Electronic Research Archive

https://www.aimspress.com/journal/era

ERA, 33(5): 3201–3209.
DOI: 10.3934/era.2025140
Received: 12 March 2025
Revised: 15 May 2025
Accepted: 20 May 2025
Published: 23 May 2025

Research article

Generalizations of Wigner's theorem from rank-1 projections to rank-n projections

Yulong Tian and Jinli Xu*

Department of Mathematics, Northeast Forestry University, Harbin 150040, China

* **Correspondence:** Email: jclixv@qq.com; Tel: 18846756606.

Abstract: We provide generalizations of the classical Wigner's theorem as well as Uhlhorn's version of Wigner's theorem by considering maps that send rank-1 projections to rank-n projections. Namely, we describe the general form of maps $\phi : P_1(H) \to P_n(K)$ multiplying n times the transition probability and maps $\phi : P_1(H) \to P_n(K)$ sending each complete orthogonal system of rank-1 projections to some complete orthogonal system of rank-n projections.

Keywords: Wigner's theorem; preserver; projections; transition probability

1. Introduction and statement of the main results

Let H, K be complex or real separable Hilbert spaces and n a positive integer. As usual, the symbols $P_n(H)$ and I_H stand for the set of all rank-n self-adjoint projections on H, and the identity operator on H, respectively. For $S, T \in P_n(H)$, we say S is orthogonal to T iff $ST = 0$ and the quantity $\mathrm{Tr}(ST)$ is the transition probability between S, T. Plainly, $S \perp T$ is equivalent to $\mathrm{Tr}(ST) = 0$. If $u \in H$ is a unit vector, then the rank-1 projection onto span $\{u\}$ will be denoted by $u \otimes u$. The transition probability associated with a pair of rank-1 projections (pure states) is the commonly used concept in quantum theory. We call a family $\{S_i\} \subseteq P_n(H)$ a complete orthogonal system of rank-n projections (briefly, COSP_n) iff

- $S_i \perp S_j$ whenever $i \neq j$.
- There is no rank-1 projection T orthogonal to each S_i.

The celebrated Wigner's theorem [1, pp.251–254] states that if $\phi : P_1(H) \to P_1(H)$ is a bijection satisfying

$$\mathrm{Tr}(\phi(S)\phi(T)) = \mathrm{Tr}(ST), \quad S, T \in P_1(H), \tag{1.1}$$

equivalently, if ϕ preserves the transition probability between S and T, then there exists a unitary or an anti-unitary $U : H \to H$ such that $\phi(A) = UAU^*$. Recently, there has been considerable interest in improving and reproving this vital result in many ways (referred to in [2–7]).

Wigner's theorem also serves as a frequently used tool for investigating the symmetries in some mathematical structures of quantum mechanics. Suppose that ϕ is a bijection on the set of all observables/the state space/the effect algebra, and such a map preserves a certain property/relation/operation relevant in quantum mechanics. The given problem is to characterize the form of such maps (symmetries), and a classical approach to this problem is to first show that ϕ preserves the rank-1 projections and the corresponding transition probability. This is the crucial step of the proof. Applying Wigner's theorem, one may immediately see that the restriction of ϕ to $P_1(H)$ has a nice behavior. Then the final step to prove that ϕ takes the desired form on the entire quantum structure is usually considered as an easier part of the proof. The interested readers are referred to [8, Chapter 2] and references therein for more examples of this approach and some background for the so-called preservers problems.

When using the above method, sometimes we may not ensure that ϕ maps $P_1(H)$ into itself, and quite often we merely know that it preserves the zero-transition probability. This motivates us to search for a stronger version of the classical Wigner's theorem. The main aim of this paper is to provide the generalizations of Wigner's theorem in which instead of assuming that ϕ maps $P_1(H)$ into itself, we assume that ϕ maps $P_1(H)$ into $P_n(K)$.

Theorem 1.1. *If $\phi : P_1(H) \to P_n(K)$ is a map satisfying*

$$\operatorname{Tr}(\phi(S)\phi(T)) = n\operatorname{Tr}(ST), \quad S,T \in P_1(H), \tag{1.2}$$

then there exists a collection $\{V_1, \dots, V_n\}$ of linear or conjugate linear isometries from H into K with mutually orthogonal ranges, such that

$$\phi(A) = \sum_{i=1}^{n} V_i A V_i^*, \quad A \in P_1(H).$$

Notice that the property (1.2) is equivalent to the following condition:

$$\|\phi(S) - \phi(T)\|_{HS} = \sqrt{n}\,\|S - T\|_{HS}, \quad S,T \in P_1(H),$$

where $\|\cdot\|_{HS}$ represents the Hilbert–Schmidt norm. Namely, our result describes the general form of maps from $P_1(H)$ into $P_n(K)$ multiplying $\sqrt{n}$ times the distance induced by this special norm. We point out that several papers [9, 10] studied the isometries of $P_n(H)$ with respect to the operator norm.

For the case of $\dim H \geq 3$, Uhlhorn [11] significantly generalized Wigner's theorem by replacing the assumption (1.1) with a weaker one: $\operatorname{Tr}(ST) = 0 \Leftrightarrow \operatorname{Tr}(\phi(S)\phi(T)) = 0$. Uhlhorn's result has been further improved in [12, 13]: It is proved that the bijectivity assumption can be relaxed when $\dim H < \infty$. Unfortunately, when $\dim H = \infty$, it is shown in [14] that there exist injective maps preserving orthogonality in both directions, which behave quite wildly. Thus, an additional hypothesis will be needed in the infinite-dimensional case.

Theorem 1.2. *Let $\dim H \geq 3$. If $\phi : P_1(H) \to P_n(K)$ is a map that sends each complete orthogonal system of rank-1 projections to some complete orthogonal system of rank-n projections, then there exists a collection $\{V_1, \dots, V_n\}$ of linear or conjugate linear isometries from H into K, which have mutually orthogonal ranges and satisfy $\sum_{i=1}^{n} V_i V_i^* = I$, such that*

$$\phi(A) = \sum_{i=1}^{n} V_i A V_i^*, \quad A \in P_1(H). \tag{1.3}$$

If $3 \leq \dim H < \infty$ and $\dim K = n \dim H$, then a map $\phi : P_1(H) \to P_n(K)$ that preserves orthogonality only in one direction automatically sends each COSP_1 to some COSP_n. Therefore, a generalization (without bijectivity either) of Uhlhorn's theorem in matrix algebra is a direct consequence of Theorem 1.2.

Corollary 1.3. *Let $3 \leq \dim H < \infty$ and $\dim K = n \dim H$. If $\phi : P_1(H) \to P_n(K)$ is a map that preserves orthogonality in one direction, then ϕ has the form* (1.3)*.*

2. Proofs

In what follows, we denote by $C(H)$, $F(H)$, and $F_s(H)$ the set of compact operators, finite-rank operators, and finite-rank self-adjoint operators on H. The following lemma will be used to prove Theorem 1.1.

Lemma 2.1. *If $\phi : F_s(H) \to F_s(K)$ is a linear map that sends rank-1 projections to rank-n projections and satisfies*

$$\mathrm{Tr}(\phi(S)\phi(T)) = n\,\mathrm{Tr}(ST), \quad S, T \in F_s(H), \tag{2.1}$$

then there exists a collection $\{V_1, \ldots, V_n\}$ of linear or conjugate linear isometries from H into K with mutually orthogonal ranges, such that

$$\phi(A) = \sum_{i=1}^{n} V_i A V_i^*, \quad A \in F_s(H).$$

To prove Lemma 2.1, we need the following lemmas. For $S, T \in F_s(H)$, we write $S \leq T$ if $T - S$ is positive.

Lemma 2.2. *Let $\phi : F_s(H) \to F_s(K)$ be a linear map that preserves projections. If $S, T \in F_s(H)$ are projections with $S \geq T$, then $\phi(S) \geq \phi(T)$.*

Proof. Since S, T are projections with $S \geq T$, there exists some projection R orthogonal to T, such that $S = T + R$. Thus, $\phi(S) = \phi(T) + \phi(R) \geq \phi(T)$. □

Lemma 2.3. *(see [15, Theorem 1.9.1]) Let $\mathcal{M}$ be a dense subspace of a normed space $\mathcal{V}$, and $\mathcal{W}$ a Banach space. If $\phi : \mathcal{M} \to \mathcal{W}$ is a continuous linear map, then ϕ has a unique continuous linear extension $\phi' : \mathcal{V} \to \mathcal{W}$.*

Proof of Lemma 2.1. By Eq (2.1), we see that ϕ sends orthogonal rank-1 projections to orthogonal rank-n projections. Clearly, any finite-rank projection is the sum of mutually orthogonal rank-1 projections. Consequently, ϕ preserves the projections.

Assume that the underlying space H is complex. Extend ϕ to a complex linear map from $F(H)$ into $F(K)$ by setting

$$\tilde{\phi}(A + iB) := \phi(A) + i\phi(B), \quad A, B \in F_s(H).$$

Let $A = \sum_i \alpha_i P_i$, $\alpha_i \in \mathbb{R}$, $P_i \in P_1(H)$, denote the spectral decomposition of any operator $A \in F_s(H)$. Then $\tilde{\phi}(P_i)\tilde{\phi}\left(P_j\right) = 0$ for each $i \neq j$, and hence $\tilde{\phi}\left(A^2\right) = \tilde{\phi}(A)^2$. Replacing A by $A + B$, with $A, B \in F_s(H)$, we obtain that $\tilde{\phi}(AB + BA) = \tilde{\phi}(A)\tilde{\phi}(B) + \tilde{\phi}(B)\tilde{\phi}(A)$. Then it follows that

$$\tilde{\phi}\left((A + iB)^2\right) = \tilde{\phi}\left(A^2\right) - \tilde{\phi}\left(B^2\right) + i\tilde{\phi}(AB + BA)$$

$$
\begin{aligned}
&= \tilde{\phi}(A)^2 - \tilde{\phi}(B)^2 + i\left(\tilde{\phi}(A)\tilde{\phi}(B) + \tilde{\phi}(B)\tilde{\phi}(A)\right) \\
&= \left(\tilde{\phi}(A) + i\tilde{\phi}(B)\right)^2 = \tilde{\phi}(A+iB)^2 .
\end{aligned}
$$

This implies that $\tilde{\phi}$ is a Jordan homomorphism. Since $\tilde{\phi}$ preserves the self-adjoint operators, we infer that $\tilde{\phi}$ is a (continuous) Jordan $*$-homomorphism. It is known that $F(H)$ is dense in the C^*-algebra $C(H)$. By Lemma 2.3, $\tilde{\phi}$ can be uniquely extended to a Jordan $*$-homomorphism from $C(H)$ into $C(K)$. According to [8, Theorem A.6], each Jordan $*$-homomorphism of the C^*-algebra is a direct sum of a $*$-antihomomorphism and a $*$-homomorphism. Every $*$-homomorphism of $C(H)$ is in fact a direct sum of inner homomorphisms (see [16, Theorem 10.4.7]). Then $\tilde{\phi}$ has the asserted form.

The case when H is real demands an other approach (this idea is borrowed from [17, Theorem 2.2] below). Assume that $\{u_i\}_{i\in\Omega}$ is an orthonormal basis for H and denote $\operatorname{rng}\phi(u_i\otimes u_i) = K_i$, $i\in\Omega$. For any $i, j\in\Omega$ with $i\neq j$, since $\left[\left(u_i+u_j\right)\otimes\left(u_i+u_j\right)\right]/2$ is a projection with range lying within that of $u_i\otimes u_i + u_j\otimes u_j$, it follows by Lemma 2.2 that

$$
\frac{1}{2}\phi\left(\left(u_i\otimes u_j + u_j\otimes u_i\right) + \left(u_i\otimes u_i + u_j\otimes u_j\right)\right) \le I_{K_i\oplus K_j}\oplus 0.
$$

Therefore, we may write

$$
P_{ij} = \phi\left(u_i\otimes u_j + u_j\otimes u_i\right) = \begin{bmatrix} P'_{ii} & P'_{ij} \\ P'_{ji} & P'_{jj} \end{bmatrix} \oplus 0
$$

for some linear operator $P'_{ij}: K_j \to K_i$. For any nonzero $\alpha\in\mathbb{R}$, consider

$$
\begin{aligned}
Q_1 &= \left(\alpha^2+1\right)^{-1}\left(\alpha^2 u_1\otimes u_1 + \alpha\left(u_1\otimes u_2 + u_2\otimes u_1\right) + u_2\otimes u_2\right) \in P_1(H), \\
Q_2 &= \left(\alpha^2+1\right)^{-1}\left(u_1\otimes u_1 - \alpha\left(u_1\otimes u_2 + u_2\otimes u_1\right) + \alpha^2 u_2\otimes u_2\right) \in P_1(H).
\end{aligned}
$$

By directly computing, $Q_1 Q_2 = 0$. It follows that

$$
\begin{aligned}
0 &= \left(\frac{\alpha^2+1}{\alpha}\right)^2 \phi(Q_1)\phi(Q_2) \\
&= \left(\phi\left(\alpha u_1\otimes u_1 + \frac{1}{\alpha}u_2\otimes u_2\right) + P_{12}\right)\left(\phi\left(\frac{1}{\alpha}u_1\otimes u_1 + \alpha u_2\otimes u_2\right) - P_{12}\right) \\
&= \left(\begin{bmatrix} \alpha I_{K_1} & 0 \\ 0 & \frac{1}{\alpha} I_{K_2} \end{bmatrix} \oplus 0 + P_{12}\right)\left(\begin{bmatrix} \frac{1}{\alpha} I_{K_1} & 0 \\ 0 & \alpha I_{K_2} \end{bmatrix} \oplus 0 - P_{12}\right) \\
&= \begin{bmatrix} I_{K_1} & 0 \\ 0 & I_{K_2} \end{bmatrix} \oplus 0 - P_{12}^2 - \begin{bmatrix} \alpha P'_{11} & \alpha P'_{12} \\ \frac{1}{\alpha} P'_{21} & \frac{1}{\alpha} P'_{22} \end{bmatrix} \oplus 0 + \begin{bmatrix} \frac{1}{\alpha} P'_{11} & \alpha P'_{12} \\ \frac{1}{\alpha} P'_{21} & \alpha P'_{22} \end{bmatrix} \oplus 0 \\
&= \begin{bmatrix} I_{K_1} & 0 \\ 0 & I_{K_2} \end{bmatrix} \oplus 0 - P_{12}^2 - \begin{bmatrix} \left(\alpha - \frac{1}{\alpha}\right) P'_{11} & 0 \\ 0 & \left(\frac{1}{\alpha} - \alpha\right) P'_{22} \end{bmatrix} \oplus 0.
\end{aligned}
$$

Because this equation holds true for any nonzero $\alpha\in\mathbb{R}$, we see that P'_{11} and P'_{22} are zeroes. Hence, we obtain

$$
P'_{12}P'_{21} = I_{K_1} \quad \text{and} \quad P'_{21}P'_{12} = I_{K_2}.
$$

It follows that $P'_{12} = {P'_{21}}^{-1} = {P'_{21}}^{*}$ and we have similar conclusions for all P_{ij}.

Since all K_i are isomorphic to $\mathbb{R}^n$, there exists an isomorphism from $H \otimes \mathbb{R}^n$ to $\bigoplus_{\Omega} K_i$, given by $\sum_i u_i \otimes \eta_i \to (\cdots \eta_i \cdots)$. Write $K = (H \otimes \mathbb{R}^n) \oplus K_s$ and replace ϕ by the mapping

$$A \to \left(I_{K_1} \oplus \left(\bigoplus_{i \in \Omega, i \neq 1} P'_{1i} \right) \oplus I_{K_s} \right) \phi(A) \left(I_{K_1} \oplus \left(\bigoplus_{i \in \Omega, i \neq 1} {P'_{1i}}^{-1} \right) \oplus I_{K_s} \right)$$

such that

$$\phi(u_1 \otimes u_i + u_i \otimes u_1) = [(u_1 \otimes u_i + u_i \otimes u_1) \otimes I_{\mathbb{R}^n}] \oplus 0_{K_s}, \quad i \in \Omega.$$

We are going to prove that

$$\phi\left(u_i \otimes u_j + u_j \otimes u_i\right) = \left[\left(u_i \otimes u_j + u_i \otimes u_j\right) \otimes I_{\mathbb{R}^n}\right] \oplus 0_{K_s} \quad \text{whenever } i, j \in \Omega \text{ with } i \neq j.$$

To see this, let $Z = \left[\left(u_1 + u_i + u_j\right) \otimes \left(u_1 + u_i + u_j\right)\right]/3$. Then Z is a rank-1 projection such that, up to unitary similarity, $\phi(Z)$ is equal to a direct sum of 0 and

$$Y = 3^{-1} \begin{bmatrix} I_{\mathbb{R}^n} & I_{\mathbb{R}^n} & I_{\mathbb{R}^n} \\ I_{\mathbb{R}^n} & I_{\mathbb{R}^n} & P'_{ij} \\ I_{\mathbb{R}^n} & {P'_{ij}}^{-1} & I_{\mathbb{R}^n} \end{bmatrix}.$$

As $Y^2 = Y$, it follows that $I_{\mathbb{R}^n} + 2P'_{ij} = 3P'_{ij}$. Thus $P'_{ij} = {P'_{ij}}^{-1} = I_{\mathbb{R}^n}$.

Since $\operatorname{span}_{\mathbb{R}} \left\{u_i \otimes u_j + u_j \otimes u_i : i, j \in \Omega\right\}$ is dense in $F_s(H)$ when H is a real Hilbert space, we can prove that

$$\phi(A) = U\left[(A \otimes I_{\mathbb{R}^n}) \oplus 0_{K_s}\right] U^*, \quad A \in F_s(H),$$

where $U : K \to K$ is a unitary. We arrive at the conclusion. □

Proof of Theorem 1.1. Since the whole $F_s(H)$ is real linearly generated by $P_1(H)$, we may extend ϕ to a real-linear map $\tilde{\phi} : F_s(H) \to F_s(K)$ by setting

$$\tilde{\phi}\left(\sum_i \lambda_i S_i\right) := \sum_i \lambda_i \phi(S_i),$$

where $\{\lambda_i\} \subseteq \mathbb{R}$ and $\{S_i\} \subseteq P_1(H)$ are finite subsets. We claim that $\tilde{\phi}$ is well-defined. Assume that $\sum_i \lambda_i S_i = \sum_j \mu_j T_j$, $\left\{\mu_j\right\} \subseteq \mathbb{R}$, $\left\{T_j\right\} \subseteq P_1(H)$. Then for each $A \in P_1(H)$, it follows by Eq (1.2) that

$$\begin{aligned} \operatorname{Tr}\left(\sum_i \lambda_i \phi(S_i) \phi(A)\right) &= \sum_i \lambda_i \operatorname{Tr}(\phi(S_i)\phi(A)) = \sum_i \lambda_i n \operatorname{Tr}(S_i A) = n \operatorname{Tr}\left(\sum_i \lambda_i S_i A\right) \\ &= n \operatorname{Tr}\left(\sum_j \mu_j T_j A\right) = \sum_j \mu_j n \operatorname{Tr}\left(T_j A\right) = \sum_j \mu_j \operatorname{Tr}\left(\phi\left(T_j\right)\phi(A)\right) \\ &= \operatorname{Tr}\left(\sum_j \mu_j \phi\left(T_j\right)\phi(A)\right). \end{aligned}$$

This implies that

$$\operatorname{Tr}\left(\left(\sum_i \lambda_i \phi(S_i) - \sum_j \mu_j \phi\left(T_j\right)\right)\phi(A)\right) = 0.$$

Based on the linearity of the function Tr, we can replace $\phi(A)$ by its linear combination. Then we obtain

$$\mathrm{Tr}\left(\left(\sum_i \lambda_i\phi(S_i) - \sum_j \mu_j\phi\left(T_j\right)\right)\left(\sum_i \lambda_i\phi(S_i) - \sum_j \mu_j\phi\left(T_j\right)\right)\right) = 0.$$

Since the square of Hermitian operator $\left(\sum_i\lambda_i\phi(S_i) - \sum_j\mu_j\phi\left(T_j\right)\right)^2$ is positive with zero trace, we deduce that

$$\left(\sum_i \lambda_i\phi(S_i) - \sum_j \mu_j\phi\left(T_j\right)\right)^2 = 0 = \sum_i \lambda_i\phi(S_i) - \sum_j \mu_j\phi\left(T_j\right).$$

It means that $\tilde{\phi}$ is well-defined. Then the form of this linear map $\tilde{\phi}$ is given by Lemma 2.1. □

Proof of Theorem 1.2. First, let us recall Gleason's theorem [18]. A positive and trace-class operator $\sigma : H \to H$ with $\mathrm{Tr}(\sigma) = 1$ is called a density operator. We restate Gleason's theorem as follows: Suppose that $\dim H \geq 3$ and $f : P_1(H) \to [0,1]$ is a function such that for each complete orthogonal system of rank-1 projections $\{S_i\} \subseteq P_1(H)$, one has

$$\sum_i f(S_i) = 1.$$

Then there is a density operator $\sigma : H \to H$ for which

$$f(S) = \mathrm{Tr}(\sigma S), \quad S \in P_1(H).$$

To prove Theorem 1.2, we need to choose an arbitrary density operator $\varrho : K \to K$ and define the function $f_\varrho : P_1(H) \to [0,1]$ by

$$f_\varrho(S) := \mathrm{Tr}(\varrho\phi(S)), \quad S \in P_1(H).$$

It follows from our assumption that Gleason's theorem can be used. Therefore, for each density operator $\varrho : K \to K$, there exists a density operator $\sigma : H \to H$ such that

$$f_\varrho(S) = \mathrm{Tr}(\sigma S), \quad S \in P_1(H).$$

In particular, pick $\varrho = \phi(T)/n$ for some fixed $T \in P_1(H)$. Then we obtain

$$f_\varrho(S) = \frac{1}{n}\mathrm{Tr}(\phi(T)\phi(S)) = \mathrm{Tr}(\sigma_T S), \quad S \in P_1(H),$$

where σ_T is the density operator corresponding to T. Taking $S = T$, we infer that

$$\mathrm{Tr}(\sigma_T T) = 1.$$

It is easy to verify that if u is a unit vector such that $T = u \otimes u$, then

$$\mathrm{Tr}(\sigma_T T) = \langle \sigma_T u, u \rangle.$$

As $0 \leq \sigma_T \leq I$, it follows by the operator theory that $\sigma_T u = u$. Therefore, 1 is an eigenvalue of σ_T and u belongs to the corresponding eigenspace. Under the decomposition $H = \operatorname{span}\{u\} \oplus \{u\}^{\perp}$, the operator σ_T has the following matrix representation:

$$\sigma_T = \begin{bmatrix} 1 & 0 \\ 0 & X \end{bmatrix},$$

where X is the positive operator acting on $\{u\}^{\perp}$ with zero trace. Thus, $X = 0$, which means $\sigma_T = T$.

Hence, for each $S \in P_1(H)$, we have $\operatorname{Tr}(\phi(S)\phi(T)) = n \operatorname{Tr}(ST)$. Since T was chosen arbitrarily, we deduce that ϕ multiplies n times the transition probability. Then Theorem 1.1 tells us the form of the map ϕ. □

3. Discussion

The conclusion in Theorem 1.2 does not hold when $\dim H = 2$, as demonstrated in the following example: In fact, we can identify H with $\mathbb{C}^2$ and hence $F(H) = M_2(\mathbb{C})$, the set of 2×2 complex matrices. All the rank-1 projections in $M_2(\mathbb{C})$ are in 1-to-1 correspondence with the unit vectors in the Bloch sphere in $\mathbb{R}^3$, *i.e.*:

$$P_1\left(\mathbb{C}^2\right) = \left\{2^{-1}\begin{bmatrix} 1 + x_1 & x_2 + ix_3 \\ x_2 - ix_3 & 1 - x_1 \end{bmatrix} : x_1, x_2, x_3 \in \mathbb{R} \text{ with } x_1^2 + x_2^2 + x_3^2 = 1\right\}.$$

It is straightforward to compute the orthogonal complement of

$$A = 2^{-1}\begin{bmatrix} 1 + x_1 & x_2 + ix_3 \\ x_2 - ix_3 & 1 - x_1 \end{bmatrix} \quad \text{is} \quad I - A = 2^{-1}\begin{bmatrix} 1 - x_1 & -x_2 - ix_3 \\ -x_2 + ix_3 & 1 + x_1 \end{bmatrix}.$$

Consider the bijective transformation $\phi : P_1\left(\mathbb{C}^2\right) \to P_1\left(\mathbb{C}^2\right)$, which fix all rank-1 projections, but change the role of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Obviously, the only COSP_1 in $M_2(\mathbb{C})$ that contains $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ and hence ϕ preserves orthogonality. However, this discontinuous transformation ϕ can not be extended to any linear transformation (in fact, any Jordan $*$-homomorphism also) on the whole matrix space $M_2(\mathbb{C})$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This study was funded by Fundamental Research Funds for the Central Universities of China (Grant No. 2572022DJ07).

Conflict of interest

The authors declare there is no conflicts of interest.

References

1. E. P. Wigner, *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren*, Frederik Vieweg und Sohn, Braunschweig, 1931.
2. L. Molnár, Transformations on the set of all n-dimensional subspaces of a Hilbert space preserving principal angles, *Commun. Math. Phys.*, **217** (2001), 409–421. https://doi.org/10.1007/PL00005551
3. G. P. Gehér, Wigner's theorem on Grassmann spaces, *J. Funct. Anal.*, **273** (2017), 2994–3001. https://doi.org/10.1016/j.jfa.2017.06.011
4. P. Šemrl, Maps on Grassmann spaces preserving the minimal principal angle, *Acta Sci. Math.*, **90** (2024), 109–122. https://doi.org/10.1007/s44146-023-00093-8
5. G. P. Gehér, An elementary proof for the non-bijective version of Wigner's theorem, *Phys. Lett. A*, **378** (2014), 2054–2057. https://doi.org/10.1016/j.physleta.2014.05.039
6. M. Pankov, L. Plevnik, A non-injective version of Wigner's theorem, *Oper. Matrices*, **17** (2023), 517–524. https://doi.org/10.7153/oam-2023-17-33
7. P. Šemrl, Automorphisms of Hilbert space effect algebras, *Phys. Lett. A*, **48** (2015), 195301. https://doi.org/10.1088/1751-8113/48/19/195301
8. L. Molnár, *Selected Preserver Problems on Algebraic Structures of Linear Operators and on Function Spaces*, Springer, Berlin, 2007.
9. U. Uhlhorn, Representation of symmetry transformations in quantum mechanics, *Ark. Fysik.*, **23** (1963), 307–340.
10. P. Šemrl, G. P. Gehér, Isometries of Grassmann spaces, *J. Funct. Anal.*, **270** (2016), 1585–1601. https://doi.org/10.1016/j.jfa.2015.11.018
11. L. Molnár, J. Jamison, F. Botelho, Surjective isometries on Grassmann spaces, *J. Funct. Anal.*, **265** (2013), 2226–2238. https://doi.org/10.1016/j.jfa.2013.07.017
12. M. Pankov, T. Vetterlein, A geometric approach to Wigner-type theorems, *Bull. London. Math. Soc.*, **53** (2021), 1653–1662. https://doi.org/10.1112/blms.12517
13. P. Šemrl, Wigner symmetries and Gleason's theorem, *J. Phys. A*, **54** (2021), 315301. https://doi.org/10.1088/1751-8121/ac0d35
14. P. Šemrl, Orthogonality preserving transformations on the set of n-dimensional subspaces of a Hilbert space, *Illinois J. Math.*, **48** (2004), 567–573. https://doi.org/10.1215/ijm/1258138399
15. E. R. Megginson, *An Introduction to Banach Space Theory*, Springer, New York, 2012.
16. R. V. Kadison, J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras. Volume II: Advanced Theory*, Academic press, New York, 1986.

17. C. K. Li, M. C. Tsai, Y. S. Wang, N. C. Wang, Nonsurjective zero product preservers between matrix spaces over an arbitrary field, *Linear Multilinear Algebra*, **72** (2024), 2406–2425. https://doi.org/10.1080/03081087.2023.2263139

18. M. A. Gleason, Measures on the closed subspaces of a Hilbert space, *Indiana Univ. Math. J.*, **6** (1957), 885–893.

AIMS Press