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Research article

Dynamical analysis of a nonlocal delays spatial malaria model with Wolbachia-infected male mosquitoes release

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Abstract: Malaria continues to pose a considerable threat to global health. This study investigates the use of releasing Wolbachia-infected male mosquitoes as a method to mitigate the spread of malaria. We have formulated a reaction-diffusion model with nonlocal delays that includes the Wolbachia release strategy. The basic reproduction number R_0 is defined within our model framework, serving as a critical threshold parameter that dictates the dynamic behavior of the model. A thorough dynamic analysis of the model reveals that when $R_0 < 1$, a globally attractive infection-free steady state is established. In contrast, if $R_0 > 1$, the disease persists uniformly. Numerical simulations are conducted to validate the theoretical results and to further illustrate the effectiveness of the Wolbachia release strategy on transmission and control of malaria. These simulations underscore the potential of using Wolbachia-infected male mosquitoes to significantly reduce spread of malaria.

Keywords: malaria; reaction-diffusion model; Wolbachia; nonlocal delays; basic reproduction number; dynamic analysis

1. Introduction

Malaria remains a major global health challenge, primarily affecting tropical and subtropical regions. The disease is caused by Plasmodium parasites, transmitted through bites of infected Anopheles mosquitoes. Symptoms include fever, chills, and flu-like illness, and without treatment, malaria can lead to severe complications and death [1]. According to the World Health Organization (WHO), in 2023 there were an estimated 263 million malaria cases and 597,000 malaria deaths worldwide. The African region carries a disproportionately high share of the global malaria burden, accounting for 94% of malaria cases and 95% of malaria deaths [2]. Despite ongoing efforts to control malaria, it continues to be a major global health issue that is especially prevalent in the sub-Saharan African region [3].

Mathematical models are essential for depicting the dynamics of disease transmission and developing effective control strategies [4–6]. Early models often assumed a homogeneous environment, ignoring spatial variations such as population density, mosquito distribution, or local ecological conditions. However, newer studies have focused on the importance of spatial heterogeneity in malaria transmission and the need for reaction-diffusion models that incorporate these variations [7]. Spatial heterogeneity significantly impacts malaria transmission, especially in regions with diverse ecological factors. Moreover, delays, such as the incubation periods of malaria in humans and mosquitoes, are crucial components of modern models. These time lags affect the stability and transmission dynamics of malaria. By including these delays in reaction-diffusion malaria models, researchers can better simulate real-world scenarios and assess the effectiveness of control measures over time [8]. For articles on the variable malaria model, please refer to [9, 10].

Recent studies have shown promising results with the use of Wolbachia in controlling mosquito-borne diseases, including malaria and dengue. Field trials conducted in various parts of the world have demonstrated that releasing Wolbachia-infected Aedes aegypti mosquitoes can significantly reduce dengue transmission [11]. Similarly, the potential of Wolbachia to control malaria transmission has been explored through laboratory and field studies involving Anopheles mosquitoes, the primary malaria vectors [12]. Moreover, studies in malaria-endemic areas, including regions in Indonesia and Australia, have demonstrated that Wolbachia-infected mosquitoes can successfully integrate into local mosquito populations. This indicates a viable and potentially large-scale strategy for managing malaria [13, 14]. Consequently, it is of great importance to investigate the effects of Wolbachia-based methods that disrupt insect reproduction on malaria control efforts.

This paper aims to study the effect of releasing Wolbachia-infected male mosquitoes on malaria control. The paper proceeds with the following structure. The next section details the development of our mathematical model. The well-posedness of the model is studied in Section 3. Section 4 delves into the introduction of the basic reproduction number, denoted as R_0. Section 5 focuses on analyzing the threshold dynamics of our model, which are contingent upon the value of R_0. Section 6 presents numerical simulations to elucidate the theoretical results, explore the impact of the release ratio of Wolbachia-infected males on transmission and control of malaria, and perform a sensitivity analysis. The paper concludes with a summary in the final section.

2. Model formulation

2.1. Age-structured model with mosquitoes

The mosquito population has two subclasses: the aquatic population and the winged population. Let $L_1(x, a_1, t)$ represent wild mosquitoes at time t, position x, and chronological age a_1. $A(x, t)$ aquatic mosquitoes at time t and position x. $W(x, t)$ represents female winged mosquitoes at time t and position x. Ω is a bounded region with a smooth boundary $\partial\Omega$. In the aquatic stage, limited habitat space and food resources lead to intense intraspecific competition among the larvae. Consistent with the approach proposed in [15], we utilize the following classic age-structured equations and introduce an additional nonlinear factor to characterize the evolutionary dynamics of mosquito populations under such competition. For $t \geq 0, a_1 > 0$

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a_1}\right) L_1(x, a_1, t) = \nabla \cdot (d(x, a_1)\nabla L_1(x, a_1, t)) - g(L_1(x, a_1, t)), & x \in \Omega, \\ (d(x, a_1)\nabla L_1(x, a_1, t)) \cdot \mathbf{n} = 0, & x \in \partial\Omega, \end{cases} \tag{2.1}$$

where ∇ indicates the gradient, $\mathbf{n}$ is the outward unit normal vector on $\partial\Omega$, $d(x, a_1)$ denotes the diffusion coefficient, and $g(L_1)$ represents a continuous function of L. The expressions for $d(x, a_1)$ and $g(L_1)$ are as follows:

$$d(x, a_1) = \begin{cases} 0, & a_1 \in (0, \tau_1], \\ d_w(x), & a_1 \in (\tau_1, +\infty). \end{cases}$$

$$g(L_1(x, a_1, t)) = \begin{cases} \mu_a(x)L_1(x, a_1, t) + c(x)L_1^2(x, a_1, t), & a_1 \in (0, \tau_1], \\ \mu_w(x)L_1(x, a_1, t), & a_1 \in (\tau_1, +\infty). \end{cases}$$

Here, d_w signifies the diffusion coefficient for adult mosquitoes. Meanwhile, c indicates the intraspecific competition rate among the larvae, and μ_a and μ_w correspond to the natural mortality rates of larvae and adult mosquitoes, respectively. Then $A(x, t)$ and $W(x, t)$ can be determined by integrating the population density over the specified age ranges, as detailed below

$$A(x, t) = \int_0^{\tau_1} L_1(x, a_1, t)\mathrm{d}a_1, W(x, t) = \int_{\tau_1}^{+\infty} L_1(x, a_1, t)\mathrm{d}a_1. \tag{2.2}$$

2.2. *Releasing Wolbachia-infected male mosquitoes*

The female male proportion is assumed to be 1 : 1. Assuming that Wolbachia-infected male mosquitoes are deployed under a proportional release strategy, that is, the released amount of Wolbachia-infected male mosquitoes is proportional to the current wild male mosquitoes. Here, $q(x)$ represents the ratio of released Wolbachia-infected males to wild males. Let $W_c(x, t)$ be the Wolbachia-infected male mosquitoes. Then we have $W_c(x, t) = q(x)W(x, t)$.

We assume that Wolbachia-infected male mosquitoes have the same mating ability as wild male mosquitoes. Thus, $\frac{W_c}{W+W_c} = \frac{q(x)}{1+q(x)}$ describes the probability of a wild female mosquito mating with Wolbachia-infected male mosquitoes. Then $1 - \frac{q(x)}{1+q(x)} = \frac{1}{1+q(x)}$ is the probability of a wild female mosquito mating with wild male mosquitoes. Denote $\rho(x)$ as the egg-laying rate of each wild females mosquito. We have

$$L_1(x, 0, t) = \frac{1}{2}\rho(x)\frac{1}{1 + q(x)}W(x, t).$$

From (2.1), we can get

$$\begin{cases} \dfrac{\partial}{\partial t}A(x, t) = -\mu_a(x)A(x, t) - c(x)\displaystyle\int_0^{\tau_1} L_1^2(x, a_1, t)\mathrm{d}a_1 + L_1(x, 0, t) - L_1(x, \tau_1, t), & x \in \Omega, \\ \dfrac{\partial}{\partial t}W(x, t) = \nabla \cdot (d_m \nabla W(x, t)) - \mu_w(x)W(x, t) + L_1(x, \tau_1, t) - L_1(x, +\infty, t), & x \in \Omega. \end{cases} \tag{2.3}$$

It is natural to assume that $L_1(x,+\infty,t) = 0$. By the characteristic line method, we can obtain the expression of $L_1(x,\tau_1,t)$. If we set $w_1(x,a_1,s) = L_1(x,a_1,a_1+s)$, $a_1 \in (0,\tau_1]$, we have

$$\begin{cases} \dfrac{\partial}{\partial a_1} w_1(x,a_1,s) = \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a_1}\right) L_1(x,a_1,t)|_{t=a_1+s} \\ \qquad = -\mu_a(x)L_1(x,a_1,a_1+s) - c(x)L_1^2(x,a_1,a_1+s) \\ \qquad = -\mu_a(x)w_1(x,a_1,s) - c(x)w_1^2(x,a_1,s), & x \in \Omega, \\ w_1(x,0,s) = \dfrac{\rho(x)}{2(1+q(x))} W(x,s), & x \in \Omega. \end{cases}$$

We can obtain

$$w_1(x,a_1,s) = \frac{w_1(x,0,s)e^{-\mu_a(x)a_1}}{1 + w_1(x,0,s)\frac{c(x)}{\mu_a(x)}\left(1 - e^{-\mu_a(x)a_1}\right)},$$

If we let $a_1 = \tau_1$, then $s = t - \tau_1$, and we have

$$\begin{aligned} L_1(x,\tau_1,t) &= \frac{L_1(x,0,s)e^{-\mu_a(x)\tau_1}}{1 + L_1(x,0,s)\frac{c(x)}{\mu_a(x)}\left(1 - e^{-\mu_a(x)\tau_1}\right)} \\ &= \frac{\frac{\rho(x)}{2(1+q(x))} W(x,t-\tau_1)e^{-\mu_a(x)\tau_1}}{1 + \frac{\rho(x)c(x)}{2(1+q(x))\mu_a(x)} W(x,t-\tau_1)\left(1 - e^{-\mu_a(x)\tau_1}\right)}. \end{aligned}$$

Thus, one has

$$\begin{cases} \dfrac{\partial}{\partial t} A(x,t) = -\mu_a(x)A(x,t) - c(x)\displaystyle\int_0^{\tau_1} L_1^2(x,a_1,t)\mathrm{d}a_1 + \dfrac{\rho(x)}{2(1+q(x))} W(x,t) \\ \qquad - \dfrac{\frac{\rho(x)}{2(1+q(x))} W(x,t-\tau_1)e^{-\mu_a(x)\tau_1}}{1 + \frac{\rho(x)c(x)}{2(1+q(x))\mu_a(x)} W(x,t-\tau_1)\left(1 - e^{-\mu_a(x)\tau_1}\right)}, \\ \dfrac{\partial}{\partial t} W(x,t) = \nabla\cdot(d_m \nabla W(x,t)) - \mu_w(x)W(x,t) + \dfrac{\frac{\rho(x)}{2(1+q(x))} W(x,t-\tau_1)e^{-\mu_a(x)\tau_1}}{1 + \frac{\rho(x)c(x)}{2(1+q(x))\mu_a(x)} W(x,t-\tau_1)\left(1 - e^{-\mu_a(x)\tau_1}\right)}. \end{cases} \tag{2.4}$$

2.2.1. Human mosquito model

Let $S_w(x,t)$, $E_w(x,t)$, and $I_w(x,t)$ represent susceptible, exposed, and infectious (female) winged mosquitoes. Thus we have $W(x,t) = S_w(x,t)+E_w(x,t)+I_w(x,t)$. Let $S_h(x,t)$, $E_h(x,t)$, $I_h(x,t)$ and $R_h(x,t)$ represent susceptible, exposed, infectious and recovered humans, respectively. The total density of human population can be expressed by $N_h(x,t) = S_h(x,t) + E_h(x,t) + I_h(x,t) + R_h(x,t)$. Assume that $N_h(x,t)$ satisfies the following equation:

$$\begin{cases} \dfrac{\partial}{\partial t} N_h(x,t) = \nabla\cdot(d_h(x)\nabla N_h(x,t)) + \Lambda(x) - \mu_h(x)N_h(x,t), & x \in \Omega, \\ (d_h(x)\nabla N_h(x,t))\cdot\mathbf{n} = 0, & x \in \partial\Omega, \end{cases} \tag{2.5}$$

where Λ and μ_h correspond to the influx rate and natural mortality rate of humans, respectively. According to Lemma 1 in [7], system (2.5) has a globally attractive positive steady state, denoted by $N_h^*(x)$. In this study, we assume $N_h(x,t) \equiv N_h^*(x)$ for $\forall\ t \geq 0$ and $x \in \Omega$.

Let a_w be infection age of winged mosquitoes and $L_w(x, a_w, t)$ be the density of infected winged mosquitoes. Then

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a_w}\right) L_w(x, a_w, t) = \nabla \cdot (d_w(x)\nabla L_w(x, a_w, t)) - \mu_w(x) L_w(x, a_w, t), & x \in \Omega, \\ (d_w(x)\nabla L_w(x, a_w, t)) \cdot \mathbf{n} = 0, & x \in \partial\Omega. \end{cases} \tag{2.6}$$

So

$$E_w(x, t) = \int_0^{\tau_w} L_w(x, a_w, t)\mathrm{d}a_w, I_w(x, t) = \int_{\tau_w}^{+\infty} L_w(x, a_w, t)\mathrm{d}a_w. \tag{2.7}$$

From (2.6), we can get

$$\begin{cases} \frac{\partial}{\partial t} E_w(x, t) = \nabla \cdot (d_w(x)\nabla E_w(x, t)) - \mu_w(x)E_w(x, t) + L_w(x, 0, t) - L_w(x, \tau_w, t), & x \in \Omega, \\ \frac{\partial}{\partial t} I_w(x, t) = \nabla \cdot (d_w(x)\nabla I_w(x, t)) - \mu_w(x)I_w(x, t) + L_w(x, \tau_w, t) - L_w(x, +\infty, t), & x \in \Omega. \end{cases} \tag{2.8}$$

It is natural to assume that $L_w(x, +\infty, t) = 0$. By the characteristic line method, we can obtain the expression of $L_w(x, \tau_w, t)$. Set $w_2(x, a_w, s) = L_w(x, a_w, a_w + s)$, $a_w \in (0, \tau_w]$. We have

$$\begin{cases} \frac{\partial}{\partial a_w} w_2(x, a_w, s) = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a_w}\right) L_w(x, a_w, t)|_{t=a_w+s} \\ \qquad = \nabla \cdot (d_w(x)\nabla w_2(x, a_w, s))) - \mu_w(x) w_2(x, a_w, s), & x \in \Omega, \\ w_2(x, 0, s) = \frac{b(x)\beta_{hw}(x)}{N_h^*(x)} S_w(x, s) I_h(x, s), & x \in \Omega. \end{cases}$$

We can obtain

$$w_2(x, a_w, s) = \int_\Omega \Gamma_w(x, y, a_w) \left(\frac{b(y)\beta_{hw}(y)}{N_h^*(y)} S_w(y, s) I_h(y, s)\right) \mathrm{d}y,$$

where Γ_w is the Green function of the operator $\nabla\cdot(d_w(\cdot)\nabla) - \mu_w(\cdot)$ associated with the Neumann boundary condition, $b(x)$ represents bite rate of mosquitoes, $\beta_{hw}(x)$ represents the transmission probability from infectious humans to adult mosquitoes.

Let $a_w = \tau_w$, then $s = t - \tau_w$, and

$$L_w(x, \tau_w, t) = \int_\Omega \Gamma_w(x, y, \tau_w) \left(\frac{b(y)\beta_{hw}(y)}{N_h^*(y)} S_w(y, t - \tau_w) I_h(y, t - \tau_w)\right) \mathrm{d}y.$$

We then have

$$\begin{cases} \frac{\partial}{\partial t} E_w(x, t) = \nabla \cdot (d_w(x)\nabla E_w(x, t)) - \mu_w(x)E_w(x, t) + \frac{b(x)\beta_{hw}(x)}{N_h^*(x)} S_w(x, t) I_h(x, t) \\ \qquad - \int_\Omega \Gamma_w(x, y, \tau_w) \left(\frac{b(y)\beta_{hw}(y)}{N_h^*(y)} S_w(y, t - \tau_w) I_h(y, t - \tau_w)\right) \mathrm{d}y, & x \in \Omega, \\ \frac{\partial}{\partial t} I_w(x, t) = \nabla \cdot (d_w(x)\nabla I_w(x, t)) - \mu_w(x)I_w(x, t) \\ \qquad + \int_\Omega \Gamma_w(x, y, \tau_w) \left(\frac{b(y)\beta_{hw}(y)}{N_h^*(y)} S_w(y, t - \tau_w) I_h(y, t - \tau_w)\right) \mathrm{d}y, & x \in \Omega. \end{cases} \tag{2.9}$$

Let a_h be the infection age of humans. Similar to the derivation of $\frac{\partial E_w(x,t)}{\partial t}$ and $\frac{\partial I_w(x,t)}{\partial t}$, we can obtain the following expressions of $\frac{\partial E_h(x,t)}{\partial t}$ and $\frac{\partial I_h(x,t)}{\partial t}$:

$$\begin{cases} \frac{\partial}{\partial t}E_h(x,t) = \nabla\cdot(d_h(x)\nabla E_h(x,t)) - \mu_h(x)E_h(x,t) + \frac{b(x)\beta_{wh}(x)}{N_h^*(x)}S_h(x,t)I_w(x,t) \\ \qquad - \int_\Omega \Gamma_h(x,y,\tau_h)\left(\frac{b(y)\beta_{wh}(y)}{N_h^*(y)}S_h(y,t-\tau_w)I_w(y,t-\tau_w)\right)\mathrm{d}y, \qquad x\in\Omega, \\ \frac{\partial}{\partial t}I_h(x,t) = \nabla\cdot(d_h(x)\nabla I_h(x,t)) - (\mu_h(x)+r_h(x))I_w(x,t) \\ \qquad + \int_\Omega \Gamma_h(x,y,\tau_h)\left(\frac{b(y)\beta_{wh}(y)}{N_h^*(y)}S_h(y,t-\tau_h)I_(y,t-\tau_h)\right)\mathrm{d}y, \qquad x\in\Omega, \end{cases} \tag{2.10}$$

where Γ_h is the Green function of the operator $\nabla\cdot(d_h(\cdot)\nabla)-\mu_h(\cdot)$ associated with the Neumann boundary condition, d_h represents the diffusion coefficient for humans, β_{wh} represents the transmission probability from infectious mosquitoes to humans, and r_h and μ_h represent recovery rate and death rate of infectious humans, respectively.

In a word, we can obtain the following nonlocal delays reaction-diffusion system:

$$\begin{aligned} \frac{\partial}{\partial t}A(x,t) &= -\mu_a(x)A(x,t) - c(x)\int_0^{\tau_1} L_1^2(x,a_1,t)\mathrm{d}a + \frac{\rho(x)}{2(1+q(x))}W(x,t) \\ &\quad - \frac{\frac{\rho(x)}{2(1+q(x))}W(x,t-\tau_1)e^{-\mu_a(x)\tau_1}}{1+\frac{\rho(x)c(x)}{2(1+q(x))\mu_a(x)}W(x,t-\tau_1)\left(1-e^{-\mu_a(x)\tau_1}\right)}, \\ \frac{\partial}{\partial t}S_w(x,t) &= \nabla\cdot(d_w(x)\nabla S_w(x,t)) + \frac{\frac{\rho(x)}{2(1+q(x))}W(x,t-\tau_1)e^{-\mu_a(x)\tau_1}}{1+\frac{\rho(x)c(x)}{2(1+q(x))\mu_a(x)}W(x,t-\tau_1)\left(1-e^{-\mu_a(x)\tau_1}\right)} \\ &\quad - \mu_w(x)S_w(x,t) - \frac{b(x)\beta_{hw}(x)}{N_h^*(x)}S_w(x,t)I_h(x,t), \\ \frac{\partial}{\partial t}E_w(x,t) &= \nabla\cdot(d_w(x)\nabla E_w(x,t)) - \mu_w(x)E_w(x,t) + \frac{b(x)\beta_{hw}(x)}{N_h^*(x)}S_w(x,t)I_h(x,t) \\ &\quad - \int_\Omega \Gamma_w(x,y,\tau_w)\left(\frac{b(y)\beta_{hw}(y)}{N_h^*(y)}S_w(y,t-\tau_w)I_h(y,t-\tau_w)\right)\mathrm{d}y, \\ \frac{\partial}{\partial t}I_w(x,t) &= \nabla\cdot(d_w(x)\nabla I_w(x,t)) - \mu_w(x)I_w(x,t) \\ &\quad + \int_\Omega \Gamma_w(x,y,\tau_w)\left(\frac{b(y)\beta_{hw}(y)}{N_h^*(y)}S_w(y,t-\tau_w)I_h(y,t-\tau_w)\right)\mathrm{d}y, \\ \frac{\partial}{\partial t}S_h(x,t) &= \nabla\cdot(d_h(x)\nabla S_h(x,t)) + \Lambda(x) - \mu_h(x)S_h(x,t) - \frac{b(x)\beta_{wh}(x)}{N_h^*(x)}S_h(x,t)I_w(x,t), \\ \frac{\partial}{\partial t}E_h(x,t) &= \nabla\cdot(d_h(x)\nabla E_h(x,t)) - \mu_h(x)E_h(x,t) + \frac{b(x)\beta_{wh}(x)}{N_h^*(x)}S_h(x,t)I_w(x,t) \\ &\quad - \int_\Omega \Gamma_h(x,y,\tau_h)\left(\frac{b(y)\beta_{wh}(y)}{N_h^*(y)}S_h(y,t-\tau_w)I_w(y,t-\tau_w)\right)\mathrm{d}y, \end{aligned}$$

$$\frac{\partial}{\partial t}I_h(x,t) = \nabla\cdot(d_h(x)\nabla I_h(x,t)) - (\mu_h(x)+r_h(x))I_h(x,t)$$
$$+\int_\Omega \Gamma_h(x,y,\tau_h)\left(\frac{b(y)\beta_{wh}(y)}{N_h^*(y)}S_h(y,t-\tau_h)I_w(y,t-\tau_h)\right)\mathrm{d}y,$$
$$\frac{\partial}{\partial t}R_h(x,t) = \nabla\cdot(d_h(x)\nabla R_h(x,t)) + r_h(x)R_h(x,t) - \mu_h(x)R_h(x,t),$$
$$(d_w(x)\nabla S_w(x,t))\cdot\mathbf{n} = (d_w(x)\nabla E_w(x,t))\cdot\mathbf{n} = (d_w(x)\nabla I_w(x,t))\cdot\mathbf{n} = 0, x\in\partial\Omega,$$
$$(d_h(x)\nabla S_h(x,t))\cdot\mathbf{n} = (d_h(x)\nabla E_h(x,t))\cdot\mathbf{n} = (d_h(x)\nabla I_h(x,t))\cdot\mathbf{n} = (d_h(x)\nabla R_h(x,t))\cdot\mathbf{n} = 0, x\in\partial\Omega.$$

We know that $A(x,t), E_h(x,t)$, and $R_h(x,t)$ are decoupled from the other equations. Then it can be applied to the following system:

$$\begin{cases}
\frac{\partial}{\partial t}S_w(x,t) = \nabla\cdot(d_w(x)\nabla S_w(x,t)) - \mu_w(x)S_w(x,t) + \dfrac{\rho(x)p(x)e^{-\mu_a(x)\tau_1}W(x,t-\tau_1)}{1+\rho(x)p(x)\iota(x)\left(1-e^{-\mu_a(x)\tau_1}\right)W(x,t-\tau_1)} \\
\qquad -\beta_w(x)S_w(x,t)I_h(x,t), \\
\frac{\partial}{\partial t}E_w(x,t) = \nabla\cdot(d_w(x)\nabla E_w(x,t)) - \mu_w(x)E_w(x,t) + \beta_w(x)S_w(x,t)I_h(x,t) \\
\qquad -\int_\Omega \Gamma_w(x,y,\tau_w)\left(\beta_w(y)S_w(y,t-\tau_w)I_h(y,t-\tau_w)\right)\mathrm{d}y, \\
\frac{\partial}{\partial t}I_w(x,t) = \nabla\cdot(d_w(x)\nabla I_w(x,t)) - \mu_w(x)I_w(x,t) \\
\qquad +\int_\Omega \Gamma_w(x,y,\tau_w)\beta_w(y)S_w(y,t-\tau_w)I_h(y,t-\tau_w)\mathrm{d}y, \\
\frac{\partial}{\partial t}S_h(x,t) = \nabla\cdot(d_h(x)\nabla S_h(x,t)) + \Lambda(x) - \mu_h(x)S_h(x,t) - \beta_h(x)S_h(x,t)I_w(x,t), \\
\frac{\partial}{\partial t}I_h(x,t) = \nabla\cdot(d_h(x)\nabla I_h(x,t)) - (\mu_h(x)+r_h(x))I_h(x,t) \\
\qquad +\int_\Omega \Gamma_h(x,y,\tau_h)\beta_h(y)S_h(y,t-\tau_h)I_w(y,t-\tau_h)\mathrm{d}y, \\
(d_w(x)\nabla S_w(x,t))\cdot\mathbf{n} = (d_w(x)\nabla E_w(x,t))\cdot\mathbf{n} = (d_w(x)\nabla I_w(x,t))\cdot\mathbf{n} = 0, x\in\partial\Omega, \\
(d_h(x)\nabla S_h(x,t))\cdot\mathbf{n} = (d_h(x)\nabla I_h(x,t))\cdot\mathbf{n} = 0, x\in\partial\Omega,
\end{cases} \tag{2.11}$$

where $p(x) = \frac{1}{2(1+q(x))}$, $\iota(x) = \frac{c(x)}{\mu_a(x)}$, $\beta_w(x) = \frac{b(x)\beta_{hw}(x)}{N_h^*(x)}$, $\beta_h(x) = \frac{b(x)\beta_{wh}(x)}{N_h^*(x)}$.

3. The well-posedness

Let $\mathcal{X} = C(\overline{\Omega},\mathbb{R}^5)$ be the Banach space with the supremum norm $\|\cdot\|_{\mathcal{X}}$, and $\mathcal{X}^+ = C(\overline{\Omega},\mathbb{R}^5_+)$. Let $\tau = \max\{\tau_1,\tau_w,\tau_h\}$. Define $\mathcal{C} = C([-\tau,0],\mathcal{X})$ as the Banach space with the norm $\|\varphi\| = \max\limits_{\theta\in[-\tau,0]}\|\varphi(\theta)\|_{\mathcal{X}}$ for $\varphi\in\mathcal{C}$, and $\mathcal{C}^+ = C([-\tau,0],\mathcal{X}^+)$. Then $(\mathcal{X},\mathcal{X}^+)$ and $(\mathcal{C},\mathcal{C}^+)$ are strongly ordered Banach spaces. For $\varsigma>0$ and a function $z:[-\tau,\varsigma)\to\mathcal{X}$, we define $z_t\in\mathcal{C}$ by $z_t(\theta) = z(t+\theta)$, $\theta\in[-\tau,0]$.

Denote $\mathcal{T}_w(t)$, $\mathcal{T}_s(t)$, and $\mathcal{T}_h(t) : C(\overline{\Omega}, \mathbb{R}) \to C(\overline{\Omega}, \mathbb{R})$ are the evolution operators associated with

$$\begin{aligned}
&\frac{\partial v_1}{\partial t} = \nabla \cdot (d_w(x)\nabla v_1) - \mu_w(x)v_1, && x \in \Omega,\\
&\frac{\partial v_2}{\partial t} = \nabla \cdot (d_h(x)\nabla v_2) - \mu_h(x)v_2, && x \in \Omega,\\
&\frac{\partial v_3}{\partial t} = \nabla \cdot (d_h(x)\nabla v_3) - (\mu_h(x) + r_h(x))v_3, && x \in \Omega,\\
&(d_w(x)\nabla v_1) \cdot \mathbf{n} = (d_h(x)\nabla v_2) \cdot \mathbf{n} = (d_h(x)\nabla v_3) \cdot \mathbf{n} = 0, && x \in \partial\Omega.
\end{aligned}$$

Then, for $\vartheta \in C(\overline{\Omega}, \mathbb{R})$, one has

$$\begin{aligned}
\mathcal{T}_w(t)\vartheta(x) &= \int_\Omega \Gamma_w(x, y, \tau_w)\beta_w(y)\vartheta(y)\mathrm{d}y,\\
\mathcal{T}_s(t)\vartheta(x) &= \int_\Omega \Gamma_s(x, y, \tau_h)\beta_h(y)\vartheta(y)\mathrm{d}y,\\
\mathcal{T}_h(t)\vartheta(x) &= \int_\Omega \Gamma_h(x, y, \tau_h)\beta_h(y)\vartheta(y)\mathrm{d}y,
\end{aligned}$$

where Γ_s is the Green function of the operator $\nabla \cdot (d_h(\cdot)\nabla) - \mu_h(\cdot)$ associated with the Neumann boundary condition. We can see that $\mathcal{T}_j(t)$ is strongly positive and compact, where $j = w, s, h$. Set $\mathcal{T} = \mathrm{diag}\{\mathcal{T}_w(t), \mathcal{T}_w(t), \mathcal{T}_w(t), \mathcal{T}_s(t), \mathcal{T}_h(t)\}$. Then $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is a semigroup generated by the operator $\mathcal{A} = \mathrm{diag}\ \{\mathcal{A}_1, \mathcal{A}_1, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$ defined on $D(\mathcal{A}) = D(\mathcal{A}_1) \times D(\mathcal{A}_1) \times D(\mathcal{A}_1) \times D(\mathcal{A}_2) \times D(\mathcal{A}_3)$, in which

$$\begin{aligned}
&D(\mathcal{A}_1) = \left\{\hat{\vartheta} \in C^2(\bar{\Omega}) : (d_w(x)\nabla\hat{\vartheta}) \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\right\},\\
&\mathcal{A}_1\hat{\vartheta} = \nabla \cdot (d_w(x)\nabla\hat{\vartheta}) - \mu_w(x)\hat{\vartheta},\ \hat{\vartheta} \in D(\mathcal{A}_1),
\end{aligned}$$

$$\begin{aligned}
&D(\mathcal{A}_2) = \left\{\hat{\vartheta} \in C^2(\bar{\Omega}) : (d_h(x)\nabla\hat{\vartheta}) \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\right\},\\
&\mathcal{A}_2\hat{\vartheta} = \nabla \cdot (d_h(x)\nabla\hat{\vartheta}) - \mu_h(x)\hat{\vartheta},\ \hat{\vartheta} \in D(\mathcal{A}_2),
\end{aligned}$$

$$\begin{aligned}
&D(\mathcal{A}_3) = \left\{\hat{\vartheta} \in C^2(\bar{\Omega}) : (d_h(x)\nabla\hat{\vartheta}) \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\right\},\\
&\mathcal{A}_3\hat{\vartheta} = \nabla \cdot (d_h(x)\nabla\hat{\vartheta}) - (\mu_h(x) + r_h(x))\hat{\vartheta},\ \hat{\vartheta} \in D(\mathcal{A}_3).
\end{aligned}$$

For $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5) \in C^+$, define $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_5) : C^+ \to \mathcal{X}$ as

$$\begin{aligned}
\mathcal{F}_1\varphi(x) &= \frac{\rho(x)p(x)e^{-\mu_a(x)\tau_1}(\varphi_1(x, -\tau_1) + \varphi_2(x, -\tau_1) + \varphi_3(x, -\tau_1))}{1 + \rho(x)p(x)\iota(x)\left(1 - e^{-\mu_a(x)\tau_1}\right)(\varphi_1(x, -\tau_1) + \varphi_2(x, -\tau_1) + \varphi_3(x, -\tau_1))} - \beta_w(x)\varphi_1(x, 0)\varphi_5(x, 0),\\
\mathcal{F}_2\varphi(x) &= \beta_w(x)\varphi_1(x, 0)\varphi_5(x, 0) - \int_\Omega \Gamma_w(x, y, \tau_w)\left(\beta_w(y)\varphi_1(y, -\tau_w)\varphi_5(y, -\tau_w)\right)\mathrm{d}y,\\
\mathcal{F}_3\varphi(x) &= \int_\Omega \Gamma_w(x, y, \tau_w)\left(\beta_w(y)\varphi_1(y, -\tau_w)\varphi_5(y, -\tau_w)\right)\mathrm{d}y,\\
\mathcal{F}_4\varphi(x) &= \Lambda(x) - \beta_h(x)\varphi_3(x, 0)\varphi_4(x, 0),\\
\mathcal{F}_5\varphi(x) &= \int_\Omega \Gamma_h(x, y, \tau_h)\left(\beta_h(y)\varphi_4(y, -\tau_h)\varphi_3(y, -\tau_h)\right)\mathrm{d}y.
\end{aligned}$$

Then system (2.11) can be rewritten as an abstract functional equation as follows:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = \mathcal{A}u + \mathcal{F}(u_t), t > 0, \\ u_0 = \varphi \in C^+. \end{cases}$$

where $u := (S_w, E_w, I_w, S_h, I_h)$.

According to Corollary 8.1.3 in [16] and Corollary 4 in [17], we can get the following lemma.

Lemma 3.1. *System (2.11) with an initial value function $\varphi \in C^+$ has a unique mild solution $u(\cdot, t, \varphi)$ on its maximal interval of existence $[0, t_\varphi)$. Moreover, $u(\cdot, t, \varphi) \in C^+$ for any $t \in [0, t_\varphi)$, and $u(\cdot, t, \varphi)$ is a classical solution of system (2.11) for $t > \tau$.*

Lemma 3.2. *System (2.11) with an initial value function $\varphi \in C^+$ has a unique global classical solution $u(\cdot, t, \varphi)$ with $u^0 = \varphi$ for $t \in [0, \infty)$. Moreover, the solution semiflow $\Upsilon(t) = u_t(\cdot) : C^+ \to C^+$ has a compact global attractor in C^+.*

Proof. From the first and fourth equations of (2.11), we have

$$\begin{cases} \dfrac{\partial}{\partial t} S_w(x,t) \le \nabla \cdot (d_m \nabla S_w(x,t)) + \dfrac{e^{-\mu_a(x)\tau_1}}{\iota(x)\left(1 - e^{-\mu_a(x)\tau_1}\right)} - \mu_w(x) S_w(x,t), \\ \dfrac{\partial}{\partial t} S_h(x,t) \le \nabla \cdot (d_h(x) \nabla S_h(x,t)) + \Lambda(x) - \mu_h(x) S_h(x,t), \\ (d_w(x) \nabla S_w(x,t)) \cdot \mathbf{n} = (d_h(x) \nabla S_h(x,t)) \cdot \mathbf{n} = 0, x \in \partial\Omega. \end{cases}$$

So, for any $\varphi \in C^+$, $K_w, K_h > 0$ and $t_0 = t_0(\varphi) > 0$ exist such that $S_w(\cdot, t, \varphi) \le K_w$ and $S_h(\cdot, t, \varphi) \le K_h$ for $\forall t \ge t_0$.

Assume $M_1(t) = \int_\Omega S_w(x,t)\mathrm{d}x$, $M_2(t) = \int_\Omega E_w(x,t)\mathrm{d}x$, $M_3(t) = \int_\Omega I_w(x,t)\mathrm{d}x$, $M_4(t) = \int_\Omega S_h(x,t)\mathrm{d}x$, $M_5(t) = \int_\Omega I_h(x,t)\mathrm{d}x$. Integrating $S_w(x,t)$ in system (2.11), we can obtain

$$\begin{aligned} \frac{\mathrm{d}M_1(t)}{\mathrm{d}t} &\le \int_\Omega \left(\frac{e^{-\mu_a(x)\tau_1}}{\iota(x)(1 - e^{-\mu_a(x)\tau_1})} - \mu_w(x) S_w(x,t) \right) \mathrm{d}x - \int_\Omega \beta_w(x) S_w(x,t) I_h(x,t) \mathrm{d}x \\ &\le K_1 \mid \Omega \mid - \underline{\mu_w} M_1(t) - \int_\Omega \beta_w(x) S_w(x,t) I_h(x,t) \mathrm{d}x, \ t \ge 0, \end{aligned}$$

where $K_1 = \frac{e^{-\underline{\mu_a}\tau_1}}{\underline{\iota}(1 - e^{-\underline{\mu_a}\tau_1})}$. Thus

$$\int_\Omega \beta_w(x) S_w(x,t) I_h(x,t) \mathrm{d}x \le K_1 \mid \Omega \mid - \underline{\mu_w} M_1(t) - \frac{\mathrm{d}M_1(t)}{\mathrm{d}t}, \ t \ge 0.$$

Integrating $E_w(x,t)$ in system (2.11), we can obtain

$$\begin{aligned} \frac{\mathrm{d}M_2(t)}{\mathrm{d}t} &\le \int_\Omega \beta_w(x) S_w(x,t) I_h(x,t) \mathrm{d}x - \underline{\mu_w} M_2(t) \\ &\le K_1 \mid \Omega \mid - \underline{\mu_w} M_1(t) - \frac{\mathrm{d}M_1(t)}{\mathrm{d}t} - \underline{\mu_w} M_2(t), \ t \ge 0. \end{aligned}$$

We can then get

$$\frac{\mathrm{d}}{\mathrm{d}t}(M_1(t)+M_2(t)) \le K_1 \mid \Omega \mid -\underline{\mu_w}(M_1(t)+M_2(t)),\ t \ge 0.$$

It implies that

$$M_2(t) \le \frac{K_1 \mid \Omega \mid}{\underline{\mu_w}} + e^{-\underline{\mu_w}t}(M_1(0)+M_2(0)),\ t \ge 0.$$

So, $M_2(t)$ is uniformly bounded.

Since $\Gamma_w(\cdot,\cdot,\cdot)$ is bounded, by integrating $I_w(x,t)$ in system (2.11), we can obtain

$$\begin{aligned}\frac{\mathrm{d}M_3(t)}{\mathrm{d}t} &\le -\underline{\mu_w}M_3(t) + K_2\int_\Omega (\beta_w(y)S_w(y,t-\tau_w)I_h(y,t-\tau_w))\,\mathrm{d}y\\ &\le -\underline{\mu_w}M_3(t) + K_2\left(K_1 \mid \Omega \mid -\underline{\mu_a}M_1(t-\tau_w) - \frac{\mathrm{d}M_1(t-\tau_w)}{\mathrm{d}t}\right),\ t \ge \tau,\end{aligned}$$

for some positive constant K_2. One then has

$$\frac{\mathrm{d}}{\mathrm{d}t}(M_3(t)+K_2M_1(t-\tau_w)) \le K_1K_2 \mid \Omega \mid -\underline{\mu_1}(M_3(t)+K_2M_1(t-\tau_w)),\ t \ge \tau,$$

where $\underline{\mu_1} = \min\{\underline{\mu_w}, \underline{\mu_a}\}$. It imply that there are a $\varphi-$dependent positive constant K_3 and a $\varphi-$independent positive constant K_4 such that

$$M_3(t) \le M_3(t) + K_2M_1(t-\tau_w) \le K_3(\varphi)e^{-\underline{\mu_1}t} + K_4,\ t \ge \tau.$$

So, $M_3(t)$ is uniformly bounded.

Similarly, we can obtain

$$\frac{\mathrm{d}M_4(t)}{\mathrm{d}t} \le \overline{\Lambda} \mid \Omega \mid -\underline{\mu_h}M_4(t) - \int_\Omega \beta_h(x)S_h(x,t)I_w(x,t)\mathrm{d}x,\ t \ge 0.$$

Thus

$$\int_\Omega \beta_h(x)S_h(x,t)I_w(x,t)\mathrm{d}x \le \overline{\Lambda} \mid \Omega \mid -\underline{\mu_h}M_4(t) - \frac{\mathrm{d}M_4(t)}{\mathrm{d}t},\ t \ge 0.$$

Since $\Gamma_h(\cdot,\cdot,\cdot)$ is bounded, by integrating $I_h(x,t)$ in system (2.11), we can obtain

$$\frac{\mathrm{d}M_5(t)}{\mathrm{d}t} \le -(\underline{\mu_h}+\underline{r_h})M_5(t) + K_5\left(\overline{\Lambda} \mid \Omega \mid -\underline{\mu_h}M_4(t-\tau_h) - \frac{\mathrm{d}M_4(t-\tau_h)}{\mathrm{d}t}\right),\ t \ge \tau,$$

for some positive constant K_5. One then has

$$\frac{\mathrm{d}}{\mathrm{d}t}(M_5(t)+K_5M_4(t-\tau_w)) \le K_5\overline{\Lambda} \mid \Omega \mid -\underline{\mu_h}(M_5(t)+K_5M_4(t-\tau_w)),\ t \ge \tau.$$

It imply that there are a $\varphi-$dependent positive constant K_6 and a $\varphi-$independent positive constant K_7

$$M_5(t) \le K_6(\varphi)e^{-\underline{\mu_h}t} + K_7,\ t \ge \tau.$$

So, $M_5(t)$ is uniformly bounded. By the comparison theorem for delayed parabolic equations [16], we find that there exist φ–independent positive number K_8, K_9, K_{10} and $t_1 = t_1(\varphi) > t_0(\varphi) + \tau$ such that $E_w(\cdot, t, \varphi) \le K_8$, $I_w(\cdot, t, \varphi) \le K_9$ and $I_h(\cdot, t, \varphi) \le K_{10}$ for any $t \ge t_1$. So, the solution of system (2.11) is global and the solution semiflow $\Upsilon(t) = u_t(\cdot) : C^+ \to C^+$ is point dissipative by Theorem 2.1.8 in [16]. Based on Theorem 3.4.8 in [18], $\Upsilon(t)$ has a compact global attractor in C^+ for $t \ge 0$.

□

4. The basic reproduction number

Set $E_w = I_w = I_h = 0$ in system (2.11). Then, $S_w(x, t)$ satisfies the following system:

$$\begin{cases} \dfrac{\partial}{\partial t}S_w(x,t) = \nabla \cdot (d_w(x)\nabla S_w(x,t)) - \mu_w(x)S_w(x,t) + \dfrac{\rho(x)p(x)e^{-\mu_a(x)\tau_1}S_w(x,t-\tau_1)}{1+\rho(x)p(x)\iota(x)\left(1-e^{-\mu_a(x)\tau_1}\right)S_w(x,t-\tau_1)}, \\ (d_w(x)\nabla S_w(x,t)) \cdot \mathbf{n} = 0, x \in \partial\Omega. \end{cases} \tag{4.1}$$

By Theorem 11 in [19] and Lemma 2.1 in [20], system (4.1) admits a globally attractive positive steady state $W^*(x)$. It is easy to know that system (2.11) has an infection-free steady state $E_0(x) = (W^*(x), 0, 0, N_h^*(x), 0)$. Linearizing system (2.11) at $E_0(x)$, we can get

$$\begin{cases} \dfrac{\partial}{\partial t}I_w(x,t) = \nabla \cdot (d_w(x)\nabla I_w(x,t)) - \mu_w(x)I_w(x,t) + \displaystyle\int_\Omega \Gamma_w(x,y,\tau_w)\beta_w(y)W^*(y)I_h(y,t-\tau_w)\mathrm{d}y, \\ \dfrac{\partial}{\partial t}I_h(x,t) = \nabla \cdot (d_h(x)\nabla I_h(x,t)) - (\mu_h(x)+r_h(x))I_h(x,t) + \displaystyle\int_\Omega \Gamma_h(x,y,\tau_h)b(y)\beta_{wh}(y)I_w(y,t-\tau_h)\mathrm{d}y, \\ (d_w(x)\nabla I_w(x,t)) \cdot \mathbf{n} = (d_h(x)\nabla I_h(x,t)) \cdot \mathbf{n} = 0, x \in \partial\Omega. \end{cases} \tag{4.2}$$

Consider the following linear nonlocal and cooperative reaction-diffusion system:

$$\begin{cases} \dfrac{\partial}{\partial t}I_w(x,t) = \nabla \cdot (d_w(x)\nabla I_w(x,t)) - \mu_w(x)I_w(x,t) + \displaystyle\int_\Omega \Gamma_w(x,y,\tau_w)\beta_w(y)W^*(y)I_h(y,t)\mathrm{d}y, \\ \dfrac{\partial}{\partial t}I_h(x,t) = \nabla \cdot (d_h(x)\nabla I_h(x,t)) - (\mu_h(x)+r_h(x))I_h(x,t) + \displaystyle\int_\Omega \Gamma_h(x,y,\tau_h)b(y)\beta_{wh}(y)I_w(y,t)\mathrm{d}y, \\ (d_w(x)\nabla I_w(x,t)) \cdot \mathbf{n} = (d_h(x)\nabla I_h(x,t)) \cdot \mathbf{n} = 0, x \in \partial\Omega. \end{cases} \tag{4.3}$$

Substituting $I_w(x,t) = e^{\lambda t}\psi_1(x)$ and $I_h(x,t) = e^{\lambda t}\psi_2(x)$ into (4.2) and (4.3), we have

$$\begin{cases} \lambda\psi_1(x) = \nabla \cdot (d_w(x)\nabla\psi_1(x)) - \mu_w(x)\psi_1(x) + e^{-\lambda\tau_w}\displaystyle\int_\Omega \Gamma_w(x,y,\tau_w)\beta_w(y)W^*(y)\psi_1(y)\mathrm{d}y, \\ \lambda\psi_2(x) = \nabla \cdot (d_h(x)\nabla\psi_2(x)) - (\mu_h(x)+r_h(x))\psi_2(x) + e^{-\lambda\tau_h}\displaystyle\int_\Omega \Gamma_h(x,y,\tau_h)b(y)\beta_{wh}(y)\psi_2(y)\mathrm{d}y, \\ (d_w(x)\nabla\psi_1(x)) \cdot \mathbf{n} = (d_h(x)\nabla\psi_2(x)) \cdot \mathbf{n} = 0, x \in \partial\Omega, \end{cases} \tag{4.4}$$

and

$$\begin{cases}\lambda\psi_1(x) = \nabla\cdot(d_w(x)\nabla\psi_1(x)) - \mu_w(x)\psi_1(x) + \displaystyle\int_\Omega \Gamma_w(x,y,\tau_w)\beta_w(y)W^*(y)\psi_1(y)\mathrm{d}y,\\ \lambda\psi_2(x) = \nabla\cdot(d_h(x)\nabla\psi_2(x)) - (\mu_h(x)+r_h(x))\psi_2(x) + \displaystyle\int_\Omega \Gamma_h(x,y,\tau_h)b(y)\beta_{wh}(y)\psi_2(y)\mathrm{d}y,\\ (d_w(x)\nabla\psi_1(x))\cdot\mathbf{n} = (d_h(x)\nabla\psi_2(x))\cdot\mathbf{n} = 0, x\in\partial\Omega.\end{cases} \tag{4.5}$$

By similar methods to Theorem 7.6.1 in [21], we can see that (4.5) has a principal eigenvalue $\lambda_0(W^*)$ with strongly positive eigenfunctions. $\lambda_0(W^*)$ varies continuously under small perturbations W^*. Similar to Lemma 2.1 in [22], we can obtain the next lemma.

Lemma 4.1. *The eigenvalue problem (4.4) admits principal eigenvalues associated with a strongly positive eigenfunction, denoted $\tilde{\lambda}_0(W^*,\tau)$. Moreover, $\tilde{\lambda}_0(W^*,\tau)$ and $\lambda_0(W^*)$ have the same sign.*

Assume $\phi(x) := (\phi_w(x), \phi_h(x))$. Assume that $\phi(x)$ is the spatial distribution of the initial infective female winged mosquitos and humans. It is easy to see that the distribution of those infective individuals at time $t > 0$ is described by $\mathbb{T}(t)\phi(x) := (\mathcal{T}_w(t)\phi_w(x), \mathcal{T}_h(t)\phi_h(x))$. Then the distribution of new infectious female winged mosquitos can be expressed as

$$\int_\Omega \Gamma_w(x,y,\tau_w)\beta_w(y)W^*(y)\mathcal{T}_h(t)\phi_h(y)\mathrm{d}y.$$

Therefore, the total distribution of infectious female winged mosquitos during the infective period is

$$\int_{\tau_w}^{+\infty}\int_\Omega \Gamma_w(x,y,\tau_w)\beta_w(y)W^*(y)\mathcal{T}_h(t-\tau_w)\phi_h(y)\mathrm{d}y\mathrm{d}t = \int_0^{+\infty}\int_\Omega \Gamma_w(x,y,\tau_w)\beta_w(y)W^*(y)\mathcal{T}_h(t)\phi_h(y)\mathrm{d}y\mathrm{d}t.$$

Similarly, the total distribution of infectious humans during the infective period is

$$\int_{\tau_h}^{+\infty}\int_\Omega \Gamma_h(x,y,\tau_h)b(y)\beta_{wh}(y)\mathcal{T}_w(t-\tau_h)\phi_w(y)\mathrm{d}y\mathrm{d}t = \int_0^{+\infty}\int_\Omega \Gamma_h(x,y,\tau_h)b(y)\beta_{wh}(y)\mathcal{T}_w(t)\phi_w(y)\mathrm{d}y\mathrm{d}t.$$

Define the operator $\mathbb{F}$ on $C(\overline{\Omega},\mathbb{R})\times C(\overline{\Omega},\mathbb{R})$ as

$$\mathbb{F}(\phi)(x) = (\mathbb{F}_1(\phi)(x), \mathbb{F}_2(\phi)(x)),\ \phi := (\phi_1,\phi_2)\in C(\overline{\Omega},\mathbb{R})\times C(\overline{\Omega},\mathbb{R}),\ x\in\overline{\Omega},$$

where

$$\mathbb{F}_1(\phi)(x) = \int_\Omega \Gamma_w(x,y,\tau_w)\beta_w(y)W^*(y)\phi_1(y)\mathrm{d}y$$

and

$$\mathbb{F}_2(\phi)(x) = \int_\Omega \Gamma_h(x,y,\tau_h)b(y)\beta_{wh}(y)\phi_2(y)\mathrm{d}y.$$

Then, we can define the next reproduction operator as

$$\mathbb{L} = \int_0^{+\infty}\mathbb{F}(\mathbb{T}(t)\phi)\mathrm{d}t = \mathbb{F}\left(\int_0^{+\infty}\mathbb{T}(t)\phi\mathrm{d}t\right).$$

So, Based on the result in [23], the basic reproduction number R_0 can be defined by

$$R_0 := r(\mathbb{L}). \tag{4.6}$$

Additionally, we can get the following lemma.

Lemma 4.2. *$R_0 - 1$ has the same sign as $\lambda_0(W^*)$.*

5. Dynamic analysis

Theorem 5.1. *If $R_0 < 1$, then the infection-free steady state $E_0(x)$ of system (2.11) is globally attractive in C^+.*

Proof. When $R_0 < 1$, according to Lemmas 4.1 and 4.2, we have $\tilde{\lambda}_0(W^*, \tau) < 0$, Since $\lim_{\zeta\to 0} \tilde{\lambda}_0(W^* + \zeta, \tau) = \tilde{\lambda}_0(W^*, \tau)$, a small enough $\zeta_0 > 0$ exists such that $\tilde{\lambda}_0(W^* + \zeta_0, \tau) < 0$.

Recall that $W^*(\cdot)$ is globally attractive for system (4.1), we have $\limsup_{t\to\infty} S_w(\cdot, t) \leq W^*(\cdot)$. So, for some positive constant t_0, we have $S_w(\cdot, t) \leq W^*(\cdot) + \zeta_0$ for $t \geq t_0$. Then, for all $t \geq t_0$, we can get

$$\begin{cases} \dfrac{\partial}{\partial t} I_w(x,t) \leq \nabla \cdot (d_w(x)\nabla I_w(x,t)) - \mu_w(x) I_w(x,t) + \displaystyle\int_\Omega \Gamma_w(x,y,\tau_w)\beta_w(y)(W^*(y) + \zeta_0) + I_h(y, t-\tau_w)\mathrm{d}y, \\ \dfrac{\partial}{\partial t} I_h(x,t) \leq \nabla \cdot (d_h(x)\nabla I_h(x,t)) - (\mu_h(x) + r_h(x)) I_h(x,t) + \displaystyle\int_\Omega \Gamma_h(x,y,\tau_h) b(y)\beta_{wh}(y) I_w(y, t-\tau_h)\mathrm{d}y, \\ (d_w(x)\nabla I_w(x,t)) \cdot \mathbf{n} = (d_h(x)\nabla I_h(x,t)) \cdot \mathbf{n} = 0, x \in \partial\Omega. \end{cases} \tag{5.1}$$

Set $\tilde{\psi}$ be the strongly positive eigenfunction corresponding to $\tilde{\lambda}_0(W^* + \zeta_0, \tau)$. Then, for $t > 0$, the following linear system

$$\begin{cases} \dfrac{\partial}{\partial t} \tilde{u}_1(x,t) = \nabla \cdot (d_w(x)\nabla \tilde{u}_1(x,t)) - \mu_w(x) \tilde{u}_1(x,t) + \displaystyle\int_\Omega \Gamma_w(x,y,\tau_w)\beta_w(y)(W^*(y) + \zeta_0) + \tilde{u}_2(y, t-\tau_w)\mathrm{d}y, \\ \dfrac{\partial}{\partial t} \tilde{u}_2(x,t) = \nabla \cdot (d_h(x)\nabla \tilde{u}_2(x,t)) - (\mu_h(x) + r_h(x)) \tilde{u}_2(x,t) + \displaystyle\int_\Omega \Gamma_h(x,y,\tau_h) b(y)\beta_{wh}(y) \tilde{u}_1(y, t-\tau_h)\mathrm{d}y, \\ (d_w(x)\nabla \tilde{u}_1(x,t)) \cdot \mathbf{n} = (d_h(x)\nabla \tilde{u}_2(x,t)) \cdot \mathbf{n} = 0, x \in \partial\Omega, \end{cases} \tag{5.2}$$

admits a solution $\tilde{u}(x,t) = e^{\tilde{\lambda}_0(W^*+\zeta_0,\tau)t}\tilde{\psi}(x)$. Then, for any given $\varphi \in C^+$, there is some positive constant $\tilde{m}$, and one has $(I_w(\cdot, t, \varphi), I_h(\cdot, t, \varphi)) \leq \tilde{m}\tilde{u}(\cdot, t)$ for all $t \in [t_0 - \tau, t_0]$. According to the comparison principle, we can obtain

$$(I_w(x,t,\varphi), I_h(x,t,\varphi)) \leq \tilde{m} e^{\tilde{\lambda}_0(W^*+\zeta_0,\tau)(t-t_0)}\tilde{\psi}(x), \ \forall\, t \geq t_0, x \in \overline{\Omega}.$$

Therefore, $\lim_{t\to\infty}(I_w(x,t,\varphi), I_h(x,t,\varphi)) = (0,0)$ uniformly for $x \in \overline{\Omega}$. Then $S_w(x,t), E_w(x,t), S_h(x,t)$ in system (2.11) are asymptotic to the following equations

$$\begin{aligned} &\frac{\partial}{\partial t} \hat{u}_1(x,t) = \nabla \cdot (d_w(x)\nabla \hat{u}_1(x,t)) - \mu_w(x)\hat{u}_1(x,t) + \frac{\rho(x)p(x)e^{-\mu_a(x)\tau_1}\hat{u}_1(x,t-\tau_1)}{1 + \rho(x)p(x)\iota(x)\left(1 - e^{-\mu_a(x)\tau_1}\right)\hat{u}_1(x,t-\tau_1)}, \\ &\frac{\partial}{\partial t} \hat{u}_2(x,t) = \nabla \cdot (d_w(x)\nabla \hat{u}_2(x,t)) - \mu_w(x)\hat{u}_2(x,t), \\ &\frac{\partial}{\partial t} \hat{u}_3(x,t) = \nabla \cdot (d_h(x)\nabla \hat{u}_3(x,t)) + \Lambda(x) - \mu_h(x)\hat{u}_3(x,t), \\ &(d_w(x)\nabla \hat{u}_1(x,t)) \cdot \mathbf{n} = (d_w(x)\nabla \hat{u}_2(x,t)) \cdot \mathbf{n} = (d_w(x)\nabla \hat{u}_3(x,t)) \cdot \mathbf{n} = 0, x \in \partial\Omega. \end{aligned}$$

Therefore, $\lim_{t\to\infty} E_w(x,t,\varphi) = 0$, $\lim_{t\to\infty} S_w(x,t,\varphi) = W^*(x)$, $\lim_{t\to\infty} S_h(x,t,\varphi) = N_h^*(x)$ uniformly for $x \in \overline{\Omega}$. This completes the proof.

□

Lemma 5.1. *For the solution $(S_w(x,t,\varphi), E_w(x,t,\varphi), I_w(x,t,\varphi), S_h(x,t,\varphi), I_h(x,t,\varphi))$ of system (2.11) with an initial value function $\varphi \in C^+$.*

(i) *For any $t > 0$, $x \in \bar{\Omega}$, one has $S_w(x,t,\varphi) > 0$ and $S_h(x,t,\varphi) > 0$. Furthermore, for $\forall\ \varphi \in C^+$, there is $\varphi-$independent positive constant ζ such that*

$$\liminf_{t\to\infty} S_w(x,t,\varphi) \geq \zeta,\ \liminf_{t\to\infty} S_h(x,t,\varphi) \geq \zeta,\ uniformly\ for\ x \in \bar{\Omega}. \tag{5.3}$$

(ii) *Assume that there exists some $t^* \geq 0$ such that $I_w(\cdot,t^*,\varphi) \not\equiv 0$ and $I_h(\cdot,t^*,\varphi) \not\equiv 0$, then the solution satisfies*

$$I_w(x,t,\varphi) > 0, I_h(x,t,\varphi) > 0,\ \ \forall\ t > t^*,\ x \in \bar{\Omega}.$$

Proof. (i) From Lemma 4.1, for any $t > 0$, $x \in \bar{\Omega}$ and an initial value function $\varphi \in C^+$, there is an $\breve{N}_1 > 0$ such that $S_h(x,t,\varphi) \leq \breve{N}_1$ and $I_h(x,t,\varphi) \leq \breve{N}_1$. Let $v_w(\cdot,t,\varphi)$ satisfy

$$\begin{aligned}\frac{\partial}{\partial t} v_w(x,t) =& \nabla \cdot (d_w(x)\nabla v_w(x,t)) - \mu_w(x)v_w(x,t) - \beta_w(x)\breve{N}_1 v_w(x,t)\\ &+ \frac{\rho(x)p(x)e^{-\mu_a(x)\tau_1} v_w(x,t-\tau_1)}{1+\rho(x)p(x)\iota(x)\left(1-e^{-\mu_a(x)\tau_1}\right)v_w(x,t-\tau_1)}.\end{aligned} \tag{5.4}$$

It follows from the comparison principle that

$$S_w(\cdot,t,\varphi) \geq v_w(\cdot,t,\varphi) > 0$$

for any $t > 0$ and $\varphi \in C^+$. By Lemma 2.1 in [20], system (5.4) admits a globally attractive positive steady state $v_w^*(x)$. Set $\zeta_w = \min_{x\in\bar{\Omega}} v_w^*(x)$, then we have

$$\liminf_{t\to\infty} S_w(x,t,\varphi) \geq \zeta_w,\ uniformly\ for\ x \in \bar{\Omega}.$$

Similarly, from Lemma 4.1, for any $t > 0$, $x \in \bar{\Omega}$, $\varphi \in C^+$, there is an $\breve{N}_2 > 0$ such that $I_w(x,t,\varphi) \leq \breve{N}_2$.

$$\frac{\partial}{\partial t} S_h(x,t) \geq \nabla \cdot (d_h(x)\nabla S_h(x,t)) + \Lambda(x) - (\mu_h(x) + \beta_h(x)\breve{N}_2)S_h(x,t). \tag{5.5}$$

It follows from the comparison principle that $S_h(\cdot,t,\varphi) > 0$, and there is an $\zeta < \zeta_w$ such that $\liminf_{t\to\infty} S_h(x,t,\varphi) \geq \zeta$ uniformly for $x \in \bar{\Omega}$. This implies that (i) holds.

(ii) From system (2.11), we can obtain I_w and I_h, which satisfy

$$\begin{cases}\frac{\partial}{\partial t} I_w(x,t) \geq \nabla \cdot (d_w(x)\nabla I_w(x,t)) - \mu_w(x)I_w(x,t),\\ \frac{\partial}{\partial t} I_h(x,t) \geq \nabla \cdot (d_h(x)\nabla I_h(x,t)) - (\mu_h(x) + r_h(x))I_h(x,t),\\ (d_w(x)\nabla I_w(x,t)) \cdot \mathbf{n} = (d_h(x)\nabla I_h(x,t)) \cdot \mathbf{n} = 0, x \in \partial\Omega.\end{cases} \tag{5.6}$$

According to the maximum principle, assume some $t^* \geq 0$ exists such that $I_w(\cdot,t^*,\varphi) \not\equiv 0$ and $I_h(\cdot,t^*,\varphi) \not\equiv 0$, and then we have

$$I_w(x,t,\varphi) > 0, I_h(x,t,\varphi) > 0,\ \ \forall\ t > t^*,\ x \in \bar{\Omega}.$$

This implies that (ii) holds. □

Theorem 5.2. *The disease is uniformly persistent when $R_0 > 1$; that is, $\varrho > 0$ exists such that for any initial value function $\varphi \in C^+$ with $\varphi_3(\cdot, 0) \not\equiv 0$ and $\varphi_5(\cdot, 0) \not\equiv 0$, we have*

$$\liminf_{t\to+\infty}(S_w(x,t,\varphi), E_w(x,t,\varphi), I_w(x,t,\varphi), S_h(x,t,\varphi), I_h(x,t,\varphi)) \geq (\varrho,\varrho,\varrho,\varrho,\varrho) \tag{5.7}$$

uniformly for $x \in \bar{\Omega}$.

Proof. Let $(S_w(x,t,\varphi), E_w(x,t,\varphi), I_w(x,t,\varphi), S_h(x,t,\varphi), I_h(x,t,\varphi))$ be the solution of system (2.11) with an initial value function $\varphi \in C^+$. Set

$$\mathbb{S}_0 = \{\varphi \in C^+ : \varphi_3(\cdot) \not\equiv 0 \ and \ \varphi_5(\cdot) \not\equiv 0\}, \partial\mathbb{S}_0 := C^+\backslash\mathbb{S}_0 = \{\varphi \in C^+ : \varphi_3(\cdot) \equiv 0, \ or \ \varphi_5(\cdot) \equiv 0\}.$$

From Lemma 5.1, if $\varphi \in \partial\mathbb{S}_0$, then $I_w(x,t,\varphi) > 0$ and $I_h(x,t,\varphi) > 0$ for $x \in \bar{\Omega}$, $t > 0$. So, we have $\Upsilon(t)\mathbb{S}_0 \subset \mathbb{S}_0$. Denote $S_\partial = \{\varphi \in \partial\mathbb{S}_0 : \Upsilon(t)\varphi \in \partial\mathbb{S}_0, t \geq 0\}$. $\omega(\varphi)$ as the omega limit set of the forward orbit $\gamma^+(\varphi) := \{\Upsilon(t)\varphi : t \geq 0\}$. We divide the proof into the following two claims.

Claim 1. For any $\varphi \in S_\partial$, the omega limit set $\omega(\varphi) = E_0(x)$.

For any $\varphi \in S_\partial$, we have $\Upsilon(t)\varphi \in \partial\mathbb{S}_0$. Hence, either $I_w(\cdot,t,\varphi) \equiv 0$ or $I_h(\cdot,t,\varphi) \equiv 0$ for all $t \geq 0$. If $I_w(\cdot,t,\varphi) \equiv 0$ for all $t \geq 0$, then, from the third equation of (2.11), we have $I_h(\cdot,t,\varphi) \equiv 0$ for all $t \geq 0$. Then $E_w(\cdot,t,\varphi)$ satisfies

$$\begin{cases} \dfrac{\partial}{\partial t}E_w(x,t) = \nabla\cdot(d_w(x)\nabla E_w(x,t)) - \mu_w(x)E_w(x,t), \\ (d_w(x)\nabla E_w(x,t))\cdot\mathbf{n} = 0, x \in \partial\Omega. \end{cases}$$

Thus, $\lim_{t\to\infty} E_w(x,t,\varphi) = 0$, uniformly for $x \in \bar{\Omega}$. Then, $S_w(\cdot,t,\varphi)$ is asymptotic to the linear system (4.1). So, $\lim_{t\to\infty} S_w(x,t,\varphi) = W^*(x)$ uniformly for $x \in \bar{\Omega}$. It is easy to know that $S_h(\cdot,t,\varphi)$ satisfies system (2.5). Then, $\lim_{t\to\infty} S_h(x,t,\varphi) = N_h^*(x)$ uniformly for $x \in \bar{\Omega}$. In words,

$$\lim_{t\to\infty}(S_w(x,t,\varphi), E_w(x,t,\varphi), I_w(x,t,\varphi), S_h(x,t,\varphi), I_h(x,t,\varphi)) = E_0(x), \ uniformly\ for\ x \in \bar{\Omega}.$$

Assume $I_w(\cdot,t_3,\varphi) \not\equiv 0$ for some $t_3 > 0$. Then $I_h(\cdot,t,\varphi) \equiv 0$ for $t > t_3$. From Lemma 5.1, we can obtain $I_w(\cdot,t,\varphi) > 0$ for $t > t_3$. Since $I_h(\cdot,t,\varphi) \equiv 0$ for $t > t_3$, we have $I_w(\cdot,t,\varphi) \equiv 0$ for $t > t_3$ from the fifth equation of (2.11). This contradicts our assumption. Therefore, $\omega(\varphi) = E_0(x)$ for any $\varphi \in S_\partial$.

Claim 2. $E_0(x)$ is a uniform weak repeller for $\mathbb{S}_0$, i.e.,

$$\limsup_{t\to+\infty}\|\Upsilon(t)\varphi - E_0(\cdot)\| \geq \sigma^*, \ for\ any\ \varphi \in \mathbb{S}_0. \tag{5.8}$$

Since $R_0 > 1$, from Lemma 4.2, $\lambda_0(W^*) > 0$. For any given $\sigma \in (0, \min\{\underline{W}^*, \underline{N_h}^*\}]$, set $\lambda_0(W^*,\sigma)$ be the principal eigenvalue of the nonlocal elliptic eigenvalue problem as follows:

$$\begin{cases} \lambda\psi_1(x) = \nabla\cdot(d_w(x)\nabla\psi_1(x)) - \mu_w(x)\psi_1(x) + \displaystyle\int_\Omega \Gamma_w(x,y,\tau_w)\beta_w(y)(W^*(y)-\sigma)\psi_1(y)dy, \\ \lambda\psi_2(x) = \nabla\cdot(d_h(x)\nabla\psi_2(x)) - (\mu_h(x)+r_h(x))\psi_2(x) + \displaystyle\int_\Omega \Gamma_h(x,y,\tau_h)\beta_h(y)(N_h^*(y)-\sigma)\psi_2(y)dy, \\ (d_w(x)\nabla\psi_1(x))\cdot\mathbf{n} = (d_h(x)\nabla\psi_2(x))\cdot\mathbf{n} = 0, x \in \partial\Omega. \end{cases} \tag{5.9}$$

We can then obtain $\lim_{\sigma\to 0^+}\lambda_0(W^*,\sigma)=\lambda_0(W^*)$. So, we can find some sufficiently small constant $\sigma^*\in(0,\min\{\underline{W}^*,\underline{N_h}^*\}]$ such that $\lambda_0(W^*,\sigma^*)>0$.

If (5.8) is not true, then $\varphi^*\in\mathbb{S}_0$ exists such that $\limsup_{t\to+\infty}\|\Upsilon(t)\varphi-E_0(\cdot)\|<\sigma^*$. Then some $t_4>0$ exists such that $\|\Upsilon(t)\varphi-E_0(\cdot)\|<\sigma^*$ for any $t\geq t_4$. So, $S_w(\cdot,t,\varphi^*)>W^*(x)-\sigma^*$, $S_h(\cdot,t,\varphi^*)>N_h^*(x)-\sigma^*$,$0<E_w(\cdot,t,\varphi^*)<\sigma^*$, $0<I_w(\cdot,t,\varphi^*)<\sigma^*$, and $0<I_h(\cdot,t,\varphi^*)<\sigma^*$ for any $t\geq t_4$. Then we have

$$\begin{cases}\dfrac{\partial}{\partial t}I_w(x,t)\geq\nabla\cdot(d_w(x)\nabla I_w(x,t))-\mu_w(x)I_w(x,t)+\displaystyle\int_\Omega\Gamma_w(x,y,\tau_w)\beta_w(y)(W^*(y)-\sigma^*)I_h(y,t-\tau_w)\mathrm{d}y,\\ \dfrac{\partial}{\partial t}I_h(x,t)\geq\nabla\cdot(d_h(x)\nabla I_h(x,t))-(\mu_h(x)+r_h(x))I_h(x,t)\\ \qquad\qquad+\displaystyle\int_\Omega\Gamma_h(x,y,\tau_h)\beta_h(y)(N_h^*(y)-\sigma^*)I_w(y,t-\tau_h)\mathrm{d}y,\\ (d_w(x)\nabla I_w(x,t))\cdot\mathbf{n}=(d_h(x)\nabla I_h(x,t))\cdot\mathbf{n}=0,x\in\partial\Omega,\end{cases}\tag{5.10}$$

for all $t\geq t_5:=t_4+\tau$.

Set $\psi=(\psi_1,\psi_2)$ be the positive eigenfunction associated with the principal eigenvalue $\tilde{\lambda}_0(W^*,\tau,\sigma^*)$ of the following eigenvalue problem:

$$\begin{cases}\lambda\psi_1(x)=\nabla\cdot(d_w(x)\nabla\psi_1(x))-\mu_w(x)\psi_1(x)+e^{-\lambda\tau_w}\displaystyle\int_\Omega\Gamma_w(x,y,\tau_w)\beta_w(y)(W^*(y)-\sigma^*)\psi_1(y)\mathrm{d}y,\\ \lambda\psi_2(x)=\nabla\cdot(d_h(x)\nabla\psi_2(x))-(\mu_h(x)+r_h(x))\psi_2(x)\\ \qquad\qquad+e^{-\lambda\tau_h}\displaystyle\int_\Omega\Gamma_h(x,y,\tau_h)\beta_h(y)(N_h^*(y)-\sigma^*)\psi_2(y)\mathrm{d}y,\\ (d_w(x)\nabla\psi_1(x))\cdot\mathbf{n}=(d_h(x)\nabla\psi_2(x))\cdot\mathbf{n}=0,x\in\partial\Omega.\end{cases}\tag{5.11}$$

From Lemma 4.1, $\tilde{\lambda}_0(W^*,\tau,\sigma^*)$ and $\lambda_0(W^*,\sigma^*)$ have the same sign. So, $\tilde{\lambda}_0(W^*,\tau,\sigma^*)>0$. Consider the following the linear system:

$$\begin{cases}\dfrac{\partial}{\partial t}v_1(x,t)=\nabla\cdot(d_w(x)\nabla v_1(x,t))-\mu_w(x)v_1(x,t)+\displaystyle\int_\Omega\Gamma_w(x,y,\tau_w)\beta_w(y)(W^*(y)-\sigma^*)v_2(y,t-\tau_w)\mathrm{d}y,\\ \dfrac{\partial}{\partial t}v_2(x,t)=\nabla\cdot(d_h(x)\nabla v_2(x,t))-(\mu_h(x)+r_h(x))v_2(x,t)\\ \qquad\qquad+\displaystyle\int_\Omega\Gamma_h(x,y,\tau_h)\beta_h(y)(N_h^*(y)-\sigma^*)v_1(y,t-\tau_h)\mathrm{d}y,\\ (d_w(x)\nabla v_1(x,t))\cdot\mathbf{n}=(d_h(x)\nabla v_2(x,t))\cdot\mathbf{n}=0,x\in\partial\Omega,\end{cases}\tag{5.12}$$

for $t\geq t_5$. It is easy to see that system (5.12) has a solution $(v_1(x,t),v_2(x,t))=e^{\tilde{\lambda}_0(W^*,\tau,\sigma^*)t}((\psi_1(x),\psi_2(x)))$. Together (5.10) with the comparison principle, we can see that a sufficiently small positive number $\mathcal{L}$ exists such that

$$(I_w(x,t),I_h(x,t))\geq\mathcal{L}(v_1(x,t),v_2(x,t))=\mathcal{L}e^{\tilde{\lambda}_0(W^*,\tau,\sigma^*)t}((\psi_1(x),\psi_2(x))),\ \forall t\geq t_5,\ x\in\bar{\Omega}.$$

Since $\tilde{\lambda}_0(W^*,\tau,\sigma^*)>0$, we get

$$\lim_{t\to+\infty}I_w(\cdot,t)=\infty,\lim_{t\to+\infty}I_h(\cdot,t)=\infty,$$

which is a contradiction. Therefore, (5.8) is true.

So, $E_0(\cdot)$ is an isolated invariant set C^+, and $W^S(E_0(\cdot)) \cap \mathbb{S}_0 = \emptyset$, in which $W^S(E_0(\cdot))$ is a stable set of $E_0(\cdot)$. Define a continuous function $\mathfrak{f} : C^+ \to \mathbb{R}^+$ with the following form:

$$\mathfrak{f}(\varphi) = \min\{\min_{x\in\bar{\Omega}} \varphi_3(x), \min_{x\in\bar{\Omega}} \varphi_5(x)\}, \forall \in \varphi \in C^+.$$

Clearly, $\mathfrak{f}^{-1}(0, +\infty) \subset \mathbb{S}_0$. From Lemma 5.1 (ii), we can see that if $\mathfrak{f}(\varphi) = 0$ for $\varphi \in \mathbb{S}_0$, or $\mathfrak{f}(\varphi) > 0$, then $\mathfrak{f}(\Upsilon(t)(\varphi)) > 0$ for any $t > 0$. So, $\mathfrak{f}$ is a generalized distance function for the semiflow $\Upsilon(t) : C^+ \to C^+$. By Theorem 3 in [24], we can see that $\varrho_0 > 0$ exists such that $\min_{\vartheta\in\omega(\varphi)} \mathfrak{f}(\vartheta) \geq \varrho_0$ for any $\varphi \in \mathbb{S}_0$. This implies uniform persistence.

□

6. Numerical simulations

In this section, numerical simulations are conducted to validate the analytical outcomes. By [25], we choose $H_*(x) = 53(\text{km}^2)^{-1}$, the periodicity is set to $T = 12$ months, and $\Omega = (0, \pi)$. The parameter values are as follows: $N_h^* = 50$, $\mu_h = 0.00157$, $\Lambda = N_h^* \times \mu_h$, $r_h = 0.08$, $c = 0.000001$, $\rho = 50$, $\mu_a = 9$, $\mu_w = 2$, $\tau_1 = 0.5605$, $\tau_w = 17.25/30.4$, $\tau_h = 15/30.4$, $d_w = 0.0125$, $d_h = 0.1$,

$$q(x) = 5(1 - 0.01\cos(2x)), \beta_{hw}(x) = 0.6(1 - 0.01\cos(2x)), \beta_{wh}(x) = 0.2(1 - 0.01\cos(2x)).$$

6.1. Dynamic behavior of model (2.11)

We set the initial functions as follows:

$$S_w(x, \psi) = 5(1 - \cos(2x)), E_w(x, \psi) = 0.2(1 - \cos(2x)), I_w(x, \psi) = 0.3(1 - \cos(2x)),$$

$$S_h(x, \psi) = 4(1 - \cos(2x)), I_h(x, \psi) = 0.2(1 - \cos(2x)), \psi \in [-\tau, 0], x \in [0, \pi].$$

Choose $b = 5$. Numerical simulation determines that $R_0 \approx 0.2844 < 1$. Based on Theorem 5.1, $E_0(x)$ is globally attractive. As shown in Figure 1, the disease will eventually vanish. In other hand, we set $b = 18$. The basic reproduction number can be calculated to be $R_0 \approx 1.024 > 1$. The dynamic behavior of model (2.11) is illustrated in Figure 2, corroborating the theoretical result of Theorem 5.2. This indicates that an outbreak of the disease is likely to occur.

6.2. The effect of the release ratio of Wolbachia-infected males on R_0

Let $q(x) = q^*(1 - k\cos(2x))$, and set $b = 18$, $\beta_{hw}(x) = 0.6$, $\beta_{wh}(x) = 0.2$. Firstly, fix $k = 0.01$. We investigate the correlation between q^* and R_0. From Figure 3(a), as q^* increases, R_0 gradually decreases and crosses the threshold of 1. Next, fix $q^* = 1.7$. We analyze the influence of spatial heterogeneity, represented by k, on R_0. Figure 3(b) demonstrates that R_0 is an increasing function with respect to k. It should be noted that $q(x)$ represents a homogeneous distribution when $k = 0$. Moreover, set $q = 1.5$, we explore the effects of the diffusion rates d_w and d_h on R_0 as shown in Figure 4.

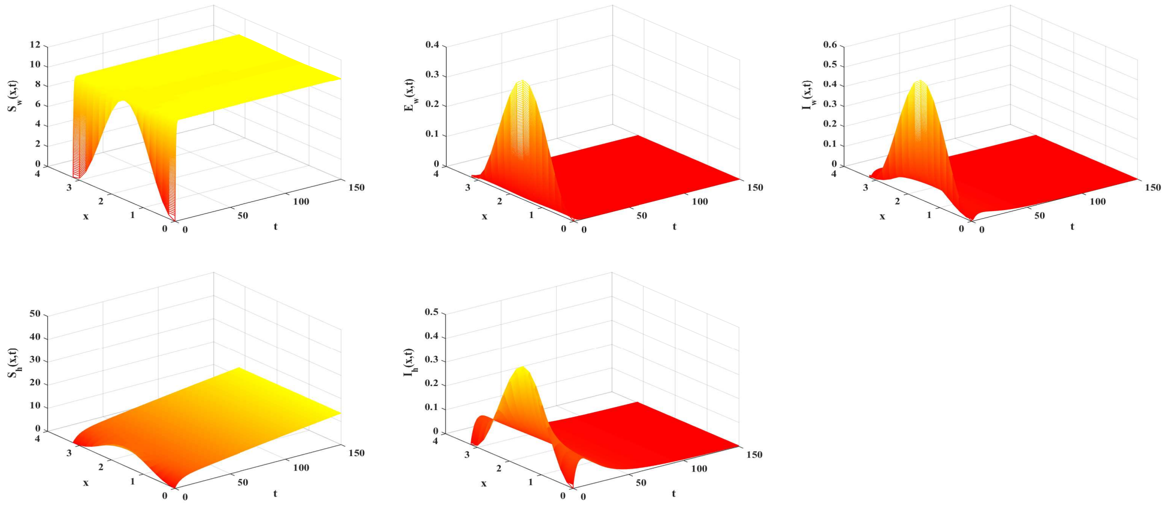


Figure 1. Evolution of numerical solutions for model (2.11) for $R_0 < 1$.

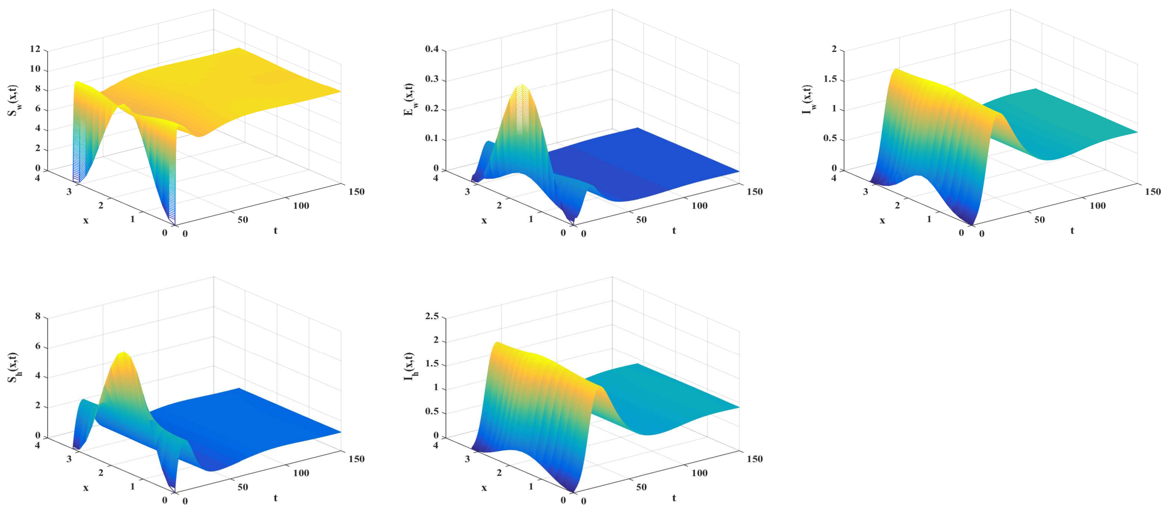
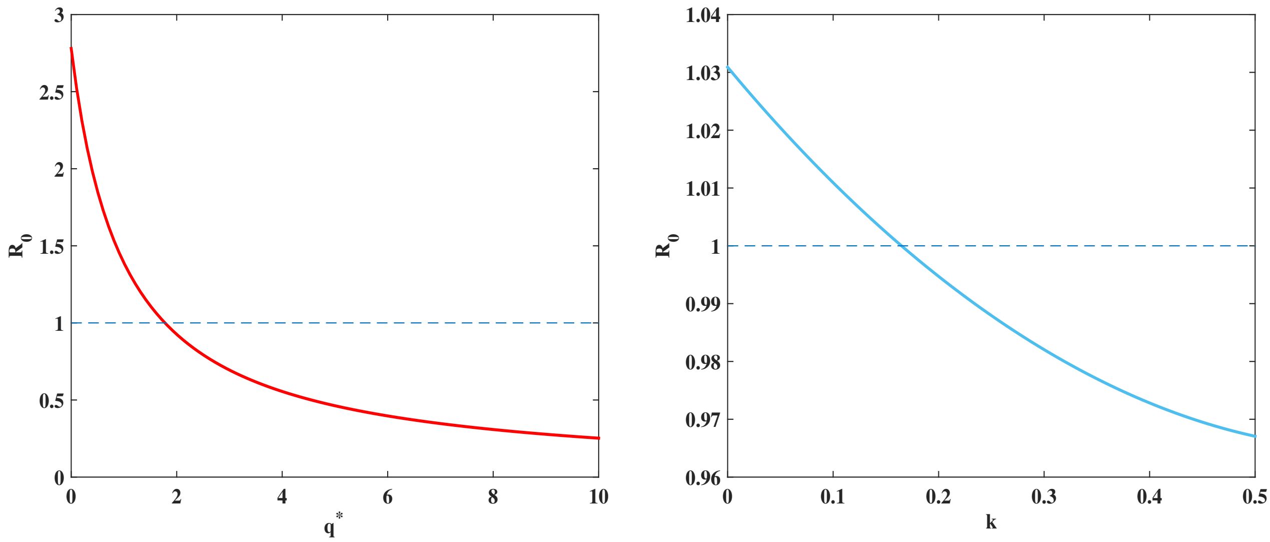
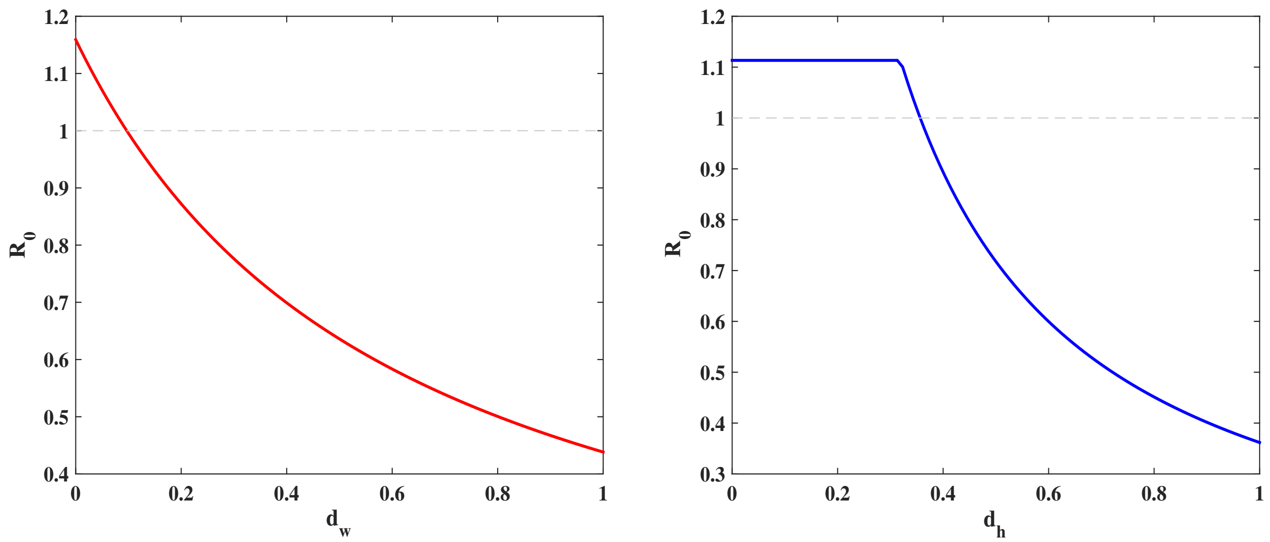


Figure 2. Evolution of numerical solutions for model (2.11) for $R_0 > 1$.



(a) $q(x) = q^*(1 - 0.01\cos(2x))$ (b) $q(x) = 1.7(1 - k\cos(2x))$,

Figure 3. Relationship of R_0 with $q(x)$.



(a) (b)

Figure 4. Relationships of R_0 with d_w and d_h.

6.3. Sensitivity analysis

In this part, we explore the situation where the parameters of model (2.11) are constant. By [26], the basic reproduction number R_0 can be calculated as

$$R_0 = \sqrt{\frac{b^2 \beta_{hw} \beta_{wh} W^* e^{-\tau_w \mu_w} e^{-\tau_h \mu_h}}{N_h^* \mu_w (\mu_h(x) + r_h(x))}}.$$

Figure 5 displays scatter plots that illustrate the correlation between the parameters and the basic reproduction number R_0. Figure 6 displays a bar chart illustrating the partial rank correlation coefficient (PRCC) values for different parameters in relation to R_0. The bars represent the correlation coefficients for each parameter, with some bars above and some below the zero line, indicating positive and negative correlations, respectively. The parameters β_{hw} and β_{wh} show the strong positive correlations, while q, r_h and μ_w show the most negative correlations. The parameter b shows a moderate positive correlation with R_0. Additionally, some parameters τ_1, τ_w, and μ_h show moderate negative correlations. The variations in the remaining parameters have a moderate impact on R_0. The contour diagrams in Figure 7 illustrate the joint effect of two parameters on the basic reproduction number R_0.

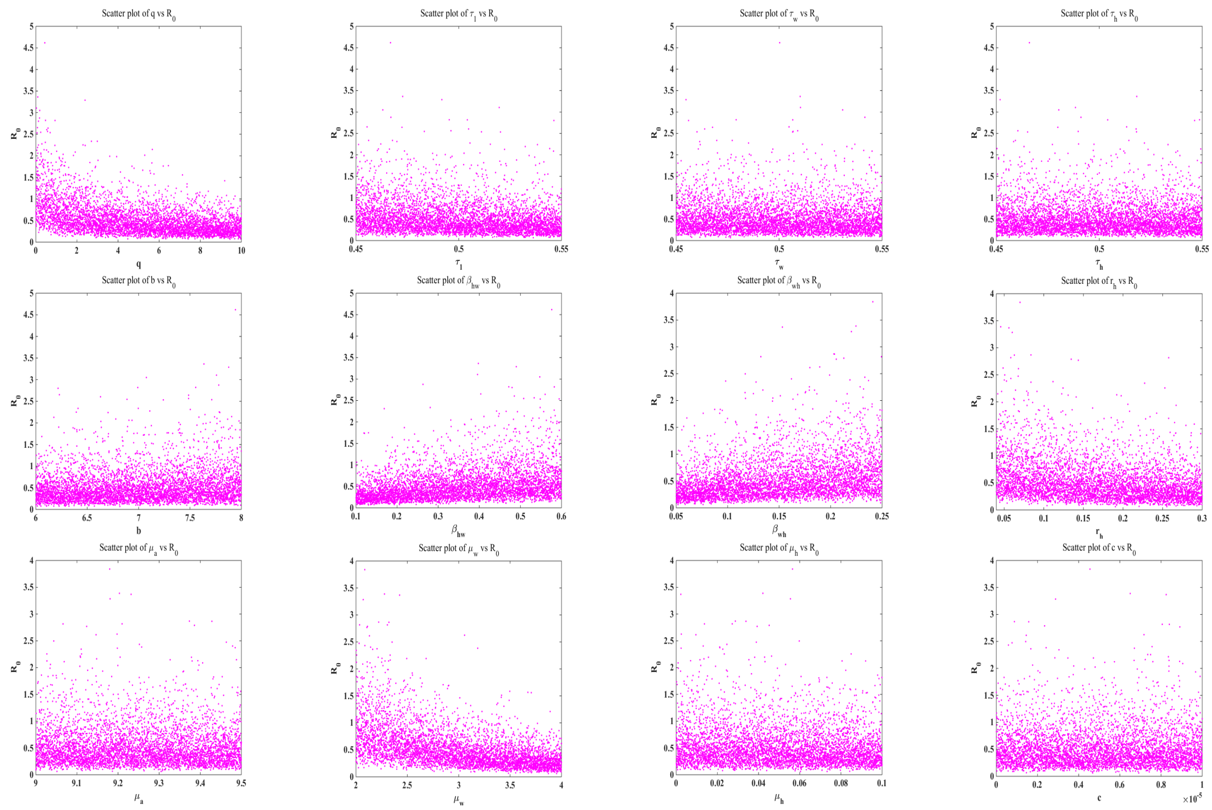


Figure 5. Scatter plots of parameters via R_0.

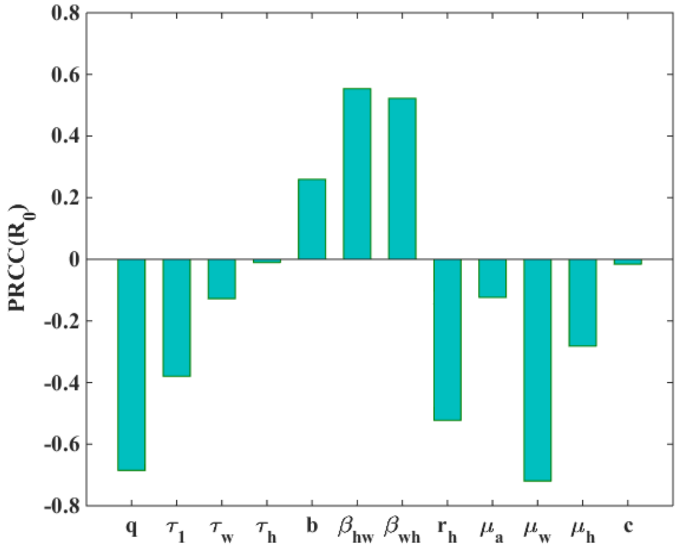


Figure 6. Sensitivity analysis of R_0 for each parameter.

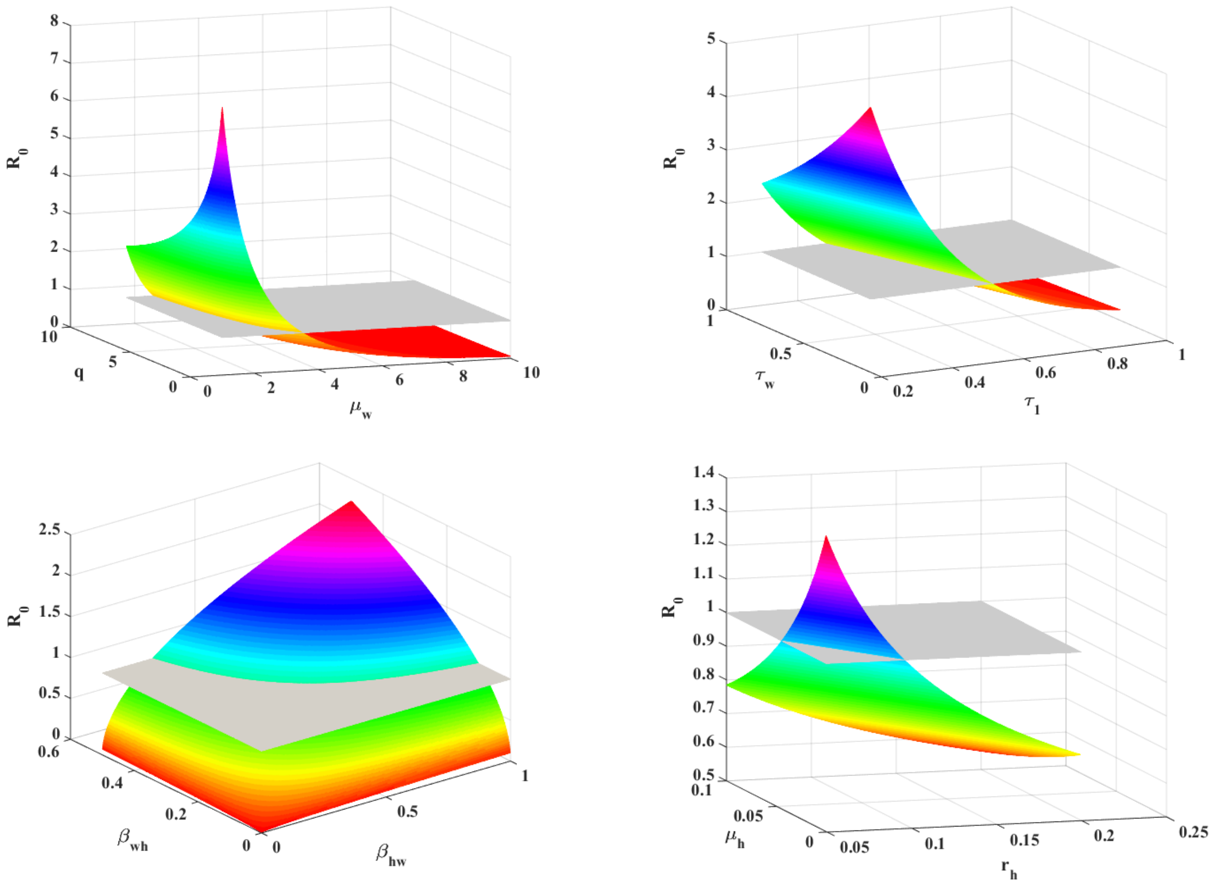


Figure 7. The contour plots of R_0.

7. Conclusions

This paper has presented a comprehensive mathematical model for the spatial spread of malaria, incorporating the effects of nonlocal delays and the release of Wolbachia-infected male mosquitoes as a control strategy. The model's dynamics were primarily governed by the basic reproduction number R_0, which served as a threshold parameter. Through rigorous theoretical analysis, we established that when $R_0 < 1$, the infection-free steady state is globally attractive, indicating that malaria can be effectively controlled under this condition. Conversely, when $R_0 > 1$, the disease persists uniformly, suggesting that malaria will persist in the population. The simulations confirmed the theoretical predictions and provided insights into the impact of the release ratio of Wolbachia-infected males on the transmission and control of malaria. Future research could further explore the integration of Wolbachia release with other vector control methods to optimize malaria elimination efforts.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

References

1. N. J. White, S. Pukrittayakamee, Malaria, *Lancet,* **394** (2019), 1059–1070. https://doi.org/10.1016/S0140-6736(19)31752-7
2. Malaria, *World Health Organization (WHO)*, 2024. Available from: https://www.who.int/news-room/fact-sheets/detail/malaria.
3. D. Lek, M. Shrestha, K. Lhazeen, T. Tobgyel, S. Kandel, G. Dahal, et al., Malaria elimination challenges in countries approaching the last mile: A discussion among regional stakeholders, *Malaria J.*, **23** (2024), 401. https://doi.org/10.1186/s12936-024-05215-3
4. W. Wang, M. Zhou, X. Fan, T. Zhang, Global dynamics of a nonlocal PDE model for Lassa haemorrhagic fever transmission with periodic delays, *Comput. Appl. Math.*, **43** (2024), 140. https://doi.org/10.1007/s40314-024-02662-1

5. W. Wang, X. Wang, X. Fan, Threshold dynamics of a reaction-advection-diffusion waterborne disease model with seasonality and human behavior change, *Int. J. Biomath.*, **18** (2024), 2350106. https://doi.org/10.1142/S1793524523501061

6. P. Wu, Global well-posedness and dynamics of spatial diffusion HIV model with CTLs response and chemotaxis, *Math. Comput. Simul.*, **228** (2024), 402–417. https://doi.org/10.1016/j.matcom.2024.09.020

7. Y. Lou, X. Q. Zhao, A reaction-diffusion malaria model with incubation period in the vector population, *J. Math. Biol.*, **62** (2011), 543–568. https://doi.org/10.1007/s00285-010-0346-8

8. Z. Bai, R. Peng, X. Q. Zhao, A reaction-diffusion malaria model with seasonality and incubation period, *J. Math. Biol.*, **77** (2018), 201–228. https://doi.org/10.1007/s00285-017-1193-7

9. Z. Xu, On the global attractivity of a nonlocal and vector-bias malaria model, *Appl. Math. Lett.*, **121** (2021), 107459. https://doi.org/10.1016/j.aml.2021.107459

10. B. He, Q. R. Wang, Threshold dynamics of a vector-bias malaria model with time-varying delays in environments of almost periodicity, *Nonlinear Anal. Real World Appl.*, **78** (2024), 104078. https://doi.org/10.1016/j.nonrwa.2024.104078

11. A. A. Hoffmann, B. L. Montgomery, J. Popović, I. Iturbe-Ormaetxe, P. H. Johnson, F. Muzzi, et al., Successful establishment of Wolbachia in Aedes populations to suppress dengue transmission, *Nature*, **476** (2011), 454–457. https://doi.org/10.1038/nature10356

12. G. Bian, D. Joshi, Y. Dong, P. Lu, G. Zhou, X. Pan, et al., Wolbachia invades Anopheles stephensi populations and induces refractoriness to Plasmodium infection, *Science*, **340** (2013), 748–751. https://doi.org/10.1126/science.1236192

13. C. L. Jeffries, G. G. Lawrence, G. Golovko, M. Kristan, J. Orsborne, K. Spence, et al., Novel Wolbachia strains in Anopheles malaria vectors from Sub-Saharan Africa, *Wellcome Open Res.*, **3** (2018), 113. https://doi.org/10.12688/wellcomeopenres.14765.2

14. G. L. Hughes, A. Rivero, J. L. Rasgon, Wolbachia can enhance Plasmodium infection in mosquitoes: Implications for malaria control?, *PLoS Pathogens*, **10** (2014), e1004182. https://doi.org/10.1371/journal.ppat.1004182

15. K. Liu, Y. Lou, A periodic delay differential system for mosquito control with Wolbachia incompatible insect technique, *Nonlinear Anal. Real World Appl.*, **73** (2023), 103867. https://doi.org/10.1016/j.nonrwa.2023.103867

16. J. Wu, *Theory and Applications of Partial Functional Differential Equations*, Springer, New York, 1996.

17. R. H. Martin, H. L. Smith, Abstract functional-differential equations and reaction-diffusion systems. *Trans. Am. Math. Soc.*, **321** (1990), 1–44. https://doi.org/10.1090/S0002-9947-1990-0967316-X

18. J. K. Hale, Asymptotic behavior of dissipative systems, *Am. Math. Soc.*, **25** (1988). https://dhttp://dx.doi.org/10.1090/surv/025

19. Y. Zha, W. Jiang, Transmission dynamics of a reaction-advection-diffusion dengue fever model with seasonal developmental durations and intrinsic incubation periods, *J. Math. Biol.*, **88** (2024), 74. https://doi.org/10.1007/s00285-024-02089-6

20. L. Zhang, Z. Wang, X. Q. Zhao, Threshold dynamics of a time periodic reaction-diffusion epidemic model with latent period, *J. Differ. Equations*, **258** (2015), 3011–3036. https://doi.org/10.1016/j.jde.2014.12.032
21. H. L. Smith, *Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems*, American Mathematical Society, Providence, 2008.
22. W. Wang, X. Zhao, A nonlocal and time-delayed reaction-diffusion model of dengue transmission, *SIAM J. Appl. Math.*, **71** (2011), 147–168. https://doi.org/10.1137/090775890
23. H. R. Thieme, Spectral bound and reproduction number for infinite-dimensional population structure and time heterogeneity, *SIAM J. Appl. Math.*, **70** (2009), 188–211. https://doi.org/10.1137/080732870
24. H. L. Smith, X. Q. Zhao, Robust persistence for semidynamical systems, *Nonlinear Anal.*, **47** (2001), 6169–6179. https://doi.org/10.1016/S0362-546X(01)00678-2
25. R. Wu, X. Q. Zhao, A reaction-diffusion model of vector-borne disease with periodic delays, *J. Nonlinear Sci.*, **29** (2019), 29–64. https://doi.org/10.1007/s00332-018-9475-9
26. W. Wang, X. Q. Zhao, Basic reproduction numbers for reaction-diffusion epidemic models, *SIAM J. Appl. Dyn. Syst.*, **11** (2012), 1652–1673. https://doi.org/10.1137/120872942