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Research article

The alternating direction implicit difference scheme and extrapolation method for a class of three dimensional hyperbolic equations with constant coefficients

Zhaoxiang Zhang[1], **Xuehua Yang**[1,*] **and Song Wang**[2]

[1] School of Science, Hunan University of Technology, Zhuzhou 412007, China

[2] School of Electrical Engineering, Computing and Mathematical Sciences, Curtin University, GPO Box U1987 Perth, WA 6845, Australia

* **Correspondence:** Email: hunanshidayang@163.com

Abstract: This paper conducts a study on the alternating direction implicit (ADI) difference schemes for a class of three dimensional hyperbolic equations with constant coefficients. The central difference methods are employed in the temporal and spatial direction. The solvability, stability, and convergence of the proposed ADI schemes are proven. Moreover, the Richardson extrapolation method is established to enhance the accuracy of the algorithm. Numerical examples are presented for the errors and convergence orders of the established ADI schemes and extrapolation schemes. By comparing the results of numerical examples , it can be concluded that the proposed Richardson extrapolation method can effectively improve the accuracy of the numerical solutions and reduce the errors.

Keywords: constant coefficient hyperbolic equation; three dimensional; finite difference method; ADI scheme; richardson extrapolation

1. Introduction

The class of three-dimensional hyperbolic partial differential equations (PDEs) with constant coefficients studied in this paper is as follows:

$$u_{tt} - a^2 u_{xx} - b^2 u_{yy} - c^2 u_{zz} = f(x, y, z, t), \qquad (x, y, z) \in \Omega, t \in [0, T], \tag{1.1}$$

$$u(x, y, z, 0) = \varphi(x, y, z), u_t(x, y, z, 0) = \psi(x, y, z), \qquad (x, y, z) \in \bar{\Omega}, \tag{1.2}$$

$$u(x, y, z, t) = \alpha(x, y, z, t), \qquad (x, y, z) \in \Gamma, t \in [0, T], \tag{1.3}$$

where a, b, and c are constants, $\Omega = (0, L_1) \times (0, L_2) \times (0, L_3)$, Γ is the boundary of Ω, and when $(x, y, z) \in \Gamma$, $\alpha(x, y, z, 0) = \varphi(x, y, z)$, $\partial\alpha(x, y, z, 0)/\partial t = \psi(x, y, z)$.

The hyperbolic partial differential equations (1.1)–(1.3) hold substantial theoretical and practical significance, serving as a cornerstone for both the evaluation of numerical methods and the simulation of complex natural and physical phenomena. These models provide a rigorous framework for assessing the feasibility and performance of numerical approaches [1–3], as well as for modeling intricate dynamic processes observed in nature [4, 5].

The applications of hyperbolic models span a wide range of disciplines, extending far beyond mathematics. In biology, for instance, the hyperbolic equations are extensively employed to describe population dynamics and the propagation of neural signals [6–9]. The models play a pivotal role in the analysis of vibrational systems and the exploration of fundamental physical principles. For example, Fedotov et al. [10] utilized hyperbolic and pseudo-hyperbolic equations to investigate vibration theory, offering insights into the characterization and analysis of vibrational phenomena. Paul [11] examined the application of Huygens' principle to hyperbolic equations, while Alinhac [12] delved into the blowup behavior of nonlinear hyperbolic equations. The hyperbolic models find significant applications in engineering [13, 14], where they are used to address a variety of practical problems.

This problem (1.1)–(1.3) is formulated for a three-dimensional wave equation with boundary conditions of the first kind, which has been a subject of considerable research interest in the mixed problem for a hyperbolic equation [15, 16]. Gordienko [17] studied the mixed problem for real wave equations satisfying uniform Lopatinskii conditions, systematically establishing all viable reduction methods to transform it into a mixed problem for symmetric hyperbolic systems with dissipative boundary conditions. Malyshev [18] researched initial-boundary value problems for second-order hyperbolic PDEs with complex first-order boundary conditions. These investigations collectively highlight the significance of this research topic in contemporary mathematical physics.

The ADI method has long been recognized as a powerful and efficient numerical technique in scientific computing and engineering, particularly for solving PDEs [19–22]. Its unique ability to decompose high-dimensional problems into a series of one-dimensional problems has made it a preferred choice for tackling complex systems, especially those involving high dimension problems in space. The model in Eqs (1.1)–(1.3) has been widely investigated using ADI methods [23–27], demonstrating both theoretical significance and ADI unique advantages for high-dimensional simulations of the model. Nie and Cao [28] developed an improved ADI scheme based on a staggered grid to address transient heat conduction problems in mixed representations. Zhao [29] proposed a novel Douglas ADI method for solving two-dimensional heat equations with interfaces, further demonstrating the versatility of ADI techniques in handling complex boundary conditions.

Over the years, the difference and ADI difference method has garnered significant attention from researchers due to its computational efficiency and stability, which are critical for solving large scale problems in various fields [30–34]. Zhou et al. [35] employed an efficient ADI approach to solve three-dimensional heat equations with irregular boundaries and interfaces. Shen et al. [36] provided the high-order ADI difference method and extrapolation method for 2D nonlinear parabolic evolution equations. Chen et al. [37] investigated numerical methods for two-dimensional integro-differential equations with two fractional Riemann-Liouville (R-L) integral kernels. Liu et al. [38] utilized the ADI method to transform three-dimensional problems into one-dimensional problems, enabling the efficient solution of three-dimensional integro-differential equations with weakly singular kernels. Moreover, the ADI method has found extensive use in solving nonlocal evolution equations [39–43], it demonstrates the wide adaptability of ADI across different models.

The Richardson extrapolation scheme plays a significant role in refining numerical solutions and achieving higher-order accuracy, a widely applied technique in the numerical solution of linear differential equations [44–47] and nonlinear differential equations [48–51]. In the present study, we explore the ADI method and its Richardson extrapolation scheme for solving a class of three-dimensional constant-coefficient hyperbolic partial differential equations. By leveraging the ADI method within the framework of the original hyperbolic equation, we decompose the three-dimensional problem into a sequence of one-dimensional subproblems, solving them iteratively.

The main contributions of this article are as follows.

- We prove strictly the solvability, stability and convergence of the proposed ADI difference schemes for the three-dimensional problem, which is the first time to provide a strict theoretical analysis of the three-dimensional problem with constant coefficients. Although there were some papers on the ADI method, but they did not provide strict theoretical analysis for three-dimensional problems.
- We construct the Richardson extrapolation for the original model, present the theoretical analysis, and compare the extrapolated results with the initial data, which fill the gap in the ADI extrapolation method for three-dimensional problems.

The paper is organized as follows. Section 2 introduces notations and lemmas. Section 3 presents the discretization and finite difference scheme. Section 4 proves the solvability, stability and convergence. Section 5 discusses the Richardson extrapolation scheme. Section 6 provides numerical experiments to validate the method. Finally, Section 7 concludes with a summary.

2. Notations and lemmas

Take positive integers m_1, m_2, m_3, and n. Denote $h_1 = L_1/m_1$, $h_2 = L_2/m_2$, $h_3 = L_3/m_3$, and $\tau = T/n$. Define $x_i = ih_1(0 \le i \le m_1)$, $y_j = jh_2(0 \le j \le m_2)$, $z_k = kh_3(0 \le k \le m_3)$, and $t_l = l\tau(0 \le l \le n)$. Define

$$\begin{aligned}
&\Omega_h = \{(x_i, y_j, z_k) \mid 0 \le i \le m_1, 0 \le j \le m_2, 0 \le k \le m_3\},\\
&\Omega_\tau = \{t_l \mid 0 \le l \le n\}, \quad \omega = \{(i, j, k) \mid (x_i, y_j, z_k) \in \Omega\},\\
&\gamma = \{(i, j, k) \mid (x_i, y_j, z_k) \in \Gamma\}, \quad \bar{\omega} = \omega \cup \gamma,\\
&f_{ijk}^{l+\frac{1}{2}} = f(x_i, y_j, z_k, t_{l+\frac{1}{2}}), \quad t_{l+\frac{1}{2}} = \frac{1}{2}(t_l + t_{l+1}).
\end{aligned}$$

Let $u = \{u_{ijk}^l \mid 0 \le i \le m_1, 0 \le j \le m_2, 0 \le k \le m_3, 0 \le l \le n\}\}$ be a grid function defined on $\Omega_h \times \Omega_\tau$,

and we hereby introduce the following notations:

$$v_{ijk}^{l+\frac{1}{2}} = \frac{1}{2}(v_{ijk}^{l} + v_{ijk}^{l+1}), \quad \delta_t v_{ijk}^{l+\frac{1}{2}} = \frac{1}{\tau}(v_{ijk}^{l+1} - v_{ijk}^{l}),$$
$$v_{ijk}^{\bar{l}} = \frac{1}{2}(v_{ijk}^{l-1} + v_{ijk}^{l+1}), \quad \Delta_t v_{ijk}^{l} = \frac{1}{2\tau}(v_{ijk}^{l+1} - v_{ijk}^{l-1}),$$
$$\delta_x^2 v_{ijk}^n = \frac{1}{h_1^2}(v_{i-1,j,k}^n - 2v_{ijk}^n + v_{i+1,j,k}^n),$$
$$\delta_y^2 v_{ijk}^n = \frac{1}{h_2^2}(v_{i,j-1,k}^n - 2v_{ijk}^n + v_{i,j+1,k}^n),$$
$$\delta_z^2 v_{ijk}^n = \frac{1}{h_3^2}(v_{i,j,k-1}^n - 2v_{ijk}^n + v_{i,j,k+1}^n),$$
$$\Delta_h v_{ijk}^n = \delta_x^2 v_{ijk}^n + \delta_y^2 v_{ijk}^n + \delta_z^2 v_{ijk}^n.$$

Let $v^l = \{v_{ijk}^l \mid (i, j, k) \in \omega\}$ be a grid function on Ω_h. Assume the spatial grid $V_h = \{v \mid v = \{v_{ijk}^l \mid (i, j, k) \in \bar{v}\} \in \Omega_h$, $\mathring{V}_h = \{v \mid v = \{v_{ijk} \mid (i, j, k) \in \bar{\omega}\}$ and $v_{ijk} = 0$ if $(i, j, k) \in \gamma\}$.

Let $\iota, \mu \in \mathring{V}_h$ and introduce the following inner product and norm:

$$(\iota, \mu) = h_1 h_2 h_3 \sum_{i=1}^{m_1-1} \sum_{j=1}^{m_2-1} \sum_{k=1}^{m_3-1} \iota_{ijk}\mu_{ijk}, \quad \|\iota\| = \sqrt{(\iota, \iota)},$$
$$(\delta_x\iota, \delta_x\mu) = h_1 h_2 h_3 \sum_{i=1}^{m_1} \sum_{j=1}^{m_2-1} \sum_{k=1}^{m_3-1} (\delta_x\iota_{i-\frac{1}{2},j,k})(\delta_x\mu_{i-\frac{1}{2},j,k}),$$
$$(\delta_x\delta_y\iota, \delta_x\delta_y\mu) = h_1 h_2 h_3 \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \sum_{k=1}^{m_3-1} (\delta_x\delta_y\iota_{i-\frac{1}{2},j-\frac{1}{2},k})(\delta_x\delta_y\mu_{i-\frac{1}{2},j-\frac{1}{2},k}),$$
$$(\delta_x\delta_y\delta_z\iota, \delta_x\delta_y\delta_z\mu) = h_1 h_2 h_3 \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \sum_{k=1}^{m_3} (\delta_x\delta_y\delta_z\iota_{i-\frac{1}{2},j-\frac{1}{2},k-\frac{1}{2}})(\delta_x\delta_y\delta_z\mu_{i-\frac{1}{2},j-\frac{1}{2},k-\frac{1}{2}}),$$
$$\|\delta_x\delta_y\iota\|^2 = \sqrt{(\delta_x\delta_y\iota, \delta_x\delta_y\iota)}, \quad \|\delta_y\delta_z\iota\|^2 = \sqrt{(\delta_y\delta_z\iota, \delta_y\delta_z\iota)},$$
$$\|\delta_x\delta_z\iota\|^2 = \sqrt{(\delta_x\delta_z\iota, \delta_x\delta_z\iota)}, \quad \|\delta_x\delta_y\delta_z\iota\|^2 = \sqrt{(\delta_x\delta_y\delta_z\iota, \delta_x\delta_y\delta_z\iota)},$$
$$\|\delta_x\iota\|^2 = \sqrt{(\delta_x\iota, \delta_x\iota)}, \quad \|\delta_y\iota\|^2 = \sqrt{(\delta_y\iota, \delta_y\iota)},$$
$$\|\delta_z\iota\|^2 = \sqrt{(\delta_z\iota, \delta_z\iota)}, \quad \|\Delta_h\iota\|^2 = \sqrt{(\Delta_h\iota, \Delta_h\iota)},$$
$$|\iota|_1 = \sqrt{\|\delta_x\iota\|^2 + \|\delta_y\iota\|^2 + \|\delta_z\iota\|^2}, \quad \|\iota\|_1 = \sqrt{\|\iota\|^2 + |\iota|_1^2},$$
$$\|v\|_\infty = \max_{0\le i\le m_1, 0\le j\le m_2, 0\le k\le m_3} |v_{ijk}|.$$

Lemma 2.1. *[52–54] For arbitrary $\iota, \mu \in \mathring{V}_h$, it always follows that*

$$(\delta_x^2\iota, \mu) = -(\delta_x\iota, \delta_x\mu), \quad (\delta_y^2\iota, \mu) = -(\delta_y\iota, \delta_y\mu), \quad (\delta_z^2\iota, \mu) = -(\delta_z\iota, \delta_z\mu),$$
$$(\delta_x^2\delta_y^2\iota, \mu) = (\delta_x\delta_y\iota, \delta_x\delta_y\mu), \quad (\delta_x^2\delta_z^2\iota, \mu) = (\delta_x\delta_z\iota, \delta_x\delta_z\mu), \quad (\delta_y^2\delta_z^2\iota, \mu) = (\delta_y\delta_z\iota, \delta_y\delta_z\mu),$$
$$(\delta_x^2\delta_y^2\delta_z^2\iota, \mu) = -(\delta_x\delta_y\delta_z\iota, \delta_x\delta_y\delta_z\mu).$$

Lemma 2.2. *[53] Suppose that $g \in C^{(4)}[t_l, t_l + \tau]$. Then, we have*

$$g''(t_l + \tau) = \frac{2}{h}\left[\frac{g(t_l + \tau) - g(t_l)}{h} - g'(t_l) + \frac{h^2}{3}g'''(t_l)\right] + \frac{5}{12}h^2 g^{(4)}(t_l + \xi_0\tau), \quad \xi_0 \in (0, 1).$$

Lemma 2.3. *[53] If the function $g \in C^4[t_l - \tau, t_l + \tau]$, then*

$$g''(t_l) = \frac{1}{h^2}[g(t_l - \tau) - 2g(t_l) + g(t_l + \tau)] - \frac{h^2}{12}g^{(4)}(\xi_1), \quad t_l - \tau < \xi_1 < t_l + \tau,$$
$$g(t_l) = \frac{1}{2}[g(t_l - \tau) + g(t_l + \tau)] - \frac{h^2}{2}g''(\xi_2), \quad t_l - \tau < \xi_2 < t_l + \tau.$$

Lemma 2.4. *[53] Assume $v \in \mathring{V}_h$. It follows that*

$$\|v\| \le C|v|_1,$$

wherein, C denotes a positive constant that may differ across various scenarios. It invariably hinges upon the solution and the prescribed data, while remaining independent of the time step τ and the grid spacing h.

Lemma 2.5. *[53] (Gronwall's inequality-E) Let $\{F^k\}_{k=0}^{\infty}$ be a non-negative sequence, $\{g^k\}_{k=0}^{\infty}$ be non-negative and monotonically increasing (not necessarily strictly monotonically increasing), satisfying the condition that*

$$F^k \le c\tau \sum_{l=0}^{k-1} F^l + g^k, \quad k = 0, 1, 2, ...$$

Then, we have

$$F^k \le e^{ck\tau} g^k, \quad k = 0, 1, 2, ...$$

Lemma 2.6. *Assume that $v \in \mathring{V}_h$. Then, we conclude that*

$$\|v\|_\infty \le \frac{1}{2}\sqrt{\frac{L_1L_2L_3}{L_1L_2 + L_1L_3 + L_2L_3}}|v|_1.$$

Proof. For $1 \le i \le m_1 - 1, 1 \le j \le m_2 - 1, and 1 \le k \le m_3 - 1$, it holds that

$$v_{ijk} = \sum_{i'=1}^{i}\sum_{j=1}^{m_2-1}\sum_{k=1}^{m_3-1}(v_{i',j,k} - v_{i'-1,j,k}) = h_1\sum_{i'=1}^{i}\sum_{j=1}^{m_2-1}\sum_{k=1}^{m_3-1}\delta_x v_{i'-\frac{1}{2},j,k},$$
$$v_{ijk} = -\sum_{i'=i+1}^{m_1}\sum_{j=1}^{m_2-1}\sum_{k=1}^{m_3-1}(v_{i',j,k} - v_{i'-1,j,k}) = -h_1\sum_{i'=i+1}^{m_1}\sum_{j=1}^{m_2-1}\sum_{k=1}^{m_3-1}\delta_x v_{i'-\frac{1}{2},j,k}.$$

Square the above two equations respectively, and then apply the Cauchy-Schwarz inequality. We can obtain that

$$v_{ijk}^2 \le (h_1\sum_{i'=1}^{i}1^2)h_1\sum_{i'=1}^{i}\sum_{j=1}^{m_2-1}\sum_{k=1}^{m_3-1}(\delta_x v_{i'-\frac{1}{2},j,k})^2 = x_i h_1\sum_{i'=1}^{i}\sum_{j=1}^{m_2-1}\sum_{k=1}^{m_3-1}(\delta_x v_{i'-\frac{1}{2},j,k})^2, \tag{2.1}$$

$$v_{ijk}^2 \le (h_1 \sum_{i'=i+1}^{m_1} 1^2) h_1 \sum_{i'=i+1}^{m_1} \sum_{j=1}^{m_2-1} \sum_{k=1}^{m_3-1} (\delta_x v_{i'-\frac{1}{2},j,k})^2 = (L_1 - x_i) h_1 \sum_{i'=i+1}^{m_1} \sum_{j=1}^{m_2-1} \sum_{k=1}^{m_3-1} (\delta_x v_{i'-\frac{1}{2},j,k})^2. \tag{2.2}$$

Multiply (2.1) by $L_1 - x_i$, multiply (2.2) by x_i, and then add the obtained results together to get

$$L_1 v_{ijk}^2 \le x_i(L_1 - x_i) \sum_{i'=1}^{m_1} \sum_{j=1}^{m_2-1} \sum_{k=1}^{m_3-1} (\delta_x v_{i'-\frac{1}{2},j,k})^2 = x_i(L_1 - x_i)\|\delta_x v\|^2, \quad 1 \le i \le m_1 - 1.$$

By applying

$$x_i(L_1 - x_i) \le \frac{L_1^2}{4},$$

it can be derived that

$$L_1 v_{ijk}^2 \le \frac{L_1^2}{4}\|\delta_x v\|^2, \tag{2.3}$$

and, following the same argumentation, one arrives at the conclusion that

$$L_2 v_{ijk}^2 \le \frac{L_2^2}{4}\|\delta_y v\|^2, \tag{2.4}$$

$$L_3 v_{ijk}^2 \le \frac{L_3^2}{4}\|\delta_z v\|^2. \tag{2.5}$$

From $|v|_1 = \sqrt{\|\delta_x v\|^2 + \|\delta_y v\|^2 + \|\delta_z v\|^2}$ and the above three equations, we can obtain that

$$(L_1L_2 + L_2L_3 + L_1L_3)|v_{ijk}|^2 \le \frac{L_1L_2L_3}{4}|v|_1^2.$$

That is,

$$|v_{ijk}| \le \frac{1}{2}\sqrt{\frac{L_1L_2L_3}{L_1L_2 + L_2L_3 + L_1L_3}}|v|_1.$$

Moreover, because

$$v_{0,j,k} = v_{i,0,k} = v_{i,j,0} = v_{m_1,j,k} = v_{i,m_2,k} = v_{i,j,m_3} = 0,$$

then

$$\|v\|_\infty \le \frac{1}{2}\sqrt{\frac{L_1L_2L_3}{L_1L_2 + L_2L_3 + L_1L_3}}|v|_1.$$

□

3. Establishment of the ADI difference scheme

Define

$$M_1 = \frac{a^2b^2\tau^2\delta_x^2\delta_y^2}{2} + \frac{b^2c^2\tau^2\delta_y^2\delta_z^2}{2} + \frac{a^2c^2\tau^2\delta_x^2\delta_z^2}{2} - \frac{a^2b^2c^2\tau^2\delta_x^2\delta_y^2\delta_z^2}{4}.$$

Consider Eqs (1.1)–(1.3) on (x_i, y_j, z_k, t_1),

$$u_{tt}(x_i, y_j, z_k, t_1) - a^2u_{xx}(x_i, y_j, z_k, t_1) - b^2u_{yy}(x_i, y_j, z_k, t_1) - c^2u_{zz}(x_i, y_j, z_k, t_1) = f_{ijk}^1, \quad (i, j, k) \in \omega.$$

(3.1)

From the combination of (1.1) and (1.2), we get

$$u_{ttt}(x,y,z,0) = a^2\psi_{xx}(x,y,z) + b^2\psi_{yy}(x,y,z) + c^2\psi_{zz}(x,y,z) + f_t(x,y,z,0) \equiv \rho(x,y,z).$$

Based on Lemma 2.2, it follows that

$$\begin{aligned} u_{ttt}(x_i,y_j,z_k,t_1) &= \frac{2}{\tau}(\delta_t U^1_{ijk} - u_t(x_i,y_j,z_k,t_0) + \frac{\tau^2}{3}u_{ttt}(x_i,y_j,z_k,t_1)) + \frac{5\tau^2}{12}u_{tttt}(x_i,y_j,z_k,\zeta_{ijk}\tau) \\ &= \frac{2}{\tau}(\delta_t U^1_{ijk} - \psi_{ijk} + \frac{\tau^2}{3}\rho_{ijk}) + \frac{5}{12}\tau^2 u_{tttt}(x_i,y_j,z_k,\zeta_{ijk}\tau), \end{aligned}$$

where $\psi_{ijk} = \psi(x_i,y_j,z_k)$, $\rho_{ijk} = \rho(x_i,y_j,z_k)$, $\zeta_{ijk} \in (0,1)$.

Using Lemma 2.3, we have

$$\begin{aligned} u_{xx}(x_i,y_j,z_k,t_1) &= \delta_x^2 U^1_{ijk} - \frac{1}{12}u_{xxxx}(\eta_{ijk},y_j,z_k,t_1), \quad \eta_{ijk} \in (x_i - h_1, x_i + h_1), \\ u_{yy}(x_i,y_j,z_k,t_1) &= \delta_y^2 U^1_{ijk} - \frac{1}{12}u_{yyyy}(x_i,\xi_{ijk},z_k,t_1), \quad \xi_{ijk} \in (y_j - h_2, y_j + h_2), \\ u_{zz}(x_i,y_j,z_k,t_1) &= \delta_z^2 U^1_{ijk} - \frac{1}{12}u_{zzzz}(x_i,y_j,\lambda_{ijk},t_1), \quad \lambda_{ijk} \in (z_k - h_3, z_k + h_3). \end{aligned}$$

Next, substitute the above four formulas into (3.1) and add $M_1 U^1_{ijk}$ on both sides at the same time, so there is

$$\frac{2}{\tau}(\delta_t U^1_{ijk} - \psi_{ijk} + \frac{\tau^2}{3}\rho_{ijk}) - a^2\delta_x^2 U^1_{ijk} - b^2\delta_y^2 U^1_{ijk} - c^2\delta_z^2 U^1_{ijk} + M_1 U^1_{ijk} = f^1_{ijk} + (R_1)^0_{ijk}, \quad (i,j,k) \in \omega, \tag{3.2}$$

where

$$\begin{aligned} (R_1)^0_{ijk} =& \frac{5\tau^2}{12}u_{tttt}(x_i,y_j,z_k,\zeta_{ijk}\tau) - \frac{a^2h_1^2}{12}u_{xxxx}(\eta_{ijk},y_j,z_k,t_1) - \frac{b^2h_2^2}{12}u_{yyyy}(x_i,\xi_{ijk},z_k,t_1) \\ &- \frac{c^2h_3^2}{12}u_{zzzz}(x_i,y_j,\lambda_{ijk},t_1) - M_1 U^1_{ijk}, \quad (i,j,k) \in \omega. \end{aligned} \tag{3.3}$$

Then, there exists a constant c_1 such that

$$|(R_1)^0_{ijk}| \le c_1(\tau^2 + h_1^2 + h_2^2 + h_3^2), \quad (i,j,k) \in \omega.$$

Consider Eqs (1.1)–(1.3) on $(x_i,y_j,z_k,t_l)(1 \le l \le n-1)$,

$$\begin{aligned} u_{tt}(x_i,y_j,z_k,t_l) - a^2u_{xx}(x_i,y_j,z_k,t_l) - b^2u_{yy}(x_i,y_j,z_k,t_l) - c^2u_{zz}(x_i,y_j,z_k,t_l) = f^l_{ijk}, \\ (i,j,k) \in \omega. \end{aligned} \tag{3.4}$$

From Lemma 2.3, it can be concluded that

$$
\begin{aligned}
u_{tt}(x_i, y_j, z_k, t_l) &= \delta_t^2 U_{ijk}^l - \frac{\tau^2}{12} u_{tttt}(x_i, y_j, z_k, \bar{\zeta}_{ijk}^l), \quad \bar{\zeta}_{ijk}^l \in (-1, 1), \\
u_{xx}(x_i, y_j, z_k, t_l) &= \frac{1}{2}[u_{xx}(x_i, y_j, z_k, t_{l-1}) + u_{xx}(x_i, y_j, z_k, t_{l+1})] - \frac{\tau^2}{2} u_{xxtt}(x_i, y_j, z_k, s_{ijk}^l) \\
&= \frac{1}{2}[\delta_x^2 U_{ijk}^{l-1} + \delta_x^2 U_{ijk}^{l+1}] - \frac{h_1^2}{24}[u_{xxxx}(\bar{\eta}_{ijk}^l, y_j, z_k, t_{l-1}) + u_{xxxx}(\bar{\eta}_{ijk}^l, y_j, z_k, t_{l+1})] \\
&\quad - \frac{\tau^2}{2} u_{xxtt}(x_i, y_j, z_k, s_{ijk}^l), \quad s_{ijk}^l \in (t_l - \tau, t_l + \tau), \bar{\eta}_{ijk}^l \in (x_i - h_1, x_i + h_1). \\
u_{yy}(x_i, y_j, z_k, t_l) &= \frac{1}{2}[u_{yy}(x_i, y_j, z_k, t_{l-1}) + u_{yy}(x_i, y_j, z_k, t_{l+1})] - \frac{\tau^2}{2} u_{yytt}(x_i, y_j, z_k, s_{ijk}^l) \\
&= \frac{1}{2}[\delta_x^2 U_{ijk}^{l-1} + \delta_x^2 U_{ijk}^{l+1}] - \frac{h_1^2}{24}[u_{yyyy}(x_i, \bar{\xi}_{ijk}^l, z_k, t_{l-1}) + u_{yyyy}(x_i, \bar{\xi}_{ijk}^l, z_k, t_{l+1})] \\
&\quad - \frac{\tau^2}{2} u_{yytt}(x_i, y_j, z_k, s_{ijk}^l), \quad s_{ijk}^l \in (t_l - \tau, t_l + \tau), \bar{\xi}_{ijk}^l \in (y_j - h_2, y_j + h_2). \\
u_{zz}(x_i, y_j, z_k, t_l) &= \frac{1}{2}[u_{zz}(x_i, y_j, z_k, t_{l-1}) + u_{zz}(x_i, y_j, z_k, t_{l+1})] - \frac{\tau^2}{2} u_{zztt}(x_i, y_j, z_k, s_{ijk}^l) \\
&= \frac{1}{2}[\delta_x^2 U_{ijk}^{l-1} + \delta_x^2 U_{ijk}^{l+1}] - \frac{h_1^2}{24}[u_{zzzz}(x_i, y_j, \bar{\lambda}_{ijk}^l, t_{l-1}) + u_{zzzz}(x_i, y_j, \bar{\lambda}_{ijk}^l, t_{l+1})] \\
&\quad - \frac{\tau^2}{2} u_{zztt}(x_i, y_j, z_k, s_{ijk}^l), \quad s_{ijk}^l \in (t_l - \tau, t_l + \tau), \bar{\lambda}_{ijk}^l \in (z_k - h_3, z_k + h_3).
\end{aligned}
$$

Then, substitute the above four formulas into (3.4), and add $M_1 U_{ijk}^{\bar{l}}$

$$
\delta_t^2 U_{ijk}^l - a^2 \delta_x^2 U_{ijk}^{\bar{l}} - b^2 \delta_y^2 U_{ijk}^{\bar{l}} - c^2 \delta_z^2 U_{ijk}^{\bar{l}} + M_1 U_{ijk}^{\bar{l}} = f_{ijk}^l + (R_1)_{ijk}^l, \quad (i, j, k) \in \omega, 1 \le l \le n-1, \tag{3.5}
$$

where

$$
\begin{aligned}
(R_1)_{ijk}^l &= \tau^2 [\frac{1}{12} u_{tttt}(x_i, y_j, z_k, t_l + \bar{\zeta}\tau) - \frac{a^2}{2} u_{xxtt}(x_i, y_j, z_k, s_{ijk}^l) \\
&\quad - \frac{b^2}{2} u_{yytt}(x_i, y_j, z_k, s_{ijk}^l) - \frac{c^2}{2} u_{zztt}(x_i, y_j, z_k, s_{ijk}^l)] + M_1 U_{ijk}^{\bar{l}} \\
&\quad - h_1^2 [\frac{a^2}{24} u_{xxxx}(\bar{\eta}_{ijk}^l, y_j, z_k, t_{l-1}) + \frac{a^2}{24} u_{xxxx}(\bar{\eta}_{ijk}^l, y_j, z_k, t_{l+1})] \\
&\quad - h_2^2 [\frac{b^2}{24} u_{yyyy}(x_i, \bar{\xi}_{ijk}^l, z_k, t_{l-1}) + \frac{b^2}{24} u_{yyyy}(x_i, \bar{\xi}_{ijk}^l, z_k, t_{l+1})] \\
&\quad - h_3^2 [\frac{c^2}{24} u_{zzzz}(x_i, y_j, \bar{\lambda}_{ijk}^l, t_{l-1}) + \frac{c^2}{24} u_{zzzz}(x_i, y_j, \bar{\lambda}_{ijk}^l, t_{l+1})], \\
&\qquad (i, j, k) \in \omega, 1 \le l \le n-1.
\end{aligned} \tag{3.6}
$$

A constant c_2 exists such that

$$
\begin{aligned}
&|(R_1)_{ijk}^l| \le c_2(\tau^2 + h_1^2 + h_2^2 + h_3^2), \quad (i, j, k) \in \omega, 1 \le l \le n-1, \\
&|\Delta_t (R_1)_{ijk}^l| \le c_2(\tau^2 + h_1^2 + h_2^2 + h_3^2), \quad (i, j, k) \in \omega, 2 \le l \le n-2,
\end{aligned}
$$

and

$$\Delta_t(R_1)_{ijk}^l = \frac{1}{2\tau}[(R_1)_{ijk}^{l+1} - (R_1)_{ijk}^{l-1}].$$

Notice the initial boundary value condition (1.1)–(1.3):

$$\begin{aligned} U_{ijk}^0 &= \phi(x_i, y_j, z_k), \quad (i, j, k) \in \omega, \\ U_{ijk}^l &= \psi(x_i, y_j, z_k), \quad (i, j, k) \in \gamma, 0 \le l \le n. \end{aligned} \tag{3.7}$$

Ignoring the minimum terms $(R_1)_{ijk}^0$ and $(R_1)_{ijk}^l$, the following ADI difference scheme is established for the solution of problem (1.1)–(1.3):

$$\frac{2}{\tau}(\delta_t u_{ijk}^1 - \psi_{ijk} + \frac{\tau^2}{3}\rho_{ijk}) - a^2\delta_x^2 u_{ijk}^1 - b^2\delta_y^2 u_{ijk}^1 - c^2\delta_z^2 u_{ijk}^1 + M_1 u_{ijk}^1 = f_{ijk}^1, \quad (i, j, k) \in \omega, \tag{3.8}$$

$$\delta_t^2 u_{ijk}^l - a^2\delta_x^2 u_{ijk}^{\bar{l}} - b^2\delta_y^2 u_{ijk}^{\bar{l}} - c^2\delta_z^2 u_{ijk}^{\bar{l}} + M_1 u_{ijk}^{\bar{l}} = f_{ijk}^l, \quad (i, j, k) \in \omega, 1 \le l \le n-1, \tag{3.9}$$

$$u_{ijk}^0 = \psi(x_i, y_j, z_k) \quad (i, j, k) \in \omega, \tag{3.10}$$

$$u_{ijk}^l = \alpha(x_i, y_j, z_k, t_l), \quad (i, j, k) \in \gamma, 1 \le l \le n. \tag{3.11}$$

Equation (3.8) can be transformed into

$$\left(I - \frac{a^2\tau^2}{2}\delta_x^2\right)\left(I - \frac{b^2\tau^2}{2}\delta_y^2\right)\left(I - \frac{c^2\tau^2}{2}\delta_z^2\right)u_{ijk}^1 = u_{ijk}^0 + \tau\psi_{ijk} - \frac{\tau^3}{3}\rho_{ijk} + \frac{\tau^2}{2}f_{ijk}^1, \quad (i, j, k) \in \omega.$$

Letting

$$\begin{aligned} \bar{u}_{ijk} &= \left(I - \frac{b^2\tau^2}{2}\delta_y^2\right)u_{ijk}^*, \\ u_{ijk}^* &= \left(I - \frac{c^2\tau^2}{2}\delta_z^2\right)u_{ijk}^1, \end{aligned}$$

then

$$\left(I - \frac{a^2\tau^2}{2}\delta_x^2\right)\bar{u}_{ijk} = u_{ijk}^0 + \tau\psi_{ijk} - \frac{\tau^3}{3}\rho_{ijk} + \frac{\tau^2}{2}f_{ijk}^1, \tag{3.12}$$

$$\left(I - \frac{b^2\tau^2}{2}\delta_y^2\right)u_{ijk}^* = \bar{u}_{ijk}, \tag{3.13}$$

$$\left(I - \frac{c^2\tau^2}{2}\delta_z^2\right)u_{ijk}^1 = u_{ijk}^*. \tag{3.14}$$

For any fixed $j(1 \le j \le m_2 - 1)$ and $k(1 \le k \le m_3 - 1)$, take the boundary conditions

$$\bar{u}_{0,j,k} = \left(I - \frac{b^2\tau^2}{2}\delta_y^2\right)u_{0,j,k}^* = \left(I - \frac{b^2\tau^2}{2}\delta_y^2\right)\left(I - \frac{c^2\tau^2}{2}\delta_z^2\right)u_{0,j,k}^1, \tag{3.15}$$

$$\bar{u}_{m_1,j,k} = \left(I - \frac{b^2\tau^2}{2}\delta_y^2\right)u_{m_1,j,k}^* = \left(I - \frac{b^2\tau^2}{2}\delta_y^2\right)\left(I - \frac{c^2\tau^2}{2}\delta_z^2\right)u_{m_1,j,k}^1. \tag{3.16}$$

Using (3.15) and (3.16), solve the equation

$$\left(I - \frac{a^2\tau^2}{2}\delta_x^2\right)\bar{u}_{ijk} = u_{ijk}^0 + \tau\psi_{ijk} - \frac{\tau^3}{3}\rho_{ijk} + \frac{\tau^2}{2}f_{ijk}^1, \quad 1 \le i \le m_1 - 1,$$

from which we obtain $\{\bar{u}_{ijk} \mid 1 \le i \le m_1 - 1\}$.

For arbitrary fixed $i(1 \le i \le m_1 - 1)$ and $k(1 \le k \le m_3 - 1)$, take the boundary conditions

$$u^*_{i,0,k} = \left(I - \frac{c^2\tau^2}{2}\delta_z^2\right) u^1_{i,0,k}, \tag{3.17}$$

$$u^*_{i,m_2,k} = \left(I - \frac{c^2\tau^2}{2}\delta_z^2\right) u^1_{i,m_2,k}. \tag{3.18}$$

Using (3.17) and (3.18), we solve the equation

$$\left(I - \frac{b^2\tau^2}{2}\delta_y^2\right) u^*_{ijk} = \bar{u}_{ijk}, \quad 1 \le j \le m_2,$$

from which we obtain $\{u^*_{ijk} \mid 1 \le j \le m_2\}$.

For fixed $i(1 \le i \le m_1 - 1)$ and $j(1 \le j \le m_2 - 1)$, with the boundary conditions

$$u^1_{i,j,0,} = \alpha(x_i, y_j, z_0, t_1), \tag{3.19}$$

$$u^1_{i,j,m_3} = \alpha(x_i, y_j, z_{m_3}, t_1), \tag{3.20}$$

we solve the equation

$$\left(I - \frac{c^2\tau^2}{2}\delta_z^2\right) u^1_{ijk} = u^*_{ijk}, \quad 1 \le k \le m_3,$$

from which we obtain $\{u^1_{ijk} \mid 1 \le k \le m_3 - 1\}$.

Also, Eq (3.9) can be rewritten as

$$\left(I - \frac{a^2\tau^2}{2}\delta_x^2\right)\left(I - \frac{b^2\tau^2}{2}\delta_y^2\right)\left(I - \frac{c^2\tau^2}{2}\delta_z^2\right) u^{\bar{l}}_{ijk} = u^l_{ijk} + \frac{\tau^2}{2} f^l_{ijk}, \quad (i, j, k) \in \omega, 1 \le l \le n - 1.$$

Letting

$$\hat{u}_{ijk} = \left(I - \frac{b^2\tau^2}{2}\delta_y^2\right) \hat{u}^*_{ijk},$$

$$\hat{u}^*_{ijk} = \left(I - \frac{c^2\tau^2}{2}\delta_z^2\right) u^{\bar{l}}_{ijk},$$

then

$$\left(I - \frac{a^2\tau^2}{2}\delta_x^2\right) \hat{u}_{ijk} = u^l_{ijk} + \frac{\tau^2}{2} f^l_{ijk}, \tag{3.21}$$

$$\left(I - \frac{b^2\tau^2}{2}\delta_y^2\right) \hat{u}^*_{ijk} = \hat{u}_{ijk}, \tag{3.22}$$

$$\left(I - \frac{c^2\tau^2}{2}\delta_z^2\right) u^{\bar{l}}_{ijk} = \hat{u}^*_{ijk} \tag{3.23}$$

When the values of the l-th layer and the $l - 1$-th layer are known, for any fixed $j(1 \le j \le m_2 - 1)$ and $k(1 \le k \le m_3 - 1)$, take the boundary conditions

$$\hat{u}_{0,j,k} = \left(I - \frac{b^2\tau^2}{2}\delta_y^2\right) \hat{u}^*_{0,j,k} = \left(I - \frac{b^2\tau^2}{2}\delta_y^2\right)\left(I - \frac{c^2\tau^2}{2}\delta_z^2\right) \hat{u}^{\bar{l}}_{0,j,k}, \tag{3.24}$$

$$\hat{u}_{m_1,j,k} = \left(I - \frac{b^2\tau^2}{2}\delta_y^2\right)\hat{u}^*_{m_1,j,k} = \left(I - \frac{b^2\tau^2}{2}\delta_y^2\right)\left(I - \frac{c^2\tau^2}{2}\delta_z^2\right)\hat{u}^{\bar{l}}_{m_1,j,k}. \tag{3.25}$$

Using (3.24) and (3.25), we solve the equation

$$\left(I - \frac{a^2\tau^2}{2}\delta_x^2\right)\hat{u}_{ijk} = u^l_{ijk} + \frac{\tau^2}{2}f^l_{ijk}, \quad 1 \le i \le m_1 - 1,$$

from which we obtain $\{\hat{u}_{ijk} \mid 1 \le i \le m_1 - 1\}$.

With $i(1 \le i \le m_1 - 1)$ and $k(1 \le k \le m_3 - 1)$ being any fixed values, take the boundary conditions

$$\hat{u}^*_{i,0,k} = \left(I - \frac{c^2\tau^2}{2}\delta_z^2\right)u^{\bar{l}}_{i,0,k}, \tag{3.26}$$

$$\hat{u}^*_{i,m_2,k} = \left(I - \frac{c^2\tau^2}{2}\delta_z^2\right)u^{\bar{l}}_{i,m_2,k}. \tag{3.27}$$

Using (3.26) and (3.27), we solve the equation

$$\left(I - \frac{b^2\tau^2}{2}\delta_y^2\right)\hat{u}^*_{ijk} = \hat{u}_{ijk}, \quad 1 \le j \le m_2 - 1,$$

from which we obtain $\{\hat{u}^*_{ijk} \mid 1 \le j \le m_2 - 1\}$.

For fixed $i(1 \le i \le m_1 - 1)$ and $j(1 \le j \le m_2 - 1)$, with the boundary conditions

$$u^{\bar{l}}_{i,j,0} = \frac{1}{2}[\alpha(x_i, y_j, z_0, t_{l+1}) + \alpha(x_i, y_j, z_0, t_{l-1})], \tag{3.28}$$

$$u^{\bar{l}}_{i,j,m_3} = \frac{1}{2}[\alpha(x_i, y_j, z_{m_3}, t_{l+1}) + \alpha(x_i, y_j, z_{m_3}, t_{l-1})], \tag{3.29}$$

we solve the equation

$$\left(I - \frac{c^2\tau^2}{2}\delta_z^2\right)u^{\bar{l}}_{ijk} = \hat{u}^*_{ijk}, \quad 1 \le k \le m_3 - 1,$$

from which we obtain $\{u^{\bar{l}}_{ijk} \mid 1 \le k \le m_3 - 1\}$. Then, we utilize

$$u^{l+1}_{ijk} = 2u^{\bar{l}}_{ijk} - u^{l-1}_{ijk}, \quad (i, j, k) \in \omega$$

from which we obtain $\{u^{l+1}_{ijk} \mid (i, j, k) \in \omega\}$.

4. Solvability, stability and convergence

Theorem 4.1. *(Solvability) The solution of the ADI difference scheme* (3.8)–(3.11) *exists and is unique.*

Proof. Note $u^l = \{u^l_{ijk} \mid (i, j, k) \in \omega\}$. It is known from (3.10) and (3.11) that u^0 has been given. The difference scheme regarding u^1 is

$$\frac{2}{\tau}(\delta_t u^1_{ijk} - \psi_{ijk} + \frac{\tau^2}{3}\rho_{ijk}) - a^2\delta_x^2 u^1_{ijk} - b^2\delta_y^2 u^1_{ijk} - c^2\delta_z^2 u^1_{ijk} + M_l u^1_{ijk} = f^1_{ijk} + (R_1)^0_{ijk}, \quad (i, j, k) \in \omega,$$
$$u^l_{ijk} = 0, \quad (i, j, k) \in \gamma.$$

Consider its homogeneous system of equations

$$\begin{aligned}&\frac{2}{\tau^2}u^l_{ijk} - a^2\delta_x^2 u^1_{ijk} - b^2\delta_y^2 u^1_{ijk} - c^2\delta_z^2 u^1_{ijk} + M_l u^1_{ijk} = 0, \quad (i,j,k)\in\omega,\\ &u^l_{ijk} = 0, \quad (i,j,k)\in\gamma.\end{aligned} \tag{4.1}$$

Using u^1 and (4.1) to calculate the inner product, we obtain

$$\begin{aligned}&\frac{2}{\tau^2}\|u^1\|^2 - a^2(\delta_x^2 u^1, u^1) - b^2(\delta_y^2 u^1, u^1) - c^2(\delta_z^2 u^1, u^1) + \frac{a^2c^2\tau^2}{2}(\delta_x^2\delta_z^2 u^1, u^1)\\ &+ \frac{b^2c^2\tau^2}{2}(\delta_y^2\delta_z^2 u^1, u^1) + \frac{a^2b^2\tau^2}{2}(\delta_x^2\delta_y^2 u^1, u^1) - \frac{a^2b^2c^2\tau^2}{2}(\delta_x^2\delta_y^2\delta_z^2 u^1, u^1) = 0,\end{aligned}$$

so

$$\begin{aligned}&\frac{2}{\tau^2}\|u^1\|^2 + a^2\|\delta_x u^1\|^2 + b^2\|\delta_y u^1\|^2 + c^2\|\delta_z u^1\|^2 + \frac{a^2c^2\tau^2}{2}\|\delta_x\delta_z u^1\|^2\\ &+ \frac{b^2c^2\tau^2}{2}\|\delta_y\delta_z u^1\|^2 + \frac{a^2b^2\tau^2}{2}\|\delta_x\delta_y u^1\|^2 + \frac{a^2b^2c^2\tau^2}{2}\|\delta_x\delta_y\delta_z u^1\|^2 = 0,\end{aligned}$$

and it is easy to get

$$u^1_{ijk} = 0, \quad (i,j,k)\in\bar{\omega}.$$

Now that $u^{l-1}, u^l (1 \le l \le n-1)$ have been determined, the difference scheme for u^{l+1} is

$$\begin{aligned}&\delta_t^2 u^l_{ijk} - a^2\delta_x^2 u^{\bar{l}}_{ijk} - b^2\delta_y^2 u^{\bar{l}}_{ijk} - c^2\delta_z^2 u^{\bar{l}}_{ijk} + \frac{1}{2}M_1 u^{l+1}_{ijk} = 0, \quad (i,j,k)\in\omega,\\ &u^{l+1}_{ijk} = 0, \quad (i,j,k)\in\gamma.\end{aligned}$$

Consider its homogeneous system of equations

$$\begin{aligned}&\frac{1}{\tau^2}u^{l+1}_{ijk} - \frac{a^2}{2}\delta_x^2 u^{l+1}_{ijk} - \frac{b^2}{2}\delta_y^2 u^{l+1}_{ijk} - \frac{c^2}{2}\delta_z^2 u^{l+1}_{ijk} + \frac{1}{2}M_l u^{l+1}_{ijk} = 0, \quad (i,j,k)\in\omega,\\ &u^{l+1}_{ijk} = 0, \quad (i,j,k)\in\gamma.\end{aligned}$$

Using the equation with u^{l+1}_{ijk} and above for the inner product, we can obtain

$$\begin{aligned}&\frac{1}{\tau^2}\|u^{l+1}\| - \frac{a^2}{2}(\delta_x^2 u^{l+1}, u^{l+1}) - \frac{b^2}{2}(\delta_y^2 u^{l+1}, u^{l+1}) - \frac{c^2}{2}(\delta_z^2 u^{l+1}, u^{l+1}) + \frac{a^2c^2\tau^2}{4}(\delta_x^2\delta_z^2 u^{l+1}, u^{l+1})\\ &+ \frac{b^2c^2\tau^2}{4}(\delta_y^2\delta_z^2 u^{l+1}, u^{l+1}) + \frac{a^2b^2\tau^2}{4}(\delta_x^2\delta_y^2 u^{l+1}, u^{l+1}) - \frac{a^2b^2c^2\tau^4}{8}(\delta_x^2\delta_y^2\delta_z^2 u^{l+1}, u^{l+1}) = 0.\end{aligned}$$

Thus,

$$\begin{aligned}&\frac{1}{\tau^2}\|u^{l+1}\| + \frac{a^2}{2}\|\delta_x u^{l+1}\|^2 + \frac{b^2}{2}\|\delta_y u^{l+1}\|^2 + \frac{c^2}{2}\|\delta_z u^{l+1}\|^2 + \frac{a^2c^2\tau^2}{4}\|\delta_x\delta_z u^{l+1}\|^2\\ &+ \frac{b^2c^2\tau^2}{4}\|\delta_y\delta_z u^{l+1}\|^2 + \frac{a^2b^2\tau^2}{4}\|\delta_x\delta_y u^{l+1}\|^2 + \frac{a^2b^2c^2\tau^4}{8}\|\delta_x\delta_y\delta_z u^{l+1}\|^2 = 0.\end{aligned}$$

It is easy to conclude

$$u^{l+1}_{ijk} = 0, \quad (i,j,k)\in\bar{\omega}.$$

Based on the induction principle, the system of difference equations (3.8)–(3.11) is uniquely solvable. □

The following is the analysis and proof process regarding stability.

Theorem 4.2. *(Stability) Let* $\{u^l_{ijk}|(i,j,k) \in \bar{\omega}\}$ *be the solution of the following system of difference equations:*

$$\begin{aligned}
&\frac{2}{\tau}\cdot\delta_t u^{\frac{1}{2}}_{ijk} - a^2\delta_x^2 u^1_{ijk} - b^2\delta_y^2 u^1_{ijk} - c^2\delta_z^2 u^1_{ijk} + M_l u^1_{ijk} = g^0_{ijk}, \quad (i,j,k)\in\omega,\\
&\delta_t^2 u^l_{ijk} - a^2\delta_x^2 u^{\bar{l}}_{ijk} - b^2\delta_y^2 u^{\bar{l}}_{ijk} - c^2\delta_z^2 u^{\bar{l}}_{ijk} + M_l u^{\bar{l}}_{ijk} = g^{\bar{l}}_{ijk}, \quad (i,j,k)\in\omega, 1\le l\le n-1,\\
&u^0_{ijk} = \psi_{ijk}, \quad (i,j,k)\in\omega,\\
&u^l_{ijk} = 0, \quad (i,j,k)\in\gamma.
\end{aligned} \tag{4.2}$$

Then, there is

$$\begin{aligned}
\frac{1}{2}(|u^{l+1}|_1^2 + |u^l|_1^2) \le \frac{1}{C_3^2}e^{2T}[6C_4^2|u^0|_{A1}^2 + 3M_2 + 2C_d^2(6\|g^0\|^2 + 4\max_{1\le m\le l}\|g^m\|^2\\
+ \tau\sum_{m=2}^{l-1}\|\Delta_t g^m\|^2)], \qquad 1\le l\le n-1,
\end{aligned}$$

where

$$\begin{aligned}
&|u|_{A1}^2 = a^2\|\delta_x u\|^2 + b^2\|\delta_y u\|^2 + c^2\|\delta_z u\|^2,\\
&M_2 = a^2b^2\tau^2\|\delta_x\delta_y u^0\|^2 + a^2c^2\tau^2\|\delta_x\delta_z u^0\|^2 + b^2c^2\tau^2\|\delta_y\delta_z u^0\|^2 + \frac{a^2b^2c^2\tau^4}{2}\|\delta_x\delta_y\delta_z u^0\|^2,\\
&C_3 = \sqrt{\min\{a^2,b^2,c^2\}}, \quad C_4 = \sqrt{\max\{a^2,b^2,c^2\}}, \quad C_d = \frac{C}{C_3}.
\end{aligned}$$

Proof. Taking the inner product of both sides of Eq (4.2) and $\delta_t u^{\frac{1}{2}}$, we obtain

$$\begin{aligned}
&\frac{2}{\tau}(\delta_t u^{\frac{1}{2}}, \delta_t u^{\frac{1}{2}}) - a^2(\delta_x^2 u^1, \delta_t u^{\frac{1}{2}}) - b^2(\delta_y^2 u^1, \delta_t u^{\frac{1}{2}}) - c^2(\delta_z^2 u^1, \delta_t u^{\frac{1}{2}}) + \frac{a^2c^2\tau^2}{2}(\delta_x^2\delta_z^2 u^1, \delta_t u^{\frac{1}{2}})\\
&+ \frac{b^2c^2\tau^2}{2}(\delta_y^2\delta_z^2 u^1, \delta_t u^{\frac{1}{2}}) + \frac{a^2b^2\tau^2}{2}(\delta_x^2\delta_y^2 u^1, \delta_t u^{\frac{1}{2}}) - \frac{a^2b^2c^2\tau^2}{4}(\delta_x^2\delta_y^2\delta_z^2 u^1, \delta_t u^{\frac{1}{2}}) = (g^0, \delta_t u^{\frac{1}{2}}).
\end{aligned}$$

From the above equation, we obtain

$$\begin{aligned}
&\frac{2}{\tau}\|\delta_t u^{\frac{1}{2}}\|^2 + \frac{a^2}{2\tau}(\|\delta_x u^1\|^2 - \|\delta_x u^0\|^2) + \frac{b^2}{2\tau}(\|\delta_y u^1\|^2 - \|\delta_y u^0\|^2)\\
&+ \frac{c^2}{2\tau}(\|\delta_z u^1\|^2 - \|\delta_z u^0\|^2) + \frac{a^2b^2}{4}(\|\delta_x\delta_y u^1\|^2 - \|\delta_x\delta_y u^0\|^2)\\
&+ \frac{a^2c^2}{4}(\|\delta_x\delta_z u^1\|^2 - \|\delta_x\delta_z u^0\|^2) + \frac{b^2c^2}{4}(\|\delta_y\delta_z u^1\|^2 - \|\delta_y\delta_z u^0\|^2)\\
&+ \frac{a^2b^2c^2}{8}(\|\delta_x\delta_y\delta_z u^1\|^2 - \|\delta_x\delta_y\delta_z u^0\|^2) = (g^0, \delta_t u^{\frac{1}{2}}).
\end{aligned} \tag{4.3}$$

When $u \in V_h$, we have

$$|u|_{A1}^2 = a^2\|\delta_x u\|^2 + b^2\|\delta_y u\|^2 + c^2\|\delta_z u\|^2,$$

because

$$C_3^2 = min\{a^2, b^2, c^2\}, \quad C_4^2 = max\{a^2, b^2, c^2\}.$$

Then,

$$C_3^2|u|_1^2 \le |u|_{A1}^2 \le C_4^2|u|_1^2,$$

thus $|u|_{A1}^2$ and $|u|_1^2$ are equivalent. If $C_d^2 = \frac{C_4^2}{C_3^2}$ is set, then it can be inferred from Lemma 2.4 that

$$||u||^2 \le C_d^2|u|_{A1}^2.$$

After substituting $|u|_{A1}^2$ into (4.3) and simplifying both sides of the equation, we obtain

$$\begin{aligned}
&2||\delta_x u^{\frac{1}{2}}||^2 + \frac{1}{4}(|u^1|_{A1}^2 + |u^0|_{A1}^2) + \frac{a^2b^2\tau^2}{4}(||\delta_x\delta_y u^1||^2 + ||\delta_x\delta_y u^0||^2)\\
&+ \frac{a^2c^2\tau^2}{4}(||\delta_x\delta_z u^1||^2 + ||\delta_x\delta_z u^0||^2) + \frac{b^2c^2\tau^2}{4}(||\delta_y\delta_z u^1||^2 + ||\delta_y\delta_z u^0||^2)\\
&+ \frac{a^2b^2c^2\tau^4}{8}(||\delta_x\delta_y\delta_z u^1||^2 + ||\delta_x\delta_y\delta_z u^0||^2) \le |u^0|_{A1}^2 + \frac{a^2b^2\tau^2}{2}||\delta_x\delta_y u^0||^2\\
&+ \frac{a^2c^2\tau^2}{2}||\delta_x\delta_z u^0||^2 + \frac{b^2c^2\tau^2}{2}||\delta_y\delta_z u^0||^2 + \frac{b^2c^2\tau^2}{4}||\delta_x\delta_y\delta_z u^0||^2 + 2C_d^2||g^0||^2,
\end{aligned}$$

Then, let

$$\begin{aligned}
E^l = &||\delta_x u^{l+\frac{1}{2}}||^2 + \frac{1}{2}(|u^{l+1}|_{A1}^2 + |u^l|_{A1}^2) + \frac{a^2b^2\tau^2}{2}(||\delta_x\delta_y u^{l+1}||^2 + ||\delta_x\delta_y u^l||^2)\\
&+ \frac{a^2c^2\tau^2}{2}(||\delta_x\delta_z u^{l+1}||^2 + ||\delta_x\delta_z u^l||^2) + \frac{b^2c^2\tau^2}{2}(||\delta_y\delta_z u^{l+1}||^2 + ||\delta_y\delta_z u^l||^2)\\
&+ \frac{a^2b^2c^2\tau^4}{4}(||\delta_x\delta_y\delta_z u^{l+1}||^2 + ||\delta_x\delta_y\delta_z u^l||^2),
\end{aligned}$$

so

$$\begin{aligned}
E^0 \le &2|u^0|_{A1}^2 + a^2b^2\tau^2||\delta_x\delta_y u^0||^2 + a^2c^2\tau^2||\delta_x\delta_z u^0||^2\\
&+ b^2c^2\tau^2||\delta_y\delta_z u^0||^2 + \frac{a^2b^2c^2\tau^4}{2}||\delta_x\delta_y\delta_z u^0||^2 + 4C_d^2||g^0||^2.
\end{aligned} \tag{4.4}$$

Taking the inner product of both sides of Eq (4.2) with respect to $\Delta_t u^l$, we have

$$\begin{aligned}
&(\delta_t^2 u^l, \Delta_t u^l) - a^2(\delta_x^2 u^{\bar{l}}, \Delta_t u^l) - b^2(\delta_y^2 u^{\bar{l}}, \Delta_t u^l) - c^2(\delta_z^2 u^{\bar{l}}, \Delta_t u^l) + \frac{a^2c^2\tau^2}{2}(\delta_x^2\delta_z^2 u^{\bar{l}}, \Delta_t u^l)\\
&+ \frac{a^2b^2\tau^2}{2}(\delta_x^2\delta_y^2 u^{\bar{l}}, \Delta_t u^l) + \frac{b^2c^2\tau^2}{2}(\delta_y^2\delta_z^2 u^{\bar{l}}, \Delta_t u^l) - \frac{a^2b^2c^2\tau^4}{4}(\delta_x^2\delta_y\delta_z^2 u^{\bar{l}}, \Delta_t u^l) = (g^l, \Delta_t u^l),
\end{aligned}$$

through calculation and simplification, we can obtain

$$\begin{aligned}
&\frac{1}{2\tau}(\|\delta_t u^{l+\frac{1}{2}}\|^2 - \|\delta_t u^{l-\frac{1}{2}}\|^2) + \frac{1}{2\tau}(\frac{|u^{l+1}|_{A1}^2 + |u^l|_{A1}^2}{2} - \frac{|u^l|_{A1}^2 + |u^{l-1}|_{A1}^2}{2}) \\
&+ \frac{a^2c^2\tau^2}{2\tau}(\frac{\|\delta_x\delta_z u^{l+1}\|^2 + \|\delta_x\delta_z u^l\|^2}{2} - \frac{\|\delta_x\delta_z u^l\|^2 + \|\delta_x\delta_z u^{l-1\|^2}}{2}) \\
&+ \frac{a^2b^2\tau^2}{2\tau}(\frac{\|\delta_x\delta_y u^{l+1}\|^2 + \|\delta_x\delta_y u^l\|^2}{2} - \frac{\|\delta_x\delta_y u^l\|^2 + \|\delta_x\delta_y u^{l-1\|^2}}{2}) \\
&+ \frac{b^2c^2\tau^2}{2\tau}(\frac{\|\delta_y\delta_z u^{l+1}\|^2 + \|\delta_y\delta_z u^l\|^2}{2} - \frac{\|\delta_y\delta_z u^l\|^2 + \|\delta_y\delta_z u^{l-1\|^2}}{2}) \\
&+ \frac{a^2b^2c^2\tau^4}{4\tau}(\frac{\|\delta_x\delta_y\delta_z u^{l+1}\|^2 + \|\delta_x\delta_y\delta_z u^l\|^2}{2} - \frac{\|\delta_x\delta_y\delta_z u^l\|^2 + \|\delta_x\delta_y\delta_z u^{l-1\|^2}}{2}) \\
&= (g^l, \Delta_t u^l),
\end{aligned}$$

thus, it follows that

$$\frac{1}{2\tau}(E^l - E^{l-1}) = (g^l, \Delta_t u^l), \tag{4.5}$$

and when $l = 1$,

$$\begin{aligned}
E^1 &= E^0 - 2\tau(g^l, \Delta_t u^l) \\
&= E^0 - (g^l, u^2 - u^0) \\
&\le E^0 + \frac{1}{4}(|u^1|_{A1}^2 + |v^0|_{A1}^2) + 2C_d\|g^l\|^2,
\end{aligned}$$

so

$$\frac{1}{2}(|u^2|_{A1}^2 + |u^1|_{A1}^2) \le 2E^0 + \frac{1}{2}(|u^2|_{A1}^2 + |u^0|_{A1}^2) + 4C_d\|g^l\|^2.$$

When $l \ge 2$, replace m with l in (4.5), and sum m from 1 to l. Then, we obtain

$$\begin{aligned}
E^l &= E^0 - 2\tau\sum_{m=1}^{l}(g^m, \Delta_t u^m) \\
&= E^0 - \sum_{m=1}^{l}(g^m, u^{m+1} - u^{m-1}) \\
&= E^0 - \sum_{m=2}^{l+1}(g^{m-1}, u^m) + \sum_{m=1=0}^{l-1}(g^{m+1}, u^m) \\
&= E^0 + \sum_{m=2}^{l-1}(\Delta_t g^m, u^m) - (g^{l-1}, u^l) - (g^l, u^{l+1}) + (g^1, u^0) + (g^2, u^1), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 2 \le l \le n-1,
\end{aligned}$$

observing that

$$E^l \ge \frac{1}{2}(|u^{l+1}|_{A1}^2 + |u^l|_{A1}^2),$$

we can obtain

$$\begin{aligned}\frac{1}{2}(|u^{l+1}|_{A1}^2+|u^l|_{A1}^2) &\le E^0+\sum_{m=2}^{l-1}(\Delta_t g^m,v^m)-(g^{l-1},u^l)-(g^l,u^{l+1})+(g^1,u^0)+(g^2,u^1)\\ &\le E^0+\tau\sum_{m=2}^{l-1}(C_x^2\|\Delta_t g^m\|^2+|u^m|_{A1}^2)+C_d^2(\|g^{l-1}\|^2+\|g^l\|^2+\|g^1\|^2+\|g^2\|^2)\\ &\quad+\frac{1}{4}(|u^l|_{A1}^2+|u^{l+1}|_{A1}^2+|u^0|_{A1}^2+|u^1|_{A1}^2), \qquad 2\le l\le n-1.\end{aligned}$$

Furthermore, we can obtain

$$\begin{aligned}\frac{|u^{l+1}|_{A1}^2+|u^l|_{A1}^2}{2} &\le 2E^0+2C_d^2(\|g^{l-1}\|^2+\|g^l\|^2+\|g^1\|^2+\|g^2\|^2+\tau\sum_{m=2}^{l-1}\|\Delta_t g^m\|^2)\\ &\quad+\frac{|u^0|_{A1}^2+|u^1|_{A1}^2}{2}+2\tau\sum_{m=2}^{l-1}\frac{|u^{m+1}|_{A1}^2+|u^m|_{A1}^2}{2}, \quad 2\le l\le n-1.\end{aligned}$$

According to Lemma 2.5, we can obtain

$$\begin{aligned}\frac{|u^{l+1}|_{A1}^2+|u^l|_{A1}^2}{2} &\le e^{2(l-1)\tau}[3E^0+2C_d^2(4\max_{1\le m\le l}\|g^m\|^2+\tau\sum_{m=2}^{l-1}\|\Delta_t g^m\|^2)]\\ &\le e^{2T}[3\times(2|u^0|_{A1}^2+M+4C_d^2\|g^0\|^2)+2C_x^2(4\max_{1\le m\le l}\|g^m\|^2+\tau\sum_{m=2}^{l-1}\|\Delta_t g^m\|^2)]\\ &\le e^{2T}[6|u^0|_{A1}^2+3M+2C_d^2(6\|g^0\|^2+4\max_{1\le m\le l}\|g^m\|^2+\tau\sum_{m=2}^{l-1}\|\Delta_t g^m\|^2)], \quad 1\le l\le n-1,\end{aligned}$$

moreover, since

$$C_3^2|u|_1^2\le|u|_{A1}^2\le C_4^2|u|_1^2,$$

then

$$\begin{aligned}\frac{1}{2}(|u^{l+1}|_1^2+|u^l|_1^2) &\le \frac{1}{C_3^2}e^{2T}[6C_4^2|u^0|_{A1}^2+3M_2+2C_d^2(6\|g^0\|^2+4\max_{1\le m\le l}\|g^m\|^2\\ &\quad+\tau\sum_{m=2}^{l-1}\|\Delta_t g^m\|^2)], \qquad 1\le l\le n-1.\end{aligned}$$

By noting the definition of E^1 and (4.4), we can see that the above formula also holds for $l=0$. The proof of stability is completed. □

The rigorous proof of convergence is derived as follows.

Theorem 4.3. *(Convergence) Let* $\{u(x,y,z,t)\mid(x,y,z)\in\bar{\Omega},0\le t\le T\}$ *be the solution of the problem* (1.1)–(1.3)*, and* $\{u_{ijk}^l\mid(i,j,k)\in\bar{\omega},0\le l\le n\}$ *be the solution of the difference scheme* (3.8)–(3.11)*. Denote*

$$e_{ijk}^l=u(x_i,y_j,z_k,t_l)-u_{ijk}^l, \quad (i,j,k)\in\bar{\omega},0\le l\le n.$$

Then, we have

$$\|e^l\|_\infty \le \frac{L_1L_2L_3C_d}{C_3}e^T\sqrt{\frac{6c_1^2+4c_2^2+Tc_2^2}{L_2L_3+L_1L_3+L_1L_2}}(\tau^2+h_1^2+h_2^2+h_3^2), \quad 1\le l\le n.$$

Proof. Subtracting (3.2), (3.5), and (3.7) from (3.8)–(3.9), we can obtain the following error equation system:

$$\frac{2}{\tau}\delta_t e_{ijk}^{\frac{1}{2}} - a^2\delta_x^2 e_{ijk}^1 - b^2\delta_y^2 e_{ijk}^1 - c^2\delta_z^2 e_{ijk}^1 + M_1 e_{ijk}^1 = (R_1)_{ijk}^0, \quad (i,j,k)\in\omega, \tag{4.6}$$

$$\delta_t^2 e_{ijk}^l - a^2\delta_x^2 e_{ijk}^{\bar{l}} - b^2\delta_y^2 e_{ijk}^{\bar{l}} - c^2\delta_z^2 e_{ijk}^{\bar{l}} + M_1 e_{ijk}^{\bar{l}} = (R_{1ed})_{ijk}^l, \quad (i,j,k)\in\omega, 1\le l\le n-1 \tag{4.7}$$

$$e_{ijk}^0 = 0, \quad (i,j,k)\in\omega, \tag{4.8}$$

$$e_{ijk}^l = 0, \quad (i,j,k)\in\gamma, 1\le l\le n-1. \tag{4.9}$$

From the conclusions in the proof of stability, along with (4.8) and (4.9), we can obtain

$$\begin{aligned}\frac{1}{2}(|e^{l+1}|_1^2+|e^l|_1^2) &\le \frac{1}{C_3^2}e^{2T}[2C_d^2(6\|(R_1)\|^2+4\max_{1\le m\le l}\|(R_1)\|^2+\tau\sum_{m=2}^{l-1}\|\Delta_t(R_1)\|^2)]\\ &\le 2\frac{C_d^2}{C_3^2}e^{2T}[6L_1L_2L_3c_1^2(\tau^2+h_1^2+h_2^2+h_3^2)^2\\ &\quad + 4L_1L_2L_3c_2^2(\tau^2+h_1^2+h_2^2+h_3^2)^2+4(l-2)\tau c_2^2(\tau^2+h_1^2+h_2^2+h_3^2)^2]\\ &\le 2\frac{C_d^2}{C_3^2}L_1L_2L_3e^{2T}(6c_1^2+4c_2^2+Tc_2^2)(\tau^2+h_1^2+h_2^2+h_3^2)^2, \quad 0\le l\le n-1.\end{aligned}$$

Thus

$$|e^l|_1^2 \le 4\frac{C_d^2}{C_3^2}L_1L_2L_3e^{2T}(6c_1^2+4c_2^2+Tc_2^2)(\tau^2+h_1^2+h_2^2+h_3^2)^2, \quad 0\le l\le n,$$

$$|e^l|_1 \le 2\frac{C_d}{C_3}e^T\sqrt{L_1L_2L_3(6c_1^2+4c_2^2+Tc_2^2)}(\tau^2+h_1^2+h_2^2+h_3^2), \quad 0\le l\le n.$$

Then, according to Lemma 2.6, we can obtain

$$\begin{aligned}\|e\|_\infty &\le \frac{1}{2}\sqrt{\frac{L_1L_2L_3}{L_2L_3+L_1L_2+L_1L_3}}|e^l|_1\\ &\le \frac{L_1L_2L_3C_d}{C_3}e^T\sqrt{\frac{6c_1^2+4c_2^2+Tc_2^2}{L_2L_3+L_1L_2+L_1L_3}}(\tau^2+h_1^2+h_2^2+h_3^2), \quad 1\le l\le n.\end{aligned}$$

The proof of convergence is completed. □

5. Richardson extrapolation scheme

Theorem 5.1. *Assume that the problems*

$$\begin{cases}\frac{2}{\tau}\delta_t q_{1ijk}^{\frac{1}{2}} - a^2\delta_x^2 q_{1ijk}^1 - b^2\delta_y^2 q_{1ijk}^1 - c^2\delta_z^2 q_{1ijk}^1 = -p_{1ijk}^0, & (i,j,k)\in\omega,\\ \delta_t^2 q_{1ijk}^l - a^2\delta_x^2 q_{1ijk}^{\bar{l}} - b^2\delta_y^2 q_{1ijk}^{\bar{l}} - c^2\delta_z^2 q_{1ijk}^{\bar{l}} = -p_{1ijk}^l, & (i,j,k)\in\omega, 1\le l\le n-1,\\ q_{1ijk}^0 = 0, & (i,j,k)\in\omega,\\ q_{1ijk}^l = 0, & (i,j,k)\in\gamma, 1\le l\le n-1.\end{cases} \tag{5.1}$$

$$\begin{cases} \frac{2}{\tau}\delta_t q_{2ijk}^{\frac{1}{2}} - a^2\delta_x^2 q_{2ijk}^1 - b^2\delta_y^2 q_{2ijk}^1 - c^2\delta_z^2 q_{2ijk}^1 = -p_{2ijk}^0, \quad (i,j,k)\in\omega, \\ \delta_t^2 q_{2ijk}^l - a^2\delta_x^2 q_{2ijk}^{\bar{l}} - b^2\delta_y^2 q_{2ijk}^{\bar{l}} - c^2\delta_z^2 q_{2ijk}^{\bar{l}} = -p_{2ijk}^l, \quad (i,j,k)\in\omega, 1\le l\le n-1, \\ q_{2ijk}^0 = 0, \quad (i,j,k)\in\omega, \\ q_{2ijk}^l = 0, \quad (i,j,k)\in\gamma, 1\le l\le n-1. \end{cases} \tag{5.2}$$

$$\begin{cases} \frac{2}{\tau}\delta_t q_{3ijk}^{\frac{1}{2}} - a^2\delta_x^2 q_{3ijk}^1 - b^2\delta_y^2 q_{3ijk}^1 - c^2\delta_z^2 q_{3ijk}^1 = -p_{3ijk}^0, \quad (i,j,k)\in\omega, \\ \delta_t^2 q_{3ijk}^l - a^2\delta_x^2 q_{3ijk}^{\bar{l}} - b^2\delta_y^2 q_{3ijk}^{\bar{l}} - c^2\delta_z^2 q_{3ijk}^{\bar{l}} = -p_{3ijk}^l, \quad (i,j,k)\in\omega 1\le l\le n-1, \\ q_{3ijk}^0 = 0, \quad (i,j,k)\in\omega, \\ q_{3ijk}^l = 0, \quad (i,j,k)\in\gamma, 1\le l\le n-1. \end{cases} \tag{5.3}$$

and

$$\begin{cases} \frac{2}{\tau}\delta_t q_{4ijk}^{\frac{1}{2}} - a^2\delta_x^2 q_{4ijk}^1 - b^2\delta_y^2 q_{4ijk}^1 - c^2\delta_z^2 q_{4ijk}^1 = -p_{4ijk}^0, \quad (i,j,k)\in\omega, \\ \delta_t^2 q_{4ijk}^l - a^2\delta_x^2 q_{4ijk}^{\bar{l}} - b^2\delta_y^2 q_{4ijk}^{\bar{l}} - c^2\delta_z^2 q_{4ijk}^{\bar{l}} = -p_{4ijk}^l, \quad (i,j,k)\in\omega, 1\le l\le n-1, \\ q_{4ijk}^0 = 0, \quad (i,j,k)\in\omega, \\ q_{4ijk}^l = 0, \quad (i,j,k)\in\gamma, 1\le l\le n-1. \end{cases} \tag{5.4}$$

have smooth solutions, where

$$\begin{aligned} p_{1ijk}^0 &= \frac{5}{12}u_{tttt}(x_i,y_j,z_k,t_1), \\ p_{2ijk}^0 &= -\frac{a^2}{12}u_{xxxx}(x_i,y_j,z_k,t_1), \\ p_{3ijk}^0 &= -\frac{b^2}{12}u_{yyyy}(x_i,y_j,z_k,t_1), \\ p_{4ijk}^0 &= -\frac{c^2}{12}u_{zzzz}(x_i,y_j,z_k,t_1), \end{aligned}$$

$$\begin{aligned} p_{1ijk}^l &= \frac{1}{12}u_{tttt}(x_i,y_j,z_k,t_l) - \frac{1}{2}u_{xxtt}(x_i,y_j,z_k,t_l), \\ &\quad - \frac{1}{2}u_{yytt}(x_i,y_j,z_k,t_l) - \frac{1}{2}u_{zztt}(x_i,y_j,z_k,t_l), \\ p_{2ijk}^l &= -\frac{a^2}{24}[u_{xxxx}(x_i,y_j,z_k,t_{l+1}) + u_{xxxx}(x_i,y_j,z_k,t_{l-1})], \\ p_{3ijk}^l &= -\frac{b^2}{24}[u_{yyyy}(x_i,y_j,z_k,t_{l+1}) + u_{yyyy}(x_i,y_j,z_k,t_{l-1})], \\ p_{4ijk}^l &= -\frac{c^2}{24}[u_{zzzz}(x_i,y_j,z_k,t_{l+1}) + u_{zzzz}(x_i,y_j,z_k,t_{l-1})]. \end{aligned}$$

Then we have

$$\begin{aligned} u_{ijk}^l(h_1,h_2,h_3,\tau) &= u(x_i,y_j,z_k,t_l) + \tau^2 q_{1ijk}^l(x_i,y_j,z_k,t_l) + h_1^2 q_{2ijk}^l(x_i,y_j,z_k,t_l) \\ &\quad + h_2^2 q_{3ijk}^l(x_i,y_j,z_k,t_l) + h_3^2 q_{4ijk}^l(x_i,y_j,z_k,t_l) + O(\tau^3 + h_1^4 + h_2^4 + h_3^4), \end{aligned}$$

$$\max\left|u(x_i,y_j,z_k,t_l) - \left[\frac{4}{3}u_{2i,2j,2k}^{2l}(\frac{h_1}{2},\frac{h_2}{2},\frac{h_3}{2},\frac{\tau}{2}) - \frac{1}{3}u_{ijk}^l(h_1,h_2,h_3,\tau)\right]\right| = O(\tau^3 + h_1^4 + h_2^4 + h_3^4).$$

Proof. By subtracting $M_1 e^1_{ijk}$ and $M_1 e^l_{ijk}$ from both sides of the error equation system (4.6)–(4.9), we can obtain

$$\begin{cases} \frac{2}{\tau}\delta_t e^{\frac{1}{2}}_{ijk} - a^2\delta_x^2 e^1_{ijk} - b^2\delta_y^2 e^1_{ijk} - c^2\delta_z^2 e^1_{ijk} \\ = p^0_{1ijk}\tau^2 + p^0_{2ijk}h_1^2 + p^0_{3ijk}h_2^2 + p^0_{4ijk}h_3^2 + O(\tau^3 + h_1^4 + h_2^4 + h_3^4), \quad (i,j,k) \in \omega, \\ \delta_t^2 e^l_{ijk} - a^2\delta_x^2 e^{\bar{l}}_{ijk} - b^2\delta_y^2 e^{\bar{l}}_{ijk} - c^2\delta_z^2 e^{\bar{l}}_{ijk} \\ = p^l_{1ijk}\tau^2 + p^l_{2ijk}h_1^2 + p^l_{3ijk}h_2^2 + p^l_{4ijk}h_3^2 + O(\tau^4 + h_1^4 + h_2^4 + h_3^4), \quad (i,j,k) \in \omega, 1 \le l \le n-1, \\ e^0_{ijk} = 0, \quad (i,j,k) \in \omega, \\ e^l_{ijk} = 0, \quad (i,j,k) \in \gamma, 1 \le l \le n-1. \end{cases} \tag{5.5}$$

where

$$\begin{aligned} O(\tau^3 + h_1^4 + h_2^4 + h_3^4) = {} & \frac{3\tau^3}{20}u_{ttttt}(x_i, y_j, z_k, \hat{\zeta}_{ijk}\tau) - \frac{a^2h_1^4}{360}u_{xxxxxx}(\hat{\eta}_{ijk}, y_j, z_k, t_1) \\ & - \frac{b^2h_2^4}{360}u_{yyyyyy}(x_i, \hat{\xi}_{ijk}, z_k, t_1) - \frac{c^2h_3^4}{360}u_{zzzzzz}(x_i, y_j, \hat{\lambda}_{ijk}, t_1), \\ & \hat{\zeta}_{ijk} \in (0,1), \hat{\eta}_{ijk} \in (x_i - h_1, x_i + h_1), \hat{\xi}_{ijk} \in (y_j - h_2, y_j + h_2), \\ & \hat{\lambda}_{ijk} \in (z_k - h_3, z_k + h_3), \\ O(\tau^4 + h_1^4 + h_2^4 + h_3^4) = {} & \tau^4[\frac{1}{360}u_{tttttt}(x_i, y_j, z_k, \dot{\zeta}^l_{ijk}) - \frac{a^2}{24}u_{xxtttt}(x_i, y_j, z_k, \dot{s}^l_{ijk}) \\ & - \frac{b^2}{24}u_{yytttt}(x_i, y_j, z_k, \dot{s}^l_{ijk}) - \frac{c^2}{24}u_{zztttt}(x_i, y_j, z_k, \dot{s}^l_{ijk})] \\ & - h_1^4[\frac{a^2}{720}u_{xxxxxx}(\dot{\eta}^l_{ijk}, y_j, z_k, t_{l+1}) + \frac{a^2}{720}u_{xxxxxx}(\dot{\eta}^l_{ijk}, y_j, z_k, t_{l-1})] \\ & - h_2^4[\frac{b^2}{720}u_{yyyyyy}(x_i, \dot{\xi}^l_{ijk}, z_k, t_{l+1}) + \frac{b^2}{720}u_{yyyyyy}(x_i, \dot{\xi}^l_{ijk}, z_k, t_{l-1})] \\ & - h_3^4[\frac{c^2}{720}u_{zzzzzz}(x_i, y_j, \dot{\lambda}^l_{ijk}, t_{1+1}) + \frac{c^2}{720}u_{zzzzzz}(x_i, y_j, \dot{\lambda}^l_{ijk}, t_{1-1})], \\ & \hat{\zeta}_{ijk} \in (t_l - \tau, t_l + \tau), \hat{\eta}_{ijk} \in (x_i - h_1, x_i + h_1), \hat{\xi}_{ijk} \in (y_j - h_2, y_j + h_2), \\ & \hat{\lambda}_{ijk} \in (z_k - h_3, z_k + h_3). \end{aligned}$$

Denote

$$r^l_{ijk} = e^l_{ijk} + \tau^2 p^l_{1ijk} + h_1^2 p^l_{2ijk} + h_2^2 p^l_{3ijk} + h_3^2 p^l_{4ijk}, \quad 0 \le i \le m_1, 0 \le j \le m_2, 0 \le k \le m_3, 0 \le l \le n.$$

Multiply (5.1) by τ^2, (5.2) by h_1^2, (5.3) by h_2^2, and (5.4) by h_3^2, respectively. Then, add the resultant expressions to (5.5), yielding

$$\begin{aligned} & \frac{2}{\tau}\delta_t r^{\frac{1}{2}}_{ijk} - a^2\delta_x^2 r^1_{ijk} - b^2\delta_y^2 r^1_{ijk} - c^2\delta_z^2 r^1_{ijk} = O(\tau^3 + h_1^4 + h_2^4 + h_3^4), \quad (i,j,k) \in \omega, \\ & \delta_t^2 r^l_{ijk} - a^2\delta_x^2 r^{\bar{l}}_{ijk} - b^2\delta_y^2 r^{\bar{l}}_{ijk} - c^2\delta_z^2 r^{\bar{l}}_{ijk} = O(\tau^4 + h_1^4 + h_2^4 + h_3^4), \quad (i,j,k) \in \omega, 1 \le l \le n-1, \\ & r^0_{ijk} = 0, \quad (i,j,k) \in \omega, \\ & r^l_{ijk} = 0, \quad (i,j,k) \in \gamma, 1 \le l \le n-1. \end{aligned}$$

As can be seen from the proof of stability in Subsection 4.2,

$$r_{ijk}^{l} = O(\tau^3 + h_1^4 + h_2^4 + h_3^4), \quad (i, j, k) \in \omega, 1 \le l \le n,$$

that is

$$u(x_i, y_j, z_k, t_l) - u_{ijk}^{l}(h_1, h_2, h_3, \tau) + \tau^2 p_{1ijk}^{l} + h_1^2 p_{2ijk}^{l} + h_2^2 p_{3ijk}^{l} + h_3^2 p_{4ijk}^{l} = O(\tau^3 + h_1^4 + h_2^4 + h_3^4),$$
$$0 \le i \le m_1 - 1, 0 \le j \le m_2 - 1, 0 \le k \le m_3 - 1, 0 \le l \le n - 1.$$

By transposing the terms, we can obtain

$$u_{ijk}^{l}(h_1, h_2, h_3, \tau) = u(x_i, y_j, z_k, t_l) + \tau^2 p_{1ijk}^{l} + h_1^2 p_{2ijk}^{l} + h_2^2 p_{3ijk}^{l} + h_3^2 p_{4ijk}^{l} + O(\tau^3 + h_1^4 + h_2^4 + h_3^4),$$
$$0 \le i \le m_1 - 1, 0 \le j \le m_2 - 1, 0 \le k \le m_3 - 1, 0 \le l \le n - 1. \tag{5.6}$$

By the same token, we have

$$\begin{aligned} &u_{2i,2j,2k}^{2l}\left(\frac{h_1}{2}, \frac{h_2}{2}, \frac{h_3}{2}, \frac{\tau}{2}\right) \\ &= u(x_i, y_j, z_k, t_l) + \frac{\tau}{2}^2 p_{1ijk}^{l} + \frac{h_1^2}{2} p_{2ijk}^{l} + \frac{h_2}{2}^2 p_{3ijk}^{l} + \frac{h_3}{2}^2 p_{4ijk}^{l} + O\left(\left(\frac{\tau}{2}\right)^3 + \left(\frac{h_1}{2}\right)^4 + \left(\frac{h_2}{2}\right)^4 + \left(\frac{h_3}{2}\right)^4\right), \\ &\qquad 0 \le i \le m_1 - 1, 0 \le j \le m_2 - 1, \quad 0 \le k \le m_3 - 1, 0 \le l \le n - 1. \end{aligned} \tag{5.7}$$

Multiply both sides of (5.6) by 4/3 and both sides of (5.7) by 1/3, respectively. Subsequently, subtract the expression obtained from the latter operation from that of the former. As a result, we arrive at

$$\begin{aligned} &\frac{4}{3} u_{2i,2j,2k}^{2l}\left(\frac{h_1}{2}, \frac{h_2}{2}, \frac{h_3}{2}, \frac{\tau}{2}\right) - \frac{1}{3} u_{ijk}^{l}(h_1, h_2, h_3, \tau) \\ &= u(x_i, y_j, z_k, t_l) + O\left(\left(\frac{\tau}{2}\right)^3 + \left(\frac{h_1}{2}\right)^4 + \left(\frac{h_2}{2}\right)^4 + \left(\frac{h_3}{2}\right)^4\right), \\ &\qquad 0 \le i \le m_1 - 1, 0 \le j \le m_2 - 1, 0 \le k \le m_3 - 1, 0 \le l \le n - 1. \end{aligned}$$

The proof of the theorem is completed. □

6. Numerical example

In this section, we present two numerical examples to compute the errors and convergence orders using the ADI finite difference scheme. The overall convergence accuracy under extrapolation methods is also calculated, which validates the theoretical analysis.

We take $h_1 = h_2 = h_3 = h$, and denote the maximum absolute error as

$$E_\infty(h_1, h_2, h_3, \tau) = \max_{(i,j,k)\in\omega, 0\le l\le n} \left| u(x_i, y_j, z_k, t_l) - u_{ijk}^{l} \right|.$$

The maximum absolute error of the Richardson extrapolation method is

$$\bar{E}_\infty(h_1, h_2, h_3, \tau) = \max_{(i,j,k)\in\omega, 0\le l\le n} \left| u(x_i, y_j, z_k, t_l) - \left[\frac{4}{3} u_{2i,2j,2k}^{2l}\left(\frac{h_1}{2}, \frac{h_2}{2}, \frac{h_3}{2}, \frac{\tau}{2}\right) - \frac{1}{3} u_{ijk}^{l}(h_1, h_2, h_3, \tau)\right]\right|.$$

6.1. Example 1

We consider

$$
\begin{aligned}
&u_{tt} - u_{xx} - 2u_{yy} - 4u_{zz} = -\frac{3}{4}e^{\frac{x+y+z}{2}-t}, \quad 0 < x, y, z < 1, 0 < t \leq 1,\\
&u(x, y, z, 0) = e^{\frac{x+y+z}{2}}, u_t(x, y, z, 0) = -e^{\frac{x+y+z}{2}}, \quad 0 \leq x, y, z \leq 1,\\
&u(0, y, z, t) = e^{\frac{y+z}{2}-t}, u(1, y, z, t) = e^{\frac{1+y+z}{2}-t}, \quad 0 \leq y, z \leq 1, 0 < t \leq 1,\\
&u(x, 0, z, t) = e^{\frac{x+z}{2}-t}, u(x, 1, z, t) = e^{\frac{1+x+z}{2}-t}, \quad 0 \leq x, z \leq 1, 0 < t \leq 1,\\
&u(x, y, 0, t) = e^{\frac{x+y}{2}-t}, u(x, y, 1, t) = e^{\frac{1+x+y}{2}-t}, \quad 0 \leq x, y \leq 1, 0 < t \leq 1.
\end{aligned}
\tag{6.1}
$$

The exact solution of this problem is $u(x, y, z, t) = e^{\frac{x+y+z}{2}-t}$.

Tables 1 and 2 show the maximum errors and convergence orders for different step sizes using the finite difference method and for Richardson extrapolation scheme.

Table 1. Maximum errors and convergence orders for Example 1.

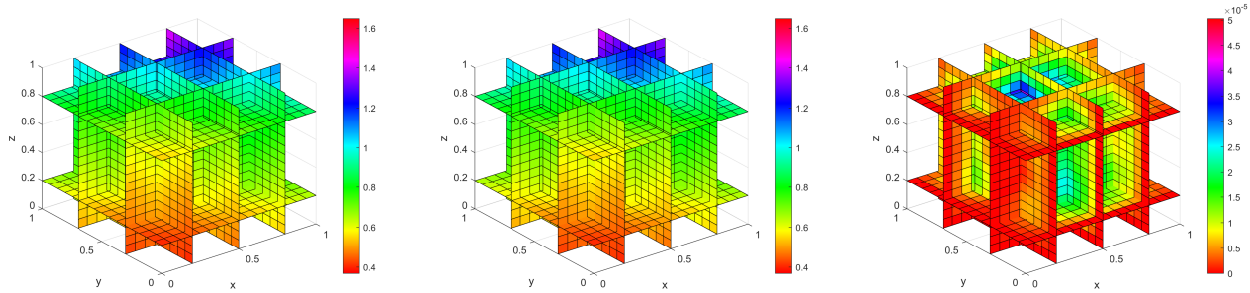
(h,τ)	$E_\infty(h_1, h_2, h_3, \tau)$	$\log_2\left(\frac{E(2h_1,2h_2,2h_3,2\tau)}{E(h_1,h_2,h_3,\tau)}\right)$
$(\frac{1}{16}, \frac{1}{16})$	1.4367e-04	*
$(\frac{1}{32}, \frac{1}{32})$	3.9257e-05	1.8718
$(\frac{1}{64}, \frac{1}{64})$	1.0036e-05	1.9678
$(\frac{1}{128}, \frac{1}{128})$	2.5559e-06	1.9733

Table 2. Maximum errors and convergence orders for extrapolation scheme for Example 1.

(h,τ)	$\bar{E}_\infty(h_1, h_1, h_1, \tau)$	$\log_2\left(\frac{\bar{E}(2h_1,2h_2,2h_3,2\tau)}{\bar{E}(h_1,h_2,h_3,\tau)}\right)$
$(\frac{1}{16}, \frac{1}{16})$	1.7385e-05	*
$(\frac{1}{32}, \frac{1}{32})$	2.0485e-06	3.0851
$(\frac{1}{64}, \frac{1}{64})$	2.2440e-07	3.1904

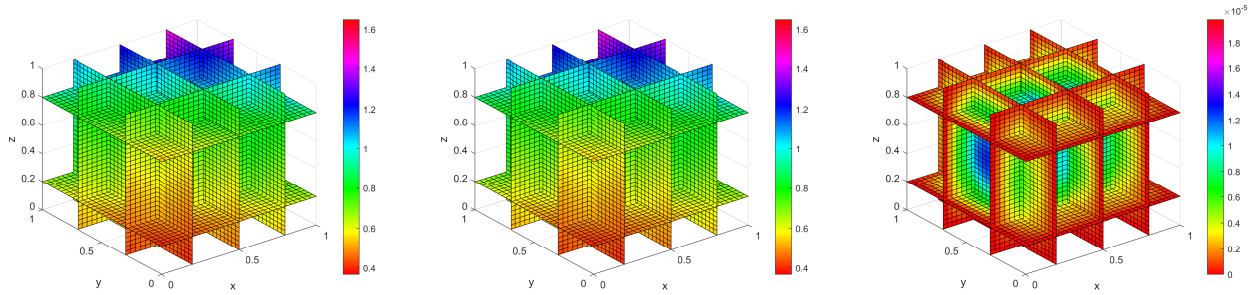
As is evident from the aforementioned table, with the progressive refinement of the mesh partitioning, the maximum error exhibits a gradual decline. The precision of the initial finite difference scheme is of the second order. In contrast, following the implementation of the Richardson extrapolation technique, the precision is elevated to the third order.

Figures 1–4 presented herein are the numerical solutions corresponding to diverse step sizes, the exact solutions, and the error graphs derived from the numerical experiments of Example 1 with $1/h = 1/\tau = 16, 32, 64, 128$.



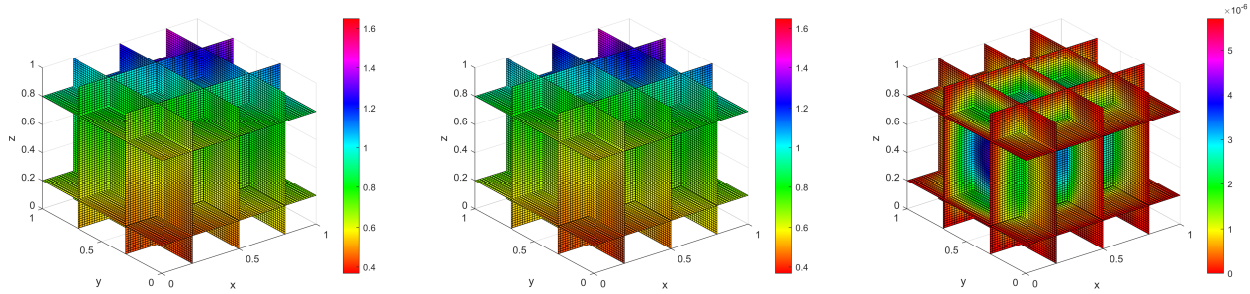
(a) Numerical solution (b) Exact solution (c) Error

Figure 1. Numerical solution, exact solution, and error graph of the final layer ($\frac{1}{h} = \frac{1}{\tau} = 16$) for Example 1.



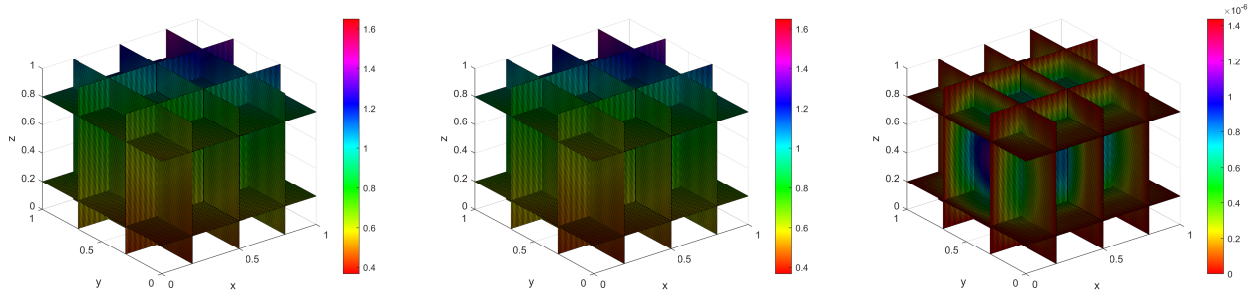
(a) Numerical solution (b) Exact solution (c) Error

Figure 2. Numerical solution, exact solution, and error graph of the final layer ($\frac{1}{h} = \frac{1}{\tau} = 32$) for Example 1.



(a) Numerical solution (b) Exact solution (c) Error

Figure 3. Numerical solution, exact solution, and error graph of the final layer ($\frac{1}{h} = \frac{1}{\tau} = 64$) for Example 1.



(a) Numerical solution (b) Exact solution (c) Error

Figure 4. Numerical solution, exact solution, and error graph of the final layer ($\frac{1}{h} = \frac{1}{\tau} = 128$) for Example 1.

6.2. *Example 2*

We consider

$$\begin{aligned}
&u_{tt} - u_{xx} - 3u_{yy} - 5u_{zz} = (2+t^2)sinx + (2+3t^2)siny + (2+5t^2)sinz, \quad 0 < x, y, z < 1, 0 < t \le 1,\\
&u(x,y,z,0) = 0, u_t(x,y,z,0) = 0, \quad 0 \le x, y, z \le 1,\\
&u(0,y,z,t) = t^2(siny + sinz), u(1,y,z,t) = t^2(sin1 + siny + sinz), \quad 0 \le y, z \le 1, 0 < t \le 1,\\
&u(x,0,z,t) = t^2(sinx + sinz), u(x,1,z,t) = t^2(sin1 + sinx + sinz), \quad 0 \le x, z \le 1, 0 < t \le 1,\\
&u(x,y,0,t) = t^2(sinx + siny), u(x,y,1,t) = t^2(sin1 + sinx + siny), \quad 0 \le x, y \le 1, 0 < t \le 1.
\end{aligned} \tag{6.2}$$

The exact solution is $u(x,y,z,t) = t^2(sinx + siny + sinz)$.

The following two tables shows the maximum error and convergence order of computational Example 2 under different step sizes, as well as the maximum error and convergence order for the extrapolation scheme.

Table 3. Maximum errors and convergence orders for Example 2.

(h,τ)	$E_\infty(h_1,h_2,h_3,\tau)$	$\log_2\left(\frac{E(2h_1,2h_2,2h_3,2\tau)}{E(h_1,h_2,h_3,\tau)}\right)$
$(\frac{1}{16},\frac{1}{16})$	6.4577e-04	*
$(\frac{1}{32},\frac{1}{32})$	1.7401e-04	1.8919
$(\frac{1}{64},\frac{1}{64})$	4.4204e-05	1.9769
$(\frac{1}{128},\frac{1}{128})$	1.1212e-05	1.9791

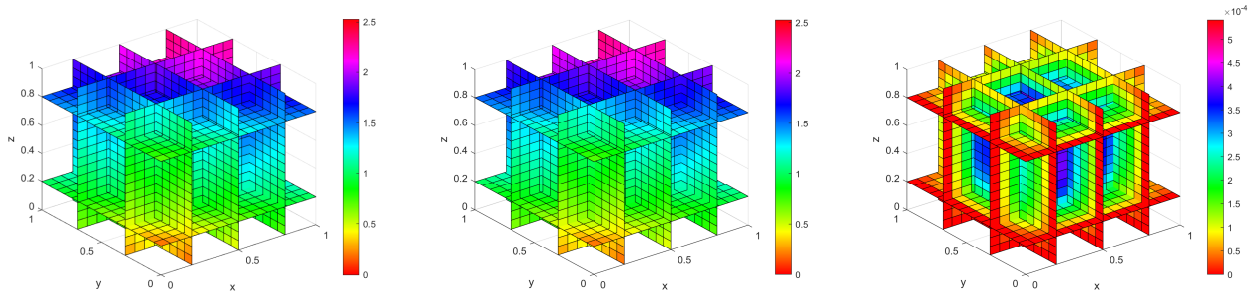
Table 4. Maximum errors and convergence orders for extrapolation scheme for Example 2.

(h,τ)	$\bar{E}_\infty(h_1,h_2,h_3,\tau)$	$\log_2\left(\frac{\bar{E}(2h_1,2h_2,2h_3,2\tau)}{\bar{E}(h_1,h_2,h_3,\tau)}\right)$
$(\frac{1}{16},\frac{1}{16})$	9.3597e-05	*
$(\frac{1}{32},\frac{1}{32})$	9.7157e-06	3.2681
$(\frac{1}{64},\frac{1}{64})$	9.6377e-07	3.3337

As is discernible from the aforementioned Tables 3 and 4, for Example 2, the errors and convergence

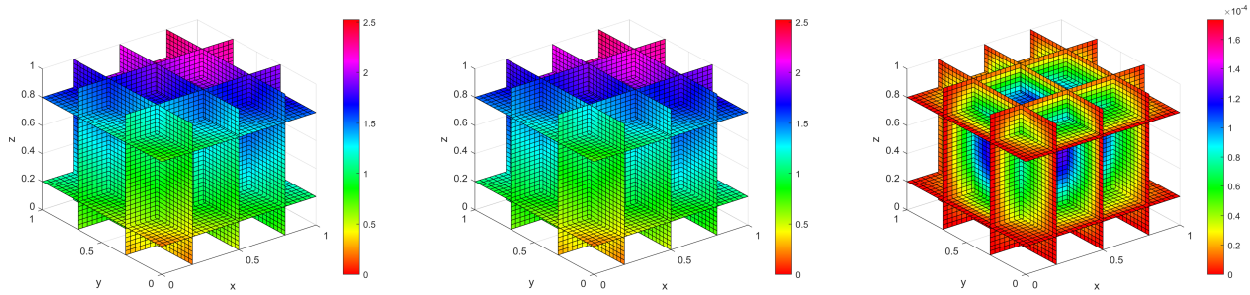
orders corresponding to diverse meshing scenarios are elucidated. Evidently, within the context of the original finite difference scheme, the error ratio hovers approximately around order 2. Subsequent to the utilization of the Richardson extrapolation technique, the convergence order approximates order 3. In addition, paralleling the situation in Example 1, as the mesh refinement progresses, the maximum error consistently exhibits a decreasing trend.

In Figures 5–8 the numerical solutions, exact solutions, and error graphs of Example 2 are presented under different step size ratios with $1/h = 1/\tau = 16, 32, 64, 128$.



(a) Numerical solution (b) Exact solution (c) Error

Figure 5. Numerical solution, exact solution, and error graph of the final layer ($\frac{1}{h} = \frac{1}{\tau} = 16$) for Example 2.

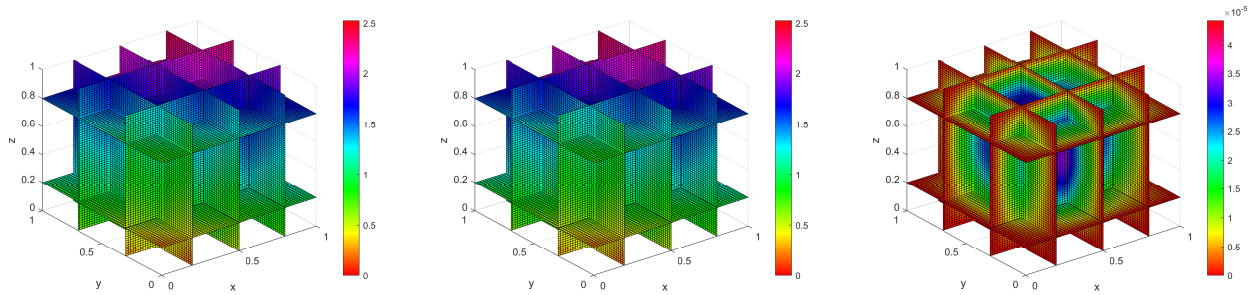


(a) Numerical solution (b) Exact solution (c) Error

Figure 6. Numerical solution, exact solution, and error graph of the final layer ($\frac{1}{h} = \frac{1}{\tau} = 32$) for Example 2.

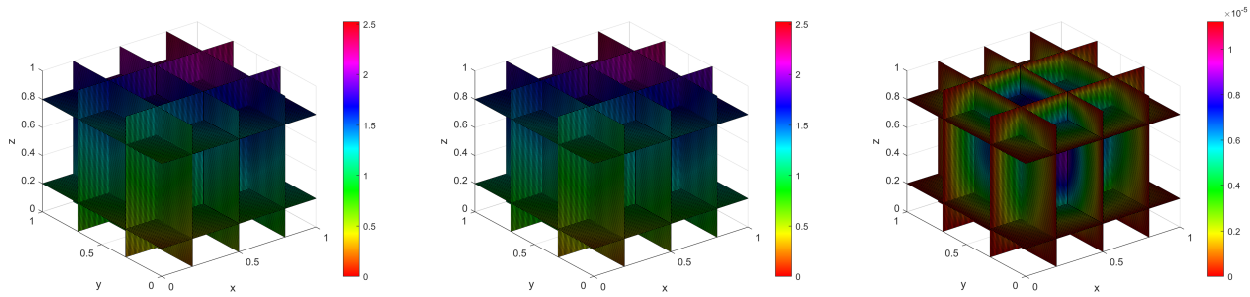
7. Conclusions

We investigate the ADI difference scheme and the Richardson extrapolation scheme for a class of three-dimensional hyperbolic equations. By employing Lemma 2.2 and the central difference method, we discretize the equation (1.1)–(1.3) and construct an ADI finite difference scheme. We prove solvability, stability, and convergence, and it is demonstrated that the convergence order is $O(\tau^2 + h_1^2 + h_2^2 + h_3^2)$. By further studying the Richardson extrapolation scheme for the model, the accuracy is significantly enhanced, and the convergence order is improved.



(a) Numerical solution (b) Exact solution (c) Error

Figure 7. Numerical solution, exact solution, and error graph of the final layer ($\frac{1}{h} = \frac{1}{\tau} = 64$) for Example 2.



(a) Numerical solution (b) Exact solution (c) Error

Figure 8. Numerical solution, exact solution, and error graph of the final layer ($\frac{1}{h} = \frac{1}{\tau} = 128$) for Example 2.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

References

1. M. Dehghan, A. Ghesmati, Combination of meshless local weak and strong (MLWS) forms to solve the two dimensional hyperbolic telegraph equation, *Eng. Anal. Bound. Elem.*, **32** (2010), 324–336. https://doi.org/10.1016/j.enganabound.2009.10.010
2. A. Saadatmandi, M. Dehghan, Numerical solution of hyperbolic telegraph equation using the Chebyshev tau method, *Numer. Methods Partial Differ. Equations*, **26** (2010), 239–252. https://doi.org/10.1002/num.20442
3. R. Jiwari, S. Pandit, R. Mittal, A differential quadrature algorithm to solve the two dimensional linear hyperbolic telegraph equation with Dirichlet and Neumann boundary conditions, *Appl. Math. Comput.*, **218** (2012), 7279–7294. https://doi.org/10.1016/j.amc.2012.01.006
4. K. Zhukovsky, Operational approach and solutions of hyperbolic heat conduction equations, *Axioms*, **5** (2016), 28. https://doi.org/10.3390/axioms5040028
5. M. S. Yudin, Inversion effects on wind and surface pressure in atmospheric front propagation simulation with a hyperbolic model, *Carpath. J. Earth. Env.*, **96** (2017), e012002. https://doi.org/10.1088/1755-1315/96/1/012002
6. T. Hillen, K. Hadeler, Hyperbolic systems and transport equations in mathematical biology, *Carpath. J. Earth. Env.*, (2005), 257–279. https://doi.org/10.1007/3-540-27907-5_11
7. W. Alt, Inversion effects on wind and surface pressure in atmospheric front propagation simulation with a hyperbolic model, *Carpath. J. Earth. Env.*, (2003), 431–461. https://doi.org/10.1088/1755-1315/96/1/012002
8. K. K. Sharma, P. Singh, Hyperbolic partial differential-difference equation in the mathematical modeling of neuronal firing and its numerical solution, *Carpath. J. Earth. Env.*, **201** (2008), 229–238. https://doi.org/10.1016/j.amc.2007.12.051

9. K. Hadeler, K. Dietz, Nonlinear hyperbolic partial differential equations for the dynamics of parasite populations, *Hyperbolic Part. Differ. Equations*, (1983), 415–430. https://doi.org/10.1016/B978-0-08-030254-6.50016-1
10. I. Fedotov, M. Shatalov, J. Marais, Hyperbolic and pseudo-hyperbolic equations in the theory of vibration, *Acta. Mech.*, **227** (2016), 3315–3324. https://doi.org/10.1007/s00707-015-1537-6
11. G. Paul, *Huygens' Principle and Hyperbolic Equations*, Academic Press, 2014.
12. S. Alinhac, *Blowup for Nonlinear Hyperbolic Equations*, Springer Science & Business Media, 2013.
13. M. Dehghan, A. Ghesmati, Solution of the second-order one-dimensional hyperbolic telegraph equation by using the dual reciprocity boundary integral equation (DRBIE) method, *Eng. Anal. Bound. Elem.*, **34** (2010), 51–59. https://doi.org/10.1016/j.enganabound.2009.07.002
14. A. Häck, *Kinetic and Hyperbolic Equations with Applications to Engineering Processes*, Ph.D thesis, Dissertation, RWTH Aachen University, 2017.
15. O. A. Ladyzhenskaya, *The Mixed Problem for a Hyperbolic Equation*, Gostekhizdat, MoscowX, 1953.
16. R. Sakamoto, *Hyperbolic Boundary Value Problems*, Cup Archive, 1982.
17. V. M. Gordienko, Dissipativity of boundary condition in a mixed problem for the three-dimensional wave equation, *Sib. Electron. Math. Rep.*, **10** (2013), 311–323. https://www.mathnet.ru/eng/semr/v10/p311
18. A. N. Malyshev, A mixed problem for a second-order hyperbolic equation with a complex first-order boundary condition, *Sibirskii Matematicheskii Zhurnal*, **24** (1983), 102–121. https://doi.org/10.1007/BF00970317
19. K. Liu, Z. He, H. Zhang, X. Yang, A Crank-Nicolson ADI compact difference scheme for the three-dimensional nonlocal evolution problem with a weakly singular kernel, *Comput. Appl. Math.,* **44** (2025), 164. https://doi.org/10.1007/s40314-025-03125-x
20. Y. Shi, X. Yang, The pointwise error estimate of a new energy-preserving nonlinear difference method for supergeneralized viscous Burgers' equation, *Comput. Appl. Math.*, **44** (2025), 257. https://doi.org/10.1007/s40314-025-03222-x
21. Y. Shi, X. Yang, A time two-grid difference method for nonlinear generalized viscous Burgers' equation. *J. Math. Chem.,* **62** (2024), 1323–1356. https://doi.org/10.1007/s10910-024-01592-x
22. C. Li, H. Zhang, X. Yang, A new linearized ADI compact difference method on graded meshes for a nonlinear 2D and 3D PIDE with a WSK, *Comput. Math. Appl.*, **176** (2024), 349–370. https://doi.org/10.1016/j.camwa.2024.11.006
23. K. Li, W. Liao, Y. Lin, A compact high order alternating direction implicit method for three-dimensional acoustic wave equation with variable coefficient, *J. Comput. Appl. Math.*, **361** (2019), 113–129. https://doi.org/10.1016/j.cam.2019.04.013
24. W. Liao, On the dispersion, stability and accuracy of a compact higher-order finite difference scheme for 3D acoustic wave equation, *J. Comput. Appl. Math.*, **270** (2014), 571–583. https://doi.org/10.1016/j.cam.2013.08.024

25. W. Liao, P. Yong, H. Dastour, J. Huang, Efficient and accurate numerical simulation of acoustic wave propagation in a 2D heterogeneous media, *Appl. Math. Comput.*, **321** (2018), 385–400. https://doi.org/10.1016/j.amc.2017.10.052

26. D. Deng, D. Liang, The time fourth-order compact ADI methods for solving two-dimensional nonlinear wave equations, *Appl. Math. Comput.*, **329** (2018), 188–209. https://doi.org/10.1016/j.amc.2018.02.010

27. S. Das, W. Liao, A. Gupta, An efficient fourth-order low dispersive finite difference scheme for a 2-D acoustic wave equation, *J. Appl. Math. Comput.*, **258** (2014), 151–167. https://doi.org/10.1016/j.cam.2013.09.006

28. B. D. Nie, B. Y. Cao, Three mathematical representations and an improved ADI method for hyperbolic heat conduction, *Int. J. Heat Mass Transfer*, **135** (2019), 974–984. https://doi.org/10.1016/j.ijheatmasstransfer.2019.02.026

29. S. Zhao, A matched alternating direction implicit (ADI) method for solving the heat equation with interfaces, *J. Sci. Comput.*, **63** (2015), 118–137. https://doi.org/10.1007/s10915-014-9887-0

30. W. Wang, H. Zhang, X. Jiang, X. Yang, A high-order and efficient numerical technique for the nonlocal neutron diffusion equation representing neutron transport in a nuclear reactor, *Ann. Nucl. Energy*, **195** (2024), 110163. https://doi.org/10.1016/j.anucene.2023.110163

31. W. Wang, H. Zhang, Z. Zhou, X. Yang, A fast compact finite difference scheme for the fourth-order diffusion-wave equation, *Int. J. Comput. Math.*, **101** (2024), 170–193. https://doi.org/110.1080/00207160.2024.2323985

32. J. Wang, X. Jiang, X. Yang, H. Zhang, A compact difference scheme for mixed-type time-fractional Black-Scholes equation in European option pricing, *Math. Method Appl. Sci.*, **48** (2025), 6818–6829. https://doi.org/10.1002/mma.10717

33. J. Wang, X. Jiang, H. Zhang, X. Yang, A new fourth-order nonlinear difference scheme for the nonlinear fourth-order generalized Burgers-type equation, *J. Appl. Math. Comput.*, (2025), 1–31. https://doi.org/10.1007/s12190-025-02467-3

34. X. Jiang, J. Wang, W. Wang, H. Zhang, A predictor-corrector compact difference scheme for a nonlinear fractional differential equation, *Fractal Fract.*, **7** (2023), 521. https://doi.org/10.3390/fractalfract7070521

35. H. Zhou, M. Huang, W. Ying, ADI schemes for heat equations with irregular boundaries and interfaces in 3D with applications, preprint, arXiv:2309.00979. https://doi.org/10.48550/arXiv.2309.00979

36. X. Shen, X. Yang, H. Zhang, The high-order ADI difference method and extrapolation method for solving the two-dimensional nonlinear parabolic evolution equations, *Mathematics*, **12** (2024), 3469. https://doi.org/10.3390/math12223469

37. Z. Chen, H. Zhang, H. Chen, ADI compact difference scheme for the two-dimensional integro-differential equation with two fractional Riemann-Liouville integral kernels, *Fractal Fract.*, **8** (2024), 707. https://doi.org/10.3390/fractalfract8120707

38. T. Liu, H. Zhang, X. Yang, The ADI compact difference scheme for three-dimensional integro-partial differential equation with three weakly singular kernels, *J. Appl. Math. Comput.*, (2025), 1–29. https://doi.org/10.1007/s12190-025-02386-3

39. Z. Zhou, H. Zhang, X. Yang, A BDF2 ADI difference scheme for a three-dimensional non-local evolution equation with multi-memory kernels, *Comput. Appl. Math.*, **43** (2024), 418. https://doi.org/10.1007/s40314-024-02931-z

40. L. Qiao, W. Qiu, D. Xu, Crank-Nicolson ADI finite difference/compact difference schemes for the 3D tempered integrodifferential equation associated with Brownian motion, *Numerical Algorithms*, **93** (2023), 1083–1104. https://doi.org/10.1007/s11075-022-01454-0

41. W. Qiu, X. Zheng, K. Mustapha, Numerical approximations for a hyperbolic integrodifferential equation with a non-positive variable-sign kernel and nonlinear-nonlocal damping, *Appl. Numer. Math.,* **213** (2025), 61–76. https://doi.org/10.1016/j.apnum.2025.02.018

42. W. Qiu, Y. Li, X. Zheng, Numerical analysis of nonlinear Volterra integrodifferential equations for viscoelastic rods and plates. *Calcolo*, **61** (2024), 50. https://doi.org/10.1007/s10092-024-00607-y

43. L. Chen, Z. Wang, S. Vong, A second-order weighted ADI scheme with nonuniform time grids for the two-dimensional time-fractional telegraph equation. *J. Appl. Math. Comput.,* **70** (2024), 5777–5794. https://doi.org/10.1007/s12190-024-02200-6

44. K. G. Tay, S. L. Kek, R. Abdul-Kahar, M. Azlan, M. Lee, A Richardson's Extrapolation spreadsheet calculator for Numerical Differentiation, *Spreadsheets Educ.*, **6** (2013). http://epublications.bond.edu.au/ejsie/vol6/iss2/5

45. A. Alekseev, A. Bondarev, A comparison of the Richardson extrapolation and the approximation error estimation on the ensemble of numerical solutions, in *International Conference on Computational Science*, (2021), 554–566. https://doi.org/10.1007/978-3-030-77980-1_42

46. J. Wang, X. Jiang, H. Zhang, A BDF3 and new nonlinear fourth-order difference scheme for the generalized viscous Burgers' equation, *Appl. Math. Lett.*, **151** (2024), 109002. https://doi.org/10.1016/j.aml.2024.109002

47. X. Yang, W. Wang, Z. Zhou, H. Zhang, An efficient compact difference method for the fourth-order nonlocal subdiffusion problem, *Taiwan. J. Math.*, **29** (2025), 35–66. https://doi.org/10.11650/tjm/240906

48. D. Ruan, X. Wang, A high-order Chebyshev-type method for solving nonlinear equations: local convergence and applications, *Electron. Res. Arch.*, **33** (2025), 1398–1413. https://doi.org/10.3934/era.2025065

49. X. Wang, N. Shang, Local convergence analysis of a novel derivative-free method with and without memory for solving nonlinear systems, *Int. J. Comput. Math.*, 2025. https://doi.org/10.1080/00207160.2025.2464701

50. D. Ruan, X. Wang, Y. Wang, Local convergence of seventh-order iterative method under weak conditions and its applications, *Eng. Comput.*, 2025. https://doi.org/10.1108/EC-08-2024-0775

51. X. Wang, W. Li, Fractal behavior of King's optimal eighth-order iterative method and its numerical application, *Math. Commun.*, **29** (2024), 217–236. https://hrcak.srce.hr/321249

52. J. C. Lopez-Marcos, A difference scheme for a nonlinear partial integrodifferential equation, *SIAM J. Numer. Anal.*, **27** (1990), 20–31. https://doi.org/10.1137/0727002

53. Z. Sun, *Numerical Methods for Partial Differential Equations*, Science Press, 2005.

54. Z. Sun, *The Method of Order Reduction and its Application to the Numerical Solutions of Partial Differential Equations*, Science Press, 2009.

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