Electronic Research Archive

http://www.aimspress.com/journal/ERA

ERA, 33(5): 3225–3245.
DOI: 10.3934/era.2025142
Received: 12 February 2025
Revised: 15 May 2025
Accepted: 16 May 2025
Published: 23 May 2025

Research article

Existence and uniqueness of solutions for (*p, q*)-difference equations with integral boundary conditions

Özlem Batıt Özen[1], Erbil Çetin[1], Öyküm Ülke[2], Aynur Şahin[3,*] and Fatma Serap Topal[1]

[1] Department of Mathematics, Ege University, Bornova 35100, Izmir, Türkiye
[2] National Defence University, Air NCO Vocational HE School, Gaziemir 35410, Izmir, Türkiye
[3] Department of Mathematics, Faculty of Sciences, Sakarya University, Sakarya 54050, Türkiye

*** Correspondence:** Email: ayuce@sakarya.edu.tr.

Abstract: In this paper, we explored the existence and uniqueness of solutions for a boundary value problem involving (p,q)-difference equations with integral conditions. By employing well-established fixed-point theorems, we established new and significant results in this area. To further illustrate the applicability of our findings, we presented three concrete examples that demonstrate the validity of the theoretical results.

Keywords: (p,q)-difference equations; boundary value problems; integral conditions; existence results; fixed point theorems

1. Introduction

Fractional calculus has gained considerable attention in recent years due to its ability to model complex phenomena with memory and hereditary properties. While classical calculus focuses on integer-order derivatives and integrals, fractional calculus extends these concepts to non-integer orders, offering a broader framework for analyzing real-world problems in various fields, such as physics, biology, and engineering (see, e.g., [1–4]).

Quantum calculus, often referred to as q-calculus, has emerged as an essential mathematical framework with numerous applications in various scientific fields, particularly in physics. It provides tools for studying phenomena in quantum mechanics, special functions, and other areas of theoretical and applied physics (see, e.g., [5–8]). Over time, advancements in quantum calculus have led to the development of postquantum calculus, a more generalized framework that extends the concepts of q-calculus. While quantum calculus is primarily concerned with q-numbers that rely on a single base q, postquantum calculus introduces p and q-numbers, incorporating two independent parameters p and q.

This generalization has significantly expanded the scope and applicability of the calculus, offering a more flexible and robust mathematical structure for modeling complex systems. The (p,q)-calculus has drawn considerable interest from both mathematicians and physicists, who have explored its potential in a variety of research domains. These studies have addressed topics ranging from generalized special functions to discrete dynamic systems, quantum theory, and number theory. For detailed discussions and applications, readers are referred to works such as [9–14].

One of the prominent areas where (p,q)-calculus has found significant application is the study of (p,q)-difference equations. These equations, which generalize classical difference equations, play a vital role in analyzing discrete dynamic systems and exploring their underlying mathematical properties. The flexibility introduced by the independent parameters p and q enables a deeper understanding of such systems, facilitating new theoretical insights and practical applications across diverse fields.

Boundary value problems (BVPs) involving fractional (p,q)-difference equations form a crucial branch of this research, as they address the existence of solutions satisfying both fractional difference equations and prescribed boundary conditions. Such problems frequently arise in discrete systems where boundary constraints or endpoint behaviors play a significant role (see, e.g., [15–18]).

In [15], Gençtürk obtained some existence results of solutions for the following boundary value problem

$$D_q^2 u(t) + f\big(t, u(t)\big) = 0, \quad 0 < t < 1$$

$$u(0) = \int_0^1 u(t) d_{p,q} t,\ u(1) = \int_0^1 t\, u(t)\, d_{p,q} t,$$

for q-difference equation with integral conditions.

In [16], Qin and Sun investigated the existence of positive solutions for the following boundary value problem of a class of fractional (p,q)-difference equation involving the Riemann–Liouville fractional derivative

$$D_{p,q}^{\alpha} u(t) + f\big(p^{\alpha} t, u(p^{\alpha} t)\big) = 0, \ \ 0 < t < 1$$

$$u(0) = u(1) = 0.$$

Motivated by these works, this paper proposes a new framework that combines (p,q)-difference equations involving the Caputo fractional derivative with nonlocal boundary conditions. We investigate the existence and uniqueness of solutions for a fractional (p,q)-difference boundary value problem given by the following:

$$\begin{cases} {}^{c}D_{p,q}^{\alpha}u(t) + f\big(t,u(t)\big) = 0, \quad 0 < t < 1 \\ u(0) = u'(0) = u''(0) = \cdots = u^{(n-3)}(0) = 0, \\ u^{(n-2)}(0) = \int_0^1 u(t) d_{p,q}t, \\ u(1) = \int_0^1 t\, u(t)\, d_{p,q}t, \end{cases} \tag{1.1}$$

where $n-1 < \alpha \leq n$ with $n \geq 3$, ${}^{c}D_{p,q}^{\alpha}$ denotes the Caputo-type fractional (p,q)-derivative operator, while $D_{p,q}$ denotes the first-order (p,q)-difference operator, and $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. By employing fixed-point theorems and related analytical tools, we establish sufficient conditions under which solutions exist and are unique. This work contributes to the theoretical foundation of fractional (p,q)-difference equations and provides a basis for further exploration of their applications in discrete mathematical modeling.

The rest of our paper is organized as follows: In this section, we present necessary definitions, properties, and lemmas. In Section 2, we will give some sufficient lemmas and theorems, which are used in the main results. In Section 3, some results on the existence and uniqueness of positive solutions are obtained. Also, some examples illustrating the obtained results are presented. Our results generalize many known results in the literature of BVPs.

Now, we will give some fundamental theorems, lemmas, and definitions of the (p,q)-calculus, which can be found in [11,19]. Let $[a,b] \subset \mathbb{R}$ be an interval with $a < b$, and $0 < q < p \leq 1$ be constants with $p + q \neq 1$,

$$[k]_{p,q} = \frac{p^k - q^k}{p - q}, \quad k \in \mathbb{N},$$

$$[k]_{p,q}! = \begin{cases} [k]_{p,q}[k-1]_{p,q} \dots [1]_{p,q} = \prod_{i=1}^{k} \frac{p^i - q^i}{p - q}, & k \in \mathbb{N}, \\ 1, & k = 0. \end{cases}$$

The (p,q)-analogue of the power function $(a-b)_{p,q}^{(n)}$ with $n \in \mathbb{N}_0 = \{0,1,2,\dots\}$ is given by

$$(a-b)_{p,q}^{(0)} := 1, \qquad (a-b)_{p,q}^{(n)} := \prod_{k=0}^{n-1} (ap^k - bq^k), \quad a,b \in \mathbb{R}.$$

The (p,q)-gamma and (p,q)-beta functions are defined by

$$\Gamma_{p,q}(x) := \begin{cases} \frac{(p-q)_{p,q}^{(x-1)}}{(p-q)^{x-1}} = \frac{\left(1 - \frac{q}{p}\right)_{p,q}^{(x-1)}}{\left(1 - \frac{q}{p}\right)^{x-1}}, & x \in \mathbb{R} \backslash \{0,-1,-2,\dots\}, \\ [x-1]_{p,q}!\ , & x \in \mathbb{N}, \end{cases}$$

and

$$B_{p,q}(x,y) := \int_0^1 t^{x-1}\,(1-qt)_{p,q}^{(y-1)} d_{p,q}t = p^{\frac{1}{2}(y-1)(2x+y-2)} \frac{\Gamma_{p,q}(x)\Gamma_{p,q}(y)}{\Gamma_{p,q}(x+y)},$$

respectively.

Definition 1.1. [11] *Let* $0 < q < p \leq 1$. *Then, the* (p,q)*-derivative of the function* f *is defined as*

$$D_{p,q}f(t) = \frac{f(pt) - f(qt)}{(p-q)t}, \quad t \neq 0,$$

and $D_{p,q}f(0) = \lim_{t \to 0} D_{p,q}f(t)$, *provided that* f *is differentiable at 0.*

Definition 1.2. [11] *Let* $0 < q < p \leq 1$, f *be an arbitrary function, and* t *be a real number. The* (p,q)*-integral of* f *is defined as*

$$\int_0^t f(s) d_{p,q}s = (p-q)t \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}} t\right) \tag{1.2}$$

provided that the series of the right-hand side in (1.2) converges.

Definition 1.3. [11] *For* $\alpha > 0$, $0 < q < p \leq 1$ *and* f *defined on* $I_{p,q}^T := \left\{\frac{q^k}{p^{k+1}} T : k \in \mathbb{N}_0\right\} \cup \{0\}$, *the fractional* (p,q)*-integral of* f *is defined by*

$$I_{p,q}^{\alpha} f(t) = \frac{1}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \int_0^t (t-qs)_{p,q}^{(\alpha-1)} f\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s.$$

Lemma 1.1. *The* (p,q)*-fractional integral operator* $I_{p,q}^{\alpha}$ *is monotone, that is, if* $f_1(t) \leq f_2(t)$ *for all* t, *then* $I_{p,q}^{\alpha} f_1(t) \leq I_{p,q}^{\alpha} f_2(t)$ *for all* t.

Definition 1.4. [19] *For* $\alpha > 0$, $0 < q < p \leq 1$, *and* $f: I_{p,q}^T \to \mathbb{R}$, *the fractional* (p,q)*-difference operator of Caputo type of order* α *is defined by*

$${}^{C}D_{p,q}^{\alpha} f(t) = I_{p,q}^{N-\alpha} D_{p,q}^{N} f(t) = \frac{1}{p^{\binom{N-\alpha}{2}} \Gamma_{p,q}(N-\alpha)} \int_0^t (t-qs)_{p,q}^{(N-\alpha-1)} D_{p,q}^{N} f\left(\frac{s}{p^{N-\alpha-1}}\right) d_{p,q}s,$$

and ${}^{C}D_{p,q}^{0} f(t) = f(t)$, *where* $N - 1 < \alpha < N$, $N \in \mathbb{N}$.

Theorem 1.1. [19] *Letting* $\alpha \in (N-1, N)$, $N \in \mathbb{N}, 0 < q < p \leq 1$, *and* $f: I_{p,q}^T \to \mathbb{R}$ *leads to*

$${}^{C}D_{p,q}^{\alpha} f(t) = \frac{(p-q)t^{N-\alpha}}{p^{\binom{N-\alpha}{2}} \Gamma_{p,q}(N-\alpha)} \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} \left(1 - \left(\frac{q}{p}\right)^{k+1}\right)_{p,q}^{(N-\alpha-1)} D_{p,q}^{N} f\left(\frac{q^k}{p^{k+N-\alpha}} t\right).$$

Theorem 1.2. [19] *Let* $\alpha \in (N-1, N), N \in \mathbb{N}, 0 < q < p \leq 1$, *and* $f: I_{p,q}^T \to \mathbb{R}$. *Then*

$$I_{p,q}^{\alpha}\,{}^{C}D_{p,q}^{\alpha}f(t) = f(t) - \sum_{k=0}^{N-1}\frac{t^k}{p^{\binom{k}{2}}[k]_{p,q}!}\left[D_{p,q}^{k}f(0)\right].$$

Lemma 1.2. (Leibniz Formula) [19] *Let* $f: I_{p,q}^{T} \times I_{p,q}^{T} \to \mathbb{R}$, *then*

$$D_{p,q}\left(\int_0^t f(t,s)d_{p,q}s\right) = \int_0^{qt} {}_tD_{p,q}f(t,s)\,d_{p,q}s + f(pt,t).$$

Also, *we obtain the following formula*

$$\frac{d}{dt}\left(\int_0^t f(t,s)d_{p,q}s\right) = \int_0^t \frac{\partial f(t,s)}{\partial t}d_{p,q}s + f(t,t).$$

2. Preliminaries

This section deals with relevant prerequisites that are essential for investigations into this study. We also establish some significant results that will be needed to prove our main results.

Lemma 2.1. *For any* $g \in C([0,1],\mathbb{R})$, *the boundary value problem*

$$\begin{cases} {}^{c}D_{p,q}^{\alpha}u(t) + g(t) = 0, \quad 0 < t < 1 \\ u(0) = u'(0) = u''(0) = \cdots = u^{(n-3)}(0) = 0, \\ u^{(n-2)}(0) = \int_0^1 u(t)d_{p,q}t, \\ u(1) = \int_0^1 t\,u(t)\,d_{p,q}t \end{cases} \tag{2.1}$$

is equivalent to the following integral equation

$$u(t) = \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^1 H(t,qs)\,g\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s, \tag{2.2}$$

where

$$\begin{aligned} H(t,qs) = {} & G(t,qs) \\ & + \frac{t^{n-2}-t^{n-1}}{\Delta\,(n-2)!}\left\{\left(1 - B_{p,q}(n+1,1)\right)\int_0^1 G(t,qs)d_{p,q}t \right. \\ & \left. + B_{p,q}(n,1)\int_0^1 t\,G(t,qs)d_{p,q}t\right\} \\ & + \frac{t^{n-1}}{\Delta}\left\{\left(1 - \frac{B_{p,q}(n-1,1) - B_{p,q}(n,1)}{(n-2)!}\right)\int_0^1 t\,G(t,qs)d_{p,q}t \right. \\ & \left. + \frac{B_{p,q}(n,1) - B_{p,q}(n+1,1)}{(n-2)!}\int_0^1 G(t,qs)d_{p,q}t\right\} \end{aligned}$$

such that

$$\Delta=\begin{vmatrix} 1-\dfrac{B_{p,q}(n-1,1)-B_{p,q}(n,1)}{(n-2)!} & -B_{p,q}(n,1) \\ \dfrac{B_{p,q}(n+1,1)-B_{p,q}(n,1)}{(n-2)!} & 1-B_{p,q}(n+1,1) \end{vmatrix}$$

and

$$G(t,s)=\begin{cases} -(t-s)_{p,q}^{(\alpha-1)}+t^{n-1}(1-s)_{p,q}^{(\alpha-1)}, & 0\le s\le t, \\ t^{n-1}(1-s)_{p,q}^{(\alpha-1)}, & t\le s\le 1. \end{cases}$$

Proof. Since

$$ {}^{c}D_{p,q}^{\alpha}u(t)=-g(t),$$
$$I_{p,q}^{\alpha}\,{}^{c}D_{p,q}^{\alpha}u(t)=-I_{p,q}^{\alpha}g(t)$$

then we get

$$u(t)\ =-I_{p,q}^{\alpha}g(t)+c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1}$$
$$=-\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^t(t-qs)_{p,q}^{(\alpha-1)}g\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s+c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1}.$$

Since the boundary condition $u(0)=0,$ we have $c_0=0.$ Using Leibniz formula, we have

$$u'(t)=-\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^t(\alpha-1)(t-qs)_{p,q}^{(\alpha-2)}g\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s$$
$$-\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}(t-qt)_{p,q}^{(\alpha-1)}g\left(\frac{t}{p^{\alpha-1}}\right)+c_1+2c_2t+\cdots+(n-1)c_{n-1}t^{n-2},$$

and since the boundary condition $u'(0)=0,$ we get $c_1=0.$
Similarly, since

$$u''(t)=-\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^t(\alpha-1)(\alpha-2)(t-qs)_{p,q}^{(\alpha-3)}g\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s$$
$$-\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}(\alpha-1)(t-qt)_{p,q}^{(\alpha-2)}g\left(\frac{t}{p^{\alpha-1}}\right)$$
$$-\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\left[(1-q)_{p,q}^{(\alpha-1)}(\alpha-1)t^{(\alpha-2)}g\left(\frac{t}{p^{\alpha-1}}\right)+(1-q)_{p,q}^{(\alpha-1)}t^{(\alpha-1)}\frac{1}{p^{\alpha-1}}g'\left(\frac{t}{p^{\alpha-1}}\right)\right]$$
$$+2c_2+\cdots+(n-1)(n-2)c_{n-1}t^{n-3},$$

and the boundary condition $u''(0) = 0$, we obtain $u''(0) = 2c_2 = 0$. Then, we have $c_2 = 0$.

If we continue like this, by using other boundary conditions $u'''(0) = \cdots = u^{(n-3)}(0) = 0$, we get $c_3 = \cdots = c_{n-3} = 0$. By using the other boundary condition, we get

$$u^{(n-2)}(0) = (n-2)!\, c_{n-2} = \int_0^1 u(t) d_{p,q} t,$$

and so

$$c_{n-2} = \frac{1}{(n-2)!} \int_0^1 u(t) d_{p,q} t, \quad n > 2.$$

Substituting c_{n-2}, we get

$$u(t) = -\frac{1}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \int_0^t (t-qs)_{p,q}^{(\alpha-1)} g\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s + \left(\frac{1}{(n-2)!} \int_0^1 u(s) d_{p,q}s\right) t^{n-2} + c_{n-1} t^{n-1}.$$

Since

$$\begin{aligned} u(1) &= -\frac{1}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \int_0^1 (1-qs)_{p,q}^{(\alpha-1)} g\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s + \left(\frac{1}{(n-2)!} \int_0^1 u(s) d_{p,q}s\right) + c_{n-1} \\ &= \int_0^1 s\, u(s)\, d_{p,q}s \end{aligned}$$

then, from the last boundary condition, we have

$$c_{n-1} = \int_0^1 s\, u(s)\, d_{p,q}s + \frac{1}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \int_0^1 (1-qs)_{p,q}^{(\alpha-1)} g\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s - \frac{1}{(n-2)!} \int_0^1 u(s) d_{p,q}s.$$

Consequently

$$\begin{aligned} u(t) = &-\frac{1}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \int_0^t (t-qs)_{p,q}^{(\alpha-1)} g\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s + \left(\frac{1}{(n-2)!} \int_0^1 u(s) d_{p,q}s\right) t^{n-2} \\ &+ \left(\int_0^1 s\, u(s)\, d_{p,q}s + \frac{1}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \int_0^1 (1-qs)_{p,q}^{(\alpha-1)} g\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s - \frac{1}{(n-2)!} \int_0^1 u(s) d_{p,q}s\right) t^{n-1} \\ = &-\frac{1}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \int_0^t (t-qs)_{p,q}^{(\alpha-1)} g\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s + \frac{t^{n-1}}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \int_0^1 (1-qs)_{p,q}^{(\alpha-1)} g\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \\ &+ \left[\frac{t^{n-2}}{(n-2)!} - \frac{t^{n-1}}{(n-2)!}\right] \int_0^1 u(s) d_{p,q}s + t^{n-1} \int_0^1 s\, u(s) d_{p,q}s. \end{aligned}$$

Whence

$$u(t)=\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^1 G(t,qs)\,g\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s+\left[\frac{t^{n-2}-t^{n-1}}{(n-2)!}\right]\int_0^1 u(s)d_{p,q}s$$
$$+\,t^{n-1}\int_0^1 s\,u(s)d_{p,q}s, \tag{2.3}$$

where

$$G(t,s)=\begin{cases}-(t-s)_{p,q}^{(\alpha-1)}+t^{n-1}(1-s)_{p,q}^{(\alpha-1)}, & 0\le s\le t,\\ t^{n-1}(1-s)_{p,q}^{(\alpha-1)}, & t\le s\le 1.\end{cases}$$

Integrating (2.3) over $[0,1]$, we obtain

$$\int_0^1 u(t)\,d_{p,q}t=\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^1\int_0^1 G(t,qs)\,g\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s\,d_{p,q}t+\left(\int_0^1 u(s)d_{p,q}s\right)\int_0^1\left(\frac{t^{n-2}-t^{n-1}}{(n-2)!}\right)d_{p,q}t$$
$$+\left(\int_0^1 s\,u(s)d_{p,q}s\right)\int_0^1 t^{n-1}d_{p,q}t$$
$$=\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^1\int_0^1 G(t,qs)\,g\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s\,d_{p,q}t+\left[\frac{B_{p,q}(n-1,1)-B_{p,q}(n,1)}{(n-2)!}\right]\left(\int_0^1 u(s)d_{p,q}s\right)$$
$$+\,B_{p,q}(n,1)\int_0^1 s\,u(s)d_{p,q}s.$$

Thus, we have

$$\left[1-\frac{B_{p,q}(n-1,1)-B_{p,q}(n,1)}{(n-2)!}\right]\int_0^1 u(s)d_{p,q}s-B_{p,q}(n,1)\int_0^1 s\,u(s)d_{p,q}s$$
$$=\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^1\int_0^1 G(t,qs)\,g\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s\,d_{p,q}t. \tag{2.4}$$

Multiplying (2.3) by t and integrating over $[0,1]$, we get

$$\int_0^1 t\,u(t)d_{p,q}t=\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^1\left(t\int_0^t G(t,qs)\,g\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s\right)d_{p,q}t+\int_0^1\left[\left(\frac{t^{n-1}-t^n}{(n-2)!}\right)\int_0^1 u(s)d_{p,q}s\right]d_{p,q}t$$
$$+\int_0^1\left(t^n\int_0^1 s\,u(s)d_{p,q}s\right)d_{p,q}t$$
$$=\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^1\int_0^1 t\,G(t,qs)\,g\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s\,d_{p,q}t+\left[\frac{B_{p,q}(n,1)-B_{p,q}(n+1,1)}{(n-2)!}\right]\int_0^1 u(s)d_{p,q}s$$
$$+\,B_{p,q}(n+1,1)\int_0^1 s\,u(s)d_{p,q}s.$$

Hence

$$\left[\frac{B_{p,q}(n+1,1)-B_{p,q}(n,1)}{(n-2)!}\right]\int_0^1 u(s)d_{p,q}s+\left[1-B_{p,q}(n+1,1)\right]\int_0^1 s\,u(s)d_{p,q}s$$
$$=\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^1\int_0^1 t\,G(t,qs)\,g\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s\,d_{p,q}t. \tag{2.5}$$

Applying the Cramer rule to (2.4) and (2.5), saying

$$\Delta=\begin{vmatrix}1-\frac{B_{p,q}(n-1,1)-B_{p,q}(n,1)}{(n-2)!} & -B_{p,q}(n,1)\\ \frac{B_{p,q}(n+1,1)-B_{p,q}(n,1)}{(n-2)!} & 1-B_{p,q}(n+1,1)\end{vmatrix},$$

and assuming $\Delta\neq 0$, then we have

$$\int_0^1 u(s)d_{p,q}s=\frac{1}{\Delta}\begin{vmatrix}\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^1\int_0^1 G(t,qs)\,g\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s\,d_{p,q}t & -B_{p,q}(n,1)\\ \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^1\int_0^1 t\,G(t,qs)\,g\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s\,d_{p,q}t & 1-B_{p,q}(n+1,1)\end{vmatrix}$$
$$=\frac{1}{\Delta p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\left\{\int_0^1\int_0^1 G(t,qs)\,g\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s\,d_{p,q}t-B_{p,q}(n+1,1)\int_0^1\int_0^1 G(t,qs)\,g\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s\,d_{p,q}t\right.$$
$$\left.+B_{p,q}(n,1)\int_0^1\int_0^1 t\,G(t,qs)\,g\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s\,d_{p,q}t\right\}$$

and

$$\int_0^1 s\,u(s)d_{p,q}s=\frac{1}{\Delta}\begin{vmatrix}1-\frac{B_{p,q}(n-1,1)-B_{p,q}(n,1)}{(n-2)!} & \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^1\int_0^1 G(t,qs)\,g\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s\,d_{p,q}t\\ \frac{B_{p,q}(n+1,1)-B_{p,q}(n,1)}{(n-2)!} & \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^1\int_0^1 t\,G(t,qs)\,g\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s\,d_{p,q}t\end{vmatrix}$$
$$=\frac{1}{\Delta p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\left\{\int_0^1\int_0^1 t\,G(t,qs)\,g\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s\,d_{p,q}t-\frac{B_{p,q}(n-1,1)-B_{p,q}(n,1)}{(n-2)!}\int_0^1\int_0^1 t\,G(t,qs)\,g\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s\,d_{p,q}t\right.$$
$$\left.-\frac{B_{p,q}(n+1,1)-B_{p,q}(n,1)}{(n-2)!}\int_0^1\int_0^1 G(t,qs)\,g\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s\,d_{p,q}t\right\}.$$

As a result,

$$u(t)=\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^1 G(t,qs)\,g\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s$$

$$+\left[\frac{t^{n-2}-t^{n-1}}{(n-2)!}\right]\frac{1}{\Delta p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\left\{\left(1-B_{p,q}(n+1,1)\right)\int_0^1\int_0^1 G(t,qs)\,g\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s\,d_{p,q}t\right.$$

$$\left.+B_{p,q}(n,1)\int_0^1\int_0^1 t\,G(t,qs)\,g\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s\,d_{p,q}t\right\}$$

$$+\frac{t^{n-1}}{\Delta p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\left\{\left(1+\frac{B_{p,q}(n,1)-B_{p,q}(n-1,1)}{(n-2)!}\right)\int_0^1\int_0^1 t\,G(t,qs)\,g\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s\,d_{p,q}t\right.$$

$$\left.+\frac{B_{p,q}(n,1)-B_{p,q}(n+1,1)}{(n-2)!}\int_0^1\int_0^1 G(t,qs)\,g\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s\,d_{p,q}t\right\}$$

$$=\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^1 H(t,qs)\,g\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s,$$

where

$$H(t,qs)=G(t,qs)+\frac{t^{n-2}-t^{n-1}}{\Delta\,(n-2)!}\left\{\left(1-B_{p,q}(n+1,1)\right)\int_0^1 G(t,qs)d_{p,q}t+B_{p,q}(n,1)\int_0^1 t\,G(t,qs)d_{p,q}t\right\}$$

$$+\frac{t^{n-1}}{\Delta}\left\{\left(1-\frac{B_{p,q}(n-1,1)-B_{p,q}(n,1)}{(n-2)!}\right)\int_0^1 t\,G(t,qs)d_{p,q}t+\frac{B_{p,q}(n,1)-B_{p,q}(n+1,1)}{(n-2)!}\int_0^1 G(t,qs)d_{p,q}t\right\}.$$

□

Lemma 2.2. *The Green functions* $G(t,s)$ *and* $H(t,s)$ *satisfy the following inequalities:*
1) $|G(t,s)|\le 2(1-s)^{(\alpha-1)},\ for\ all\ t,s\in[0,1]$
2) $|H(t,s)|\le 2A,\ for\ all\ t,s\in[0,1]$
where

$$A=\left(1+\frac{2}{|\Delta|\,(n-2)!}\left\{\left|1-B_{p,q}(n+1,1)\right|B_{p,q}(1,1)+\left|B_{p,q}(n,1)\right|B_{p,q}(2,1)\right\}+\frac{1}{|\Delta|}\left\{\left|1-\frac{B_{p,q}(n-1,1)-B_{p,q}(n,1)}{(n-2)!}\right|B_{p,q}(2,1)+\left|\frac{B_{p,q}(n,1)-B_{p,q}(n+1,1)}{(n-2)!}\right|\right\}\right).\tag{2.6}$$

Proof. From $G(t,s)$ given in Lemma 2.1, we get

$$|G(t,s)|\le(t-s)^{(\alpha-1)}+t^{n-1}(1-s)^{(\alpha-1)}\le 2(1-s)^{(\alpha-1)},\quad \forall t,s\in[0,1].$$

Also, we obtain

$$|H(t,s)|\le |G(t,s)|+\frac{|t^{n-2}-t^{n-1}|}{|\Delta|\,(n-2)!}\left\{\left|1-B_{p,q}(n+1,1)\right|\int_0^1|G(t,s)|\,d_{p,q}t+\left|B_{p,q}(n,1)\right|\int_0^1 t\,|G(t,s)|d_{p,q}t\right\}$$

$$+\frac{t^{n-1}}{|\Delta|}\left\{\left|1-\frac{B_{p,q}(n-1,1)-B_{p,q}(n,1)}{(n-2)!}\right|\int_0^1 t\,|G(t,s)|d_{p,q}t\right.$$

$$+\left|\frac{B_{p,q}(n,1)-B_{p,q}(n+1,1)}{(n-2)!}\right|\int_0^1|G(t,s)|d_{p,q}t\Bigg\}$$

$$\le 2(1-s)^{(\alpha-1)}$$

$$+\frac{2}{|\Delta|\,(n-2)!}\left\{\left|1-B_{p,q}(n+1,1)\right|2\int_0^1(1-s)^{\alpha-1}\,d_{p,q}t+\left|B_{p,q}(n,1)\right|2(1-s)^{\alpha-1}\int_0^1 t\,d_{p,q}t\right\}$$

$$+\frac{1}{|\Delta|}\left\{\left|1-\frac{B_{p,q}(n-1,1)-B_{p,q}(n,1)}{(n-2)!}\right|2(1-s)^{\alpha-1}\int_0^1 t\,d_{p,q}t\right.$$

$$\left.+\left|\frac{B_{p,q}(n,1)-B_{p,q}(n+1,1)}{(n-2)!}\right|2(1-s)^{\alpha-1}\int_0^1 d_{p,q}t\right\}$$

$$\le 2(1-s)^{(\alpha-1)}\left(1+\frac{2}{|\Delta|\,(n-2)!}\left\{\left|1-B_{p,q}(n+1,1)\right|.B_{p,q}(1,1)+\left|B_{p,q}(n,1)\right|B_{p,q}(2,1)\right\}\right.$$

$$\left.+\frac{1}{|\Delta|}\left\{\left|1-\frac{B_{p,q}(n-1,1)-B_{p,q}(n,1)}{(n-2)!}\right|B_{p,q}(2,1)+\left|\frac{B_{p,q}(n,1)-B_{p,q}(n+1,1)}{(n-2)!}\right|B_{p,q}(1,1)\right\}\right)$$

$$=2(1-s)^{(\alpha-1)}A,$$

such that

$$A=\left(1+\frac{2}{|\Delta|\,(n-2)!}\left\{\left|1-B_{p,q}(n+1,1)\right|.B_{p,q}(1,1)+\left|B_{p,q}(n,1)\right|B_{p,q}(2,1)\right\}\right)$$
$$+\frac{1}{|\Delta|}\left\{\left|1-\frac{B_{p,q}(n-1,1)-B_{p,q}(n,1)}{(n-2)!}\right|B_{p,q}(2,1)+\left|\frac{B_{p,q}(n,1)-B_{p,q}(n+1,1)}{(n-2)!}\right|\right\}.$$

So, we get $|H(t,s)|\le 2A(1-s)^{(\alpha-1)}\le 2A$, for all $t,s\in[0,1]$.

□

Next, we give some well-known fixed-point theorems that will be the main tools for our results. For more details on fixed point theory, we refer the readers to [20–23].

Theorem 2.1. (Krasnoselskii's fixed-point theorem) [24] *Let $\mathcal{K}$ be a bounded, closed, convex, and nonempty subset of a Banach space $\mathbb{X}$. Let T_1 and T_2 be two operators such that*

a) $T_1u+T_2v\in\mathcal{K}$ whenever $u,v\in\mathcal{K}$.

b) T_1 is compact and continuous.

c) T_2 is a contraction mapping.

Then, there exists a $z\in\mathcal{K}$ such that $z=T_1z+T_2z$.

Theorem 2.2. (Banach fixed-point theorem) [25] *Let $\mathbb{X}$ be a Banach space, and let $T:\mathbb{X}\to\mathbb{X}$ be a contraction operator, i.e., there exists a constant $\lambda\in[0,1)$ such that $\|Tu-Tv\|\le\lambda\,\|u-v\|$ for any $u,v\in\mathbb{X}$. Then there exists a unique $z\in\mathbb{X}$ such that $Tz=z$.*

Definition 2.1. [26] *Let $\mathbb{X}$ be a Banach space, and let $T:\mathbb{X}\to\mathbb{X}$ be a mapping. T is called a nonlinear contraction if there exists a continuous non-decreasing function $\psi:\mathbb{R}^+\to\mathbb{R}^+$, which $\psi(0)=0$ and $\psi(x)<x$ for all $x>0$ has the following property:*

$$\|Tu-Tv\|\le\psi(\|u-v\|),\ \ \forall u,v\in\mathbb{X}.$$

Theorem 2.3. (Boyd and Wong fixed-point theorem) [27] *Assume that $\mathbb{X}$ is a Banach space, and let $T:\mathbb{X}\to\mathbb{X}$ be a nonlinear contraction. Then, T has a unique fixed point in $\mathbb{X}$.*

We will denote by $\mathbb{X}=C([0,1],\mathbb{R})$ the Banach space of all continuous functions from $[0,1]$ to $\mathbb{R}$ endowed with the norm defined by $\|u\|=\sup\{|u(t)|:\ t\in[0,1]\}$.

We regard $T:C([0,1],\mathbb{R})\to C([0,1],\mathbb{R})$ as being defined by an operator as

$$Tu(t) = \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^1 H(t,qs)\, f\left(\frac{s}{p^{\alpha-1}}, u\left(\frac{s}{p^{\alpha-1}}\right)\right) d_{p,q}s. \tag{2.7}$$

3. Existence and uniqueness of solutions

In this section, we will deal with our main results. The first result is based on Krasnoselskii's fixed-point theorem.

Theorem 3.1 *Let* $f: [0,1] \times \mathbb{R} \to \mathbb{R}$ *be a continuous function satisfying the following assumptions:*

(H_1) $|f(t,u) - f(t,v)| \le L|u-v|$, $\forall t \in [0,1]$ *and* $u, v \in \mathbb{R}$,

(H_2) $|f(t,u)| \le \mu(t)$, $\forall (t,u) \in [0,1] \times \mathbb{R}$ *and* $\mu \in L^1([0,1], \mathbb{R}^+)$.

If $\mathcal{M}L < 1$, *where*

$$\mathcal{M} := \frac{2\, B_{p,q}(1,\alpha)}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left\{ 1 + \frac{2}{|\Delta|\,(n-2)!}\left[\left|1 - B_{p,q}(n+1,1)\right| + \left|B_{p,q}(n,1)\right|\right] + \frac{1}{|\Delta|}\left(\left|1 - \frac{B_{p,q}(n-1,1) - B_{p,q}(n,1)}{(n-2)!}\right| + \left|\frac{B_{p,q}(n,1) - B_{p,q}(n+1,1)}{(n-2)!}\right|\right)\right\}$$

then the problem (1.1) has at least one solution on $[0,1]$.

Proof. Defining $\max_{t\in[0,1]} |\mu(t)| := \mu^*$ and fixing a constant $\tilde{R} \ge \mu^* \mathcal{M}$, consider $B_{\tilde{R}} = \{u \in C([0,1], \mathbb{R}) : \|u\| \le \tilde{R}\}$. Let us define the operators T_1 and T_2 on the ball $B_{\tilde{R}}$ as follows

$$T_1 u(t) := \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^1 G(t,qs)\, f\left(\frac{s}{p^{\alpha-1}}, u\left(\frac{s}{p^{\alpha-1}}\right)\right) d_{p,q}s,$$

$$\begin{aligned} T_2 u(t) := {} & \left(\frac{t^{n-2} - t^{n-1}}{(n-2)!}\right) \frac{1}{\Delta\, p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left\{ \left(1 - B_{p,q}(n+1,1)\right) \int_0^1 \int_0^1 G(t,qs)\, f\left(\frac{s}{p^{\alpha-1}}, u\left(\frac{s}{p^{\alpha-1}}\right)\right) d_{p,q}t\, d_{p,q}s \right. \\ & \left. + B_{p,q}(n,1) \int_0^1 \int_0^1 t\, G(t,qs)\, f\left(\frac{s}{p^{\alpha-1}}, u\left(\frac{s}{p^{\alpha-1}}\right)\right) d_{p,q}t\, d_{p,q}s \right\} \\ & + \frac{t^{n-1}}{\Delta\, p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left\{ \left(1 - \frac{B_{p,q}(n-1,1) - B_{p,q}(n,1)}{(n-2)!}\right) \int_0^1 \int_0^1 t\, G(t,qs)\, f\left(\frac{s}{p^{\alpha-1}}, u\left(\frac{s}{p^{\alpha-1}}\right)\right) d_{p,q}t\, d_{p,q}s \right. \\ & \left. + \frac{B_{p,q}(n,1) - B_{p,q}(n+1,1)}{(n-2)!} \int_0^1 \int_0^1 G(t,qs)\, f\left(\frac{s}{p^{\alpha-1}}, u\left(\frac{s}{p^{\alpha-1}}\right)\right) d_{p,q}t\, d_{p,q}s \right\}. \end{aligned}$$

For $u, v \in B_{\tilde{R}}$, we have

$$|T_1 u(t) + T_2 v(t)| \le \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^1 |G(t,qs)|\, \mu\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s$$

$$+\left(\frac{2}{(n-2)!}\right)\frac{1}{|\Delta|\,p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\left\{\left|1-B_{p,q}(n+1,1)\right|\int_0^1\int_0^1|G(t,qs)|\,\mu\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}t\,d_{p,q}s\right.$$

$$\left.+\left|B_{p,q}(n,1)\right|\int_0^1\int_0^1|G(t,qs)|\,\mu\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}t\,d_{p,q}s\right\}$$

$$+\frac{1}{|\Delta|\,p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\left\{\left|1-\frac{B_{p,q}(n-1,1)-B_{p,q}(n,1)}{(n-2)!}\right|\int_0^1\int_0^1|G(t,qs)|\,\mu\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}t\,d_{p,q}s\right.$$

$$\left.+\left|\frac{B_{p,q}(n,1)-B_{p,q}(n+1,1)}{(n-2)!}\right|\int_0^1\int_0^1|G(t,qs)|\,\mu\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}t\,d_{p,q}s\right\}$$

$$\leq\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\,2\,B_{p,q}(1,\alpha)\mu^{*}+\frac{2}{|\Delta|(n-2)!\,p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\left\{\left|1-B_{p,q}(n+1,1)\right|+\left|B_{p,q}(n,1)\right|\right\}2\,B_{p,q}(1,\alpha)\|\mu\|$$

$$+\frac{2\,B_{p,q}(1,\alpha)\mu^{*}}{|\Delta|\,p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\left\{\left|1-\frac{B_{p,q}(n-1,1)-B_{p,q}(n,1)}{(n-2)!}\right|+\left|\frac{B_{p,q}(n,1)-B_{p,q}(n+1,1)}{(n-2)!}\right|\right\}$$

$$=\mu^{*}\frac{2\,B_{p,q}(1,\alpha)}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\left\{1+\frac{2}{|\Delta|(n-2)!}\left[\left|1-B_{p,q}(n+1,1)\right|+\left|B_{p,q}(n,1)\right|\right]\right.$$

$$\left.+\frac{1}{|\Delta|}\left(\left|1-\frac{B_{p,q}(n-1,1)-B_{p,q}(n,1)}{(n-2)!}\right|+\left|\frac{B_{p,q}(n,1)-B_{p,q}(n+1,1)}{(n-2)!}\right|\right)\right\}$$

$$=\mu^{*}\mathcal{M}\leq\tilde{R}.$$

This implies that $T_1u+T_2v\in B_{\tilde{R}}$.

$$|T_2u_1(t)-T_2u_2(t)|$$

$$=\left|\left(\frac{t^{n-2}-t^{n-1}}{(n-2)!}\right)\frac{1}{\Delta\,p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\left\{\left(1-B_{p,q}(n+1,1)\right)\int_0^1\int_0^1G(t,qs)\,f\left(\frac{s}{p^{\alpha-1}},u_1\left(\frac{s}{p^{\alpha-1}}\right)\right)d_{p,q}t\,d_{p,q}s\right.\right.$$

$$\left.+B_{p,q}(n,1)\int_0^1\int_0^1t\,G(t,qs)\,f\left(\frac{s}{p^{\alpha-1}},u_1\left(\frac{s}{p^{\alpha-1}}\right)\right)d_{p,q}t\,d_{p,q}s\right\}$$

$$+\frac{t^{n-1}}{\Delta\,p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\left\{\left(1-\frac{B_{p,q}(n-1,1)-B_{p,q}(n,1)}{(n-2)!}\right)\int_0^1\int_0^1t\,G(t,qs)\,f\left(\frac{s}{p^{\alpha-1}},u_1\left(\frac{s}{p^{\alpha-1}}\right)\right)d_{p,q}t\,d_{p,q}s\right.$$

$$\left.+\frac{B_{p,q}(n,1)-B_{p,q}(n+1,1)}{(n-2)!}\int_0^1\int_0^1G(t,qs)\,f\left(\frac{s}{p^{\alpha-1}},u_1\left(\frac{s}{p^{\alpha-1}}\right)\right)d_{p,q}t\,d_{p,q}s\right\}$$

$$-\left(\frac{t^{n-2}-t^{n-1}}{(n-2)!}\right)\frac{1}{\Delta\,p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\left\{\left(1-B_{p,q}(n+1,1)\right)\int_0^1\int_0^1G(t,qs)\,f\left(\frac{s}{p^{\alpha-1}},u_2\left(\frac{s}{p^{\alpha-1}}\right)\right)d_{p,q}t\,d_{p,q}s\right.$$

$$\left.+B_{p,q}(n,1)\int_0^1\int_0^1t\,G(t,qs)\,f\left(\frac{s}{p^{\alpha-1}},u_2\left(\frac{s}{p^{\alpha-1}}\right)\right)d_{p,q}t\,d_{p,q}s\right\}$$

$$+\frac{t^{n-1}}{\Delta\, p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\Bigg\{\left(1-\frac{B_{p,q}(n-1,1)-B_{p,q}(n,1)}{(n-2)!}\right)\int_0^1\int_0^1 t\,G(t,qs)\,f\left(\frac{s}{p^{\alpha-1}},u_2\left(\frac{s}{p^{\alpha-1}}\right)\right)d_{p,q}t\,d_{p,q}s$$

$$+\frac{B_{p,q}(n,1)-B_{p,q}(n+1,1)}{(n-2)!}\int_0^1\int_0^1 G(t,qs)\,f\left(\frac{s}{p^{\alpha-1}},u_2\left(\frac{s}{p^{\alpha-1}}\right)\right)d_{p,q}t\,d_{p,q}s\Bigg\}\Bigg|$$

$$=\Bigg|\left(\frac{t^{n-2}-t^{n-1}}{(n-2)!}\right)\frac{1}{\Delta\, p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\Bigg\{\left(1-B_{p,q}(n+1,1)\right)\int_0^1\int_0^1 G(t,qs)\left[f\left(\frac{s}{p^{\alpha-1}},u_1\left(\frac{s}{p^{\alpha-1}}\right)\right)-f\left(\frac{s}{p^{\alpha-1}},u_2\left(\frac{s}{p^{\alpha-1}}\right)\right)\right]d_{p,q}t\,d_{p,q}s$$

$$+B_{p,q}(n,1)\int_0^1\int_0^1 t\,G(t,qs)\left[f\left(\frac{s}{p^{\alpha-1}},u_1\left(\frac{s}{p^{\alpha-1}}\right)\right)-f\left(\frac{s}{p^{\alpha-1}},u_2\left(\frac{s}{p^{\alpha-1}}\right)\right)\right]d_{p,q}t\,d_{p,q}s\Bigg\}$$

$$+\frac{t^{n-1}}{\Delta\, p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\Bigg\{\left(1-\frac{B_{p,q}(n-1,1)-B_{p,q}(n,1)}{(n-2)!}\right)\int_0^1\int_0^1 t\,G(t,qs)\left[f\left(\frac{s}{p^{\alpha-1}},u_1\left(\frac{s}{p^{\alpha-1}}\right)\right)-f\left(\frac{s}{p^{\alpha-1}},u_2\left(\frac{s}{p^{\alpha-1}}\right)\right)\right]d_{p,q}t\,d_{p,q}s$$

$$+\frac{B_{p,q}(n,1)-B_{p,q}(n+1,1)}{(n-2)!}\int_0^1\int_0^1 G(t,qs)\left[f\left(\frac{s}{p^{\alpha-1}},u_1\left(\frac{s}{p^{\alpha-1}}\right)\right)-f\left(\frac{s}{p^{\alpha-1}},u_2\left(\frac{s}{p^{\alpha-1}}\right)\right)\right]d_{p,q}t\,d_{p,q}s\Bigg\}\Bigg|$$

$$\leq\Bigg\{\frac{2}{|\Delta|\,p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)(n-2)!}\left[\left|1-B_{p,q}(n+1,1)\right|+B_{p,q}(n,1)\right]+\frac{1}{|\Delta|\,p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\left(\left|1-\frac{B_{p,q}(n-1,1)-B_{p,q}(n,1)}{(n-2)!}\right|\right.$$

$$\left.+\left|\frac{B_{p,q}(n,1)-B_{p,q}(n+1,1)}{(n-2)!}\right|\right)\Bigg\}\int_0^1\int_0^1|G(t,qs)|\left|f\left(\frac{s}{p^{\alpha-1}},u_1\left(\frac{s}{p^{\alpha-1}}\right)\right)-f\left(\frac{s}{p^{\alpha-1}},u_2\left(\frac{s}{p^{\alpha-1}}\right)\right)\right|\,d_{p,q}t\,d_{p,q}s$$

$$\leq\Bigg\{\frac{2}{|\Delta|\,p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)(n-2)!}\left[\left|1-B_{p,q}(n+1,1)\right|+B_{p,q}(n,1)\right]+\frac{1}{|\Delta|\,p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\left(\left|1-\frac{B_{p,q}(n-1,1)-B_{p,q}(n,1)}{(n-2)!}\right|\right.$$

$$\left.+\left|\frac{B_{p,q}(n,1)-B_{p,q}(n+1,1)}{(n-2)!}\right|\right)\Bigg\}\int_0^1\int_0^1 2(1-qs)_{p,q}^{(\alpha-1)}L\,|u_1-u_2|\,d_{p,q}t\,d_{p,q}s$$

$$\leq\Bigg\{\frac{2}{|\Delta|\,p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)(n-2)!}\left[\left|1-B_{p,q}(n+1,1)\right|+B_{p,q}(n,1)\right]+\frac{1}{|\Delta|\,p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\left(\left|1-\frac{B_{p,q}(n-1,1)-B_{p,q}(n,1)}{(n-2)!}\right|\right.$$

$$\left.+\left|\frac{B_{p,q}(n,1)-B_{p,q}(n+1,1)}{(n-2)!}\right|\right)\Bigg\}L\|u_1-u_2\|\;2\,B_{p,q}(1,\alpha)$$

$$\leq\mathcal{M}L\,\|u_1-u_2\|,$$

and so

$$\|T_2u_1-T_2u_2\|\leq\mathcal{M}L\,\|u_1-u_2\|.$$

Since $\mathcal{ML}<1$, then T_2 is a contraction mapping.

At present, we will show that T_1 is compact and continuous. The continuity of f together with the assumption (H_2) yields that the operator T_1 is continuous and uniformly bounded on $B_{\tilde{R}}$. Setting $\sup_{(t,u)\in[0,1]\times B_{\tilde{R}}}|f(t,u)|:=f_{max}<\infty$, also for $t_1,t_2\in[0,1]$ with $t_1\leq t_2$ and $u\in B_{\tilde{R}}$, we obtain that

$$|T_1u(t_2) - T_1u(t_1)| = \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\left|\int_0^1 \left(G(t_2,qs) - G(t_1,qs)\right) f\left(\frac{s}{p^{\alpha-1}}, u\left(\frac{s}{p^{\alpha-1}}\right)\right) d_{p,q}s\right|$$
$$\leq \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^1 |G(t_2,qs) - G(t_1,qs)|\, f_{max} d_{p,q}s.$$

It is clear that when $t_2 \to t_1$, the right-hand side of the above inequality tends to be zero, because of the continuity of $G(t,s)$. Thus, T_1 is relatively compact on $B_{\tilde{R}}$. Hence, by the Arzelà–Ascoli theorem, T_1 is compact on $B_{\tilde{R}}$.

Since all the assumptions of Theorem 2.1 are satisfied, we deduce that the problem (1.1) has at least one solution on $[0,1]$. This completes the proof.

□

Example 3.1. Consider the following boundary value problem for fractional (p,q)-difference equation of the form:

$$\begin{cases} {}^{c}D^{\frac{5}{2}}_{\frac{1}{2},\frac{1}{3}}u(t) + \frac{\Gamma_{\frac{1}{2},\frac{1}{3}}\left(\frac{7}{2}\right)}{10^3}\frac{\sin u(t)}{(1+t^2)} = 0, \quad t \in (0,1) \\ u(0) = 0, \quad u'(0) = \int_0^1 u(t) d_{p,q}t, \qquad u(1) = \int_0^1 t\, u(t)\, d_{p,q}t. \end{cases} \tag{3.1}$$

Setting constants $\alpha = \frac{5}{2}$, $p = \frac{1}{2}$, $q = \frac{1}{3}$, $n = 3$, and the function $f(t,u(t)) = \frac{\Gamma_{\frac{1}{2},\frac{1}{3}}\left(\frac{7}{2}\right)}{10^3}\frac{\sin u(t)}{(1+t^2)}$. It is easy to see that $|f(t,u) - f(t,v)| \leq \frac{\Gamma_{\frac{1}{2},\frac{1}{3}}\left(\frac{7}{2}\right)}{10^3}|u - v|$, then the condition (H_1) is satisfied with $L = \frac{\Gamma_{\frac{1}{2},\frac{1}{3}}\left(\frac{7}{2}\right)}{10^3}$. We can find easily that $|f(t,u)| \leq \frac{\Gamma_{\frac{1}{2},\frac{1}{3}}\left(\frac{7}{2}\right)}{10^3(1+t^2)} := \mu(t)$; then, this shows that the function f satisfies condition (H_2). Indeed, by calculating $\mathcal{M}$, we obtain $\mathcal{M} \cong \frac{20{,}78432}{\Gamma_{\frac{1}{2},\frac{1}{3}}\left(\frac{7}{2}\right)}$. Hence, $\mathcal{M}L < 1$. Consequently, by Theorem 3.1, we conclude that problem (3.1) has a unique solution on $[0,1]$.

The second result is based on the Banach fixed-point theorem.

Theorem 3.2. *Suppose that $f: [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying (H_1); also, the following assumption holds:*

(H_3) $\quad p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha) > 2ALB_{p,q}(1,\alpha)$,

where L is a Lipschitz constant in (H_1), and A is given by (2.6). Then the problem (1.1) has a unique solution.

Proof. We convert problem (1.1) to a fixed-point problem $u = Tu$, where $T: C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is given by (2.8). Suppose that $M := \sup_{t\in[0,1]} |f(t,0)|$ and a constant R satisfies

$$R \geq \frac{2AMB_{p,q}(1,\alpha)}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha) - 2ALB_{p,q}(1,\alpha)}.$$

First, we will show that $TB_R \subset B_R$, where $B_R = \{u \in C([0,1],\mathbb{R}): \|u\| \le R\}$. For any $u \in B_R$, we have

$$|Tu(t)| = \left|\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^1 H(t,qs)\, f\left(\frac{s}{p^{\alpha-1}}, u\left(\frac{s}{p^{\alpha-1}}\right)\right) d_{p,q}s\right|$$

$$\le \sup_{t\in[0,1]}\left|\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^1 H(t,qs)\, f\left(\frac{s}{p^{\alpha-1}}, u\left(\frac{s}{p^{\alpha-1}}\right)\right) d_{p,q}s\right|$$

$$\le \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^1 2A(1-qs)_{p,q}^{(\alpha-1)}\left|f\left(\frac{s}{p^{\alpha-1}}, u\left(\frac{s}{p^{\alpha-1}}\right)\right)\right| d_{p,q}s$$

$$= \frac{2A}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^1 (1-qs)_{p,q}^{(\alpha-1)}\left|f\left(\frac{s}{p^{\alpha-1}}, u\left(\frac{s}{p^{\alpha-1}}\right)\right) - f\left(\frac{s}{p^{\alpha-1}},0\right) + f\left(\frac{s}{p^{\alpha-1}},0\right)\right| d_{p,q}s$$

$$\le \frac{2A}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^1 (1-qs)_{p,q}^{(\alpha-1)}\left\{\left|f\left(\frac{s}{p^{\alpha-1}}, u\left(\frac{s}{p^{\alpha-1}}\right)\right) - f\left(\frac{s}{p^{\alpha-1}},0\right)\right| + \left|f\left(\frac{s}{p^{\alpha-1}},0\right)\right|\right\} d_{p,q}s$$

$$\le \frac{2A}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^1 (1-qs)_{p,q}^{(\alpha-1)}\left\{L\left|u\left(\frac{s}{p^{\alpha-1}}\right)\right| + M\right\} d_{p,q}s$$

$$\le \frac{2A}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^1 (1-qs)_{p,q}^{(\alpha-1)}\{LR+M\} d_{p,q}s$$

$$\le \frac{2A(LR+M)}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^1 (1-qs)_{p,q}^{(\alpha-1)} d_{p,q}s = \frac{2A(LR+M)}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} B_{p,q}(1,\alpha) \le R.$$

Therefore, $\|Tu\| \le R$ and $TB_R \subset B_R$.
Second, we will show that T is contraction. For any $u, v \in C([0,1],\mathbb{R})$, we have

$$|Tu(t) - Tv(t)|$$

$$= \left|\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^1 H(t,qs)\, f\left(\frac{s}{p^{\alpha-1}}, u\left(\frac{s}{p^{\alpha-1}}\right)\right) d_{p,q}s - \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^1 H(t,qs)\, f\left(\frac{s}{p^{\alpha-1}}, v\left(\frac{s}{p^{\alpha-1}}\right)\right) d_{p,q}s\right|$$

$$\le \sup_{t\in[0,1]}\left|\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^1 H(t,qs)\left[f\left(\frac{s}{p^{\alpha-1}}, u\left(\frac{s}{p^{\alpha-1}}\right)\right) - f\left(\frac{s}{p^{\alpha-1}}, v\left(\frac{s}{p^{\alpha-1}}\right)\right)\right] d_{p,q}s\right|$$

$$\le \sup_{t\in[0,1]}\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^1 |H(t,qs)|\left|f\left(\frac{s}{p^{\alpha-1}}, u\left(\frac{s}{p^{\alpha-1}}\right)\right) - f\left(\frac{s}{p^{\alpha-1}}, v\left(\frac{s}{p^{\alpha-1}}\right)\right)\right| d_{p,q}s$$

$$\le \sup_{t\in[0,1]}\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^1 L\,|H(t,qs)|\left|u\left(\frac{s}{p^{\alpha-1}}\right) - v\left(\frac{s}{p^{\alpha-1}}\right)\right| d_{p,q}s$$

$$\leq \frac{L\|u-v\|}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \sup_{t\in[0,1]} \int_0^1 |H(t,qs)|\, d_{p,q}s$$

$$\leq \frac{L\|u-v\|}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^1 2A\,(1-qs)_{p,q}^{(\alpha-1)}\, d_{p,q}s$$

$$= \frac{2AL}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \|u-v\| \int_0^1 (1-qs)_{p,q}^{(\alpha-1)}\, d_{p,q}s = \frac{2ALB_{p,q}(1,\alpha)}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \|u-v\|.$$

Thus, we get

$$\|Tu-Tv\| \leq \frac{2ALB_{p,q}(1,\alpha)}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \|u-v\|.$$

From the condition (H_3), T is a contraction. By the Banach fixed-point theorem, the boundary value problem (1.1) has a unique solution.

□

Example 3.2. Consider the following boundary value problem for the fractional (p,q)-difference equation of the form:

$$\begin{cases} {}^{c}D_{\frac{1}{2},\frac{1}{3}}^{\frac{5}{2}} u(t) + \frac{\Gamma_{\frac{1}{2},\frac{1}{3}}\left(\frac{7}{2}\right)}{10^2} \frac{u^2(t)+2|u(t)|}{1+|u(t)|} = 0, \quad t\in(0,1) \\ u(0)=0, \quad u'(0)=\int_0^1 u(t) d_{p,q}t, \qquad u(1)=\int_0^1 t\, u(t)\, d_{p,q}t. \end{cases} \tag{3.2}$$

Setting constants $\alpha=\frac{5}{2}$, $p=\frac{1}{2}$, $q=\frac{1}{3}$, $n=3$, and the function $f(t,u(t)) = \frac{\Gamma_{\frac{1}{2},\frac{1}{3}}\left(\frac{7}{2}\right)}{10^2} \frac{u^2(t)+2|u(t)|}{1+|u(t)|}$, we find that the value of A given by (2.6) is approximately 1127. By some calculations, we have $|f(t,u)-f(t,v)| \leq \frac{1}{50}\Gamma_{\frac{1}{2},\frac{1}{3}}\left(\frac{7}{2}\right)|u-v|$, then the condition (H_1) is satisfied with $L=\frac{1}{50}\Gamma_{\frac{1}{2},\frac{1}{3}}\left(\frac{7}{2}\right)$. With the same L, we can easily see that the condition (H_3)

$$\left(\frac{1}{2}\right)^{\binom{\frac{5}{2}}{2}} \Gamma_{\frac{1}{2},\frac{1}{3}}\left(\frac{5}{2}\right) > \frac{11{,}27}{25}\Gamma_{\frac{1}{2},\frac{1}{3}}\left(\frac{7}{2}\right) B_{\frac{1}{2},\frac{1}{3}}\left(1,\frac{5}{2}\right)$$

is also satisfied. Consequently, by Theorem 3.2, we conclude that the boundary value problem (3.2) has a unique solution on $[0,1]$.

The third result is derived from Boyd and Wong fixed-point theorem.

Theorem 3.3 *Suppose that*

(H_4) *There exists a continuous function* $h\colon[0,1]\to\mathbb{R}^+$ *with the property that*

$$|f(t,u)-f(t,v)| \leq h(t)\frac{|u-v|}{J+|u-v|}$$

for $\forall t \in [0,1]$ *and* $u, v \geq 0$, *where*

$$J = \frac{2A}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \int_0^1 (1 - qs)_{p,q}^{(\alpha-1)} \, h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s.$$

Then the boundary value problem (1.1) has a unique solution.

Proof. Consider the operator $T: C([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$ by

$$Tu(t) = \frac{1}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \int_0^1 H(t, qs) \, f\left(\frac{s}{p^{\alpha-1}}, u\left(\frac{s}{p^{\alpha-1}}\right)\right) d_{p,q}s,$$

where $H(t, s)$ is defined by Lemma 2.1.

Let us set a continuous non-decreasing function $\psi: \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\psi(x) = \frac{Jx}{J + x}, \quad \forall x > 0$$

with $\psi(0) = 0$ and $\psi(x) < x$, $\forall x > 0$. Using (H_4), we have

$$|Tu(t) - Tv(t)| = \left| \frac{1}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \int_0^1 H(t, qs) \, f\left(\frac{s}{p^{\alpha-1}}, u\left(\frac{s}{p^{\alpha-1}}\right)\right) d_{p,q}s - \frac{1}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \int_0^1 H(t, qs) \, f\left(\frac{s}{p^{\alpha-1}}, v\left(\frac{s}{p^{\alpha-1}}\right)\right) d_{p,q}s \right|$$

$$\leq \frac{1}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \int_0^1 |H(t, qs)| \left| f\left(\frac{s}{p^{\alpha-1}}, u\left(\frac{s}{p^{\alpha-1}}\right)\right) - f\left(\frac{s}{p^{\alpha-1}}, v\left(\frac{s}{p^{\alpha-1}}\right)\right) \right| d_{p,q}s$$

$$\leq \frac{1}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \int_0^1 |H(t, qs)| \, h\left(\frac{s}{p^{\alpha-1}}\right) \frac{\left| u\left(\frac{s}{p^{\alpha-1}}\right) - v\left(\frac{s}{p^{\alpha-1}}\right) \right|}{J + \left| u\left(\frac{s}{p^{\alpha-1}}\right) - v\left(\frac{s}{p^{\alpha-1}}\right) \right|} d_{p,q}s$$

$$\leq \frac{1}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \int_0^1 2A \, (1 - qs)_{p,q}^{(\alpha-1)} \, h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \;\; \frac{\|u - v\|}{J + \|u - v\|},$$

so

$$\|Tu - Tv\| \leq \psi(\|u - v\|).$$

Hence, we see that T is a nonlinear contraction. Therefore, by Definition 2.1 and Theorem 2.3, the operator T has a unique fixed point in $C([0,1], \mathbb{R})$, which is a unique solution to the boundary value problem (1.1).

□

Example 3.3. Consider the following boundary value problem for the fractional (p, q)-difference equation of the form:

$$\begin{cases} {}^cD^{\frac{5}{2}}_{\frac{1}{2},\frac{1}{3}}u(t) + \Gamma_{\frac{1}{2},\frac{1}{3}}\left(\frac{11}{2}\right)\, t^2 \tan^{-1}\left(\frac{|u|}{\lambda+|u|}\right) = 0, \quad t \in (0,1) \\ u(0) = 0, \quad u'(0) = \int_0^1 u(t) d_{p,q}t, \qquad u(1) = \int_0^1 t\, u(t)\, d_{p,q}t. \end{cases} \tag{3.3}$$

Setting constants $\alpha = \frac{5}{2},\ p = \frac{1}{2},\ q = \frac{1}{3},\ n = 3,$ and the function $f(t, u(t)) = h(t)\ tan^{-1}\left(\frac{|u|}{\lambda+|u|}\right)$, where $h(t) = \Gamma_{\frac{1}{2},\frac{1}{3}}\left(\frac{11}{2}\right)\, t^2$ and $\lambda \geq 19$, we obtain

$$J = \frac{2A}{\left(\frac{1}{2}\right)^{\left(\frac{5}{2}\right)} \Gamma_{\frac{1}{2},\frac{1}{3}}\left(\frac{5}{2}\right)} \int_0^1 \left(1 - \frac{1}{3}s\right)^{\left(\frac{3}{2}\right)}_{\frac{1}{2},\frac{1}{3}} \Gamma_{\frac{1}{2},\frac{1}{3}}\left(\frac{11}{2}\right) \left(\frac{s}{2^{-\frac{3}{2}}}\right)^2 d_{\frac{1}{2},\frac{1}{3}}s \cong 18{,}8,$$

where $A \cong 11{,}27$ as in Example 3.2. Since

$$\begin{aligned} |f(t,u) - f(t,v)| &= \Gamma_{\frac{1}{2},\frac{1}{3}}\left(\frac{11}{2}\right)\, t^2 \left|\tan^{-1}\frac{|u|}{\lambda+|u|} - \tan^{-1}\frac{|v|}{\lambda+|v|}\right| \\ &\leq \Gamma_{\frac{1}{2},\frac{1}{3}}\left(\frac{11}{2}\right)\, t^2 \left|\frac{|u|}{\lambda+|u|} - \frac{|v|}{\lambda+|v|}\right| \\ &\leq \Gamma_{\frac{1}{2},\frac{1}{3}}\left(\frac{11}{2}\right)\, t^2 \frac{|u-v|}{\lambda+|u-v|} \\ &\leq \Gamma_{\frac{1}{2},\frac{1}{3}}\left(\frac{11}{2}\right)\, t^2 \frac{|u-v|}{18{,}8+|u-v|} \end{aligned}$$

the condition (H_4) holds. Thus, by Theorem 3.3, the boundary value problem (3.3) has a unique solution on $[0,1]$.

4. Conclusions

In this study, we investigated the existence and uniqueness of solutions for a boundary value problem involving (p, q)-difference equations with integral conditions. By leveraging well-known fixed-point theorems, we derived new theoretical results that contribute to the ongoing research in this field. Our findings provide a strong foundation for analyzing such equations, which have wide-ranging applications in mathematical modeling and applied sciences. In the limit as $p \to 1$, our results reduce to results for the fractional q- difference integral boundary value problem.

To validate our theoretical framework, we presented three illustrative examples that demonstrate the applicability and effectiveness of our results. These examples confirm that the imposed conditions ensure the existence and uniqueness of solutions, reinforcing the reliability of our approach.

Future research can extend this work in several directions. One potential avenue is the study of (p, q)-difference equations with more general boundary conditions or nonlinear integral constraints. Additionally, exploring numerical methods for approximating solutions could provide further insights into the practical implementation of these theoretical results. Moreover, investigating the stability and behavior of solutions under perturbations would be a valuable extension to this study.

In conclusion, our work contributes to the understanding of BVPs for (p, q)-difference equations and highlights the effectiveness of fixed-point techniques in establishing solution existence and uniqueness. These results open new possibilities for further advancements in the field and their applications in various scientific and engineering disciplines.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflict of interest.

References

1. B. Daşbaşı, The fractional-order mathematical modeling of bacterial resistance against multiple antibiotics in case of local bacterial infection, *Sakarya Univ. J. Sci.*, **21** (2017), 442–453. https://doi.org/10.16984/saufenbilder.298934
2. S. Alkan, A numerical method for solution of integro-differential equations of fractional order, *Sakarya Univ. J. Sci.*, **21** (2017), 82–89. https://doi.org/10.16984/saufenbilder.296796
3. S. Çetinkaya, A. Demir, Time fractional equation with non-homogenous Dirichlet boundary conditions, *Sakarya Univ. J. Sci.*, **24** (2020), 1185–1190. https://doi.org/10.16984/saufenbilder.749168
4. İ. Devecioğlu, R. Mutlu, A conformal fractional derivative-based leaky integrate-and-fire neuron model, *Sakarya Univ. J. Sci.*, **26** (2022), 568–578. https://doi.org/10.16984/saufenbilder.1041088
5. R. J. Finkelstein, q-field theory, *Letters Math. Phys.*, **34** (1995), 169–176. https://doi.org/10.1007/BF00739095
6. P. G. O. Freund, A. V. Zabrodin, The spectral problem for the q-Knizhnik-Zamolodchikov equation and continuous q-Jacobi polynomials, *Commun. Math. Phys.*, **173** (1995), 17–42. https://doi.org/10.1007/BF02100180
7. P. P. Kulish, E. V. Damaskinsky, On the q oscillator and the quantum algebra $su_q(1,1)$, *J. Phys. A: Math. Gen.*, **23** (1990), L415. https://doi.org/10.1088/ 0305-4470/23/9/003
8. N. Allouch, J. R. Graef, S. Hamani, Boundary value problem for fractional q-difference equations with integral conditions in Banach spaces, *Fractal Fract.*, **6** (2022), 2–11. https://doi.org/10.3390/fractalfract6050237
9. R. Chakrabarti, R. Jagannathan, A (p, q)-oscillator realization of two-parameter quantum algebras, *J. Phys. A: Math. Gen.*, **24** (1991), L711. https://doi.org/10.1088/0305-4470/24/13/002
10. M. N. Hounkonnou, J. D. B. Kyemba, R (p, q)-calculus: differentiation and integration, *SUT J. Math.*, **49** (2013), 145–167. https://doi.org/10.55937/sut/1394548362
11. P. N. Sadjang, On the fundamental theorem of (p, q)-calculus and some (p, q)-Taylor formulas, *Results Math.*, **73** (2018), 1–21. https://doi.org/10.48550/arXiv.1309.3934
12. V. Sahai, S. Yadav, Representations of two parameter quantum algebras and p, q-special functions, *J. Math. Anal. Appl.*, **335** (2007), 268–279. https://doi.org/10.1016/j.jmaa.2007.01.072

13. N. Turan, M. Başarır, A. Şahin, On the solutions of a nonlinear system of q-difference equations, *Boundary Value Probl.*, **92** (2024), 1–19. https://doi.org/10.1186/s13661-024-01896-6
14. N. Turan, M. Başarır, A. Şahin, On the solutions of the second-order (p, q)-difference equation with an application to the fixed-point theory, *AIMS Math.*, **9** (2024), 10679–10697. https://doi.org/10.3934/math.2024521
15. İ. Gençtürk, Boundary value problems for a second-order (p, q)-difference equation with integral conditions, *Turkish J. Math.*, **46** (2022), 499–515. https://doi.org/10.3906/mat-2106-90
16. Z. Qin, S. Sun, Positive solutions for fractional (p, q)-difference boundary value problems, *J. Appl. Math. Comput.*, **68** (2022), 2571–2588. https://doi.org/10.1007/s12190-021-01630-w
17. P. Neang, K. Nonlaopon, J. Tariboon, S.K. Ntouyas, B. Ahmad, Existence and uniqueness results for fractional (p, q)-difference equations with separated boundary conditions, *Mathematics*, **10** (2022), 1–15. https://doi.org/10.3390/math10050767
18. N. Kamsrisuk, C. Promsakon, S. K. Ntouyas, J. Tariboon, Nonlocal boundary value problems for (p,q)-difference equations, *Differ. Equations Appl.*, **10** (2018), 183–195. https://doi.org:10.7153/dea-2018-10-11
19. J. Soontharanon, T. Sitthiwirattham, On fractional (p, q)-calculus, *Adv. Differ. Equations*, **35** (2020), 1–18. https://doi.org/10.1186/s13662-020-2512-7
20. A. Şahin, Z. Kalkan, The AA-iterative algorithm in hyperbolic spaces with applications to integral equations on time scales, *AIMS Math.*, **9** (2024), 24480–24506. https://doi.org/10.3934/math.20241192
21. Z. Kalkan, A. Şahin, A. Aloqaily, N. Mlaiki, Some fixed point and stability results in b-metric-like spaces with an application to integral equations on time scales, *AIMS Math.*, **9** (2024), 11335–11351. https://doi.org/10.3934/math.2024556
22. A. Şahin, E. Öztürk, G. Aggarwal, Some fixed-point results for the *KF*-iteration process in hyperbolic metric spaces, *Symmetry*, **15** (2023), 1–16. https://doi.org/10.3390/sym15071360
23. A. Şahin, Z. Kalkan, H. Arısoy, On the solution of a nonlinear Volterra integral equation with delay, *Sakarya Univ. J. Sci.*, **21** (2017), 1367–1376. https://doi.org/10.16984/saufenbilder.305632
24. M. A. Krasnoselskii, Two remarks on the method of successive approximations, *Usp. Mat. Nauk*, **10** (1955), 123–127.
25. S. Banach, Sur les opérations dans les ensembles abstraites et leurs applications aux équations integrals, *Fundam. Math.*, **3** (1922), 133–181. https://doi.org/10.4064/fm-3-1-133-181
26. T. Sitthiwirattham, J. Tariboon, S. K. Ntouyas, Three-point boundary value problems of nonlinear second-order q-difference equations involving different numbers of q, *J. Appl. Math.*, **2013** (2013), 1–12. http://dx.doi.org/10.1155/2013/763786
27. D. W. Boyd, J. S. Wong, On nonlinear contractions, *Proc. Am. Math. Soc.*, **20** (1969), 458–464. https://doi.org/10.2307/2035677