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Research article

The ADI difference and extrapolation scheme for high-dimensional variable coefficient evolution equations

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Abstract: This paper investigated a numerical method for a two-dimensional variable coefficient evolution equation (VCEE), utilizing the alternating direction implicit (ADI) method and an extrapolation formula. The time derivative was discretised by the backward Euler (BE) scheme on a uniform mesh and the finite difference method (FDM) was applied to spatial discretization. We proved an priori estimate and the error bound of the solution to the difference scheme using the energy analysis method, and verified the uniqueness, stability, and convergence of the proposed scheme. To further improve numerical accuracy, we introduced a Richardson extrapolation method, which enhances the global accuracy to fourth order. Finally, some numerical examples were provided to demonstrate the validity of the theoretical analysis.

Keywords: parabolic equation; variable coefficients; finite difference method; alternating direction implicit; stability; convergence

1. Introduction

Variable-coefficient parabolic partial differential equations (PDEs) are widely used in physics and engineering to model processes such as heat conduction, fluid flow, and diffusion. These equations provide a more accurate representation of real-world phenomena involving spatial and temporal variations in material properties, compared to constant-coefficient equations. With advancements in computational power and numerical methods, the study of these equations has become more efficient [1, 2].

Parabolic partial differential equations with variable coefficients arise in a wide range of physical and engineering contexts, such as heat conduction in heterogeneous media, diffusion in porous materials, and anisotropic fluid flows. They are fundamental models for describing time-evolving processes involving spatial diffusion and varying material properties.

In this paper, we study the initial boundary value problem for a two-dimensional (2D) variable coefficient parabolic evolution equation

$$r(x,y,t)\frac{\partial u}{\partial t} - a\Delta u = f(x,y,t),\ (x,y)\in\Omega,\ 0<t\le T. \tag{1.1}$$

The associated initial and boundary conditions are given by

$$\begin{aligned} u(x,y,0) &= \varphi(x,y),\ (x,y)\in\Omega,\\ u(x,y,t) &= \psi(x,y,t),\ (x,y)\in\Gamma, 0<t\le T, \end{aligned} \tag{1.2}$$

where a is a positive constant, Δ is the 2D Laplace operator, $\Omega = (0, L_x)\times\left(0, L_y\right)$, and Γ represents the boundary of Ω. For any point $(x,y)\in\Gamma$, we have $\psi(x,y,0) = \varphi(x,y)$. $r(x,y,t)$, $\varphi(x,y)$, $\psi(x,y,t)$ are all smooth functions and $r(x,y,t) > 0$. Mathematically, the problem (1.1)–(1.2) represents a class of second-order parabolic PDEs with variable coefficients. The variable coefficient $r(x,y,t)$ introduces significant challenges in both theoretical analysis and numerical simulation due to its spatial and temporal dependencies. Solving such problems is essential for accurately modeling real-world phenomena involving nonuniform media and time-dependent material properties.

The variable coefficient term $r(x,y,t)$ introduces spatial and temporal dependencies, which pose significant challenges for both theoretical analysis and numerical simulation. Solving such problems is crucial for accurately modeling physical processes in nonuniform media and time-dependent material properties [3,4]. For example, the modeling of fluid dynamics, solitons, and wave propagation [5,6], as well as epidemic modeling [7,8], rely on such equations.

Variable coefficient parabolic equations play a crucial role in science and engineering, describing physical processes whose behavior changes with both spatial and temporal variables. These equations model complex phenomena such as heat conduction, fluid flow, and diffusion, where coefficients vary over space and time, offering a more accurate representation of real-world non-homogeneous media and transient processes compared to constant-coefficient equations. Recent advancements in computational power and numerical methods have significantly improved our ability to study these equations [9–13]. Various numerical methods have been developed for solving variable coefficient parabolic equations, including finite element methods [14–17], finite difference methods [18–22], finite volume methods [23–27], and spectral methods [28–31], each contributing substantially to practical applications.

In recent years, alternating direction implicit (ADI) methods and extrapolation schemes have been applied to solve high-dimensional variable-coefficient evolution equations, significantly improving computational efficiency and accuracy. Moreover, physics-informed neural networks (PINNs) have proven effective for solving these complex problems, especially when working with sparse data [32], offering new insights for tackling these challenges [33].

The ADI methods are known for their computational efficiency and stability, and are widely applied to high-dimensional evolution equations. For example, Liao et al. [34] employed the discrete energy method to demonstrate that the ADI solution is unconditionally convergent with a second-order rate in the maximum norm. Zhou et al. [35] proposed efficient ADI schemes for three-dimensional nonlocal evolution equations with weakly singular kernels, and Li et al. [36] developed a linearized ADI compact difference method for nonlinear two- and three-dimensional partial integro-differential equations.

Several researchers have extended ADI methods to fractional and time-evolution partial differential equations. Chen et al. [37–40] introduced L1-ADI methods for time-fractional diffusion and reaction-subdiffusion equations. Wang et al. [41,42] proposed compact ADI schemes for time-fractional integro-differential and diffusion-wave equations, while Huang et al. [43,44] focused on Grünwald-Letnikov and L1-ADI schemes for time-fractional reaction-diffusion models.

Qiu et al. [45,46] contributed BDF2 and Crank-Nicolson difference schemes for Volterra integrodifferential equations, while Qiao et al. [47–49] developed fast ADI methods for multidimensional tempered fractional integro-differential equations, including applications involving Brownian motion. Wang et al. [50] introduced a high-order compact exponential ADI scheme for solving 2D fractional convection-diffusion equations.

Additionally, Richardson extrapolation techniques [51–54] have been combined with ADI methods to further improve accuracy [54]. Shen et al. [55] combined ADI and Richardson extrapolation for 2D nonlinear parabolic equations. Other related strategies include Shi et al.'s time-space two-grid interpolation method [56], Wang et al.'s high-order difference method [57–59], and the compact predictor–corrector scheme of Jiang et al. [60].

However, research on ADI methods for high-dimensional variable-coefficient parabolic equations remains limited. In this work, we perform a theoretical energy analysis to demonstrate the uniqueness and stability of a proposed ADI scheme, and employ Richardson extrapolation to enhance its accuracy.

The main contributions of this work are as follows:

- We rigorously demonstrate the stability of the proposed scheme and establish its a priori bounds.
- We develop and apply an extrapolation method to enhance the accuracy of the numerical solutions. To the best of our knowledge, this is the first instance of applying the ADI extrapolation technique to this equation.

The structure of the paper is organized as follows: The design of the BE-ADI scheme is presented in Section 2. In Section 3, we address the uniqueness, convergence, and stability of the scheme we previously mentioned. Section 4 is dedicated to the formulation of the extrapolation method. Then the numerical results are provided in Section 5. Finally, we draw conclusions in Section 6.

2. Establishment of the difference scheme

Let m_1, m_2, and n be positive integers, $h_1 = L_x/m_1$, $x_p = ph_1$, $0 \le p \le m_1$, $h_2 = L_y/m_2$, $y_q = qh_2$, $0 \le q \le m_2$, $\tau = T/n$, $t_w = w\tau$, $0 \le w \le n$. Denote $\Omega_\tau = \{t_w \mid 0 \le w \le n\}$, $\Omega_h = \Big\{(x_p, y_q) \mid 0 \le p \le m_1, 0 \le q \le m_2\Big\}$, $\omega = \Big\{(p,q) \mid (x_p, y_q) \in \Omega\Big\}$, $\gamma = \Big\{(p,q) \mid (x_p, y_q) \in \Gamma\Big\}$, $\overline{\omega} = \omega \cup \gamma$. For a grid function $v = \{v^w \mid 0 \le w \le n\}$ on Ω_τ, we introduce the notations

$$D_t v^w = \frac{1}{\tau}\left(v^{w+1} - v^w\right), \quad D_{\bar{t}} v^w = \frac{1}{\tau}\left(v^w - v^{w-1}\right).$$

Let $U_h = \left\{ v \middle| v = \left\{ v_{pq} \mid (p,q) \in \overline{\omega} \right\} \ and \ v_{pq} = 0 \ if \ \ (p,q) \in \gamma \ \right\}$. For any grid function $v \in U_h$, we denote

$$\delta_x^2 v_{pq} = \frac{1}{h_1^2}\left[v_{p-1,q} - 2v_{pq} + v_{p+1,q} \right] = \frac{1}{h_1}\left(\delta_x v_{p+\frac{1}{2},q} - \delta_x v_{p-\frac{1}{2},q} \right),$$

$$\delta_x v_{p+\frac{1}{2},q} = D_x v_{pq} = \frac{1}{h_1}\left(v_{p+1,q} - v_{pq} \right), \quad D_{\bar{x}} v_{pq} = \frac{1}{h_1}\left(v_{pq} - v_{p-1,q} \right),$$

$$\Delta_x v_{pq} = \frac{1}{2h_1}\left(v_{p+1,q} - v_{p-1,q} \right), \quad \Delta_h v_{pq} = \delta_x^2 v_{pq} + \delta_y^2 v_{pq}.$$

Analogous notations may be employed, for instance, $\delta_y^2 v_{pq}$, $\delta_y v_{p,q+\frac{1}{2}}$, $D_y v_{pq}$, $D_{\bar{y}} v_{pq}$, $\Delta_y v_{pq}^w$.

For any grid function $v, u \in U_h$, the inner product and the norms are defined as follows:

$$\langle v, u \rangle = h_1 h_2 \sum_{p=1}^{m_1-1} \sum_{q=1}^{m_2-1} vu, \ \|v\|^2 = \langle v, v \rangle,$$

$$\|\Delta_h v\|^2 = h_1 h_2 \sum_{p=0}^{m_1-1} \sum_{q=1}^{m_2-1} (\Delta_h v)^2, \|\delta_x v\|^2 = h_1 h_2 \sum_{p=0}^{m_1-1} \sum_{q=1}^{m_2-1} \left(\delta_x v_{p+\frac{1}{2},q} \right)^2,$$

$$\left\|\delta_x \delta_y v\right\|^2 = h_1 h_2 \sum_{p=0}^{m_1-1} \sum_{q=0}^{m_2-1} \left(\delta_x \delta_y v_{p+\frac{1}{2},q+\frac{1}{2}} \right)^2,$$

$$\left\|\delta_x \delta_y^2 v\right\|^2 = h_1 h_2 \sum_{p=1}^{m_1-1} \sum_{q=1}^{m_2-1} \left(\delta_x \delta_y^2 v_{p+\frac{1}{2},q+\frac{1}{2}} \right)^2,$$

$$|v|_1 = \sqrt{\|\delta_x v\|^2 + \left\|\delta_y v\right\|^2}, \ \|v\|_1 = \sqrt{\|v\|^2 + |v|^2}.$$

Some grid functions are defined as

$$U_{pq}^w = u\left(x_p, y_q, t_w \right), \ f_{pq}^w = f\left(x_p, y_q, t_w \right), \ (p,q,w) \in \Omega_h \times \Omega_\tau.$$

Lemma 2.1 ([11]). *For any grid function* $u(x,y,t) \in C^2\left([0,L_x] \times \left[0, L_y\right] \times [0,T] \right)$, *we have*

$$\frac{\partial u}{\partial t}\left(x_p, y_q, t_w \right) = D_{\bar{t}} U_{pq}^w + \tau \int_0^1 \left[\frac{\partial^2 u}{\partial t^2}\left(x_p, y_q, t_w - s\tau \right) \right] (1-s)\, ds.$$

Proof. This result can be derived from [11, p. 7]. □

Lemma 2.2. *For any grid function* $u(x,y,t) \in C^4\left([0,L_x] \times \left[0, L_y\right] \times [0,T] \right)$, *we have*

$$\frac{\partial^2 u}{\partial x^2}\left(x_p, y_q, t_w \right) = \delta_x^2 U_{pq}^w - \frac{1}{6}h_1^2 \int_0^1 \left[\frac{\partial^4 u}{\partial x^4}\left(x_p + sh_1, y_q, t_w \right) + \frac{\partial^4 u}{\partial x^4}\left(x_p - sh_1, y_q, t_w \right) \right] (1-s)^3\, ds,$$
$$\frac{\partial^2 u}{\partial y^2}\left(x_p, y_q, t_w \right) = \delta_y^2 U_{pq}^w - \frac{1}{6}h_2^2 \int_0^1 \left[\frac{\partial^4 u}{\partial y^4}\left(x_p, y_q + sh_2, t_w \right) + \frac{\partial^4 u}{\partial y^4}\left(x_p, y_q - sh_2, t_w \right) \right] (1-s)^3\, ds.$$

Proof. This result can be derived from [11, p. 7]. □

Lemma 2.3 ([11], p. 19). *For any grid function $v, u \in U_h$, we have*

$$\left\langle \delta_x^2 v, u \right\rangle = -\left\langle \delta_x v, \delta_x u \right\rangle,$$
$$\left\langle \delta_y^2 v, u \right\rangle = -\left\langle \delta_y v, \delta_y u \right\rangle,$$
$$\left\langle \delta_x^2 v + \delta_y^2 v, v \right\rangle = -|v|_1^2,$$
$$\left\langle \delta_x^2 \delta_y^2 v, v \right\rangle = \left\| \delta_x \delta_y v \right\|^2.$$

Lemma 2.4 ([11], p. 19). *(Gronwall Inequality-E) Suppose $\{F^w\}_{w=0}^{\infty}$ is a non-negative sequence and let a_1 and a_2 denote two non-negative constants which satisfy*

$$F^w \le a_1 \tau \sum_{l=0}^{w-1} F^l + a_2, \quad w = 0, 1, 2, ...$$

Then, the following inequality holds:

$$F^w \le e^{a_1 w \tau} + a_2, \quad w = 0, 1, 2, \dots$$

Lemma 2.5 ([11], p. 72). *For any grid function $v \in U_h$, $\|v\|_\infty \leqslant \frac{1}{12}\sqrt{3\left(\sqrt{2}+1\right) L_x L_y}\, \|\Delta_h v\|$.*

Next, we construct the BE-ADI difference format of (1.1)–(1.2).

Equation (1.1) can be rewritten in the following form at the grid point $\left(x_p, y_q, t_w\right)$:

$$r\left(x_p, y_q, t_w\right) \frac{\partial u}{\partial t}\left(x_p, y_q, t_w\right) - a\Delta u\left(x_p, y_q, t_w\right) = f\left(x_p, y_q, t_w\right), (p, q) \in \omega, 0 < w \le n. \tag{2.1}$$

According to Lemmas 2.1 and 2.2, it can be concluded that

$$r_{pq}^w D_{\bar{t}} U_{pq}^w - a\left(\delta_x^2 U_{pq}^w + \delta_y^2 U_{pq}^w\right) = f_{pq}^w + (R_1)_{pq}^w, (p, q) \in \omega, 0 < w \le n, \tag{2.2}$$

in which

$$\begin{aligned}
(R_1)_{pq}^w = & -\tau r_{pq}^w \int_0^1 \left[\frac{\partial^2 u}{\partial t^2}\left(x_p, y_q, t_w - s\tau\right)\right](1-s)\, ds \\
& - \frac{a}{6} h_1^2 \int_0^1 \left[\frac{\partial^4 u}{\partial x^4}\left(x_p + sh_1, y_q, t_w\right) + \frac{\partial^4 u}{\partial x^4}\left(x_p - sh_1, y_q, t_w\right)\right](1-s)^3\, ds \\
& - \frac{a}{6} h_2^2 \int_0^1 \left[\frac{\partial^4 u}{\partial y^4}\left(x_p, y_q + sh_2, t_w\right) + \frac{\partial^4 u}{\partial y^4}\left(x_p, y_q + sh_2, t_w\right)\right](1-s)^3\, ds.
\end{aligned}$$

It is evident that a constant c_1 exists, satisfying

$$\begin{aligned}
\left|(R_1)_{pq}^w\right| &\leqslant c_1\left(\tau + h_1^2 + h_2^2\right), \ (p, q) \in \omega, \ 0 \leqslant w \leqslant n, \\
\left|D_{\bar{t}} (R_1)_{pq}^w\right| &\leqslant c_1\left(\tau + h_1^2 + h_2^2\right), \ (p, q) \in \omega, \ 0 < w \leqslant n,
\end{aligned} \tag{2.3}$$

in which

$$D_{\bar{t}} (R_1)_{pq}^w = \frac{1}{\tau}\left[(R_1)_{pq}^w - (R_1)_{pq}^{w-1}\right].$$

From the initial condition (1.2), there is

$$\begin{aligned} U_{pq}^{0} &= \varphi\left(x_p, y_q\right),\ (p,q) \in \omega, \\ U_{pq}^{w} &= \psi\left(x_p, y_q, t_w\right),\ (p,q) \in \gamma, 0 < w \le n. \end{aligned} \tag{2.4}$$

By adding a small perturbation term $\frac{a^2\tau^2}{r_{pq}^w}\delta_x^2\delta_y^2 D_{\bar{t}} U_{pq}^w$ on both sides of (2.2), we have

$$r_{pq}^{w} D_{\bar{t}} U_{pq}^{w} - a\left(\delta_x^2 U_{pq}^{w} + \delta_y^2 U_{pq}^{w}\right) + \frac{a^2\tau^2}{r_{pq}^{w}}\delta_x^2\delta_y^2 D_{\bar{t}} U_{pq}^{w} = f_{pq}^{w} + (R_2)_{pq}^{w},\ (p,q) \in \omega, 0 < w \le n, \tag{2.5}$$

in which $(R_2)_{pq}^w = (R_1)_{pq}^w + (R_3)_{pq}^w$, and

$$(R_3)_{pq}^{w} = \frac{1}{2}\frac{a^2\tau^2}{r_{pq}^{w}}\delta_x^2\delta_y^2 \int_0^1 \left[\frac{\partial U}{\partial t}\left(x_p, y_q, t_w + \frac{s\tau}{2}\right) + \frac{\partial U}{\partial t}\left(x_p, y_q, t_w - \frac{s\tau}{2}\right)\right] ds.$$

The Taylor formula with an integral remainder gives that

$$0 = \frac{a^2\tau^2}{r_{pq}^{w}}\delta_x^2\delta_y^2 D_{\bar{t}} U_{pq}^{w} - (R_3)_{pq}^{w}, \quad (p,q) \in \omega, 0 < w \le n.$$

It follows from the (2.3) that there exists a positive constant c_2 such that

$$\begin{aligned} \left|(R_2)_{pq}^{w}\right| &\le c_2\left(\tau + h_1^2 + h_2^2\right), (p,q) \in \omega,\ 0 \le w \le n, \\ \left|D_{\bar{t}}(R_2)_{pq}^{w}\right| &\le c_2\left(\tau + h_1^2 + h_2^2\right), (p,q) \in \omega,\ 0 < w \le n, \end{aligned} \tag{2.6}$$

in which $D_{\bar{t}}(R_2)_{pq}^w = \frac{1}{\tau}\left[(R_2)_{pq}^w - (R_2)_{pq}^{w-1}\right]$. By neglecting the small term $(R_2)_{pq}^w$ in (2.5) and substituting the exact solution U_{pq}^w with the numerical solution u_{pq}^w, we obtain the BE-ADI scheme

$$r_{pq}^{w} D_{\bar{t}} u_{pq}^{w} - a\left(\delta_x^2 u_{pq}^{w} + \delta_y^2 u_{pq}^{w}\right) + \frac{a^2\tau^2}{r_{pq}^{w}}\delta_x^2\delta_y^2 D_{\bar{t}} u_{pq}^{w} = f_{pq}^{w}, (p,q) \in \omega, 0 < w \le n, \tag{2.7}$$

and the initial and boundary conditions are

$$\begin{aligned} u_{pq}^{0} &= \varphi\left(x_p, y_q\right),\ (p,q) \in \omega, \\ u_{pq}^{w} &= \psi\left(x_p, y_q, t_w\right),\ (p,q) \in \gamma, 0 \le w \le n. \end{aligned} \tag{2.8}$$

Equation (2.7) can be rewritten as

$$r_{pq}^{w} u_{pq}^{w} - a\tau\left(\delta_x^2 u_{pq}^{w} + \delta_y^2 u_{pq}^{w}\right) + \frac{a^2\tau^2}{r_{pq}^{w}}\delta_x^2\delta_y^2 u_{pq}^{w} = F_{pq}^{w-1}, (p,q) \in \omega, 0 < w \le n, \tag{2.9}$$

where

$$F_{pq}^{w-1} = \tau f_{pq}^{w} + r_{pq}^{w} u_{pq}^{w-1} + \frac{a^2\tau^2}{r_{pq}^{w}}\delta_x^2\delta_y^2 u_{pq}^{w-1},\ 0 < w \le n.$$

By introducing the identity operator I, (2.9) can be rewritten as:

$$\frac{1}{r_{pq}^{w}}\left(r_{pq}^{w}I - a\tau\delta_x^2\right)\left(r_{pq}^{w}I - a\tau\delta_y^2\right)u_{pq}^{w} = F_{pq}^{w-1},\ (p,q)\in\omega, 0 < w \le n.$$

By introducing an intermediate variable $u_{pq}^{*} = \left(r_{pq}^{w}L - a\tau\delta_y^2\right)u_{pq}^{w},\ (p,q)\in\omega$, the difference scheme (2.7)–(2.8) can be resolved using two independent sets of one-dimensional equations.

Algorithm 1:

Input: F_{pq}^{w-1}
Output: u_{pq}^{w}
Step 1: Fixing q $(0 < q \le m_2 - 1)$, solve for u_{pq}^{*} $(1 \le p \le m_1 - 1)$:
for $0 < p \le m_1 - 1$ **do**
  $\frac{1}{r_{pq}^{w}}\left(r_{pq}^{w}L - \alpha\tau\delta_x^2\right)u_{pq}^{*} = F_{pq}^{w-1}$
end
$u_0^{*} = \left(r_{0,q}^{w}L - \alpha\tau\delta_x^2\right)u_{0,q}$
$u_{m_1,q}^{*} = \left(r_{m_1,q}^{w}L - \alpha\tau\delta_x^2\right)u_{m_1,q}^{w}$
Step 2: Fixing p $(1 < p \le m_1 - 1)$, and solve for u_{pq}^{w} $(0 < q \le m_2 - 1)$:
for $0 < q \le m_2 - 1$ **do**
  $\left(r_{pq}^{w}L - \alpha\tau\delta_y^2\right)u_{pq}^{w} = u_{pq}^{*}$
end
$u_p^{w} = \phi(x_p, y_0, t_w)$
$u_{p,m_2}^{w} = \psi(x_p, y_{m_2}, t_w)$
return u_{pq}^{w} $(0 < q \le m_2 - 1)$

3. Theoretical analysis of the difference scheme

In this section, we apply the energy analysis method to rigorously establish the uniqueness, convergence, and stability of the solution for the BE-ADI difference scheme.

3.1. Uniqueness analysis

Lemma 3.1. *For any grid function $u \in U_h$, define $m = \frac{a\tau}{r_{pq}}$, where $r_{pq} > 0$ is smooth, then the following inequality holds:*

$$\left\langle m\left(\delta_x^2 u + \delta_y^2 u\right), u\right\rangle \leqslant -h_1h_2\sum_{p=1}^{m_1-1}\sum_{q=1}^{m_2-1} m\left((\Delta_x u)^2 + \left(\Delta_y u\right)^2\right) + C\|u\|^2,$$

with $C = \max_{(x,y)\in\Omega,\ t\in[0,T]}\left|\frac{\partial m^2(x,y,t)}{\partial x^2}\right|$.

Proof.

$$
\begin{aligned}
\left\langle m\delta_x^2 u, u\right\rangle = & \ h_1 h_2 \sum_{p=1}^{m_1-1}\sum_{q=1}^{m_2-1} m_{pq}\left(\delta_x^2 u_{pq}\right)u_{pq} \\
= & \ h_2 \sum_{p=1}^{m_1-1}\sum_{q=1}^{m_2-1} m_{pq}\left(\delta_x u_{p+\frac{1}{2},q} - \delta_x u_{p-\frac{1}{2},q}\right)u_{pq} \\
= & \ h_2 \sum_{p=0}^{m_1-1}\sum_{q=1}^{m_2-1} \left(m_{pq}u_{pq} - m_{p+1,q}u_{p+1,q}\right)\delta_x u_{p+\frac{1}{2},q} \\
= & -h_1 h_2 \sum_{p=0}^{m_1-1}\sum_{q=1}^{m_2-1} \left(m_{pq}\delta_x u_{p+\frac{1}{2},q} + \left(\delta_x m_{p+\frac{1}{2},q}\right)u_{p+1,q}\right)\delta_x u_{p+\frac{1}{2},q} \\
= & -h_1 h_2 \sum_{p=0}^{m_1-1}\sum_{q=1}^{m_2-1} m_{pq}\left(\delta_x u_{p+\frac{1}{2},q}\right)^2 - h_1 h_2 \sum_{p=0}^{m_1-1}\sum_{q=1}^{m_2-1}\left(\delta_x m_{p+\frac{1}{2},q}\right)\left(\delta_x u_{p+\frac{1}{2},q}\right)u_{p+1,q},
\end{aligned} \tag{3.1}
$$

$$
\begin{aligned}
\left\langle m\delta_x^2 u, u\right\rangle = & \ h_1 h_2 \sum_{p=1}^{m_1-1}\sum_{q=1}^{m_2-1} m_{pq}\left(\delta_x^2 u_{pq}\right)u_{pq} \\
= & \ h_2 \sum_{p=1}^{m_1-1}\sum_{q=1}^{m_2-1} m_{pq}\left(\delta_x u_{p+\frac{1}{2},q} - \delta_x u_{p-\frac{1}{2},q}\right)u_{pq} \\
= & \ h_2 \sum_{p=1}^{m_1}\sum_{q=1}^{m_2-1} \left(m_{p-1,q}u_{p-1,q} - m_{pq}u_{pq}\right)\delta_x u_{p-\frac{1}{2},q} \\
= & -h_1 h_2 \sum_{p=1}^{m_1}\sum_{q=1}^{m_2-1} \left(m_{pq}\delta_x u_{p-\frac{1}{2},q} + \left(\delta_x m_{p-\frac{1}{2},q}\right)u_{p-1,q}\right)\delta_x u_{p-\frac{1}{2},q} \\
= & -h_1 h_2 \sum_{p=1}^{m_1}\sum_{q=1}^{m_2-1} m_{pq}\left(\delta_x u_{p-\frac{1}{2},q}\right)^2 - h_1 h_2 \sum_{p=0}^{m_1-1}\sum_{q=1}^{m_2-1}\left(\delta_x m_{p+\frac{1}{2},q}\right)\left(\delta_x u_{p+\frac{1}{2},q}\right)u_{pq}.
\end{aligned} \tag{3.2}
$$

Combine (3.1) and (3.2) to obtain

$$\begin{aligned}\left\langle m\delta_x^2 u, u\right\rangle =& -\frac{1}{2}h_1h_2\left[\sum_{p=0}^{m_1-1}\sum_{q=1}^{m_2-1} m_{pq}\left(\delta_x u_{p+\frac{1}{2},q}\right)^2 + \sum_{p=1}^{m_1}\sum_{q=1}^{m_2-1} m_{pq}\left(\delta_x u_{p-\frac{1}{2},q}\right)^2\right] \\ & - h_1h_2\sum_{p=0}^{m_1-1}\sum_{q=1}^{m_2-1}\left(\delta_x m_{p+\frac{1}{2},q}\right)\left(\delta_x u_{p+\frac{1}{2},q}\right)\frac{u_{p+1,q}+u_{pq}}{2} \\ \leqslant & -\frac{1}{2}h_1h_2\sum_{p=1}^{m_1-1}\sum_{q=1}^{m_2-1} m_{pq}\left[\left(\delta_x u_{p+\frac{1}{2},q}\right)^2 + \left(\delta_x u_{p-\frac{1}{2},q}\right)^2\right] \\ & -\frac{1}{2}h_1h_2\sum_{p=0}^{m_1-1}\sum_{q=1}^{m_2-1}\left(\delta_x m_{p+\frac{1}{2},q}\right)\left(\delta_x u_{p+\frac{1}{2},q}\right)\left(u_{p+1,q}+u_{pq}\right) \\ \leq & -h_1h_2\sum_{p=1}^{m_1-1}\sum_{q=1}^{m_2-1} m_{pq}\left(\Delta_x u_{pq}\right)^2 - \frac{1}{2}h_2\sum_{p=0}^{m_1-1}\sum_{q=1}^{m_2-1}\left(\delta_x m_{p-\frac{1}{2},q} - \delta_x m_{p+\frac{1}{2},q}\right)u_{pq}^2.\end{aligned}$$

It can be obtained from $\left|\delta_x m^w_{p-\frac{1}{2},q} - \delta_x m^w_{p+\frac{1}{2},q}\right| \leq Ch_1$ that

$$\left\langle m\delta_x^2 u, u\right\rangle \leqslant -h_1h_2\sum_{p=1}^{m_1-1}\sum_{q=1}^{m_2-1} m^w_{pq}(\Delta_x u)^2 + \frac{C}{2}\|u\|^2,$$

and similarly,

$$\left\langle m\delta_y^2 u, u\right\rangle \leqslant -h_1h_2\sum_{p=1}^{m_1-1}\sum_{q=1}^{m_2-1} m^w_{pq}\left(\Delta_y u\right)^2 + \frac{C}{2}\|u\|^2.$$

Thus

$$\left\langle m\left(\delta_x^2 u + \delta_y^2 u\right), u\right\rangle \leqslant -h_1h_2\sum_{p=1}^{m_1-1}\sum_{q=1}^{m_2-1} m^w\left((\Delta_x u)^2 + \left(\Delta_y u\right)^2\right) + C\|u\|^2.$$

□

Theorem 3.1. *(Uniqueness) The difference scheme (2.7)–(2.8) has a unique solution when $C < 1$.*

Proof. Denote

$$u^w = \left\{u^w_{pq}|\ (p,q)\in\overline{\omega},\ 1\leq w\leq n\right\}.$$

From (2.8), it is known that u^0 is given. Now, assume u^{w-1} has been determined, and then the following difference scheme can be applied for u^w:

$$r^w D_{\bar{t}}u^w_{pq} - a\left(\delta_x^2 u^w_{pq} + \delta_y^2 u^w_{pq}\right) + \frac{a^2\tau^2}{r^w}\delta_x^2\delta_y^2 D_{\bar{t}}u^w_{pq} = f^w_{pq}, (p,q)\in\omega, 0 < w\leq n.$$

Consider the homogeneous system of equations

$$r^w u^w_{pq} - a\tau\left(\delta_x^2 u^w_{pq} + \delta_y^2 u^w_{pq}\right) + \frac{a^2\tau^2}{r^w}\delta_x^2\delta_y^2 u^w_{pq} = 0, (p,q)\in\omega, \tag{3.3}$$

$$u_{pq}^{w} = 0,\ (p,q) \in \gamma. \tag{3.4}$$

Taking the inner product of u^w on each side of (3.3), we obtain

$$r^w \|u^w\| - a\tau \left\langle \delta_x^2 u^w + \delta_y^2 u^w, u^w \right\rangle + \frac{a^2\tau^2}{r^w} \left\langle \delta_x^2 \delta_y^2 u^w, u^w \right\rangle = 0.$$

By Lemmas 2.3 and 3.1, it can be obtained that

$$\begin{aligned}
\|u^w\| &= \left\langle \frac{a\tau}{r^w} \left(\delta_x^2 u^w + \delta_y^2 u^w\right), u^w \right\rangle - \frac{a^2\tau^2}{(r^w)^2} \left\langle \delta_x^2 \delta_y^2 u^w, u^w \right\rangle \\
&\leqslant -h_1 h_2 \sum_{p=1}^{m_1-1} \sum_{q=1}^{m_2-1} m^w \left((\Delta_x u^w)^2 + \left(\Delta_y u^w\right)^2 \right) + C\|u^w\|^2 - \frac{a^2\tau^2}{(r^w)^2} \left\|\delta_x \delta_y u^w\right\|^2 \\
&\leqslant C\|u^w\|^2.
\end{aligned}$$

If $C < 1$, then $\|u^w\|^2 = 0$. By applying the principle of induction, we prove that the proposed difference scheme (2.7)–(2.8) is uniquely solvable. □

3.2. Stability analysis

Next, we prove the H^2 stability for the BE-ADI difference scheme.

Theorem 3.2. *Let $\left\{ v_{pq}^w \,|(p,q) \in \bar{\omega}, 0 \le w \le n \right\}$ be the solution to the difference scheme*

$$r^w D_{\bar{t}} v_{pq}^w - a\Delta_h v_{pq}^w + \frac{a^2\tau^2}{r^w} \delta_x^2 \delta_y^2 D_{\bar{t}} v_{pq}^w = g_{pq}^w,\ (p,q) \in \omega,\ 0 < w \le n, \tag{3.5}$$

$$\begin{aligned}
v_{pq}^0 &= \varphi\left(x_p, y_q\right),\ (p,q) \in \omega, \\
v_{pq}^w &= 0,\ (p,q) \in \gamma,\ 0 \leqslant w \leqslant n,
\end{aligned} \tag{3.6}$$

and on the boundary $\partial\Omega_h$, let $v_{pq}^w = 0$. Then, we have

$$\|\Delta_h v^w\|^2 \leqslant e^{2w\tau} \left[4\left\|\Delta_h v^0\right\|^2 + 2a^{-2} \left(\left\|g^1\right\|^2 + 2 \max_{1 \leqslant l \leqslant w} \left\|g^l\right\|^2 \tau \sum_{l=2}^{w} \left\|D_{\bar{t}} g^l\right\|^2 \right) \right],\ 1 \leqslant w \leqslant n.$$

Proof. Taking the inner product of both sides of (3.5) with $-\Delta_h D_{\bar{t}} v^w$, we obtain

$$\begin{aligned}
-r^w \langle D_{\bar{t}} v^w, \Delta_h D_{\bar{t}} v^w \rangle + a \langle \Delta_h v^w, \Delta_h D_{\bar{t}} v^w \rangle - \frac{a^2\tau^2}{r^w} \left\langle \delta_x^2 \delta_y^2 D_{\bar{t}} v^w, \Delta_h D_{\bar{t}} v^w \right\rangle& \\
= -\langle g^w, \Delta_h D_{\bar{t}} v^w \rangle,\ 0 < w \le n.&
\end{aligned} \tag{3.7}$$

Due to

$$\begin{aligned}
-\langle D_{\bar{t}} v^w, \Delta_h D_{\bar{t}} v^w \rangle &= |D_{\bar{t}} v^w|_1^2, \\
\langle \Delta_h v^w, \Delta_h D_{\bar{t}} v^w \rangle &= \frac{1}{2\tau} \left\langle \Delta_h v^w - \Delta_h v^{w-1} + \Delta_h v^w + \Delta_h v^{w-1},\ \Delta_h v^w - \Delta_h v^{w-1} \right\rangle \\
&= \frac{1}{2\tau} \left(\left\|\Delta_h v^w - \Delta_h v^{w-1}\right\|^2 + \|\Delta_h v^w\|^2 - \left\|\Delta_h v^{w-1}\right\|^2 \right), \\
-\left\langle \delta_x^2 \delta_y^2 D_{\bar{t}} v^w, \Delta_h D_{\bar{t}} v^w \right\rangle &= \left|\delta_x \delta_y D_{\bar{t}} v^w\right|_1^2.
\end{aligned}$$

The above three equations are substituted into (3.7) , and then we have

$$\frac{a}{2\tau}\left(\|\Delta_h v^w\|^2 - \|\Delta_h v^{w-1}\|^2\right) \leq -r^w \left|D_{\bar{t}} v^w\right|_1^2 - \langle g^w, \Delta_h D_{\bar{t}} v^w\rangle - \frac{a^2\tau^2}{r^w}\left|\delta_x\delta_y D_{\bar{t}} v^w\right|_1^2 \\ \leq -\langle g^w, \Delta_h D_{\bar{t}} v^w\rangle . \tag{3.8}$$

Substituting l for w in (3.8) and summing over l from 1 to w derives

$$\frac{a}{2\tau}\left(\|\Delta_h v^w\|^2 - \left\|\Delta_h v^0\right\|^2\right) \leqslant -\sum_{l=1}^{w}\left\langle g^l, \Delta_h D_{\bar{t}} v^l\right\rangle \\ = \frac{1}{\tau}\left(\left\langle g^1, \Delta_h v^0\right\rangle - \langle g^w, \Delta_h v^w\rangle\right) + \sum_{l=2}^{w}\left\langle D_{\bar{t}} g^l, -\Delta_h v^{l-1}\right\rangle .$$

Multiplying both sides of the above equation by $\frac{2\tau}{a}$ and rearranging terms, we obtain

$$\|\Delta_h v^w\|^2 \leqslant \left\|\Delta_h v^0\right\|^2 + \frac{2}{a}\left[\left\langle g^1, \Delta_h v^0\right\rangle - \langle g^w, \Delta_h v^w\rangle\right] + \frac{2\tau}{a}\sum_{l=2}^{w}\left\langle D_{\bar{t}} g^l, \Delta_h v^{l-1}\right\rangle \\ \leqslant \left\|\Delta_h v^0\right\|^2 + a^{-2}\left\|g^1\right\|^2 + \left\|\Delta_h v^0\right\|^2 + 2a^{-2}\|g^w\|^2 + \frac{1}{2}\|\Delta_h v^w\|^2 \\ + a^{-2}\tau\sum_{l=2}^{w}\left\|D_{\bar{t}} g^l\right\|^2 + \tau\sum_{l=2}^{w}\left\|\Delta_h v^{l-1}\right\|^2.$$

Therefore,

$$\|\Delta_h v^w\|^2 \leqslant 4\left\|\Delta_h v^0\right\|^2 + 2a^{-2}\left(\left\|g^1\right\|^2 + 2\|g^w\|^2\right) \\ + 2a^{-2}\tau\sum_{l=2}^{w}\left\|D_{\bar{t}} g^l\right\|^2 + 2\tau\sum_{l=2}^{w}\left\|\Delta_h v^{l-1}\right\|^2, 1 \leqslant w \leqslant n.$$

It follows from Lemma 2.4 that

$$\|\Delta_h v^w\|^2 \leqslant e^{2w\tau}\left[4\left\|\Delta_h v^0\right\|^2 + 2a^{-2}\left(\left\|g^1\right\|^2 + 2\max_{1\leqslant l\leqslant w}\left\|g^l\right\|^2 \tau\sum_{l=2}^{w}\left\|D_{\bar{t}} g^l\right\|^2\right)\right], 1 \leqslant w \leqslant n. \tag{3.9}$$

The proof has been concluded. □

3.3. Convergence analysis

In this section, the convergence in the maximum norm of the proposed scheme (2.7)–(2.8) will be considered.

Theorem 3.3. *Suppose* $u(x, y, t) \in C^4\left([0, L_x] \times [0, L_y] \times [0, T]\right)$ *is the solution of (1.1)–(1.2), and* $\left\{u_{pq}^w\right\}$ *is the solution of the difference scheme (2.7)–(2.8). Denote*

$$e_{pq}^w = u\left(x_p, y_q, t_w\right) - u_{pq}^w, \ (p, q) \in \overline{\omega}, 0 \leq w \leq n,$$

then we have

$$\|e^w\|_\infty \leqslant \frac{e^T P L_x L_y c_2}{12a}\sqrt{6\left(\sqrt{2}+1\right)(3+T)}\left(\tau + h_1^2 + h_2^2\right),\ 1 \leqslant w \leqslant n.$$

Proof. Subtracting (2.4), (2.5) from (2.6) and (2.7), we obtain the error equations

$$\begin{cases} r^w D_{\bar{t}} e_{pq}^w - a\left(\delta_x^2 e_{pq}^w + \delta_y^2 e_{pq}^w\right) + \frac{a^2\tau^2}{r^w}\delta_x^2\delta_y^2 D_{\bar{t}} e_{pq}^w = (R_2)_{pq}^w, \ (p,q)\in\omega, 0 < w \leqslant n, \\ e_{pq}^0 = 0, \ (p,q)\in\omega, \\ e_{pq}^w = 0, \quad (p,q)\in\gamma, \ 0\le w\le n. \end{cases} \tag{3.10}$$

From Theorem 3.2 and (2.6), we obtain

$$\begin{aligned} \|\Delta_h e^w\|^2 &\leqslant e^{2w\tau}\left[4\left\|\Delta_h e^0\right\|^2 + 2P^2a^{-2}\left(\left\|(R_2)^1\right\|^2 \ + 2\max_{1\leqslant l\leqslant w}\left\|(R_2)^l\right\|^2 \tau\sum_{l=2}^{w}\left\|D_{\bar{t}}(R_2)^l\right\|^2\right)\right] \\ &\leqslant 2e^{2T}L_xL_yP^2a^{-2}(3+T)\,c_2^2\left(\tau + h_1^2 + h_2^2\right)^2, \ 1\leqslant w\leqslant n. \end{aligned}$$

Taking the square root of both sides, we obtain the inequality

$$\|\Delta_h e^w\| \leqslant e^T P a^{-1}\sqrt{2L_xL_y(3+T)}c_2\left(\tau + h_1^2 + h_2^2\right), \ 1\leqslant w\leqslant n.$$

From Lemma 2.5, we obtain

$$\begin{aligned} \|e^w\|_\infty &\leqslant \frac{e^T P a^{-1} c_2}{12}\sqrt{3\left(\sqrt{2}+1\right)L_xL_y}\times\sqrt{2L_1L_2(3+T)}\left(\tau + h_1^2 + h_2^2\right) \\ &= \frac{e^T P L_x L_y c_2}{12a}\sqrt{6\left(\sqrt{2}+1\right)(3+T)}\left(\tau + h_1^2 + h_2^2\right), \ 1\leqslant w\leqslant n. \end{aligned} \tag{3.11}$$

The proof of the convergence theorem is complete. □

4. Richardson extrapolation

Theorem 4.1. *Assume that the problems*

$$\begin{aligned} r(x,y,t)\frac{\partial v}{\partial t} - a\Delta v = g_1(x,y,t), (x,y,t)\in\Omega\times(0,T], \\ v(x,y,0) = \varphi(x,y), (x,y)\in\Omega, \\ v(x,y,t) = \psi(x,y,t), \ (x,y)\in\Gamma\times(0,T], \end{aligned} \tag{4.1}$$

$$\begin{aligned} r(x,y,t)\frac{\partial b}{\partial t} - a\Delta b = g_2(x,y,t), \ (x,y,t)\in\Omega\times(0,T], \\ b(x,y,0) = \varphi(x,y), \ (x,y)\in\Omega, \\ b(x,y,t) = \psi(x,y,t), \ (x,y)\in\Gamma\times(0,T], \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} r(x,y,t)\frac{\partial m}{\partial t} - a\Delta m = g_3(x,y,t), (x,y,t)\in\Omega\times(0,T], \\ m(x,y,0) = \varphi(x,y), \ (x,y)\in\Omega, \\ m(x,y,t) = \psi(x,y,t), \ (x,y)\in\Gamma\times(0,T], \end{aligned} \tag{4.3}$$

admit smooth solutions, where

$$g_1(x,y,t) = -r(x,y,t)\int_0^1 \frac{\partial^2 u}{\partial t^2}(x,y,t)(1-s)\,ds + \frac{a^2}{r_{pq}^w \tau}\delta_x^2\delta_y^2 D_{\bar{t}} U_{pq}^w,$$
$$g_2(x,y,t) = -\frac{a}{6}h_1^2\int_0^1 \frac{\partial^4 u}{\partial x^4}(x,y,t)(1-s)^3\,ds,$$
$$g_3(x,y,t) = -\frac{a}{6}h_2^2\int_0^1 \frac{\partial^4 u}{\partial y^4}(x,y,t)(1-s)^3\,ds.$$

Then

$$u_{pq}^w = u\left(x_p,y_q,t_w\right) + \tau v\left(x_p,y_q,t_w\right) + h_1^2 b\left(x_p,y_q,t_w\right) + h_2^2 m\left(x_p,y_q,t_w\right) + O\left(\tau^2+h_1^4+h_2^4\right),$$
$$(p,q)\in\gamma,\ 0\le w\le n.$$

$$\max_{(p,q)\in\gamma,\ 0\le w\le n}\left|u\left(x_p,y_q,t_w\right) - \left[\frac{4}{3}u_{2p,2q}^{4w}\left(\frac{h_1}{2},\frac{h_2}{2},\frac{\tau}{4}\right) - \frac{1}{3}u_{p,q}^w\right]\right| = O\left(\tau^2+h_1^4+h_2^4\right).$$

Proof. Since

$$(R_2)_{pq}^w = \tau g_1(x,y,t) + h_1^2 g_2(x,y,t) + h_2^2 g_3(x,y,t),$$

thus, the error equations (3.10) can be written as

$$\begin{aligned} r_{pq}^w D_{\bar{t}} e_{pq}^w - a\left(\delta_x^2 e_{pq}^w + \delta_y^2 e_{pq}^w\right) + \frac{a^2\tau^2}{r_{pq}^w}\delta_x^2\delta_y^2 D_{\bar{t}} e_{pq}^w \\ = \tau g_1(x,y,t) + h_1^2 g_2(x,y,t) + h_2^2 g_3(x,y,t), \quad (p,q)\in\omega, 0<w\leqslant n, \\ e_{pq}^0 = 0, \quad (p,q)\in\omega, \\ e_{pq}^w = 0, \quad (p,q)\in\gamma, 0\leqslant w\leqslant n. \end{aligned}$$

Denote

$$V_{pq}^w = v(x_p,y_q,t_w), \quad B_{pq}^w = b(x_p,y_q,t_w), \quad M_{pq}^w = m(x_p,y_q,t_w),$$
$$(p,q)\in\omega, 0\leqslant w\leqslant n.$$

Discretizing (4.1), we obtain

$$\begin{aligned} r_{pq}^w D_{\bar{t}} V_{pq}^w - a\left(\delta_x^2 V_{pq}^w + \delta_y^2 V_{pq}^w\right) + \frac{a^2\tau^2}{r_{pq}^w}\delta_x^2\delta_y^2 D_{\bar{t}} V_{pq}^w \\ = g_1(x_p,y_q,t_w) + O(\tau+h_1^2+h_2^2), \quad (p,q)\in\omega, 0<w\leqslant n, \\ V_{pq}^0 = 0, \quad (p,q)\in\omega, \\ V_{pq}^w = 0, \quad (p,q)\in\gamma, 0\leqslant w\leqslant n. \end{aligned} \tag{4.4}$$

Discretizing (4.2) and (4.3), we obtain

$$\begin{aligned} r_{pq}^w D_{\bar{t}} B_{pq}^w - a\left(\delta_x^2 B_{pq}^w + \delta_y^2 B_{pq}^w\right) + \frac{a^2\tau^2}{r_{pq}^w}\delta_x^2\delta_y^2 D_{\bar{t}} B_{pq}^w \\ = g_2(x_p,y_q,t_w) + O(\tau+h_1^2+h_2^2), \quad (p,q)\in\omega, 0<w\leqslant n, \\ B_{pq}^0 = 0, \quad (p,q)\in\omega, \\ B_{pq}^w = 0, \quad (p,q)\in\gamma, 0\leqslant w\leqslant n. \end{aligned} \tag{4.5}$$

$$\begin{aligned} & r_{pq}^w D_{\bar t} M_{pq}^w - a\left(\delta_x^2 M_{pq}^w + \delta_y^2 M_{pq}^w\right) + \frac{a^2\tau^2}{r_{pq}^w}\delta_x^2\delta_y^2 D_{\bar t} M_{pq}^w \\ & = g_3(x_p, y_q, t_w) + O(\tau + h_1^2 + h_2^2), \quad (p,q) \in \omega, 0 < w \leqslant n, \\ & M_{pq}^0 = 0, \quad (p,q) \in \omega, \\ & M_{pq}^w = 0, \quad (p,q) \in \gamma, 0 \leqslant w \leqslant n. \end{aligned} \tag{4.6}$$

Let

$$s_{pq}^w = e_{pq}^w + \tau V_{pq}^w + h_1^2 B_{pq}^w + h_2^2 M_{pq}^w, \ (p,q) \in \omega, 0 \le w \le n.$$

Multiply both sides of (4.4) by τ, multiply (4.5) by h_1, and multiply (4.6) by h_2. Then, add the results to obtain

$$\begin{aligned} & r_{pq}^w D_{\bar t} s_{pq}^w - a\left(\delta_x^2 s_{pq}^w + \delta_y^2 s_{pq}^w\right) + \frac{a^2\tau^2}{r_{pq}^w}\delta_x^2\delta_y^2 D_{\bar t} s_{pq}^w \\ & = O\left(\tau^2 + h_1^4 + h_2^4\right), (p,q) \in \omega, 0 < w \leqslant n, \\ & s_{pq}^0 = 0, \ (p,q) \in \omega, \\ & s_{pq}^w = 0, \ (p,q) \in \gamma, 0 \leqslant w \leqslant n. \end{aligned} \tag{4.7}$$

By Theorem 3.2, we obtain

$$\left\|s_{pq}^w\right\|_\infty = O\left(\tau^2 + h_1^4 + h_2^4\right), \ 1 \leqslant w \leqslant n,$$

which implies

$$\begin{aligned} & u\left(x_p, y_q, t_w\right) - u_{pq}^w(h_1, h_2, \tau) + \tau V_{pq}^w + h_1^2 B_{pq}^w + h_2^2 M_{pq}^w \\ & = O\left(\tau^2 + h_1^4 + h_2^4\right), \ (p,q) \in \omega, 0 < w \leqslant n. \end{aligned}$$

Rearranging terms, we get

$$\begin{aligned} u_{pq}^w(h_1, h_2, \tau) = {} & u\left(x_p, y_q, t_w\right) + \tau V_{pq}^w + h_1^2 B_{pq}^w + h_2^2 M_{pq}^w \\ & + O\left(\tau^2 + h_1^4 + h_2^4\right), (p,q) \in \omega, 0 < w \leqslant n. \end{aligned} \tag{4.8}$$

In fact, by multiplying both sides of (4.4) by $\tau/4$, multiplying (4.5) by $(\frac{h_1}{2})^2$, and multiplying (4.6) by $(\frac{h_2}{2})^2$, we obtain, in a similar way, that

$$\begin{aligned} u_{2p,2q}^{4w}\left(\frac{h_1}{2}, \frac{h_2}{2}, \frac{\tau}{4}\right) = {} & u\left(x_p, y_q, t_w\right) + \frac{\tau}{4}V_{pq}^w + \left(\frac{h_1}{2}\right)^2 B_{pq}^w + \left(\frac{h_2}{2}\right)^2 M_{pq}^w \\ & + O\left(\left(\frac{\tau}{4}\right)^2 + \left(\frac{h_1}{2}\right)^4 + \left(\frac{h_2}{2}\right)^4\right), \ (p,q) \in \omega, 0 < w \leqslant n. \end{aligned} \tag{4.9}$$

Multiply Eq (4.9) by $\frac{4}{3}$ and Eq (4.8) by $\frac{1}{3}$, and then subtract the resulting equations to derive

$$\frac{4}{3}u_{2p,2q}^{4w}\left(\frac{h_1}{2}, \frac{h_2}{2}, \frac{\tau}{4}\right) - \frac{1}{3}u_{pq}^w(h_1, h_2, \tau) = u\left(x_p, y_q, t_w\right) + O\left(\tau^2 + h_1^4 + h_2^4\right), (p,q) \in \omega, 0 < w \le n.$$

□

5. Numerical experiments

In this section, we will verify the theoretical results of the BE-ADI difference scheme through two numerical examples. The convergence rates are listed to test the accuracy of the difference scheme (2.7)–(2.8). Here, $E_\infty(h_1, h_2, \tau)$ represents the maximum error at the grid nodes for step sizes h_1, h_2, and τ. For practical implementation, set $h_1 = h_2 := h$. Before presenting the numerical examples, we denote the maximum errors and global convergence rates as follows:

$$E_\infty(h_1, h_2, \tau) = \max_{(p,q,w)\in\overline{\omega}} \left| u\left(x_p, y_q, t_w\right) - u_{pq}^{w} \right|,$$
$$Rate = \log_2\left(E_\infty(2h_1, 2h_2, 2\tau)/E_\infty(h_1, h_2, \tau)\right).$$

Example 1: The following problem is considered:

$$\begin{cases} r(x,y,t)\frac{\partial u}{\partial t} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \mathrm{e}^{\frac{1}{2}(x+y)-t}\cdot\left(-e^{(x+1)(y+1)t} - \frac{1}{2}\right),\ (x,y,t)\ \in (0,1)^2 \times (0,1], \\ u(x,\ 0,\ t) = \mathrm{e}^{\frac{1}{2}x-t},\ u(x,\ 1,\ t) = \mathrm{e}^{\frac{1}{2}(x+1)-t},\ x \in [0,1], t \in (0,1], \\ u(0,\ y,\ t) = \mathrm{e}^{\frac{1}{2}y-t},\ u(1,\ y,\ t) = \mathrm{e}^{\frac{1}{2}(1+y)-t},\ y \in [0,1], t \in (0,1]. \end{cases} \tag{5.1}$$

The exact solution is $u(x, y, t) = e^{\frac{1}{2}(x+y)-t}$, and the variable coefficient is $r(x, y, t) = e^{(x+1)(y+1)t}$.

Table 1. Spatial errors and convergence rates of Example 1.

BE-ADI method	$h_1 = h_2$	$E_\infty(h_1, h_2, \tau)$	$Rate$	$Time(s)$
	1/10	1.8000e-03	-	0.0166
	1/20	4.5164e-04	1.9685	0.2497
$\tau = h^2$	1/40	1.1314e-04	1.9970	3.0398
	1/80	2.8301e-05	1.9992	56.7685

Table 2. Temporal errors and convergence rates of Example 1.

BE-ADI method	$h_1 = h_2$	$E_\infty(h_1, h_2, \tau)$	$Rate$	$Time(s)$
	1/10	1.8100e-02	-	0.0035
	1/20	9.2000e-03	0.9817	0.0114
$\tau = h$	1/40	4.6000e-03	1.0097	0.0835
	1/80	2.3000e-03	1.0066	0.7494

Table 3. Spatial errors and convergence rates for the extrapolation method of Example 1.

Extrapolation method	$h_1 = h_2$	$E_\infty(h_1, h_2, \tau)$	*Rate*	*Time*(s)
	1/10	1.1684e-05	-	0.0150
	1/20	9.2944e-07	3.6529	0.2022
$\tau = h^2$	1/40	6.6534e-08	3.8042	3.2262
	1/80	4.3957e-09	3.9199	54.1589

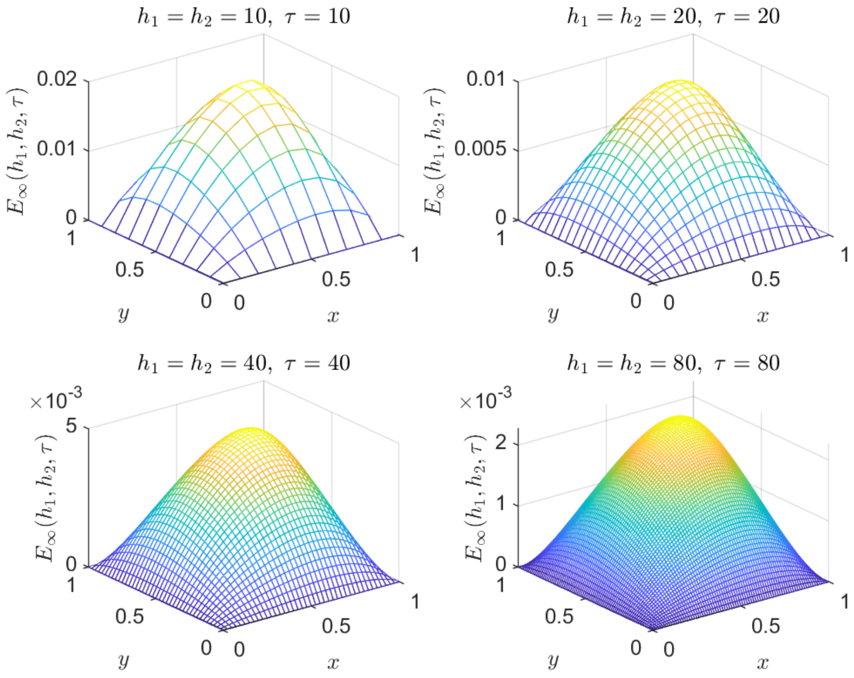

Figure 1. Error surface of Example 1 with different step sizes.

Example 2: Consider the following problem:

$$\begin{cases} r(x,y,t)\frac{\partial u}{\partial t} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = e^{\frac{1}{2}(x+y)-t} \cdot \left(-2.5\sin(\pi x)\sin(\pi y)\sin(\pi t) - 3\right), \\ \qquad (x,y,t) \in (0,1)^2 \times (0,1], \\ u(x,\ 0,\ t) = e^{\frac{1}{2}x-t},\ u(x,\ 1,\ t) = e^{\frac{1}{2}(x+1)-t},\ x \in [0,1], t \in (0,1], \\ u(0,\ y,\ t) = e^{\frac{1}{2}y-t},\ u(1,\ y,\ t) = e^{\frac{1}{2}(1+y)-t},\ y \in [0,1], t \in (0,1]. \end{cases} \tag{5.2}$$

The exact solution is $u(x,\ y,t) = e^{\frac{1}{2}(x+y)-t}$, and the variable coefficient is $r(x,\ y,t) = 2.5\sin(\pi x) \cdot \sin(\pi y)\sin(\pi t) + 2.5$.

In the two examples presented, the spatial discretization is chosen as $m = 10, 20, 40, 80$, while the temporal discretization varies depending on the difference scheme employed.

Tables 1 and 2 present the spatial and temporal maximum error estimates, their convergence rates, and computational times for the BE-ADI method described by (2.7)–(2.8) in Example 1. Table 3 shows the corresponding errors and convergence rates when the extrapolation method is applied. The error surface of the numerical solutions at $t = 1$ for various step sizes is shown in Figure 1. The results indicate that the proposed scheme maintains second-order accuracy in space and first-order accuracy in time. However, when the extrapolation method is applied, the accuracy improves significantly, enhancing the convergence rates from second to fourth order in space.

Table 4. Spatial errors and convergence rates of Example 2.

BE-ADI method	$h_1 = h_2$	$E_\infty(h_1, h_2, \tau)$	$Rate$	$Time(s)$
	1/10	1.3559e-03	-	0.0168
	1/20	3.4257e-04	1.9848	0.2671
$\tau = h^2$	1/40	8.5948e-05	1.9949	3.6716
	1/80	2.1507e-05	1.9987	65.2620

Table 5. Temporal errors and convergence rates of Example 2.

BE-ADI method	$h_1 = h_2$	$E_\infty(h_1, h_2, \tau)$	$Rate$	$Time(s)$
	1/10	1.2700e-02	-	0.0023
	1/20	6.6000e-03	0.9374	0.0197
$\tau = h$	1/40	3.4000e-03	0.9769	0.1080
	1/80	1.7000e-03	0.9897	0.8514

Table 6. Spatial errors and convergence rates for extrapolation method of Example 2.

Extrapolation method	$h_1 = h_2$	$E_\infty(h_1, h_2, \tau)$	$Rate$	$Time(s)$
	1/10	8.2519e-06	-	0.0231
	1/20	5.4054e-07	3.9322	0.2620
$\tau = h^2$	1/40	3.4063e-08	3.9881	3.7471
	1/80	2.1416e-09	3.9914	70.1477

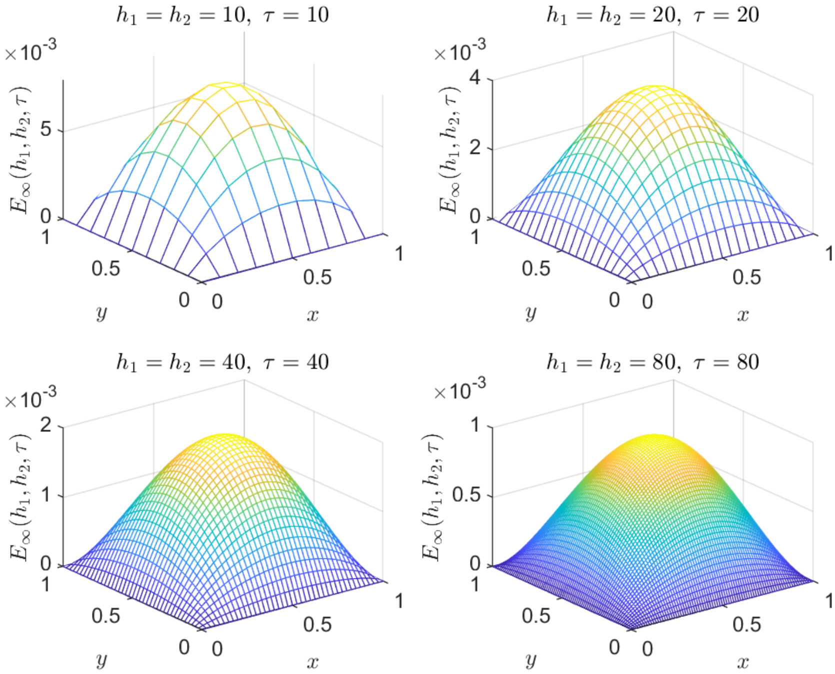

Figure 2. Error surface of Example 2 with different step sizes.

Tables 4 and 5 present the spatial and temporal maximum error estimates, along with their convergence rates and computational times, for the BE-ADI method described by (2.7)–(2.8) in Example 2. As observed, the maximum error decreases exponentially as the grid is refined, and the convergence rate indicates that the scheme achieves second-order accuracy. Table 6 presents the maximum errors and convergence rates of the numerical solutions obtained using the extrapolation method, which achieves fourth-order accuracy. Figure 2 illustrates the error surface at $t = 1$ for numerical solutions computed with various step sizes.

Figures 1 and 2 show that the error distribution is uniform, and the peak values clearly decrease as the step size decreases, indicating that this difference scheme achieves higher accuracy on finer grids.

6. Conclusions

This paper investigates the BE-ADI scheme and its fourth-order high-accuracy Richardson extrapolation scheme for 2D parabolic equations with variable coefficients. First, we presented the BE-ADI scheme with an accuracy of order $O\left(\tau + h_1^2 + h_2^2\right)$ and proved the uniqueness, convergence, and stability of this scheme using energy analysis methods. To further improve the numerical accuracy, we established a Richardson extrapolation scheme and theoretically demonstrated that it achieves an accuracy of order $O\left(\tau^2 + h_1^4 + h_2^4\right)$. Two illustrative examples were provided to validate the theoretical findings, supported by error surface plots. The results demonstrate that the proposed difference scheme and Richardson extrapolation method effectively address the challenges of low accuracy and computational complexity in the numerical solution of variable-coefficient parabolic equations.

Moreover, by reducing the three-dimensional equation into three sets of independent one-dimensional equations, the proposed method can be naturally extended to three-dimensional problems, offering an efficient and scalable approach for higher-dimensional simulations. In future work, we aim to explore the application of this approach to more complex nonlinear problems and variable-coefficient equations with non-smooth coefficients. Further improvements on adaptive mesh refinement and parallelization techniques will also be considered to enhance computational efficiency for large-scale 3D problems. By reducing the three-dimensional equation into three sets of independent one-dimensional equations, this method can also be extended to three-dimensional problems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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