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https://www.aimspress.com/journal/era

ERA, 33(5): 2972–2988.
DOI: 10.3934/era.2025130
Received: 17 February 2025
Revised: 12 April 2025
Accepted: 27 April 2025
Published: 15 May 2025

Research article

Steady-state bifurcation and regularity of nonlinear Burgers equation with mean value constraint

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Abstract: In this paper, we focus on the steady-state bifurcation problem of the nonlinear Burgers equation within a bounded domain, considering both homogeneous Dirichlet boundary conditions and homogeneous Neumann boundary conditions with a mean value constraint. Unlike previous studies, we develop an enhanced turbulence model by incorporating nonlinear higher-order terms (such as u^2 and u^3) and linear source terms λu into the one-dimensional Burgers equation. Our steady-state bifurcation analysis establishes for the first time how the coupled forward energy cascade and inverse energy transfer mechanisms collectively govern the dynamics of initial flow instability. By combining the spectral theorem for a linear compact operator with the normalized Lyapunov–Schmidt reduction method and the implicit function theorem, we derive the complete criterion for the critical bifurcation condition, the explicit form of the bifurcation solution, and its regularity.

Keywords: nonlinear Burgers equation; mean value constraint; Lyapunov–Schmidt reduction; steady-state bifurcation

1. Introduction

Consider the nonlinear Burgers equation:

$$\begin{cases} \frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2} + \lambda u - au\frac{\partial u}{\partial x} + bu^2 + cu^3, & (x,t) \in (0,\pi) \times (0,\infty), \\ u(x,0) = \psi(x), & x \in (0,\pi). \end{cases} \tag{1.1}$$

Here, $(0,\pi)$ is a continuous bounded domain in $\mathbb{R}$, and $u(x,t)$ represents the fluid velocity at position x and time t. The term λu denotes the linear source term, where $\lambda > 0$ is a system parameter. The term $\gamma\frac{\partial^2 u}{\partial x^2}$ represents the momentum diffusion effect caused by fluid viscosity, with $\gamma > 0$ being the viscosity coefficient satisfying $0 < \gamma < 1$. Furthermore, bu^2 and cu^3 describe the nonlinear effects that dominate the fluid dynamics, where $b > 0$ and $c > 0$ indicate that the nonlinear force increases with the flow velocity u. The nonlinear convective term $au\frac{\partial u}{\partial x}$ characterizes the self-convection effect of the fluid velocity field, with $a > 0$ representing the mechanism of energy transfer from large-scale vortices to small-scale vortices. Notably, when $a = 1$ and $\lambda = b = c = 0$, the equation reduces to the form introduced initially by Bateman in his paper [1], as shown below.

$$\begin{cases} \frac{\partial u}{\partial t} = \gamma\frac{\partial^2 u}{\partial x^2} - u\frac{\partial u}{\partial x}, & (x,t) \in (0,\pi) \times (0,\infty), \\ u(x,0) = \psi(x), & x \in (0,\pi). \end{cases} \tag{1.2}$$

Burgers [2] expanded on the mathematical modeling of turbulence using Eq (1.2), making a significant contribution to fluid mechanics. This equation, subsequently named the Burgers equation, is widely recognized as one of the most prominent models incorporating nonlinear propagation and diffusion effects when $\gamma \to 0$, Eq (1.2) simplifies to the inviscid Burgers equation. Conversely, when $u \to 0$, the viscous Burgers equation (1.2) can be converted to the linear heat equation.

The Burgers equation is characterized by its nonlinear convection and diffusion terms. As a one-dimensional simplified model of the Navier–Stokes equations, it provides an essential theoretical tool for studying complex phenomena such as turbulence and shock waves. Widely applied in fluid mechanics [3], nonlinear acoustics [4] and gas dynamics [5]. Furthermore, its exact solutions and simple form make it a crucial benchmark for developing and validating numerical methods, particularly in turbulence and shock wave simulations under high Reynolds number conditions.

Given that the exact solution of the Burgers equation is known, it can serve as a benchmark solution in numerical simulations of fluid dynamics, enabling the evaluation and comparison of numerical methods. Numerous studies have employed the Burgers equation for comparative analysis and improvement of numerical methods. These include classical approaches such as the Galerkin finite element method [6], cubic Hermite finite element method [7], standard finite element method [8], finite difference method [9], and finite volume method [10], along with more recent advancements. For instance, Kaur et al. [11] developed a compact finite difference scheme for the 1D nonlinear Burgers equation, achieving first-order temporal and fourth-order spatial accuracy, validated theoretically and numerically. Further advancing the field of numerical solutions, Zhang and Yu [12] developed a novel MQ quasi-interpolation meshless method, demonstrating superior accuracy to finite difference methods in solving Burgers' equation shock problems at high Reynolds numbers through theoretical convergence proofs and numerical validation. Similarly, Shi and Yang [13] first propose a temporal two-grid difference method for the nonlinear viscous Burgers equation. The Crank–Nicolson-based scheme proves more efficient than standard finite difference methods while maintaining rigorous L^2 and L^∞ stability.

Beyond these advancements in numerical methodologies, fundamental theoretical understanding of the Burgers equation's dynamical behavior has also been significantly advanced. For example, Ortiz et al. [14] integrated boundary layer theory with PINNS for viscous Burgers solutions, proving their capability to resolve both shock formation and viscous effects. Mouktonglang et al. [15] proved periodic solution uniqueness for the Rosenau–RLW–Burgers equation, showing viscous effects control convergence and oscillation dynamics. Li et al. [16] investigated the dynamic transition behavior of the generalized Burgers equation under one-dimensional periodic boundary conditions, demonstrating that the bifurcation type is determined by parameter b while elucidating the influence of length scale l, dispersion parameter δ, and viscosity coefficient υ on transition characteristics. These findings also offer insights into turbulence control mechanisms.

Nevertheless, several critical questions remain unresolved. To our knowledge, the steady-state bifurcation behavior of nonlinear Burgers equations has not been thoroughly investigated—a research gap with significant implications for elucidating the laminar-to-turbulent transition. To better capture such flow characteristics, and inspired by Li's work [16], we introduce both nonlinear higher-order terms and linear source terms λu into the primitive Burgers equation.

In mathematical physics, bifurcation refers to the fascinating phenomenon of an abrupt transition in the steady state of a dynamic system, triggered by changes in the system's parameters. Specifically, steady-state bifurcation arises from the study of the stability of nonlinear evolution equations. When the system parameters λ reach critical values, the system loses its original stability, leading to the emergence of new fixed points, limit cycles, and other dynamic behaviors. This theory is widely applied in fields such as chemistry, biology, ecology, and engineering, serving as an essential tool for analyzing changes in system stability.

Recent advances in bifurcation research of nonlinear evolution equations demonstrate the universality of bifurcation theory. For instance, Zhang et al. [17] illustrated the existence of a double eigenvalue bifurcation for nonlinear equations with singularities that are fully degenerate of the second order and nondegenerate of the third order, employing the normalized Lyapunov–Schmidt reduction method. Similarly, Wei [18] employed the same methods to investigate a nonlinear parabolic system under nonlinear boundary conditions exhibiting steady-state bifurcation behavior. Guo [19] conducted a study on the steady-state bifurcation of Langford's PDE system emanating from both simple and double eigenvalues. By integrating central manifold theory, Guo further analyzed the direction of the Hopf bifurcation within the PDE system. Pan [20] applied dynamic transition theory to the problem of convection in couple-stress fluid-saturated porous media. The study yielded significant results, including the approximate bifurcation solution, the number of global attractors, and the finding that these attractors encompass four steady-state convection solutions. Wang [21] investigated the chemotaxis-fluid coupled model, while Chen et al. [22, 23] further analyzed the existence and regularity of weak and classical solutions.

Previous studies have focused on the steady-state bifurcation behavior in two-dimensional dynamical systems, while recent work has extended these investigations to one-dimensional settings. Ma et al. [24] employed topological degree theory and bifurcation theory to analyze the Robin problem for the mean curvature equation in a one-dimensional Minkowski space. They provided a calculation of the bifurcation curvature, which not only aids in understanding the existence of solutions for specific types of Minkowski-curvature equations in higher dimensions but also elucidates the properties of these solutions. Extending these analytical approaches to 1D reaction-diffusion systems. Taylan et al. [25] investigated one-dimensional non-self-adjoint reaction-diffusion equations, revealing a unique Hopf bifurcation under zero-mean boundary conditions, and further characterized the primary transition behavior in the Burgers equation.

In this paper, we investigate the bifurcation problem of one-dimensional homogeneous boundary value problems related to the Burgers equation. We develop a simplified model to simulate turbulent behavior, which also provides a foundation for understanding high-dimensional turbulence properties. Fan and Feng [26, 27] integrate linear stability analysis with nonlinear dynamical systems approaches, employing eigenvalue analysis and reduction techniques to elucidate parameter threshold-induced dynamic transition mechanisms and external field coupling effects. Building upon these investigations of dynamic transitions, parallel theoretical advances have been made in steady-state bifurcation research. The seminal works by Li et al. [28–30] established a bifurcation-theoretic framework employing Lyapunov–Schmidt reduction and eigenvalue analysis to characterize steady-state bifurcation phenomena in diffusion systems. Inspired by these findings, we examine the influence of nonlinear, higher-order viscous effects in turbulence on the emergence of steady-state bifurcation phenomena. Using the bifurcation and transition theory developed by Ma and Wang [31,32], we analyze the steady-state bifurcation of the nonlinear Burgers equation under various homogeneous boundary conditions.

Considering the System (1.1) with Dirichlet boundary conditions as follows:

$$\begin{cases} \frac{\partial u}{\partial t} = \gamma\frac{\partial^2 u}{\partial x^2} + \lambda u - au\frac{\partial u}{\partial x} + bu^2 + cu^3, & (x,t) \in (0,\pi)\times(0,\infty), \\ u(0) = 0, \ \ u(\pi) = 0, & \\ u(x,0) = \psi(x), & x \in (0,\pi). \end{cases} \tag{1.3}$$

The Dirichlet boundary condition enforces zero fluid velocity at the boundary, corresponding physically to the no-slip condition between fluid and solid walls where complete momentum exchange occurs. Significantly, the steady-state bifurcation in System (1.3) captures the essential dynamics of the initial instability stage in wall-bounded turbulence [33], providing critical insights into the formation mechanisms of turbulent boundary layers.

Researching the System (1.1) with mean value constraint Neumman boundary conditions as follows

$$\begin{cases} \frac{\partial u}{\partial t} = \gamma\frac{\partial^2 u}{\partial x^2} + \lambda u - au\frac{\partial u}{\partial x} + bu^2 + cu^3, & (x,t) \in (0,\pi)\times(0,\infty), \\ \frac{\partial u}{\partial x}\mid_{x=0}= 0, \ \ \frac{\partial u}{\partial x}\mid_{x=\pi}= 0, & \\ \int_0^\pi u\mathrm{d}x = 0, & t \in (0,\infty), \\ u(x,0) = \psi(x), & x \in (0,\pi), \end{cases} \tag{1.4}$$

The Neumann boundary condition with mean constraint specifies a zero gradient for the velocity field u at the boundary, while the mean constraint enforces a zero spatial average of u. In turbulent flow simulations, these mean-constrained Neumann conditions rigorously satisfy the mass conservation law for shear flows. By constraining the total system momentum, they effectively suppress numerical oscillations at high Reynolds numbers and significantly enhance solution stability. Importantly, the steady-state bifurcation in System (1.4) directly captures the shear flow instability [34], with this simplified model providing crucial theoretical insight into the onset of flow instabilities.

The rest of the paper is organized as follows: Section 2 formulates the abstract operator equation for Eq (2.1) using bifurcation theory. Section 3 rigorously derives the explicit bifurcation solutions for the Burgers equation under two types of homogeneous boundary conditions and establishes their regularity. Building on these results, Section 4 reveals the universality of critical parameters, the stabilizing effect of dissipation terms, and the influence of higher-order nonlinear terms on laminar-turbulent transition under different boundary conditions, with numerical simulations demonstrating both subcritical and supercritical bifurcation scenarios.

2. Preliminary

In this paper, we focus on the existence of a bifurcation solution of (1.1), which satisfies the following equation

$$\gamma\frac{d^2u}{dx^2} + \lambda u - au\frac{du}{dx} + bu^2 + cu^3 = 0, \qquad x \in (0,\pi). \tag{2.1}$$

(2.1) with Dirichlet boundary and Neumann boundary conditions with mean value constraint, respectively, are as follows:

$$\begin{cases} \gamma\frac{d^2u}{dx^2} + \lambda u - au\frac{du}{dx} + bu^2 + cu^3 = 0, & x \in (0,\pi), \\ u(0) = u(\pi) = 0, \end{cases} \tag{2.2}$$

$$\begin{cases} \gamma\frac{d^2u}{dx^2} + \lambda u - au\frac{du}{dx} + bu^2 + cu^3 = 0, & x \in (0,\pi), \\ \frac{\partial u}{\partial x}\mid_{x=0} = \frac{\partial u}{\partial x}\mid_{x=\pi} = 0, \\ \int_0^\pi u dx = 0, & \text{in} \quad t \in (0,\infty). \end{cases} \tag{2.3}$$

First, we denote by $L^2(0,\pi)$ the Lebesgue space of square productible functions defined in $(0,\pi)$; let H be the Hilbert space $H = L^2(0,\pi)$. We define H_1 under the Dirichlet boundary and the Neumann boundary conditions with mean value constraint, respectively, as follows:

$$H_1 = \{u \in H^2[0,\pi] \mid u(0) = u(\pi) = 0\},$$

$$H_1 = \{u \in H^2[0,\pi] \mid \int_0^\pi u(x)dx = 0, \frac{du}{dx}\mid_{x=0} = \frac{du}{dx}\mid_{x=\pi} = 0\}.$$

Then, we define the linear operators $L_\lambda = -A + B_\lambda$ and the nonlinear operator $G : H_1 \to H$ by

$$-Au = \gamma\frac{d^2u}{dx^2}, \qquad B_\lambda u = \lambda u,$$

$$G(u) = -au\frac{du}{dx} + bu^2 + cu^3.$$

It is easy to see that L_λ is a completely continuous field.

We thus obtain the equivalent operator equation of the Burgers equation (2.1) as follows:

$$L_\lambda u + G(u) = 0. \tag{2.4}$$

Definition 2.1. [31] Suppose $(0,\lambda), \lambda \in R^1$ is a trivial solution of Eq (2.4). If there exists $\lambda_0 \in R^1$ such that when $\lambda < \lambda_0$ or $\lambda > \lambda_0$, Eq (2.4) has a nontrivial solution $(u_\lambda, \lambda) \neq (0,\lambda)$ and $\lim_{\lambda\to\lambda_0}(u_\lambda,\lambda) = (0,\lambda_0)$, $\lim_{\lambda\to\lambda_0}\|u_\lambda\|_{H_1} = 0$, then it is said that the Eq (2.4) undergoes a bifurcation solution at $(0,\lambda_0)$.

Definition 2.2. [31] Let H and H_1 be Hilbert spaces, and $H_1 \subset H$ being dense and compactly embedded. A linear operator $L_\lambda : H_1 \to H$ is called a completely continuous field if

$$\begin{cases} L_\lambda = -A + B_\lambda : H_1 \to H, \\ A : H_1 \to H \text{ is a linear isomorphism with eigenvalues having positive real parts,} \\ B_\lambda : H_1 \to H \text{ is a linear compact operator.} \end{cases}$$

Definition 2.3. [31] Let $u_\lambda \in H_1$ be a bifurcation solution of Eq (2.4) at $\lambda = \lambda_0$. The bifurcation solution is called regular, or nondegenerate, if the differential operator of $L_\lambda + G(\cdot, \lambda)$ at u_λ

$$L_\lambda + D_u G(u_\lambda, \lambda) : H_1 \to H,$$

is a linear isomorphism for all $0 < |\lambda - \lambda_0| < \varepsilon$, where ε is sufficiently small.

Lemma 2.1. [31] Let $L : H_1 \to H$ be a linear completely continuous field. Then the following conclusions hold:

(i) if $\{\lambda_k \mid k \geq 1\} \subset C$ are the eigenvalues (counting multiplicity) of L, then we can choose eigenvectors $\{e_k\} \subset H_1$ of L and eigenvectors $\{e_k^*\} \subset H_1^*$ of L^* such that

$$\left\langle e_i, e_j^* \right\rangle_H \begin{cases} = 0 & i \neq j, \\ \neq 0 & i = j; \end{cases}$$

(ii) if $\rho = \lambda_k = \cdots = \lambda_{n-k} (n \geq 1)$ is an eigenvalue of L with algebraic multiplicity $m = n + 1$ and geometric multiplicity $r = 1$, then for any nonzero constant $\sigma \neq 0$, we can choose eigenvectors $\{e_k, \ldots, e_{k+n}\}$ of L and eigenvectors $\{e_k^*, \ldots, e_{k+n}^*\}$ of L^* such that we have

$$\begin{cases} Le_k = \rho e_k, \\ Le_{k+1} = \rho e_{k+1} + \sigma e_k, \\ \ldots \\ Le_{k+n} = \rho e_{k+n} + \sigma e_{k+n-1}, \end{cases}$$

$$\begin{cases} L^* e_{k+n}^* = \rho e_{k+n}^*, \\ L^* e_{k+n-1}^* = \rho e_{k+n-1}^* + \sigma e_{k+n}^*, \\ \ldots \\ L^* e_k^* = \rho e_k^* + \sigma e_{k+1}^*; \end{cases}$$

(iii) H can be decomposed into the following direct sum of spaces

$$H = \bar{E}_1 \oplus \bar{E}_2, \quad E_1 = span\{e_k \mid k \geq 1\}, \quad E_2 = \{\upsilon \in H_1 \mid \langle \upsilon, e_k \rangle_H = 0, \forall k \geq 1\},$$

when $\bar{E}_1$ and $\bar{E}_2$ are the closures of E_1 and E_2 in H, respectively.

(iv) E_1 and E_2 are invariant subspaces of L

$$L : E_i \to \bar{E}_i, \quad i = 1, 2,$$

Moreover, $\ell = L\,|_{E_2}$ has an inverse $\ell^{-1} = \bar{E}_2 \to E_2 \subset \bar{E}_2$, such that

$$\lim_{n \to \infty} \left\| \ell^{-n} \right\|_H^{\frac{1}{n}} = 0, \forall u \in \bar{E}_2.$$

(v) For any $u \in H$, there exists a generalized Fourier expansion as follows

$$u = \sum_k x_k e_k + \upsilon, \quad \upsilon \in \bar{E}_2, \quad x_k = \langle u, e_k^*, \rangle_{H^*}.$$

In particular, if $L : H_1 \to H$ has a complete spectrum, there exists a complete Fourier expansion as follows:

$$u = \sum_{k}^{\infty} x_k e_k, \quad x_k = \langle u, e_k^* \rangle_H .$$

Lemma 2.2. [31] Let x_λ be the bifurcation equation of Eq (2.4), where

$$L_\lambda^1 x + P_1 G(x + \phi(x,\lambda), \lambda) = 0, \tag{2.5}$$

has a bifurcation solution at $\lambda = \lambda_0$. The bifurcation solution $u_\lambda = x + \phi(x,\lambda)$ of Eq (2.3) is regular if and only if x_λ is regular with respect to Eq (2.5).

3. Bifurcation of the Burgers equation

We are ready to state the main result and the proof process in this section. For the System (2.2), we have the following bifurcation theorem.

Theorem 3.1. System (2.2) bifurcates a bifurcation solution from $(u,\lambda) = (0,\gamma)$ under the Dirichlet boundary conditions, and the expression of the bifurcation solution is given by

$$\bar{u} = -\frac{3\pi}{8b}(\lambda - \gamma)\sin x + o(|\lambda - \gamma|^2).$$

Proof. We compute the eigenvalues and eigenvectors of L_λ.

Let ρ_k $(k = 1, 2 \dots)$ and e_k $(k = 1, 2 \dots)$ be the eigenvalues and eigenvectors of the following eigenvalue problem

$$\begin{cases} -\frac{\mathrm{d}^2 e_k}{\mathrm{d}x^2} = \rho_k e_k, & x \in (0,\pi), \\ e_k(0) = e_k(\pi) = 0, \\ \int_0^\pi e_k^2 \mathrm{d}x = 1. \end{cases} \tag{3.1}$$

We obtain the specific form of the eigenvalues $\rho_k (k = 1, 2 \dots)$ and the corresponding eigenvectors e_k $(k = 1, 2 \dots)$ of Eq (3.1)

$$\rho_k = k^2, \qquad e_k = \sqrt{\frac{2}{\pi}} \sin kx.$$

Hence, the eigenvalues and the corresponding eigenvectors of the operator L_λ in Eq (2.4) are

$$\{\beta_k = \lambda - \gamma\rho_k \mid k = 1, 2 \dots\}, \qquad \{e_k = \sqrt{\frac{2}{\pi}} \sin kx \mid k = 1, 2 \dots\}.$$

According to Lemma (2.1), it can be obtained that the eigenvectors $\{e_k \mid k = 1, 2 \dots\}$ of L_λ form an orthogonal basis for H_1. Therefore, it is easy to obtain that the first eigenvalue of L_λ and the corresponding eigenvector are

$$\beta_1(\lambda) = \lambda - \gamma, \quad \beta_j(\gamma) \neq 0, \quad j \geq 2, \quad e_1 = \sqrt{\frac{2}{\pi}} \sin x.$$

By the spectral theorem for the linear completely continuous fields (Lemma 2.1), we can decompose the space H_1 and H in a neighborhood of $\lambda = \gamma$ as follows:

$$H_1 = E_1 \oplus E_2, \quad H = E_1 \oplus \bar{E}_2,$$

where

$$E_1 = span\{e_1\}, \quad E_2 = span\{e_2, e_3, \dots\},$$

Then, the linear operator L_λ can be decomposed in a neighborhood of $\lambda = \gamma$ as

$$L_\lambda = L_\lambda^1 \oplus L_\lambda^2, \quad L_\lambda^1 : E_1 \to E_1, \quad L_\lambda^2 : E_2 \to \bar{E}_2.$$

Here, E_1denotes the finite-dimensional kernel space of operatorL_λ, whileE_2represents its corresponding infinite-dimensional complementary subspace (with its closure denoted by $\bar{E}_2$). Under this decomposition, the restricted operatorL_λ^1acting on E_1 constitutes a finite-dimensional linear operator that reduces to the zero operator at the critical parameter λ_0, whereas the restricted operator L_λ^2 acting on E_2 forms a bounded linear operator that preserves its invertibility at λ_0.

We have $u = u_1 + u_2$ with $u \in H_1$, where $u_1 \in E_1$ and $u_2 \in E_2$. Now assume

$$u_1 = x_1 e_1, \quad u_2 = \sum_{j=2}^{\infty} y_j e_j, \quad x_1, y_j \in R.$$

When the linear operator L_λ possesses a nontrivial kernel E_1 at the critical parameter λ_0, the direct solution becomes problematic due to the operator's non-invertibility. To address this, we employ the Lyapunov–Schmidt reduction method, decomposing Eq (2.2) into two subproblems on the kernel space E_1 and its complementary subspace E_2. The equation on E_2 is first solved using the implicit function theorem, thereby reducing the original problem to a finite-dimensional equation on E_1. This method essentially transforms an infinite-dimensional problem into a finite-dimensional one through dimensional reduction.

By applying the normalized Lyapunov–Schmidt reduction method, we obtain the bifurcation solution for Eq (2.2). Substituting u_1 and u_2 into Eq (2.1), we have

$$\begin{aligned} &\beta_1(\lambda)x_1 - a\langle (x_1 e_1 + \sum_{j=2}^{\infty} y_j e_j)(x_1 \frac{\mathrm{d}e_1}{\mathrm{d}x} + \sum_{j=2}^{\infty} y_j \frac{\mathrm{d}e_j}{\mathrm{d}x}), e_1\rangle \\ &+ b\langle (x_1 e_1 + \sum_{j=2}^{\infty} y_j e_j)^2, e_1\rangle + c\langle (x_1 e_1 + \sum_{j=2}^{\infty} y_j e_j)^3, e_1\rangle = 0, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} &\beta_j(\lambda)y_j - a\langle (x_1 e_1 + \sum_{j=2}^{\infty} y_j e_j)(x_1 \frac{\mathrm{d}e_1}{\mathrm{d}x} + \sum_{j=2}^{\infty} y_j \frac{\mathrm{d}e_j}{\mathrm{d}x}), e_j\rangle \\ &+ b\langle (x_1 e_1 + \sum_{j=2}^{\infty} y_j e_j)^2, e_j\rangle + c\langle (x_1 e_1 + \sum_{j=2}^{\infty} y_j e_j)^3, e_j\rangle = 0, \quad j \geq 2. \end{aligned} \tag{3.3}$$

Let us get the approximate expression for the reduction Eq (3.3) as

$$\beta_j(\lambda)y_j - a\langle x_1^2 e_1 \frac{\mathrm{d}e_1}{\mathrm{d}x}, e_j\rangle + b\langle x_1^2 e_1^2, e_j\rangle + \circ(x_1^2) = 0, \quad j \geq 2, \tag{3.4}$$

Note that

$$\langle e_1 \frac{\mathrm{d}e_1}{\mathrm{d}x}, e_2\rangle = \sqrt{\frac{1}{2\pi}}, \quad \langle e_1 \frac{\mathrm{d}e_1}{\mathrm{d}x}, e_j\rangle = 0, \ \ j \geq 3,$$
$$\langle e_1^2, e_1\rangle = \frac{8\sqrt{2}}{3\pi\sqrt{\pi}}, \quad \langle e_1^2, e_j\rangle = 0, \ \ j \geq 2.$$

Now, with Eq (3.3), we can calculate

$$y_2 = \frac{a x_1^2 \beta^{-1}(\lambda)}{\sqrt{2\pi}}, \quad y_j = \circ(x_1^2), \qquad j \geq 3,$$

substituting $y_j (j = 2, 3, \dots)$ into Eq (3.2) yields

$$\beta_1(\lambda)x_1 + b x_1^2 \langle\langle e_1^2, e_1\rangle + \circ(x_1^2) = 0,$$

namely,

$$\beta_1(\lambda)x_1 + \frac{8\sqrt{2}b}{3\pi\sqrt{\pi}} x_1^2 + \circ(x_1^2) = 0, \tag{3.5}$$

the approximate equation corresponding to Eq (3.5) is

$$\beta_1(\lambda)x_1 + \frac{8\sqrt{2}b}{3\pi\sqrt{\pi}} x_1^2 = 0, \tag{3.6}$$

therefore, Eq (3.6) has a bifurcation solution in a neighborhood of (x, λ)=$(0, \gamma)$, which indicates that Eq (3.6) undergoes a bifurcation at (x, λ)=$(0, \gamma)$, and the expression for the bifurcation solution branch is as follows:

$$x_1 = -\frac{3\pi\sqrt{\pi}\beta_1(\lambda)}{8\sqrt{2}b},$$

now, we give the expression for the bifurcation solution of Eq (2.2)

$$\bar{u} = -\frac{3\pi}{8b}(\lambda - \gamma)\sin x + \circ(|\lambda - \gamma|^2).$$

□

Remark 3.1. Theorem 3.1 establishes that under Dirichlet boundary conditions, the Burgers equation exhibits a steady-state bifurcation at the critical parameter $\lambda_0 = \gamma$, yielding a bifurcated solution that provides a one-dimensional simplified representation of the steady streak velocity profile in wall-bounded turbulence. For $\lambda < \gamma$, the negative real part of the eigenvalue guarantees system stability, whereas when $\lambda > \gamma$, the laminar base state becomes unstable and transitions to a new steady flow configuration characterized by the $\sin x$ modal structure.

Building upon the existence and analytical expressions of bifurcation solutions for System (2.2) established in Theorem 3.1, we further investigate their regularity properties. The regularity analysis of the bifurcation solution $\bar{u}$ is of fundamental importance as it directly governs the spatiotemporal evolutionary dynamics of system solutions and numerical computation accuracy. We further study the regularity of the bifurcation solution $\bar{u}$ of Eq (2.2), which gives the following conclusion.

Theorem 3.2. Let x_1 be a bifurcation solution of Eq (3.6) at $\lambda = \gamma$. So the bifurcation solution $\bar{u}$ of Eq (3.3) is regular if and only if x_1 is regular with respect to Eq (3.5).

Proof. First, considering the regularity of the bifurcation solution of Eq (3.6), we consider the derivative of Eq (3.6) with respect to x_1, as follows:

$$\beta_1(\lambda) + \frac{16\sqrt{2}b}{3\pi\sqrt{\pi}}x_1.$$

This shows that the bifurcation solution of Eq (3.6) is regular. Lemma 2.2 further establishes that the bifurcation solution $\bar{u}$ of Eq (2.4) is also regular. The bifurcation solution $\bar{u}$ of Eq (2.2) is regular. □

Then, we consider the Neumann boundary condition with mean value constraints. For system (2.3) we have the following bifurcation theorem.

Theorem 3.3. System (2.3) bifurcates a bifurcation solution from $(u, \lambda) = (0, \gamma)$ under the Neumann boundary condition with the mean value constraint, and the expression of the bifurcation solution is as follows:

$$\bar{u} = -\frac{3\pi}{4a}(\lambda - \gamma)\cos x + o(|\lambda - \gamma|^2).$$

Proof. By solving $L_\lambda = -A + B_\lambda$ for all eigenvalues and eigenvectors, following the proof of Theorem 3.1, we can obtain the eigenvalues and eigenvectors of Eq (2.4) as follows:

$$\{\beta_k(\lambda) = \lambda - \gamma k^2 \mid k = 1, 2\ldots\}, \quad \{e_k = \sqrt{\frac{2}{\pi}}\cos kx \mid k = 1, 2\ldots\}.$$

Through Lemma 2.1, we obtain that the eigenvectors $\{e_k \mid k = 1, 2\ldots\}$ of L_λ form an orthogonal basis for H_1. Hence, it can be easily obtained that

$$\beta_1(\lambda) = \lambda - \gamma, \quad \beta_j(\gamma) \neq 0, \quad j \geq 2,$$

the corresponding eigenvectors are

$$e_1 = \sqrt{\frac{2}{\pi}}\cos x.$$

Using the spectral theorem (Lemma 2.1), we can decompose the space H_1 and the operator L_λ in a neighborhood of $\lambda = \gamma$ as follows:

$$H_1 = E_1 \oplus E_2, \quad H = E_1 \oplus \bar{E}_2,$$

where

$$E_1 = span\{e_1\}, \quad E_2 = span\{e_2, e_3, \ldots\},$$

the linear operator L_λ can be decomposed near $\lambda = \gamma$ as

$$L_\lambda = L_\lambda^1 \oplus L_\lambda^2, \quad L_\lambda^1 : E_1 \to E_1, \quad L_\lambda^2 : E_2 \to \bar{E}_2,$$

Now, we let $u \in H_1$, and $u = u_1 + u_2$, and assume that

$$u_1 = x_1 e_1, \quad u_2 = \sum_{j=2}^{\infty} y_j e_j, \quad x_j, y_j \in R,$$

where $u_1 \in E_1$ and $u_2 \in E_2$.

We use the Lyapunov-Schmidt reduction method to obtain a bifurcated solution to Eq (2.4). First we substitute u_1 and u_2 into Eq (1.4) to obtain

$$\begin{aligned}\beta_1(\lambda)x_1 - a\langle(x_1e_1 + \sum_{j=2}^{\infty} y_je_j)(x_1\frac{\mathrm{d}e_1}{\mathrm{d}x} + \sum_{j=2}^{\infty} y_j\frac{\mathrm{d}e_j}{\mathrm{d}x}), e_1\rangle \\ +b\langle(x_1e_1 + \sum_{j=2}^{\infty} y_je_j)^2, e_1\rangle + c\langle(x_1e_1 + \sum_{j=2}^{\infty} y_je_j)^3, e_1\rangle = 0,\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\beta_j(\lambda)y_j - a\langle(x_1e_1 + \sum_{j=2}^{\infty} y_je_j)(x_1\frac{\mathrm{d}e_1}{\mathrm{d}x} + \sum_{j=2}^{\infty} y_j\frac{\mathrm{d}e_j}{\mathrm{d}x}), e_j\rangle \\ +b\langle(x_1e_1 + \sum_{j=2}^{\infty} y_je_j)^2, e_j\rangle + c\langle(x_1e_1 + \sum_{j=2}^{\infty} y_je_j)^3, e_j\rangle = 0, \quad j \geq 2\end{aligned} \tag{3.8}$$

the approximate equation for Eq (3.8) is

$$\beta_j(\lambda)y_j + a\frac{2\sqrt{2}}{\pi\sqrt{\pi}}x_1y_j + b\langle x_1^2e_1^2, e_j\rangle + o(x_1^2) = 0, \quad j \geq 2,$$

Note that

$$\langle e_1^2, e_2\rangle = \frac{1}{\sqrt{2\pi}}, \quad \langle e_1^2, e_j\rangle = 0, \quad j \geq 3,$$

hence, we solve for

$$y_2 = -\frac{bx_1^2}{\sqrt{2\pi}[\sqrt{2\pi}\beta_2\lambda + a(\frac{2}{\pi})^{\frac{3}{2}}x_1]}, \qquad y_j = o(x_1^2), \quad j \geq 3,$$

substitute $y_j(j = 2, 3, \dots)$ into Eq (3.7), noting that

$$\langle e_1\frac{\mathrm{d}e_1}{\mathrm{d}x}, e_1\rangle = -\frac{4\sqrt{2}}{3\pi\sqrt{\pi}}, \quad \langle e_1^2, e_1\rangle = 0,$$

we have

$$\beta_1(\lambda)x_1 - ax_1^2\langle e_1\frac{\mathrm{d}e_1}{\mathrm{d}x}, e_1\rangle + o(x_1^2) = 0,$$

that is

$$\beta_1(\lambda)x_1 + \frac{4\sqrt{2}a}{3\pi\sqrt{\pi}}x_1^2 + o(x_1^2) = 0, \tag{3.9}$$

the approximate equation corresponding to Eq (3.9) is

$$\beta_1(\lambda)x_1 + \frac{4\sqrt{2}a}{3\pi\sqrt{\pi}}x_1^2 = 0. \tag{3.10}$$

Then, Eq (3.9) has a bifurcation solution in the neighborhood of (x, λ)=$(0, \gamma)$. This indicates that Eq (3.9) undergoes a bifurcation at (x, λ)=$(0, \gamma)$, and the expression for the bifurcation solution branch is as follows

$$x_1 = -\frac{3\pi\sqrt{\pi}\beta_1(\lambda)}{4\sqrt{2}a},$$

thus, we obtain the expression for the bifurcation solution of Eq (2.3)

$$\bar{u} = -\frac{3\pi}{4a}(\lambda - \gamma)\cos x + o(|\lambda - \gamma|^2).$$

Based on the above analysis, we have successfully proven Theorem 3.3. □

Remark 3.2. Theorem 3.3 establishes that the system undergoes bifurcation at the first eigenvalue, which consequently determines the stability threshold. The mean-constrained boundary condition plays a pivotal role in the Lyapunov–Schmidt reduction, enabling the exact determination of the bifurcation solution's analytical form. This solution captures the dynamical transition of confined shear flow from steady-state destabilization to finite-amplitude coherent mode formation.

Regarding the regularity of the bifurcation solution $\bar{u}$ of Eq (2.3), we have the following conclusions.

Theorem 3.4. Let x_1 be a bifurcation solution of Eq (3.9) at $\lambda = \gamma$. The bifurcation solution $\bar{u}$ of Eq (2.4) is regular if and only if x_1 is regular with respect to Eq (3.10).

Proof. First, let us consider the regularity of the bifurcation solution of Eq (3.6). Then, we consider the derivative of Eq (3.6) with respect to x_1, which is as follows:

$$\beta_1(\lambda) + \frac{8\sqrt{2}a}{3\pi\sqrt{\pi}}x_1.$$

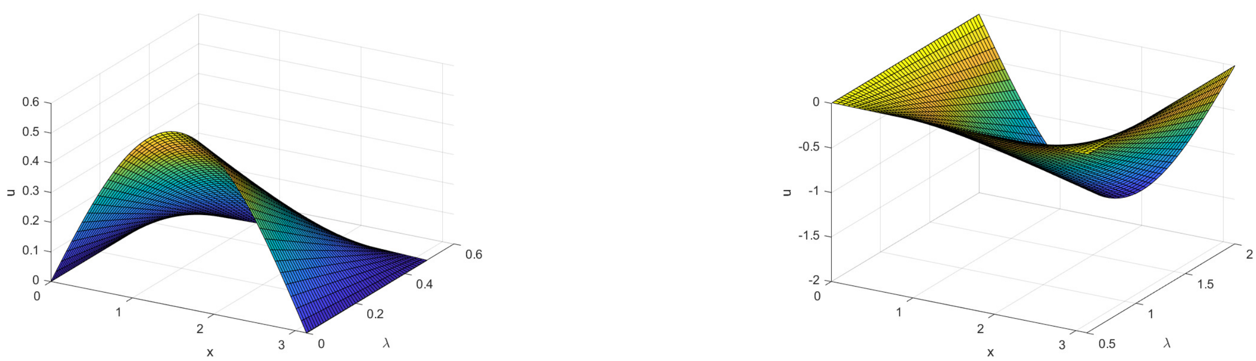
Substituting the bifurcation solution of Eq (3.6) into the above expression yields $-(\lambda - \gamma)$. If a sufficiently small decentered neighborhood $-(\lambda - \gamma) \neq 0$ of $\lambda = \gamma$ holds, this shows that the bifurcation solution of Eq (3.6) is regular. According to Lemma 2.2, we prove that the bifurcation solution $\bar{u}$ of Eq (2.4) is regular. Hence, the bifurcation solution $\bar{u}$ of Eq (2.3) is regular. □

4. Conclusions

Based on the spectral theorem analysis of the linear fully continuous field, this paper uses the normalized Lyapunov–Schmidt reduction method to rigorously derive the expressions of the bifurcation solutions of the one-dimensional nonlinear Burgers equation under Dirichlet boundary conditions and Neumann boundary conditions with mean-value constraints and demonstrates the regularity of the bifurcation solutions. We find that the bifurcation points of Systems (2.2) and (2.3) are $(0, \gamma)$, and the bifurcation solutions both maintain regularity. This also shows that when the system undergoes bifurcation, the critical parameter λ_0 has universality independent of boundary conditions, and the dissipation term plays a stabilizing role under both boundary conditions.

However, the expressions of the bifurcation solutions corresponding to the two types of boundaries are entirely different. From the structure of the bifurcation solutions, the $\sin x$ modal solution generated by the Dirichlet boundary reflects the characteristics of wall-constrained flow, with its amplitude showing an inverse proportionality to the coefficient b of the nonlinear square term clear manifestation of the energy dissipation role played by the u^2 term. The $\cos x$ modal solution generated by the Neumann boundary with mean-value constraint describes the characteristics of confined shear flow, where the amplitude is regulated by the coefficient a of the convection term, revealing the energy transport process dominated by non-convection terms. In turbulent motion, the spatial modal differences of the bifurcation solutions indicate different transition paths: the Dirichlet boundary case corresponds to the formation of wall-turbulence streak structures [35], whereas the Neumann case corresponds to the evolution process of coherent structures in confined shear flow [36].

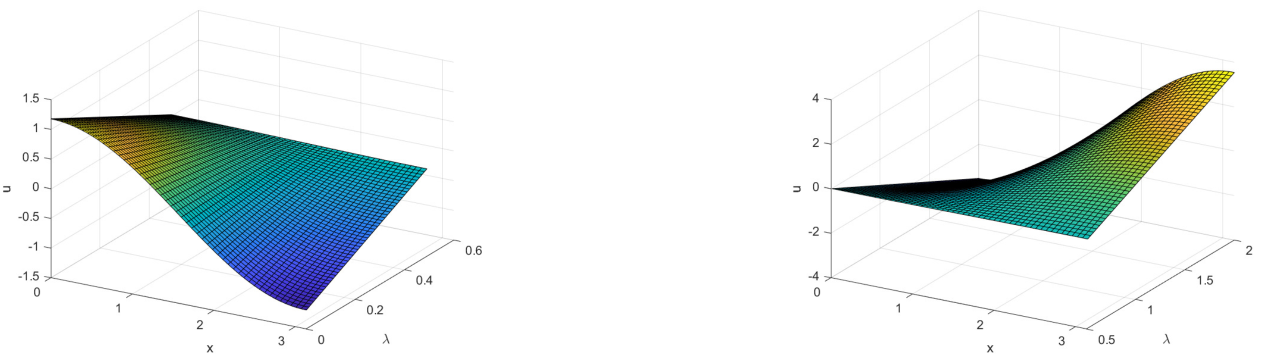
To better visualize the dependence of the bifurcation solution $\bar{u}$ on the system parameter λ, we set the viscous coefficient $\gamma = 0.5$, nonlinear convection coefficient $a = 1$, and higher-order nonlinear coefficients $b = c = 1$. Based on the conclusions of Theorems 3.1 and 3.2, we employ MATLAB to generate three-dimensional plots illustrating the bifurcation behavior of solutions $\bar{u}$ for Systems (2.2) and (2.3) at the critical parameter λ_0, demonstrating both subcritical and supercritical bifurcation scenarios.



(a) **(b)**

Figure 1. Dirichlet boundary condition: (a) subcritical $\lambda < \gamma$; (b) supercritical $\lambda > \gamma$.

Figure 1 presents a three-dimensional visualization of the System (2.2) dynamical transition: When $\lambda < \gamma$, viscous dissipation dominates, resulting in a smooth parabolic velocity profile characteristic of stable laminar flow; when $\lambda > \gamma$, the system develops $\sin x$ modulated streak structures with alternating high- and low-speed bands between near-wall regions and channel center, where nonlinear effects induce velocity profile distortion, accurately reproducing wall-turbulence features.



(a) **(b)**

Figure 2. Neumann boundary condition: (a) subcritical $\lambda < \gamma$; (b) supercritical $\lambda > \gamma$.

Figure 2 presents the three-dimensional visualization of the dynamical transition in System (2.3): when $\lambda < \gamma$, the velocity field maintains a uniform distribution; when $\lambda > \gamma$, the system develops an antisymmetric structure dominated by the $\cos x$ mode, generating streamwise vortex pairs and coherent shear-layer structures.

Under both boundary conditions, the solutions develop pronounced multi-scale features, which are characteristic hallmarks of turbulence onset. This process shares identical physical mechanisms with the transition phenomenon observed in classical fluid mechanics beyond critical Reynolds numbers. Notably, at $\gamma = 0.5$, the corresponding critical equivalent Reynolds number $Re = 2$ corresponds to the transition threshold range established in one-dimensional dynamical systems [37].

It should be noted that the one-dimensional Burgers equation is inherently limited to modeling fundamental characteristics of one-dimensional turbulent bifurcation behavior. Nevertheless, these findings establish a theoretical foundation for investigating bifurcation mechanisms in high-dimensional turbulent systems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

We are grateful to the editors and referees for their helpful comments and suggestions. This work is supported by the National Nature Science Foundation of China (NSFC) (Grant No. 11901408), the Sichuan Natural Science Youth Foundation (NSFYSE) (Grant No. 22NSFSC16338) and the Central University for Excellence in Research of Science and Technology (CERST) (Grant No. 2682022ZTPY063).

Conflict of interest

The authors declare there are no conflicts of interest.

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