}(b) - \beta_2 u^{[2]}(b) = 0, \tag{2.10}$$

where λ is a complex parameter, and the coefficients satisfy the conditions:

$$\alpha_j, \hat{\alpha}_j, r \in \mathbb{R},\ j = 1, 2,\ \beta_k \in \mathbb{C},\ k = 1, 2,\ \eta := \hat{\alpha}_1\alpha_2 - \hat{\alpha}_2\alpha_1 > 0,\ |r| \le 1,\ and\ \Im\beta_2 \ge \frac{1}{2}r^2. \tag{2.11}$$

In this paper will use the symbols $\Re$ and $\Im$ to denote the real and imaginary parts of a certain operator or parameter, respectively.

Next, define a direct sum space $\mathcal{H} = H \bigoplus \mathbb{C}$ with a new inner product

$$\langle (f, f_1)^T, (g, g_1)^T\rangle_{\mathcal{H}} = \langle f, g\rangle_H + \frac{1}{\eta} f_1\overline{g_1},$$

for any $(f, f_1)^T, (g, g_1)^T \in \mathcal{H}$.

From BC (2.8), one has

$$\lambda(\hat{\alpha}_1 u(a) - \hat{\alpha}_2 u^{[2]}(a)) = \alpha_1 u(a) - \alpha_2 u^{[2]}(a),$$

which can be written as

$$\lambda R(u) = \widetilde{R}(u),$$

by setting

$$R(u) = \hat{\alpha}_1 u(a) - \hat{\alpha}_2 u^{[2]}(a),\ \ \widetilde{R}(u) = \alpha_1 u(a) - \alpha_2 u^{[2]}(a).$$

Now consider the following set

$$D(\mathbf{L}_h) = \{U = \begin{pmatrix} u \\ R(u) \end{pmatrix} \in \mathcal{H} : u \in \Omega,\ R(u) \in \mathbb{C},\ l_j(u) = 0,\ j = 2, 3\},$$

and define the operator $\mathbf{L}_h$ on $D(\mathbf{L}_h)$ as

$$\mathbf{L}_h \begin{pmatrix} u \\ R(u) \end{pmatrix} = \begin{pmatrix} l(u) \\ \widetilde{R}(u) \end{pmatrix} = \lambda \begin{pmatrix} u \\ R(u) \end{pmatrix}.$$

Then we will present the relationship between the BVP (2.7)–(2.11) and the operator $\mathbf{L}_h$.

Definition 1. *The system of functions $u_0, u_1, ..., u_n$ is called a chain of eigenfunctions and associated functions of the BVP (2.7)–(2.11) corresponding to the eigenvalue λ_j if the conditions*

$$l(u_0) = \lambda_j u_0,\ \widetilde{R}(u_0) - \lambda_j R(u_0) = 0,\ l_2(u_0) = 0,\ l_3(u_0) = 0, \tag{2.12}$$

$$l(u_s) - \lambda_j u_s - u_{s-1} = 0,\ \widetilde{R}(u_s) - \lambda_j R(u_s) - R(u_{s-1}) = 0, \tag{2.13}$$

$$l_2(u_s) = 0,\ \ l_3(u_s) = 0,\ \ s = 1, ..., n, \tag{2.14}$$

are realized.

Then we have

Lemma 1. *Including their multiplicity, the eigenvalues of the BVP (2.7)–(2.11) and the eigenvalues of the operator $\mathbf{L}_h$ coincide. Each chain of eigenfunctions and associated functions of the BVP (2.7)–(2.11), meeting the requirements of the eigenvalue λ_j, corresponds to the chain of eigenvectors and associated vectors $U_0, U_1, ..., U_n$ of the operator $\mathbf{L}_h$ corresponding to the same eigenvalue λ_j. In this case, the equalities*

$$U_k = \begin{pmatrix} u_k \\ R(u_k) \end{pmatrix},\ \ k = 0, 1, ..., n, \tag{2.15}$$

take place.

Proof. If $U_0 \in D(\mathbf{L}_h)$ and $\mathbf{L}_h U_0 = \lambda_j U_0$, then the equalities $l(u_0) = \lambda_j u_0,\ \widetilde{R}(u_0) - \lambda_j R(u_0) = 0,\ l_2(u_0) = 0,\ l_3(u_0) = 0$ take place, i.e., u_0 is an eigenfunction of the BVP (2.7)–(2.11). Conversely, if conditions (2.12) are realized, then $\begin{pmatrix} u_0 \\ R(u_0) \end{pmatrix} = U_0 \in D(\mathbf{L}_h)$ and $\mathbf{L}_h U_0 = \lambda_j U_0$, i.e., U_0 is an eigenvector of the operator $\mathbf{L}_h$.

Furthermore, if $U_0, U_1, ..., U_n$ are a chain of the eigenvectors and associated vectors of the operator $\mathbf{L}_h$ corresponding to the eigenvalue λ_j, then by implementing the conditions $U_k \in D(\mathbf{L}_h),\ k = 0, 1, ..., n$, and equality $\mathbf{L}_h U_0 = \lambda_j U_0,\ \mathbf{L}_h U_s = \lambda_j U_s + U_{s-1},\ s = 1, ..., n$, we get the equalities (2.12)–(2.14), where $u_0, u_1, ..., u_n$ are the first components of the vectors $U_0, U_1, ..., U_n$. On the contrary, on the basis of the elements $u_0, u_1, ..., u_n$ corresponding to the BVP (2.7)-(2.11), one can construct the vectors $U_k = \begin{pmatrix} u_k \\ R(u_k) \end{pmatrix}$ for which $U_k \in D(\mathbf{L}_h),\ k = 0, 1, ..., n$, and $\mathbf{L}_h U_0 = \lambda_j U_0,\ \mathbf{L}_h U_s = \lambda_j U_s + U_{s-1},\ s = 1, ..., n$.

Now the proof is finished. □

3. Dissipative operator

The dissipative operator is defined as follows.

Definition 2. *A linear operator $\mathbf{L}_h$, acting in the Hilbert space $\mathcal{H}$ and having domain $D(\mathbf{L}_h)$, is said to be dissipative if $\Im(\mathbf{L}_h F, F) \geq 0,\ \forall F \in D(\mathbf{L}_h)$.*

Theorem 1. *The operator $\mathbf{L}_h$ is dissipative in $\mathcal{H}$.*

Proof. For $U \in D(\mathbf{L}_h)$, we have

$$2i\Im(\mathbf{L}_h U, U) = (\mathbf{L}_h U, U) - (U, \mathbf{L}_h U) = [u, u]_a^b + \frac{1}{\eta}\widetilde{R}(u)\overline{R(u)} - \frac{1}{\eta}R(u)\overline{\widetilde{R}(u)}, \tag{3.1}$$

where

$$\begin{aligned}
&\frac{1}{\eta}\widetilde{R}(u)\overline{R(u)} - \frac{1}{\eta}R(u)\overline{\widetilde{R}(u)} \\
=&\frac{1}{\eta}\Big[\Big(\alpha_1 u(a) - \alpha_2 u^{[2]}(a)\Big)\Big(\hat{\alpha}_1\overline{u(a)} - \hat{\alpha}_2\overline{u^{[2]}(a)}\Big) \\
&- \Big(\hat{\alpha}_1 u(a) - \hat{\alpha}_2 u^{[2]}(a)\Big)\Big(\alpha_1\overline{u(a)} - \alpha_2\overline{u^{[2]}(a)}\Big)\Big] \\
=&\frac{1}{\eta}\Big[(\hat{\alpha}_1\alpha_2 - \alpha_1\hat{\alpha}_2)u(a)\overline{u^{[2]}(a)} - (\hat{\alpha}_1\alpha_2 - \alpha_1\hat{\alpha}_2)u^{[2]}(a)\overline{u(a)}\Big] \\
=&u(a)\overline{u^{[2]}(a)} - u^{[2]}(a)\overline{u(a)},
\end{aligned}$$

then, applying (2.5), it follows that

$$\begin{aligned}
&2i\Im(\mathbf{L}_h U, U) = u(b)\overline{u^{[2]}(b)} - u^{[2]}(b)\overline{u(b)} + iu^{[1]}(b)\overline{u^{[1]}(b)} \\
&- \Big(u(a)\overline{u^{[2]}(a)} - u^{[2]}(a)\overline{u(a)} + iu^{[1]}(a)\overline{u^{[1]}(a)}\Big) + \Big(u(a)\overline{u^{[2]}(a)} - u^{[2]}(a)\overline{u(a)}\Big) \\
&= u(b)\overline{u^{[2]}(b)} - u^{[2]}(b)\overline{u(b)} + iu^{[1]}(b)\overline{u^{[1]}(b)} - iu^{[1]}(a)\overline{u^{[1]}(a)}.
\end{aligned} \tag{3.2}$$

From (2.9) and (2.10), it has

$$u^{[1]}(a) = ru^{[1]}(b) + \beta_1 u^{[2]}(b), \tag{3.3}$$

and

$$u(b) = ir\overline{\beta_1}u^{[1]}(b) + \beta_2 u^{[2]}(b), \tag{3.4}$$

substituting (3.3) and (3.4) into (3.2), one obtains

$$\begin{aligned} 2i\Im(\mathbf{L}_h U, U) &= (\mathbf{L}_h U, U) - (U, \mathbf{L}_h U) \\ &= i(1 - r^2)u^{[1]}(b)\overline{u^{[1]}(b)} + i(2\Im\gamma_2 - r^2)u^{[2]}(b)\overline{u^{[2]}(b)}, \end{aligned} \tag{3.5}$$

and hence

$$2\Im(\mathbf{L}_h U, U) = su^{[1]}(b)\overline{u^{[1]}(b)} + du^{[2]}(b)\overline{u^{[2]}(b)} = s|u^{[1]}(b)|^2 + d|u^{[2]}(b)|^2, \tag{3.6}$$

where

$$s = 1 - r^2, \ d = 2\Im\beta_2 - r^2.$$

Since $|r| \leq 1, \ \Im\beta_2 \geq \frac{1}{2}r^2$, we have $s \geq 0$ and $d \geq 0$, that is

$$\Im(\mathbf{L}_h U, U) \geq 0, \ \forall U \in D(\mathbf{L}_h).$$

Hence $\mathbf{L}_h$ is a dissipative operator in $\mathcal{H}$. □

Theorem 2. *If $|r| < 1, \ \Im\beta_2 > \frac{1}{2}r^2$, then the operator $\mathbf{L}_h$ has no real eigenvalue.*

Proof. Suppose λ_0 is a real eigenvalue of $\mathbf{L}_h$. Let $\Phi_0(x) = \begin{pmatrix} \phi_0(x) \\ R(\phi_0(x)) \end{pmatrix} = \begin{pmatrix} \phi(x, \lambda_0) \\ R(\phi(x, \lambda_0)) \end{pmatrix} \neq \mathbf{0}$ be a corresponding eigenvector. Since

$$\Im\left(\mathbf{L}_h\Phi_0, \Phi_0\right) = \Im\left(\lambda_0(\|\phi_0\|^2 + |R(\phi_0)|^2)\right) = 0,$$

from (3.6), it follows that

$$\begin{aligned} \Im(\mathbf{L}_h\Phi_0, \Phi_0) &= \frac{1}{2}\left(s\phi_0^{[1]}(b)\overline{\phi_0^{[1]}(b)} + d\phi_0^{[2]}(b)\overline{\phi_0^{[2]}(b)}\right) \\ &= \frac{1}{2}\begin{pmatrix}\overline{\phi_0^{[1]}(b)} & \overline{\phi_0^{[2]}(b)}\end{pmatrix}\begin{pmatrix} s & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix}\phi_0^{[1]}(b) \\ \phi_0^{[2]}(b)\end{pmatrix} = 0, \end{aligned}$$

since $|r| < 1, \ \Im\beta_2 > \frac{1}{2}r^2$, the matrix

$$\begin{pmatrix} s & 0 \\ 0 & d \end{pmatrix}$$

is positive definite. Hence $\phi_0^{[1]}(b) = 0$ and $\phi_0^{[2]}(b) = 0$, and by the BCs (2.9) and (2.10), we obtain that $\phi_0^{[1]}(a) = 0$ and $\phi_0(b) = 0$. Let $\phi_0(x) = \phi(x, \lambda_0), \tau_0(x) = \tau(x, \lambda_0)$ and $\sigma_0(x) = \sigma(x, \lambda_0)$, be three linearly independent solutions of equation $l(u) = \lambda_0 u$; then by the above results it has

$$\begin{vmatrix} \phi_0(b) & \tau_0(b) & \sigma_0(b) \\ \phi_0^{[1]}(b) & \tau_0^{[1]}(b) & \sigma_0^{[1]}(b) \\ \phi_0^{[2]}(b) & \tau_0^{[2]}(b) & \sigma_0^{[2]}(b) \end{vmatrix} = 0, \tag{3.7}$$

however, on the other hand the Wronskian of $\phi_0(x), \tau_0(x), \sigma_0(x)$ is not 0; this is a contradiction. Thus the theorem is proven. □

4. The characteristic function and Green's function

In this section, to prepare for the proof of the completeness theorems, we review the characteristic function and Green's function and use the Green's function to study the inverse of $\mathbf{L}_h$.

Let φ, ψ, and χ be the linearly independent solutions of the third-order equation (2.7) on $[a,b]$ satisfying the following initial conditions:

$$\begin{pmatrix} \varphi & \psi & \chi \\ \varphi^{[1]} & \psi^{[1]} & \chi^{[1]} \\ \varphi^{[2]} & \psi^{[2]} & \chi^{[2]} \end{pmatrix}(a,\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.1}$$

and let

$$\Psi(x,\lambda) = \begin{pmatrix} \varphi & \psi & \chi \\ \varphi^{[1]} & \psi^{[1]} & \chi^{[1]} \\ \varphi^{[2]} & \psi^{[2]} & \chi^{[2]} \end{pmatrix}(x,\lambda),\ x \in [a,b],$$

and the coefficient matrix of the BCs (2.8)–(2.10) be denoted by $(A_\lambda : B)$, where A_λ and B are both 3×3 matrices, then we have the following conclusions.

Lemma 2. *The complex number λ is an eigenvalue of the BVP (2.7)–(2.11) if and only if the characteristic function*

$$\Delta(\lambda) := \begin{vmatrix} l_1(\varphi) & l_1(\psi) & l_1(\chi) \\ l_2(\varphi) & l_2(\psi) & l_2(\chi) \\ l_3(\varphi) & l_3(\psi) & l_3(\chi) \end{vmatrix} = \mathbf{det}[A_\lambda + B\Psi(b,\lambda)] = 0.$$

Proof. If λ is an eigenvalue of BVP (2.7)–(2.11), then there exists a non-trivial solution

$$u(x,\lambda) = c_1\varphi(x,\lambda) + c_2\psi(x,\lambda) + c_3\chi(x,\lambda) \tag{4.2}$$

of (2.7), and the BCs (2.8)–(2.10) are satisfied, where $c_1,\ c_2,\ c_3 \in \mathbb{C}$ are not all zero. Since $u(x,\lambda)$ satisfies the BCs (2.8)–(2.10), we have

$$A_\lambda\left(c_1\begin{pmatrix} \varphi(a,\lambda) \\ \varphi^{[1]}(a,\lambda) \\ \varphi^{[2]}(a,\lambda) \end{pmatrix} + c_2\begin{pmatrix} \psi(a,\lambda) \\ \psi^{[1]}(a,\lambda) \\ \psi^{[2]}(a,\lambda) \end{pmatrix} + c_3\begin{pmatrix} \chi(a,\lambda) \\ \chi^{[1]}(a,\lambda) \\ \chi^{[2]}(a,\lambda) \end{pmatrix}\right)$$

$$+B\left(c_1\begin{pmatrix} \varphi(b,\lambda) \\ \varphi^{[1]}(b,\lambda) \\ \varphi^{[2]}(b,\lambda) \end{pmatrix} + c_2\begin{pmatrix} \psi(b,\lambda) \\ \psi^{[1]}(b,\lambda) \\ \psi^{[2]}(b,\lambda) \end{pmatrix} + c_3\begin{pmatrix} \chi(b,\lambda) \\ \chi^{[1]}(b,\lambda) \\ \chi^{[2]}(b,\lambda) \end{pmatrix}\right) = \mathbf{0},$$

via the initial conditions (4.1), we have

$$(A_\lambda + B\Psi(b,\lambda))(c_1, c_2, c_3)^T = \mathbf{0}. \tag{4.3}$$

Since $c_1,\ c_2$, and c_3 are not all zero, then the determinant of the coefficient matrix

$$\Delta(\lambda) := \mathbf{det}[A_\lambda + B\Psi(b,\lambda)] = 0.$$

Conversely, if $\Delta(\lambda) = 0$, then equation (4.3) has a non-zero solution $(c_1, c_2, c_3)^T$. Choose such a solution and define $u(x,\lambda)$ as in (4.2). Then $u(x,\lambda)$ satisfies the BVP ((2.7)–(2.11) and thus is an eigenfunction. Therefore, λ is an eigenvalue of the BVP (2.7)–(2.11). □

Definition 3. *Let $g(\lambda)$ be an entire function of λ, if for any $\varepsilon > 0$, there exists a positive constant $C_\varepsilon > 0$, such that*

$$|g(\lambda)| \le C_\varepsilon e^{\varepsilon|\lambda|},\ \lambda \in \mathbb{C},$$

then $g(\lambda)$ is called an entire function with growth of order ≤ 1 and minimal type.

According to Definition 3, we can easily obtain that $\varphi(b,\lambda)$, $\psi(b,\lambda)$, and $\chi(b,\lambda)$ are entire functions of λ with growth of order ≤ 1 and minimal type; therefore, $\Delta(\lambda)$ is an entire function of λ with growth of order ≤ 1 and minimal type, and then we have the following conclusion.

Corollary 1. *The entire function $\Delta(\lambda)$ is of growth order ≤ 1 and minimal type: for any $\varepsilon > 0$, there exists a positive constant C_ε such that*

$$|\Delta(\lambda)| \le C_\varepsilon e^{\varepsilon|\lambda|},\ \lambda \in \mathbb{C}, \tag{4.4}$$

and hence

$$\limsup_{|\lambda|\to\infty} \frac{ln|\Delta(\lambda)|}{|\lambda|} \le 0. \tag{4.5}$$

From Theorem 2 it follows that zero is not an eigenvalue of $\mathbf{L}_h$ (*i.e.*, $Ker\mathbf{L}_h = \mathbf{0}$), hence the operator $\mathbf{L}_h^{-1}$ exists. Now we show an analytical representation of $\mathbf{L}_h^{-1}$.

Consider the operator equation

$$\mathbf{L}_h U = F,\ F = \begin{pmatrix} f(x) \\ f_1 \end{pmatrix} \in \mathcal{H},$$

then the operator equation is equivalent to the non-homogeneous boundary value problem composed of the equation $l(u) = f(x)$ and the boundary condition $\widetilde{R}(u) - f_1 = 0$ and the BCs (2.9) and (2.10). Let $u(x)$ be the solution of the above non-homogeneous boundary value problem, and set $\varphi_0(x) = \varphi(x,0)$, $\psi_0(x) = \psi(x,0)$, $\chi_0(x) = \chi(x,0)$, then

$$u(x) = C_1\varphi_0(x) + C_2\psi_0(x) + C_3\chi_0(x) + u^*(x),$$

where $C_j,\ j = 1,2,3$ are arbitrary constants and $u^*(x)$ is a special solution.

It can be obtained by the method of constant variation

$$u(x) = C_1(x)\varphi_0(x) + C_2(x)\psi_0(x) + C_3(x)\chi_0(x),$$

where $C_j,\ j = 1,2,3$ satisfies

$$\begin{cases} C_1'(x)\varphi_0(x) + C_2'(x)\psi_0(x) + C_3'(x)\chi_0(x) = 0, \\ C_1'(x)\varphi_0^{[1]}(x) + C_2'(x)\psi_0^{[1]}(x) + C_3'(x)\chi_0^{[1]}(x) = 0, \\ -\frac{1}{w}(C_1'(x)\varphi_0^{[2]}(x) + C_2'(x)\psi_0^{[2]}(x) + C_3'(x)\chi_0^{[2]}(x)) = f(x). \end{cases}$$

Solve the equations above, one has

$$C_1'(x) = \frac{-w(x)f(x)}{D(x)} \begin{vmatrix} \psi_0(x) & \chi_0(x) \\ \psi_0^{[1]}(x) & \chi_0^{[1]}(x) \end{vmatrix},\ C_2'(x) = \frac{w(x)f(x)}{D(x)} \begin{vmatrix} \varphi_0(x) & \chi_0(x) \\ \varphi_0^{[1]}(x) & \chi_0^{[1]}(x) \end{vmatrix},$$

$$C_3'(x) = \frac{-w(x)f(x)}{D(x)} \begin{vmatrix} \varphi_0(x) & \psi_0(x) \\ \varphi_0^{[1]}(x) & \psi_0^{[1]}(x) \end{vmatrix},$$

where

$$D(x) = \mathbf{det}[\Psi(x, 0)].$$

By proper calculation, it can be obtained that

$$u^*(x) = \int_a^b K(x, \xi) f(\xi) d\xi,$$

where

$$K(x, \xi) = \begin{cases} \frac{-w(\xi)}{D(\xi)} \begin{vmatrix} \varphi_0(\xi) & \psi_0(\xi) & \chi_0(\xi) \\ \varphi_0^{[1]}(\xi) & \psi_0^{[1]}(\xi) & \chi_0^{[1]}(\xi) \\ \varphi_0(x) & \psi_0(x) & \chi_0(x) \end{vmatrix}, & a < \xi \leq x < b, \\ 0, & a < x \leq \xi < b. \end{cases} \tag{4.6}$$

Then

$$u(x) = C_1\varphi_0(x) + C_2\psi_0(x) + C_3\chi_0(x) + \int_a^b K(x, \xi) f(\xi) d\xi,$$

substituting $u(x)$ into $\widetilde{R}(u) - f_1 = 0$ and the BCs (2.9) and (2.10), one obtains

$$C_j = -\frac{1}{\Delta(0)} \int_a^b F_j(\xi) f(\xi) d\xi, \ \ j = 1, 2, 3, \tag{4.7}$$

where

$$\Delta(0) := \mathbf{det}[A_0 + B\Psi(b, 0)], \tag{4.8}$$

$$F_1(\xi) = -\begin{vmatrix} \widetilde{R}(K) - \frac{f_1}{(b-a)f(\xi)} & \widetilde{R}(\psi_0) & \widetilde{R}(\chi_0) \\ l_2(K) & l_2(\psi_0) & l_2(\chi_0) \\ l_3(K) & l_3(\psi_0) & l_3(\chi_0) \end{vmatrix}, \tag{4.9}$$

$$F_2(\xi) = -\begin{vmatrix} \widetilde{R}(\varphi_0) & \widetilde{R}(K) - \frac{f_1}{(b-a)f(\xi)} & \widetilde{R}(\chi_0) \\ l_2(\varphi_0) & l_2(K) & l_2(\chi_0) \\ l_3(\varphi_0) & l_3(K) & l_3(\chi_0) \end{vmatrix}, \tag{4.10}$$

$$F_3(\xi) = -\begin{vmatrix} \widetilde{R}(\varphi_0) & \widetilde{R}(\psi_0) & \widetilde{R}(K) - \frac{f_1}{(b-a)f(\xi)} \\ l_2(\varphi_0) & l_2(\psi_0) & l_2(K) \\ l_3(\varphi_0) & l_3(\psi_0) & l_3(K) \end{vmatrix}, \tag{4.11}$$

then $u(x)$ can be represented as

$$u(x) = -\int_a^b \frac{1}{\Delta(0)} [F_1(\xi)\varphi_0(x) + F_2(\xi)\psi_0(x) + F_3(\xi)\chi_0(x) - K(x, \xi)\Delta(0)] f(\xi) d\xi.$$

Let

$$G(x, \xi) = \frac{1}{\Delta(0)} \begin{vmatrix} \varphi_0(x) & \psi_0(x) & \chi_0(x) & K(x, \xi) \\ \widetilde{R}(\varphi_0) & \widetilde{R}(\psi_0) & \widetilde{R}(\chi_0) & \widetilde{R}(K) - \frac{f_1}{(b-a)f(\xi)} \\ l_2(\varphi_0) & l_2(\psi_0) & l_2(\chi_0) & l_2(K) \\ l_3(\varphi_0) & l_3(\psi_0) & l_3(\chi_0) & l_3(K) \end{vmatrix}, \tag{4.12}$$

then

$$u(x) = \int_a^b G(x,\xi)f(\xi)d\xi.$$

Now define the operator $\mathbf{T}_h$ as

$$\mathbf{T}_h F = \begin{pmatrix} \int_a^b G(x,\xi)f(\xi)d\xi \\ R(\int_a^b G(x,\xi)f(\xi)d\xi) \end{pmatrix}, \ \forall F = \begin{pmatrix} f(x) \\ f_1 \end{pmatrix} \in \mathcal{H}, \tag{4.13}$$

clearly, $\mathbf{T}_h$ is the inverse operator of $\mathbf{L}_h$; this implies that the root vectors (eigenvectors and associated vectors) of the operators $\mathbf{T}_h$ and $\mathbf{L}_h$ coincide.

Further, in order to better illustrate the completeness theorems, we define the operator $\mathbf{T}$ as

$$\mathbf{T}f = \int_a^b G(x,\xi)f(\xi)d\xi, \ \forall f \in H, \tag{4.14}$$

then from Lemma 1, it follows easily that the completeness of the system of root vectors (eigenfunctions and associated functions) of the BVP (2.7)–(2.11) is equivalent to the completeness of the system of root vectors of the operator $\mathbf{T}$. Since $\varphi_0(x),\ \psi_0(x),\ \chi_0(x) \in H$, then

$$\int_a^b \int_a^b |G(x,\xi)|^2 dx d\xi < +\infty, \tag{4.15}$$

hence the integral operator $\mathbf{T}$ is a Hilbert–Schmidt operator, i.e., $\mathbf{T}$ is compact.

5. Completeness theorems

In this section, we show the completeness theorems here. Before we can state our main completeness theorem, some supplementary lemmas are needed. The first lemma is known as Krein's Theorem.

Lemma 3. *([31], page 238) Let S be a compact dissipative operator in $\mathcal{H}$ with nuclear imaginary part $\Im S$. The system of all root vectors of S is complete in $\mathcal{H}$ so long as at least one of the following two conditions is fulfilled:*

$$\lim_{m\to\infty} \frac{n_+(m, \Re S)}{m} = 0, \ \lim_{m\to\infty} \frac{n_-(m, \Re S)}{m} = 0, \tag{5.1}$$

where $n_+(m, \Re S)$ and $n_-(m, \Re S)$ denote the number of eigenvalues of the real component $\Re S$ of S in the intervals $[0, m]$ and $[-m, 0]$, respectively.

Lemma 4. *Let $\mathbf{S}$ be an invertible operator. Then, $-\mathbf{S}$ is dissipative if and only if the inverse operator $\mathbf{S}^{-1}$ of $\mathbf{S}$ is dissipative.*

Proof. Assume that $-\mathbf{S}$ is dissipative. Then, for all $y \in D(\mathbf{S})$,

$$\Im(y, \mathbf{S}y) = -\Im(\mathbf{S}y, y) = \Im(-\mathbf{S}y, y) \geq 0.$$

Hence, for any $z \in D(\mathbf{S}^{-1})$,

$$\Im(\mathbf{S}^{-1}z, z) = \Im(\mathbf{S}^{-1}z, \mathbf{S}(\mathbf{S}^{-1}z)) \geq 0,$$

since $\mathbf{S}^{-1}z \in D(\mathbf{S})$. Hence $\mathbf{S}^{-1}$ is dissipative. □

Let $\mathbf{K}$ be a differential operator generated by the differential expression $l(u)$ in (2.1) and a set of boundary conditions denoted by $\mathscr{B}(u)$. Let $\mathbf{K}^*$ and $\mathbf{A}$ denote the adjoint operator and the inverse of $\mathbf{K}$, respectively. Set $\mathbf{A} = \mathbf{A}_1 + i\mathbf{A}_2$. From the preceding analysis, it follows that $\mathbf{A}$ is an integral operator satisfying

$$\mathbf{A} = \mathbf{K}^{-1}, \quad \mathbf{A}^* = (\mathbf{K}^*)^{-1}.$$

Therefore, the inverse operator of the real part of $\mathbf{A}$, denoted as $\tilde{\mathbf{K}}$, satisfies

$$\tilde{\mathbf{K}} = \mathbf{A}_1^{-1} = \left(\frac{\mathbf{A} + \mathbf{A}^*}{2}\right)^{-1} = 2\left(\mathbf{K}^{-1} + (\mathbf{K}^*)^{-1}\right)^{-1}.$$

Clearly, operator $\tilde{\mathbf{K}}$ is a differential operator, and we denote its corresponding differential expression and boundary conditions as $\widetilde{l}(u)$ and $\widetilde{\mathscr{B}}(u)$, respectively.

If $\mathbf{K}$ is a self-adjoint operator, then $\mathbf{K} = \mathbf{K}^*$. In this case,

$$\tilde{\mathbf{K}} = 2\left(\mathbf{K}^{-1} + (\mathbf{K}^*)^{-1}\right)^{-1} = \left(\mathbf{K}^{-1}\right)^{-1} = \mathbf{K},$$

and it evidently follows that $\widetilde{l}(u) = l(u)$, $\widetilde{\mathscr{B}}(u) = \mathscr{B}(u)$.

Next, consider the integral operator $\mathbf{T}_h$ defined by (4.13); we set $\mathbf{T}_h = \mathbf{T}_{h_1} + i\mathbf{T}_{h_2}$ with $\mathbf{T}_{h_1} = \Re\mathbf{T}_h$ and $\mathbf{T}_{h_2} = \Im\mathbf{T}_h$. Since $\mathbf{T}_h$ is a bounded operator, then $\mathbf{T}_{h_1}$ and $\mathbf{T}_{h_2}$ are self-adjoint ([21], page 6), and we can obtain the following results:

Lemma 5. *The operator $\mathbf{T}_{h_1}$ is the inverse of $\mathbf{L}_{h_1}$, where $\mathbf{L}_{h_1}$ is the operator generated by the differential expression in (2.1) and the unique set of boundary conditions. Of course, the operator $\mathbf{L}_{h_1}$ is self-adjoint.*

Proof. Clearly, the operator $\mathbf{L}_{h_1}$ is a differential operator. Analogous to the differential operator $\tilde{\mathbf{K}}$, we denote its corresponding differential expression as $\hat{l}(u)$.

Note that in this paper, the operator $\mathbf{L}_h$ is dissipative, but its dissipativity stems solely from the BCs (2.8)–(2.10) and is independent of the differential structure $l(u)$. Moreover, the differential expression $l(u)$ is symmetric. Consequently, we can conclude that the differential expression corresponding to operator $\mathbf{L}_{h_1}$ is $\hat{l}(u) = l(u)$; this means that the differential expression associated with $\mathbf{L}_{h_1}$ is also $l(u)$ in (2.1).

Furthermore, the self-adjoint operator $\mathbf{L}_{h_1} = \mathbf{T}_{h_1}^{-1}$ is uniquely determined by the integral operators $\mathbf{T}_h$ and $\mathbf{T}_h^*$, where both $\mathbf{T}_h$ and $\mathbf{T}_h^*$ possess unique explicit representations; it follows that the boundary conditions for $\mathbf{L}_{h_1}$ are unique.

Now the proof is finished.

□

Lemma 6. *([32], page 295 Theorem 1) If an entire function $h(\mu)$ is of order ≤ 1 and minimal type, then*

$$\lim_{\rho\to\infty} \frac{n_+(\rho, h)}{\rho} = 0, \ \lim_{\rho\to\infty} \frac{n_-(\rho, h)}{\rho} = 0, \tag{5.2}$$

where $n_+(\rho, h)$ and $n_-(\rho, h)$ denote the number of the zeros of the function $h(\mu)$ in the intervals $[0, \rho]$ and $[-\rho, 0]$, respectively.

Corollary 2. *If the operator* $\mathbf{L}_{h_1}$ *is defined as in Lemma 5, then*

$$\lim_{m\to\infty} \frac{n_+(m, \mathbf{L}_{h_1})}{m} = 0, \ \lim_{m\to\infty} \frac{n_-(m, \mathbf{L}_{h_1})}{m} = 0, \tag{5.3}$$

where $n_+(m, \mathbf{L}_{h_1})$ *and* $n_-(m, \mathbf{L}_{h_1})$ *denote the number of eigenvalues of* $\mathbf{L}_{h_1}$ *in the intervals* $[0, m]$ *and* $[-m, 0]$*, respectively.*

Proof. From Corollary 1 and Lemma 5, we can easily obtain that the characteristic function $\Delta_1(\lambda)$ of $\mathbf{L}_{h_1}$ is an entire function of λ with growth of order ≤ 1 and minimal type. Then from Lemma 6, we have

$$\lim_{\rho\to\infty} \frac{n_+(\rho, \Delta_1(\lambda))}{\rho} = 0, \ \lim_{\rho\to\infty} \frac{n_-(\rho, \Delta_1(\lambda))}{\rho} = 0.$$

From Lemma 2, we know that a complex number λ is an eigenvalue of the operator $\mathbf{L}_1$ if and only if it is the zeros of the characteristic function $\Delta_1(\lambda)$. Therefore, we can obtain that

$$\lim_{m\to\infty} \frac{n_+(m, \mathbf{L}_{h_1})}{m} = 0, \ \lim_{m\to\infty} \frac{n_-(m, \mathbf{L}_{h_1})}{m} = 0.$$

□

Now, we are ready to state and prove the completeness theorem.

Theorem 3. *The system of eigenfunctions and associated functions of the BVP (2.7)–(2.11) is complete in the Hilbert space H.*

Proof. Consider the operator $\mathbf{T}$ defined by (4.14); similarly to $\mathbf{T}_h = \mathbf{T}_{h_1} + i\mathbf{T}_{h_2}$, let $\mathbf{T} = \mathbf{T}_1 + i\mathbf{T}_2$ with $\mathbf{T}_1 = \Re\mathbf{T}$ and $\mathbf{T}_2 = \Im\mathbf{T}$. By the above discussions, the operator $-\mathbf{T}$ is a compact dissipative operator in H with a nuclear imaginary part, $-\mathbf{T}_2$.

Let r_j be the eigenvalue of $\mathbf{L}_{h_1}$; then $-\frac{1}{r_j}$ is the eigenvalue of $-\mathbf{T}_1$. From Corollary 2, we can obtain that

$$\lim_{m\to\infty} \frac{n_+(m, -\mathbf{T}_1)}{m} = 0, \ \lim_{m\to\infty} \frac{n_-(m, -\mathbf{T}_1)}{m} = 0,$$

that is

$$\lim_{m\to\infty} \frac{n_+(m, \Re(-\mathbf{T}))}{m} = 0, \ \lim_{m\to\infty} \frac{n_-(m, \Re(-\mathbf{T}))}{m} = 0.$$

Then from Lemma 3, we can get the system of all root vectors of $-\mathbf{T}$ is complete in H. And since the completeness of the system of root vectors (eigenfunctions and associated functions) of the BVP (2.7)–(2.11) is equivalent to the completeness of the system of root vectors of the operator $\mathbf{T}$, then the system of eigenfunctions and associated functions of the BVP (2.7)–(2.11) is complete in the Hilbert space H. □

Obviously, from Lemma 1 and Theorem 3, we can easily obtain that the system of eigenfunctions and associated functions of $\mathbf{L}_h$ is complete in $\mathcal{H}$.

Remark 1. *Our conclusion can be extended to the case of singular end points case by using the method in [21].*

6. Concluding remarks

In the present paper, we considered the dissipative third-order boundary value problems with distributional potentials and eigenparameter-dependent boundary conditions. By transforming the considered problem to an isospectral operator problem, we prove that the operator is dissipative and has no real eigenvalues under certain conditions. Furthermore, by applying Krein's theorem, we establish the completeness theorems for both the boundary value problem and the corresponding operator.

To our best knowledge, for third-order boundary value problems with distributional potentials and eigenparameter-dependent boundary conditions, the corresponding results have not been studied yet. The eigenvalue problems and completeness of the system of eigenfunctions and associated functions are essential to problems such as the non-classical wavelets and open quantum systems. The results here are more general than the previously known results.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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