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Research article

Boundedness of solutions in a two-species chemotaxis system

Chang-Jian Wang* **and Yuan-Hao Zang**

School of Mathematics and Statistics, Xinyang Normal University, Xinyang 464000, China

* **Correspondence:** Email: cjwang0719@126.com.

Abstract: In this paper, we consider an initial-Neumann boundary value problem for a two-species chemotaxis system

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u - \chi\nabla\cdot(u\nabla w) + u(a_1 - b_1 u^{m-1} + c_1 v), & (x,t)\in\Omega\times(0,T_{\max}), \\ \frac{\partial v}{\partial t} = \Delta v - \xi\nabla\cdot(v\nabla w) + v(a_2 - b_2 v^{l-1} - c_2 u), & (x,t)\in\Omega\times(0,T_{\max}), \\ \frac{\partial w}{\partial t} = \Delta w - (u^\alpha + v^\beta)w, & (x,t)\in\Omega\times(0,T_{\max}), \end{cases}$$

where the domain $\Omega\subset\mathbb{R}^n (n\geq 2)$ is bounded and smooth, $T_{\max}\in(0,\infty]$, and parameters $a_i, b_i, c_i, m, l, \alpha, \beta, \chi, \xi > 0$ with $m, l > 1$, $i = 1, 2$. In the current work, we provide a sufficient condition of global classical solvability to the above system. More precisely, for some suitable initial data, if $m > \max\{\frac{\alpha(n+2)}{2}, 1\}$ and $l > \max\{\frac{\beta(n+2)}{2}, 1\}$, then the system has a global classical solution. Compared to previous work, the existence result established here is more generalized, depending only on the nonlinear power exponents and spatial dimensions.

Keywords: predator-prey model; classical solutions; global existence; nonlinear consumption

1. Introduction

Chemotaxis refers to the phenomenon of directional movement of cells or organisms in response to chemical stimuli. The first system of partial differential equations with respect to chemotaxis was established by Keller and Segel [1] from a mathematical perspective. Thereafter, considering the influence of some factors (for instance, logistic terms [2,3], nonlinear diffusions [4–6], fluid effects [7,8], and the consumption mechanism [9]), many more complex variants of this model have been proposed. These models and related models also have many applications across various fields, such as ecological population models [10], pattern formation (see [11,12]), electrorheological fluids (see [13]), and image restoration (see [14–16]).

The chemotaxis-consumption system can be described as

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u - \chi \nabla \cdot (u \nabla v) + f(u), & (x,t) \in \Omega \times (0, T_{\max}), \\ \frac{\partial v}{\partial t} = \Delta v - uv, & (x,t) \in \Omega \times (0, T_{\max}), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0, T_{\max}), \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $T_{\max} \in (0, \infty]$ represents the maximum existence time of the solution, and u and v represent cell population density and oxygen concentration, respectively. In recent years, substantial theoretical results have been obtained regarding the related model [17–19]. For $f(u) = 0$, if $0 < \chi < \frac{1}{6(n+1)\|v_0\|_{L^\infty(\Omega)}}$, Tao [20] elaborated that the corresponding system is globally classically solvable by establishing the boundedness of a weighted functional. Baghaei and Khelghati [21] obtained the same results by improving the condition obtained in [20] with $0 < \chi < \frac{\pi}{\sqrt{2(n+1)}\|v_0\|_{L^\infty(\Omega)}}$. Fuest [22] considered a more generalized system with indirect consumption effect, $\frac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \nabla v)$, $\frac{\partial v}{\partial t} = \Delta v - vw$, $\frac{\partial w}{\partial t} = -\delta w + u$ with $\delta > 0$, and gave some sufficient conditions for global classical solvability with $n \leq 2$ or $\|v_0\|_{L^\infty(\Omega)} \leq \frac{1}{3n}$. For $f(u) = au - bu^2$ with $a, b > 0$, Lankeit and Wang [23] studied the influence of the size of parameter a on the global existence of solutions, including smooth solutions and weak solutions.

As demonstrated in the above models, the mechanism of resource consumption is a linear form of function u. However, based on the complexity of the external environment, the nonlinear dependence of resource dissipation on the cell density function u seems to be more reasonable sometimes. Recently, a nonlinear coupled chemotaxis-consumption problem [24] has been studied,

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w) + au - bu^m, & (x,t) \in \Omega \times (0, T_{\max}), \\ \frac{\partial v}{\partial t} = \Delta v - u^\alpha v, & (x,t) \in \Omega \times (0, T_{\max}), \\ \frac{\partial w}{\partial t} = \Delta w - u^\beta w, & (x,t) \in \Omega \times (0, T_{\max}), \end{cases} \tag{1.2}$$

where $a, b, \alpha, \beta, \xi, \chi, m$ are positive constants. In [24], we provided a sufficient condition on the existence of classical solution with $m > \max\left\{\frac{\max\{\alpha,\beta\}(n+2)}{2}, 1\right\}$. Chiyo et al. [25] studied system (1.2) involving volume-filling effect with $\alpha, \beta \in (0, 1)$, and provided a detailed characterization on the global classical solvability. Afterwards, a more generalized chemotaxis system, also called the nonlinear indirect chemotaxis-consumption system, has been discussed, and similar results on classical solutions have been demonstrated [26].

Considering interactions between two species under the stimulation of chemical signal, we get the following system:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u - \xi \nabla \cdot (u \nabla w) + f_1(u, v), & (x,t) \in \Omega \times (0, T_{\max}), \\ \frac{\partial v}{\partial t} = \Delta v - \chi \nabla \cdot (v \nabla w) + f_2(u, v), & (x,t) \in \Omega \times (0, T_{\max}), \\ \frac{\partial w}{\partial t} = \Delta w - \gamma w + \alpha u + \beta v, & (x,t) \in \Omega \times (0, T_{\max}), \end{cases} \tag{1.3}$$

where $\alpha, \beta, \gamma, \xi, \chi$ are positive constants, and the nonlinear functions f_1, f_2 are used to characterize the relationship between two species. For the case where f_1, f_2 represent the competition kinetics of two species formulated by $f_1(u, v) = \mu_1(1 - u - a_1 v), f_2(u, v) = \mu_2(1 - a_2 u - v)$ with $\mu_i, a_i > 0, i = 1, 2,$

Bai and Winkler [27] discussed the corresponding system in $\Omega \subset \mathbb{R}^n$ with $n \leq 2$ and obtained the global solvability in the classical sense. Additionally, for the case where $0 < a_1, a_2 < 1$ and $\mu_1, \mu_2 > C$ or $1 \leq a_1 < \infty, 0 < a_2 < 1$ and $\mu_2 > C$ with some $C > 0$, the long-time behavior of solutions was also studied therein. Mizukami [28] studied a quasilinear version of (1.3) and improved the hypothesis established in [27] by enlarging the ranges of μ_1, μ_2. Later, Mizukami [29] further obtained the improvement of conditions for the case $a_1, a_2 \in (0, 1)$ based on [27,28]. For the higher-dimensional case with $n \geq 2$, the global existence in the smooth sense was explored in [30,31]. If f_1, f_2 are formulated by $f_1(u, v) = \mu_1 u(1 - u - a_1 v)$ and $f_2(u, v) = \mu_2 v(1 - v + a_2 u)$, then system (1.3) turns into a predator-prey system involving chemotaxis mechanisms. Subsequently, for $n = 3$, the global classical solvability was established in [32].

More recently, when considering both species consuming nutrients, the following chemotaxis competition model has been investigated:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u - \xi_1 \nabla \cdot (u \nabla w) + \mu_1 u(1 - u - a_1 v), & (x,t) \in \Omega \times (0, T_{\max}), \\ \frac{\partial v}{\partial t} = \Delta v - \xi_2 \nabla \cdot (v \nabla w) + \mu_2 v(1 - v - a_2 u), & (x,t) \in \Omega \times (0, T_{\max}), \\ \frac{\partial w}{\partial t} = \Delta w - (u + v) w, & (x,t) \in \Omega \times (0, T_{\max}), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0, T_{\max}), \\ u(x,0) = u_0(x), v(x,0) = v_0(x), w(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{1.4}$$

where $a_i, \xi_i, \mu_i > 0, i = 1, 2$. Numerous research results have been obtained for such a model. For instance, when the initial value $\|w_0\|_{L^\infty(\Omega)}$ satisfies suitable explicit conditions, Wang et al. [33] elaborated that the system is globally classically solvable. And, they also explored the long-time stability of the system. The global classical solvability of system (1.4) with nonlinear diffusion was discussed in [34]. When removing logistic terms in system (1.4), Zhang and Tao [35] constructed the existence conditions provided that $\|w_0\|_{L^\infty(\Omega)} \leq \frac{\sqrt{\frac{2}{n}}\pi}{\max\{\xi_1, \xi_2\}}$. Ren and Liu [36] presented the global-in-time existence of weak solutions to the model involving nonlinear chemotactic sensitivity functions under the condition that $\|w_0\| \leq \bar{w}$ with $\bar{w}$ depending on the coefficients of system. Later, Ren and Liu [37] introduced a definition of weak solutions and showed that these solutions would be smooth after a certain moment $T > 0$.

The forager-exploiter model can sometimes be considered as a variant of chemotaxis-consumption model,

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + \xi \nabla \cdot (u \nabla w) + f_1(u, v), & (x,t) \in \Omega \times (0, T_{\max}), \\ \frac{\partial v}{\partial t} = \Delta v - \chi \nabla \cdot (v \nabla u) + f_2(u, v), & (x,t) \in \Omega \times (0, T_{\max}), \\ \frac{\partial w}{\partial t} = \Delta w - (u + v) w - \mu w + r(x,t), & (x,t) \in \Omega \times (0, T_{\max}), \end{cases} \tag{1.5}$$

where u and v stand for the foragers density and the exploiters density, respectively, w represents the resource concentration, and $r(x, t)$ stands for resource production rate function. Assuming system (1.5) without logistic terms, Winkler [38] provided an explicit condition with respect to $r(x, t)$ and initial data to ensure the global weak solvability. Letting $r(x, t) = r_0$ with some constant $r_0 \geq 0$, Tao and Winkler [39] explored the existence of global classical solutions to this associated system for all suitably regular initial data in one-dimensional space. For spatial dimension $n \geq 2$, if the initial data and $r(x, t)$ satisfy some smallness conditions or χ, ξ are small enough, Wang and Wang [40] established the global solvability in the classical sense for the corresponding system. In addition, if $f_1(u, v) = \eta_1(u - u^2)$

and $f_2(u,v) = \eta_2(v - v^2)$ with $\eta_1, \eta_2 > 0$, Wu and Shen [41] established the global well-posedness under the assumption that $\theta > \frac{(n-2)_+}{n+2}$ with $n \geq 1$. For the case where $f_1(u,v) = \eta_1 u(1 - u - a_1 v)$ and $f_2(u,v) = \eta_2 v(1 - v - a_2 u)$ with $\eta_1, \eta_2, a_1, a_2 > 0$, and the third equation of (1.5) is changed with $w_t = \Delta w - \frac{(u+v)w}{(1+u+v)^\theta}$, Ou and Wang [42] proved the global classical solvability provided that $\theta > 0$.

Motivated by the aforementioned works, in the current work, we are concerned with a predator-prey model involving nonlinear nutrient dissipation mechanisms and generalized logistic terms

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u - \chi \nabla \cdot (u \nabla w) + u(a_1 - b_1 u^{m-1} + c_1 v), & (x,t) \in \Omega \times (0, T_{\max}), \\ \frac{\partial v}{\partial t} = \Delta v - \xi \nabla \cdot (v \nabla w) + v(a_2 - b_2 v^{l-1} - c_2 u), & (x,t) \in \Omega \times (0, T_{\max}), \\ \frac{\partial w}{\partial t} = \Delta w - (u^\alpha + v^\beta) w, & (x,t) \in \Omega \times (0, T_{\max}), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0, T_{\max}), \\ u(x,0) = u_0(x), v(x,0) = v_0(x), w(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{1.6}$$

with homogeneous Neumann conditions $\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0$ on $\partial\Omega$, where the boundary $\Omega \subset \mathbb{R}^n (n \geq 2)$ is smooth, ν is the outward normal vector on $\partial\Omega$, and the parameters $a_i, b_i, c_i, \alpha, \beta, \chi, \xi > 0$ and $m, l > 1$ with $i = 1, 2$. The purpose of the current paper is to provide a sufficient condition on global solvability in the classical sense to system (1.6). For this purpose, suppose that the initial values u_0, v_0, and w_0 fulfill

$$u_0, v_0, w_0 \in W^{2,\infty}(\Omega) \text{ with } u_0, v_0, w_0 \geq 0, \not\equiv 0 \text{ in } \Omega. \tag{1.7}$$

We state the main result as follows.

Theorem 1.1. *Let $n \geq 2$, $a_i, b_i, c_i, \alpha, \beta, \chi, \xi > 0$ and $m, l > 1$ with $i = 1, 2$. Suppose that u_0, v_0, and w_0 satisfy* (1.7). *If $m > \max\{\frac{\alpha(n+2)}{2}, 1\}$ and $l > \max\{\frac{\beta(n+2)}{2}, 1\}$, then model* (1.6) *possesses a nonnegative solution in the sense that*

$$(u, v, w) \in \bigcap_{k>n} \left[C^0([0,\infty); W^{1,k}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)) \right]^3,$$

which is uniformly-in-time bounded, namely, we can find $C > 0$ fulfilling

$$\|u(\cdot,t)\|_{W^{1,k}(\Omega)} + \|v(\cdot,t)\|_{W^{1,k}(\Omega)} + \|w(\cdot,t)\|_{W^{1,k}(\Omega)} \leq C$$

for all $k > n$ and $t > 0$.

Comparing to the linear system explored in [33,35,36], in our conclusion we removed the dependence on the smallness condition of $\|w_0\|_{L^\infty(\Omega)}$, and showed that the existence conditions depend only on the exponents m, l, α, β and spatial dimensions n. In addition, the logistic source terms and nonlinear resource consumption considered here are more complicated than those in [42], thus the result established in this paper seems to be more generalized.

The remaining structure is carried out as follows. In Section 2, we provide some preliminary results, and introduce several useful conclusions that will be utilized in the subsequent part. In Section 3, the proof of the main conclusion is presented.

2. Preliminaries

In this part, we introduce some previously established results which will be useful later. We begin with a local existence conclusion to system (1.6), and the proof can be established through the fixed point theory.

Lemma 2.1. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain with $n \geq 2$, and $a_i, b_i, c_i, \alpha, \beta, \chi, \xi > 0, m, l > 1$ with $i = 1, 2$. For any u_0, v_0, and w_0 satisfying (1.7), system (1.6) is locally-in-time solvable in the sense that*

$$(u, v, w) \in \bigcap_{k>n} \left[C^0([0, T_{\max}); W^{1,k}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max}))\right]^3,$$

on $[0, T_{\max}]$ with $T_{\max} \in (0, +\infty]$ for all $k > n$. Furthermore, if $T_{\max} < \infty$, then

$$\limsup_{t \nearrow T_{\max}} (\|u(\cdot, t)\|_{W^{1,k}(\Omega)} + \|v(\cdot, t)\|_{W^{1,k}(\Omega)} + \|w(\cdot, t)\|_{W^{1,k}(\Omega)}) = \infty \tag{2.1}$$

Proof. As done in [43, 44], let $\psi = (u, v, w) \in \mathbb{R}^3$. Then, system (1.6) can be reformulated as the following triangular system:

$$\begin{cases} \frac{\partial \psi}{\partial t} = \nabla \cdot (A(\psi) \nabla \psi) + \sigma(\psi), & (x, t) \in \Omega \times (0, T_{\max}), \\ \frac{\partial \psi}{\partial \nu} = 0, & (x, t) \in \partial\Omega \times (0, T_{\max}), \\ \psi(\cdot, 0) = (u_0, v_0, w_0), & x \in \Omega, \end{cases} \tag{2.2}$$

where

$$A(\psi) = \begin{pmatrix} 1 & 0 & -\chi u \\ 0 & 1 & -\xi v \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \sigma(\psi) = \begin{pmatrix} u(a_1 - b_1 u^{m-1} + c_1 v) \\ v(a_2 - b_2 v^{l-1} - c_2 u) \\ -(u^\alpha + v^\beta) w \end{pmatrix}.$$

Since the matrix $A(\psi)$ is positive definite for the given initial data, this asserts that system (2.2) is generally parabolic. Then, Theorems 14.4 and 14.6 in [45] are applicable, and there exists a $T_{\max} \geq 0$ such that system (2.2) admits a solution $\psi \in \bigcap_{k>n} \left[C^0([0, T_{\max}); W^{1,k}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max}))\right]^3$. Finally, the extensibility criterion can be ensured by applying Theorems 15.5 in [45].

Lemma 2.2. *(cf. [23, 46]) Let Ω be a smooth bounded domain in $\mathbb{R}^n$ with $n \geq 1$ and any $\rho \in C^2(\overline{\Omega})$ with $\frac{\partial \rho}{\partial \nu}\big|_{\partial\Omega} = 0$. For any $\tau > 0$ and $k > 1$, there exists $C = C(\tau, k, \Omega) > 0$ such that*

$$\int_{\partial\Omega} |\nabla \rho|^{2k-2} \frac{\partial |\nabla \rho|^2}{\partial \nu} \leq \tau \int_\Omega |\nabla \rho|^{2k-2} |D^2 \rho|^2 + C \int_\Omega |\nabla \rho|^{2k}, \tag{2.3}$$

and

$$\int_\Omega |\nabla \rho|^{2k+2} \leq 2(4k^2 + n) \|\rho\|^2_{L^\infty(\Omega)} \int_\Omega |\nabla \rho|^{2k-2} |D^2 \rho|^2. \tag{2.4}$$

Lemma 2.3. *(cf. [40, 47]) For some $m_1, m_2 > 0$ and $\mu = \min\{1, \frac{\tilde{T}}{2}\}$ with $\tilde{T} \in (0, \infty]$, let $z \in C([0, \tilde{T})) \cap C^1((0, \tilde{T}))$ and $y \in L^1_{loc}([0, \tilde{T}))$ be nonnegative such that*

$$\frac{dz}{dt} + m_1 z \leq y, \quad t \in (0, \tilde{T})$$

and

$$\int_t^{t+\mu} y(s)ds \le m_2, \quad t \in (0, \tilde{T} - \mu).$$

Then, there holds

$$z(t) \le z(0) + 2m_2 + \frac{m_2}{m_1}, \quad t \in (0, \tilde{T}).$$

3. Global classical solvability

This section is dedicated to proving the main conclusion of the paper.

Lemma 3.1. *Let $n \ge 2$, and $a_i, b_i, c_i, \alpha, \beta, \chi, \xi > 0, m, l > 1$ with $i = 1, 2$. Then, there exist $K_0, K_1, K_2 > 0$ such that*

$$\|w\|_{L^\infty(\Omega)} \le K_0, \quad t \in (0, T_{\max}) \tag{3.1}$$

and

$$\int_\Omega (u + v) \le K_1, \quad t \in (0, T_{\max}), \tag{3.2}$$

as well as

$$\int_t^{t+\delta} \int_\Omega (u^m + v^l) \le K_2, \quad t \in (0, T_{\max} - \delta), \tag{3.3}$$

where $\delta = \min\{1, \frac{T_{\max}}{2}\}$.

Proof. The parabolic comparison principle enables us to obtain (3.1) from the third equation of system (1.6). Next, combining the first and second equations of (1.6), it is not hard to get

$$\frac{d}{dt}\int_\Omega (c_2 u + c_1 v) = a_1 c_2 \int_\Omega u + a_2 c_1 \int_\Omega v - b_1 c_2 \int_\Omega u^m - b_2 c_1 \int_\Omega v^l, \quad t \in (0, T_{\max}). \tag{3.4}$$

For $m, l > 1$, invoking Young's inequality, one may derive

$$-b_1 c_2 \int_\Omega u^m \le -(a_1 c_2 + c_2) \int_\Omega u + C_1 \tag{3.5}$$

and

$$-b_2 c_1 \int_\Omega v^l \le -(a_2 c_1 + c_1) \int_\Omega v + C_2, \quad t \in (0, T_{\max}), \tag{3.6}$$

with some $C_1, C_2 > 0$. Collecting (3.4)–(3.6), one may deduce

$$\frac{d}{dt}\int_\Omega (c_2 u + c_1 v) + \int_\Omega (c_2 u + c_1 v) \le C_1 + C_2, \quad t \in (0, T_{\max}). \tag{3.7}$$

Applying the ODE comparison principle to inequality (3.7), one can conclude (3.2) directly. Furthermore, integrating both sides of (3.4) from t to $t + \delta$, we can obtain

$$\int_t^{t+\delta} \int_\Omega (c_2 \frac{\partial u}{\partial t} + c_1 \frac{\partial v}{\partial t}) = \int_t^{t+\delta} \int_\Omega (a_1 c_2 u + a_2 c_1 v) - \int_t^{t+\delta} \int_\Omega (b_1 c_2 u^m + b_2 c_1 v^l), \tag{3.8}$$

with $\delta = \min\{1, \frac{T_{\max}}{2}\}$. Based on the proven conclusion in (3.2), one may see that

$$\int_t^{t+\delta}\int_\Omega (b_1c_2u^m + b_2c_1v^l) \le \int_t^{t+\delta}\int_\Omega (a_1c_2u + a_2c_1v) + \int_\Omega (c_2u + c_1v) \le C_3 \tag{3.9}$$

for all $t \in (0, T_{\max} - \delta)$. Thus, we finish the proof of this lemma.

Lemma 3.2. *Let $n \ge 2$, and $a_i, b_i, c_i, \alpha, \beta, \chi, \xi > 0, m, l > 1$ with $i = 1, 2$. For any $k > 1$, there exist $K_3, K_4, K_5 > 0$ satisfying*

$$\frac{1}{2k}\frac{d}{dt}\int_\Omega |\nabla w|^{2k} + \int_\Omega |\nabla w|^{2k} \le K_3\int_\Omega u^{\alpha(k+1)} + K_4\int_\Omega v^{\beta(k+1)} + K_5,\ t \in (0, T_{\max}). \tag{3.10}$$

Proof. Due to $\nabla w \cdot \nabla\Delta w = \frac{1}{2}\Delta|\nabla w|^2 - |D^2w|^2$, we deal with the third equation in (1.6) to deduce

$$\begin{aligned}\nabla w \cdot \nabla w_t &= \nabla w \cdot \nabla\Delta w - \nabla w \cdot \nabla\left(u^\alpha w + v^\beta w\right)\\ &= \frac{1}{2}\Delta|\nabla w|^2 - |D^2w|^2 - \nabla w \cdot \nabla\left(u^\alpha w + v^\beta w\right).\end{aligned} \tag{3.11}$$

For any $k > 1$, we can obtain from (3.11) that

$$\begin{aligned}&\frac{1}{2k}\frac{d}{dt}\int_\Omega |\nabla w|^{2k} + \int_\Omega |\nabla w|^{2k-2}|D^2w|^2 + \int_\Omega |\nabla w|^{2k}\\ &= \frac{1}{2}\int_\Omega |\nabla w|^{2k-2}\Delta|\nabla w|^2 + \int_\Omega |\nabla w|^{2k} - \int_\Omega |\nabla w|^{2k-2}\nabla w \cdot \nabla\left(u^\alpha w + v^\beta w\right)\\ &= H_1 + H_2,\end{aligned} \tag{3.12}$$

where $H_1 = \frac{1}{2}\int_\Omega |\nabla w|^{2k-2}\Delta|\nabla w|^2 + \int_\Omega |\nabla w|^{2k}$ and $H_2 = -\int_\Omega |\nabla w|^{2k-2}\nabla w \cdot \nabla\left(u^\alpha w + v^\beta w\right)$. Due to the boundedness of $\|w\|_{L^\infty(\Omega)}$ in (3.1), we employ (2.4) in Lemma 2.2 to get

$$\int_\Omega |\nabla w|^{2k+2} \le C_1\int_\Omega |\nabla w|^{2k-2}|D^2w|^2,\quad t \in (0, T_{\max}), \tag{3.13}$$

where $C_1 = 2(4k^2 + n)K_0^2 > 0$. In view of (2.3) in Lemma 2.2 and (3.13), it is not hard to deduce from Young's inequality that

$$\begin{aligned}H_1 &= \frac{1}{2}\int_{\partial\Omega} |\nabla w|^{2k-2}\frac{\partial|\nabla w|^2}{\partial\nu} - \frac{1}{2}\int_\Omega \nabla|\nabla w|^{2k-2} \cdot \nabla|\nabla w|^2 + \int_\Omega |\nabla w|^{2k}\\ &\le \frac{1}{4}\int_\Omega |\nabla w|^{2k-2}|D^2w|^2 + C_2\int_\Omega |\nabla w|^{2k} - \frac{k-1}{2}\int_\Omega |\nabla w|^{2k-4}\left|\nabla|\nabla w|^2\right|^2\\ &\le \frac{1}{4}\int_\Omega |\nabla w|^{2k-2}|D^2w|^2 + \frac{1}{4C_1}\int_\Omega |\nabla w|^{2k+2} + C_3\\ &\le \frac{1}{2}\int_\Omega |\nabla w|^{2k-2}|D^2w|^2 + C_3,\quad t \in (0, T_{\max}),\end{aligned} \tag{3.14}$$

with $C_2 > 0$ and $C_3 = (4C_1)^kC_2^{k+1}|\Omega| > 0$. Applying the inequality $|\Delta w| \le \sqrt{n}|D^2w|$, it can be inferred from (3.1) and integration by parts that

$$H_2 = -\int_\Omega |\nabla w|^{2k-2}\nabla w \cdot \nabla\left(u^\alpha w + v^\beta w\right) = \int_\Omega (u^\alpha w + v^\beta w)\nabla \cdot \left(\nabla w|\nabla w|^{2k-2}\right)$$

$$= \int_\Omega (u^\alpha w + v^\beta w)\left(\Delta w|\nabla w|^{2k-2} + (2k-2)|\nabla w|^{2k-2}|D^2 w|\right)$$
$$\le C_4 \int_\Omega (u^\alpha + v^\beta)|\nabla w|^{2k-2}|D^2 w|, \quad t \in (0, T_{\max}), \tag{3.15}$$

where $C_4 = (\sqrt{n} + 2(k-1))K_0 > 0$. Using (3.13) once more, we see

$$C_4 \int_\Omega (u^\alpha + v^\beta)|\nabla w|^{2k-2}|D^2 w| \le \frac{1}{4}\int_\Omega |\nabla w|^{2k-2}|D^2 w|^2 + C_5 \int_\Omega (u^{2\alpha} + v^{2\beta})|\nabla w|^{2k-2}$$
$$\le \frac{1}{4}\int_\Omega |\nabla w|^{2k-2}|D^2 w|^2 + \frac{1}{4C_1}\int_\Omega |\nabla w|^{2k+2} + C_6 \int_\Omega u^{\alpha(k+1)} + C_6 \int_\Omega v^{\beta(k+1)}$$
$$\le \frac{1}{2}\int_\Omega |\nabla w|^{2k-2}|D^2 w|^2 + C_6 \int_\Omega u^{\alpha(k+1)} + C_6 \int_\Omega v^{\beta(k+1)}, \quad t \in (0, T_{\max}), \tag{3.16}$$

with some $C_5, C_6 > 0$. Collecting (3.14), (3.16), and (3.12), for some $C_7 > 0$, one may get

$$\frac{1}{2k}\frac{d}{dt}\int_\Omega |\nabla w|^{2k} + \int_\Omega |\nabla w|^{2k} \le C_6 \int_\Omega u^{\alpha(k+1)} + C_6 \int_\Omega v^{\beta(k+1)} + C_7, \ t \in (0, T_{\max}). \tag{3.17}$$

Therefore, we can obtain (3.10).

Lemma 3.3. *Let $n \ge 2$ and $a_i, b_i, c_i, \alpha, \beta, \chi, \xi > 0, m, l > 1$ with $i = 1, 2$. Suppose that for any $k > \max\{\frac{(\alpha+\beta)(n+2)}{2}, 1\}$ there is $K_6 > 0$ satisfying*

$$\int_t^{t+\delta}\int_\Omega (u^{\frac{\alpha k}{\alpha+\beta}} + v^{\frac{\beta k}{\alpha+\beta}}) \le K_6, \quad t \in (0, T^*_{\max}), \tag{3.18}$$

where $\delta = \min\{1, \frac{T_{\max}}{2}\}$ and $T^*_{\max} = T_{\max} - \delta$. Then, we can find $K_7 > 0$ satisfying

$$\|\nabla w(\cdot, t)\|_{L^{2(\frac{k}{\alpha+\beta}-1)}(\Omega)} \le K_7, \quad t \in (0, T_{\max}). \tag{3.19}$$

Proof. Due to Lemma 3.2, it is not hard to find $C_1, C_2, C_3 > 0$ satisfying

$$\frac{d}{dt}\int_\Omega |\nabla w|^{2(\frac{k}{\alpha+\beta}-1)} + C_1 \int_\Omega |\nabla w|^{2(\frac{k}{\alpha+\beta}-1)} \le C_2 \int_\Omega (u^{\frac{\alpha k}{\alpha+\beta}} + v^{\frac{\beta k}{\alpha+\beta}}) + C_3, \ t \in (0, T_{\max}). \tag{3.20}$$

Since $k > \frac{(\alpha+\beta)(n+2)}{2}$, we see that $2(\frac{k}{\alpha+\beta} - 1) > n$. From (3.18) and Lemma 2.3, it is not difficult to get from (3.20) that

$$\int_\Omega |\nabla w|^{2(\frac{k}{\alpha+\beta}-1)} \le C_4, \quad t \in (0, T_{\max}), \tag{3.21}$$

with some $C_4 > 0$. Hence, we can conclude (3.19).

Lemma 3.4. *Let $n \ge 2$, and $a_i, b_i, c_i, \alpha, \beta, \chi, \xi > 0, m, l > 1$ with $i = 1, 2$. Then, we can find $K_8, K_9 > 0$ to satisfy*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \le K_8 \ \text{ and } \ \|v(\cdot, t)\|_{L^\infty(\Omega)} \le K_9, \quad t \in (0, T_{\max}). \tag{3.22}$$

Proof. Based on the variation-of-constants formula, one may derive

$$\begin{aligned} v(\cdot,t) =& e^{t\Delta}v_0 - \xi\int_0^t e^{(t-s)\Delta}\nabla\cdot(v\nabla w)ds + \int_0^t e^{(t-s)\Delta}(a_2v - b_2v^l - c_2uv)ds \\ =& e^{t\Delta}v_0 - \xi\int_0^t e^{(t-s)\Delta}\nabla\cdot(v\nabla w)ds + \int_0^t e^{(t-s)\Delta}[(a_2v - b_2v^l - c_2uv)_+ - (a_2v - b_2v^l - c_2uv)_-]ds \\ \le& e^{t\Delta}v_0 - \xi\int_0^t e^{(t-s)\Delta}\nabla\cdot(v\nabla w)ds + \int_0^t e^{(t-s)\Delta}(a_2v - b_2v^l - c_2uv)_+ds \end{aligned} \tag{3.23}$$

for all $t \in (0, T_{\max})$. Therefore, one may deduce

$$\begin{aligned} \|v(\cdot,t)\|_{L^\infty(\Omega)} \le& \|e^{t\Delta}v_0\|_{L^\infty(\Omega)} + \xi\int_0^t \left\|e^{(t-s)\Delta}\nabla\cdot(v\nabla w)\right\|_{L^\infty(\Omega)}ds \\ &+ \int_0^t \left\|e^{(t-s)\Delta}(a_2v - b_2v^l - c_2uv)_+\right\|_{L^\infty(\Omega)}ds \\ \le& C_1 + \xi\int_0^t \left\|e^{(t-s)\Delta}\nabla\cdot(v\nabla w)\right\|_{L^\infty(\Omega)}ds \\ &+ \int_0^t \left\|e^{(t-s)\Delta}(a_2v - b_2v^l - c_2uv)_+\right\|_{L^\infty(\Omega)}ds \end{aligned} \tag{3.24}$$

for all $t \in (0, T_{\max})$ with some $C_1 > 0$. From Lemma 3.3, for any $k > \max\{\frac{(\alpha+\beta)(n+2)}{2}, 1\}$, there holds

$$\|\nabla w(\cdot,t)\|_{L^{2(\frac{k}{\alpha+\beta}-1)}(\Omega)} \le K_7, \quad t \in (0, T_{\max}). \tag{3.25}$$

Define $\kappa > 0$ satisfying $n < \kappa < 2(\frac{k}{\alpha+\beta} - 1)$. Let $\gamma = \frac{2(\frac{k}{\alpha+\beta}-1)\kappa}{2(\frac{k}{\alpha+\beta}-1)-\kappa} > n$. Invoking Hölder's inequality and the L^k-interpolation inequality, we conclude from the regularization properties of the Neumann heat semigroup $(e^{t\Delta})_{t\ge0}$ (see [48]) that

$$\begin{aligned} &\xi\int_0^t \left\|e^{(t-s)\Delta}\nabla\cdot(v\nabla w)\right\|_{L^\infty(\Omega)}ds \\ &\le C_2\int_0^t (1 + (t-s)^{-\frac{1}{2}-\frac{n}{2\kappa}})e^{-\lambda(t-s)}\|(v\nabla w)\|_{L^\kappa(\Omega)}ds \\ &\le C_2\int_0^t (1 + (t-s)^{-\frac{1}{2}-\frac{n}{2\kappa}})e^{-\lambda(t-s)}\|v\|_{L^\gamma(\Omega)}\|\nabla w\|_{L^{2(\frac{k}{\alpha+\beta}-1)}(\Omega)}ds \\ &\le C_3\int_0^t (1 + (t-s)^{-\frac{1}{2}-\frac{n}{2\kappa}})e^{-\lambda(t-s)}\|v\|_{L^1(\Omega)}^{\frac{1}{\gamma}}\|v\|_{L^\infty(\Omega)}^{\frac{\gamma-1}{\gamma}}ds \\ &\le C_3K_1^{\frac{1}{\gamma}}\int_0^t (1 + (t-s)^{-\frac{1}{2}-\frac{n}{2\kappa}})e^{-\lambda(t-s)}\|v\|_{L^\infty(\Omega)}^{\frac{\gamma-1}{\gamma}}ds \end{aligned} \tag{3.26}$$

for all $t \in (0, T_{\max})$, with some $\lambda, C_2, C_3 > 0$. Let

$$I(t) = \sup_{s\in(0,t)} \|v(\cdot,s)\|_{L^\infty(\Omega)}, \quad t \in (0, T_{\max}). \tag{3.27}$$

Due to $n < \kappa < 2(\frac{k}{\alpha+\beta} - 1)$, we infer that $\frac{1}{\gamma} \in (0,1)$ and

$$\int_0^t \Big(1 + (t-s)^{-\frac{1}{2}-\frac{n}{2\kappa}}\Big) e^{-\lambda(t-s)} ds < \infty. \tag{3.28}$$

Thus, it can be deduced from (3.26)–(3.28) that

$$\xi \int_0^t \Big\| e^{(t-s)\Delta} \nabla \cdot (v \nabla w) \Big\|_{L^\infty(\Omega)} ds \le C_4 K_1^{\frac{1}{\gamma}} I^{\frac{\gamma-1}{\gamma}}(t), \quad t \in (0, T_{\max}), \tag{3.29}$$

with some $C_4 > 0$. Letting $f(y) = a_2 y - b_2 y^l$, due to $u, v \ge 0$ and $l > 1$, we know that

$$(a_2 v - b_2 v^l - c_2 uv)_+ \le (a_2 v - b_2 v^l)_+ \le f((\frac{a_2}{lb_2})^{\frac{1}{l-1}}), \tag{3.30}$$

which implies

$$\begin{aligned}\int_0^t \left\| e^{(t-s)\Delta}(a_2 v - b_2 v^l - c_2 uv)_+ \right\|_{L^\infty(\Omega)} ds &\le C_5 \int_0^t e^{-\lambda(t-s)} \left\| (a_2 v - b_2 v^l)_+ \right\|_{L^\infty(\Omega)} ds \\ &\le C_6, \ t \in (0, T_{\max}),\end{aligned} \tag{3.31}$$

where $C_5, C_6 > 0$. Substituting (3.27), (3.29), and (3.31) into (3.24), it can be concluded from Young's inequality that

$$I(t) \le C_1 + C_4 K_1^{\frac{1}{\gamma}} I^{\frac{\gamma-1}{\gamma}}(t) + C_6 \le C_7 + \frac{1}{2} I(t), \quad t \in (0, T_{\max}), \tag{3.32}$$

with some $C_7 > 0$. Therefore, from the definition of $I(t)$, there holds

$$\|v(\cdot, t)\|_{L^\infty(\Omega)} < K_8, \quad t \in (0, T_{\max}), \tag{3.33}$$

with some $K_8 > 0$. In addition, based on the variation-of-constants formula, we can also obtain

$$u(\cdot, t) = e^{t\Delta} u_0 - \chi \int_0^t e^{(t-s)\Delta} \nabla \cdot (u \nabla w) ds + \int_0^t e^{(t-s)\Delta}(a_1 u - b_1 u^m + c_1 uv) ds$$

for all $t \in (0, T_{\max})$. Due to the L^∞−boundedness of v as in (3.33), we derive from $m > 1$ that

$$(a_1 u - b_1 u^m + c_1 uv)_+ \le (a_1 u - b_1 u^m + c_1 C_8 u)_+ \le K_9 \tag{3.34}$$

for all $t \in (0, T_{\max})$ with some $K_9 > 0$. Similarly, we can use the same procedures as above to deduce the L^∞−boundedness of u. Thus, we finish the proof.

Lemma 3.5. *Let $n \ge 2$ and $a_i, b_i, c_i, \alpha, \beta, \chi, \xi > 0, m, l > 1$ with $i = 1, 2$. Then, for any $k > 1$, we can find $K_{10} > 0$ satisfying*

$$\frac{1}{2k} \frac{d}{dt} \int_\Omega (|\nabla u|^{2k} + |\nabla v|^{2k}) + \int_\Omega (|\nabla u|^{2k} + |\nabla v|^{2k}) \le K_{10} \int_\Omega |\nabla w|^{2k+2} + K_{10} \int_\Omega |\Delta w|^{k+1} + K_{10}.$$

Proof. Applying the same steps as in (3.11) and (3.12), we conclude from the second equation of system (1.6) that

$$\begin{aligned}
&\frac{1}{2k}\frac{d}{dt}\int_\Omega |\nabla v|^{2k} + \int_\Omega |\nabla v|^{2k-2}|D^2 v|^2 + \int_\Omega |\nabla v|^{2k} \\
=&\frac{1}{2}\int_\Omega |\nabla v|^{2k-2}\Delta|\nabla v|^2 + \xi\int_\Omega \nabla\cdot(|\nabla v|^{2k-2}\nabla v)(\nabla v\cdot\nabla w + v\Delta w) \\
&-\int_\Omega |\nabla v|^{2k-2}\nabla v\cdot\nabla(b_2 v^l + c_2 uv) + (a_2+1)\int_\Omega |\nabla v|^{2k} \\
=&I_1 + I_2 + I_3 + (a_2+1)\int_\Omega |\nabla v|^{2k}, \quad t\in(0,T_{\max}),
\end{aligned} \tag{3.35}$$

where the identity $\nabla v\cdot\nabla\Delta v = \frac{1}{2}\Delta|\nabla v|^2 - |D^2 v|^2$ has been used. Using similar steps as in deriving H_1 in Lemma 3.2, we can find $C_1 > 0$ such that

$$I_1 = \frac{1}{2}\int_\Omega |\nabla v|^{2k-2}\Delta|\nabla v|^2 \le \frac{1}{8}\int_\Omega |\nabla v|^{2k-2}|D^2 v|^2 + C_1, \quad t\in(0,T_{\max}). \tag{3.36}$$

For the term I_2, we can calculate that

$$\begin{aligned}
I_2 =&\xi\int_\Omega \nabla\cdot(|\nabla v|^{2k-2}\nabla v)(\nabla v\cdot\nabla w + v\Delta w) \\
=&\xi\int_\Omega (\nabla|\nabla v|^{2k-2}\cdot\nabla v)(\nabla v\cdot\nabla w) + \xi\int_\Omega v\Delta w(\nabla|\nabla v|^{2k-2}\cdot\nabla v) \\
&+\xi\int_\Omega |\nabla v|^{2k-2}\Delta v(\nabla v\cdot\nabla w) + \xi\int_\Omega v|\nabla v|^{2k-2}\Delta v\Delta w, \quad t\in(0,T_{\max}).
\end{aligned} \tag{3.37}$$

From Lemma 2.2 and (3.18), for some $C_2 > 0$ we have

$$\int_\Omega |\nabla v|^{2k+2} \le C_2\int_\Omega |\nabla v|^{2k-2}|D^2 v|^2, \quad t\in(0,T_{\max}). \tag{3.38}$$

In the following, we shall estimate each term of (3.37). For the first term, we infer from Young's inequality and (3.38) that

$$\begin{aligned}
\xi\int_\Omega (\nabla|\nabla v|^{2k-2}\cdot\nabla v)(\nabla v\cdot\nabla w) &= \xi(k-1)\int_\Omega |\nabla v|^{2k-4}(\nabla|\nabla v|^2\cdot\nabla v)(\nabla v\cdot\nabla w) \\
&\le 2\xi(k-1)\int_\Omega |\nabla v|^{2k-1}|D^2 v||\nabla w| \\
&\le \frac{1}{16}\int_\Omega |\nabla v|^{2k-2}|D^2 v|^2 + 16\xi^2(k-1)^2\int_\Omega |\nabla v|^{2k}|\nabla w|^2 \\
&\le \frac{1}{16}\int_\Omega |\nabla v|^{2k-2}|D^2 v|^2 + \frac{1}{16C_2}\int_\Omega |\nabla v|^{2k+2} + C_3\int_\Omega |\nabla w|^{2k+2} \\
&\le \frac{1}{8}\int_\Omega |\nabla v|^{2k-2}|D^2 v|^2 + C_3\int_\Omega |\nabla w|^{2k+2},
\end{aligned} \tag{3.39}$$

with some $C_3 > 0$. For the second term, we see

$$\xi\int_\Omega v\Delta w(\nabla|\nabla v|^{2k-2}\cdot\nabla v) =\xi(k-1)\int_\Omega v|\nabla v|^{2k-4}\Delta w(\nabla|\nabla v|^2\cdot\nabla v)$$

$$
\begin{aligned}
&=2\xi(k-1)\int_\Omega v|\nabla v|^{2k-4}\Delta w((D^2v\cdot\nabla v)\cdot\nabla v)\\
&\le C_4\int_\Omega |\nabla v|^{2k-2}|D^2v||\Delta w|,\quad t\in(0,T_{\max}),
\end{aligned}
\tag{3.40}
$$

with some $C_4>0$. Based on Young's inequality and (3.38), the third term can be estimated as

$$
\begin{aligned}
\xi\int_\Omega |\nabla v|^{2k-2}\Delta v(\nabla v\cdot\nabla w)\le{}& \sqrt{n}\xi\int_\Omega |\nabla v|^{2k-1}|D^2v||\nabla w|\\
\le{}&\frac{1}{16}\int_\Omega |\nabla v|^{2k-2}|D^2v|^2+C_5\int_\Omega |\nabla v|^{2k}|\nabla w|^2\\
\le{}&\frac{1}{16}\int_\Omega |\nabla v|^{2k-2}|D^2v|^2+\frac{1}{16C_2}\int_\Omega|\nabla v|^{2k+2}+C_6\int_\Omega|\nabla w|^{2k+2}\\
\le{}&\frac{1}{8}\int_\Omega |\nabla v|^{2k-2}|D^2v|^2+C_6\int_\Omega|\nabla w|^{2k+2},\quad t\in(0,T_{\max}),
\end{aligned}
\tag{3.41}
$$

with some $C_5,C_6>0$. For the last term, due to (3.22), we have

$$
\xi\int_\Omega v|\nabla v|^{2k-2}\Delta v\Delta w\le \sqrt{n}\xi\int_\Omega v|\nabla v|^{2k-2}|D^2v||\Delta w|\le C_7\int_\Omega|\nabla v|^{2k-2}|D^2v||\Delta w| \tag{3.42}
$$

for all $t\in(0,T_{\max})$, with $C_7>0$. From the nonnegativity of u and v, we can obtain

$$
\begin{aligned}
I_3={}&-\int_\Omega |\nabla v|^{2k-2}\nabla v\cdot\nabla(b_2v^l+c_2uv)\\
={}&-b_2l\int_\Omega v^{l-1}|\nabla v|^{2k}-c_2\int_\Omega u|\nabla v|^{2k}-c_2\int_\Omega v|\nabla v|^{2k-2}\nabla v\cdot\nabla u\\
\le{}&c_2\int_\Omega|\nabla v|^{2k-1}|\nabla u|\le C_8\int_\Omega|\nabla v|^{2k}+C_9\int_\Omega|\nabla u|^{2k},\quad t\in(0,T_{\max}),
\end{aligned}
\tag{3.43}
$$

with some $C_8,C_9>0$. By employing Young's inequality, for some $C_{10},C_{11}>0$, one may get

$$
C_8\int_\Omega|\nabla v|^{2k}\le\frac{1}{8C_2}\int_\Omega|\nabla v|^{2k+2}+C_{10}\le\frac18\int_\Omega|\nabla v|^{2k-2}|D^2v|^2+C_{10} \tag{3.44}
$$

and

$$
C_9\int_\Omega|\nabla u|^{2k}\le\frac{1}{8C_2}\int_\Omega|\nabla u|^{2k+2}+C_{11}\le\frac18\int_\Omega|\nabla u|^{2k-2}|D^2u|^2+C_{11} \tag{3.45}
$$

for all $t\in(0,T_{\max})$. By adding up (3.40) and (3.42), for some $C_{12},C_{13}>0$, we can further obtain

$$
\begin{aligned}
\xi\int_\Omega v\Delta w(\nabla|\nabla v|^{2k-2}\cdot\nabla v)+\xi\int_\Omega v|\nabla v|^{2k-2}\Delta v\Delta w\le{}&(C_4+C_7)\int_\Omega|\nabla v|^{2k-2}|D^2v||\Delta w|\\
\le{}&\frac18\int_\Omega|\nabla v|^{2k-2}|D^2v|^2+C_{12}\int_\Omega|\nabla v|^{2k-2}|\Delta w|^2\\
\le{}&\frac18\int_\Omega|\nabla v|^{2k-2}|D^2v|^2+\frac{1}{4C_2}\int_\Omega|\nabla v|^{2k+2}+C_{13}\int_\Omega|\Delta w|^{k+1}
\end{aligned}
$$

$$\leq \frac{3}{8}\int_\Omega |\nabla v|^{2k-2}|D^2v|^2 + C_{13}\int_\Omega |\Delta w|^{k+1},\ t \in (0, T_{\max}). \tag{3.46}$$

Thus, we can obtain from (3.35), (3.36), (3.39), (3.41), and (3.44)–(3.46) that

$$\begin{aligned}\frac{1}{2k}\frac{d}{dt}\int_\Omega |\nabla v|^{2k} + \frac{1}{8}\int_\Omega |\nabla v|^{2k-2}|D^2v|^2 + \int_\Omega |\nabla v|^{2k} &\leq C_{14}\int_\Omega |\nabla w|^{2k+2} + C_{13}\int_\Omega |\Delta w|^{k+1}\\ &+ \frac{1}{8}\int_\Omega |\nabla u|^{2k-2}|D^2u|^2 + C_{15},\ t \in (0, T_{\max}),\end{aligned} \tag{3.47}$$

with some $C_{14}, C_{15} > 0$. Additionally, employing the same derivation processes as above, we can also obtain from the first equation in (1.6) that

$$\begin{aligned}\frac{1}{2k}\frac{d}{dt}\int_\Omega |\nabla u|^{2k} + \frac{1}{8}\int_\Omega |\nabla u|^{2k-2}|D^2u|^2 + \int_\Omega |\nabla u|^{2k} &\leq C_{16}\int_\Omega |\nabla w|^{2k+2} + C_{17}\int_\Omega |\Delta w|^{k+1}\\ &+ \frac{1}{8}\int_\Omega |\nabla v|^{2k-2}|D^2v|^2 + C_{18},\ t \in (0, T_{\max}),\end{aligned} \tag{3.48}$$

with some $C_{16}, C_{17}, C_{18} > 0$. Thus, the desired conclusion can be deduced by adding up (3.47) and (3.48).

Lemma 3.6. *Let $n \geq 2$ and $a_i, b_i, c_i, \alpha, \beta, \chi, \xi > 0, m, l > 1$ with $i = 1, 2$ and $k > \max\{\frac{(\alpha+\beta)(n+2)}{2}, 1\}$, $\delta = \min\{1, \frac{T_{\max}}{2}\}$, and $T^*_{\max} = T_{\max} - \delta$. Then, we can obtain*

$$\|\nabla u(\cdot, t)\|_{L^{2(\frac{k}{\alpha+\beta}-1)}(\Omega)} + \|\nabla v(\cdot, t)\|_{L^{2(\frac{k}{\alpha+\beta}-1)}(\Omega)} \leq K_{11},\ t \in (0, T_{\max}), \tag{3.49}$$

with some $K_{11} > 0$.

Proof. Set

$$h(x, t) = -(u^\alpha + v^\beta)w,\quad (x, t) \in \Omega \times (0, T_{\max}). \tag{3.50}$$

From the boundedness of $\|w\|_{L^\infty(\Omega)}$ and (3.18), for $\delta = \min\{1, \frac{T_{\max}}{2}\}$ and $T^*_{\max} = T_{\max} - \delta$, we infer that

$$\int_t^{t+\delta}\int_\Omega |h|^{\frac{k}{\alpha+\beta}} \leq K_0\int_t^{t+\delta}\int_\Omega (u^\alpha + v^\beta)^{\frac{k}{\alpha+\beta}} \leq C_1\int_t^{t+\delta}\int_\Omega (u^{\frac{\alpha k}{\alpha+\beta}} + v^{\frac{\beta k}{\alpha+\beta}}) + C_2 \leq C_3 \tag{3.51}$$

for all $t \in (0, T^*_{\max})$, with $C_i > 0, i = 1, ..., 3$. Let w solve the problem

$$\begin{cases} \frac{\partial w}{\partial t} = \Delta w + h(x, t), & (x, t) \in \Omega \times (0, T_{\max}),\\ \frac{\partial w}{\partial \nu} = 0, & (x, t) \in \Omega \times (0, T_{\max}),\\ w(x, 0) = w_0, & x \in \Omega. \end{cases} \tag{3.52}$$

Thus, we deduce from (3.51) and [49, Lemma 2.5] that

$$\int_t^{t+\delta}\int_\Omega |\Delta w|^{\frac{k}{\alpha+\beta}} \leq C_4,\ t \in (0, T^*_{\max}), \tag{3.53}$$

with some $C_4 > 0$. Replacing k in Lemma 3.5 with $\frac{k}{\alpha+\beta} - 1$, we have

$$\frac{1}{2(\frac{k}{\alpha+\beta}-1)}\frac{d}{dt}\int_\Omega (|\nabla u|^{2(\frac{k}{\alpha+\beta}-1)} + |\nabla v|^{2(\frac{k}{\alpha+\beta}-1)}) + \int_\Omega (|\nabla u|^{2(\frac{k}{\alpha+\beta}-1)} + |\nabla v|^{2(\frac{k}{\alpha+\beta}-1)})$$
$$\leq K_{10}\int_\Omega |\nabla w|^{\frac{2k}{\alpha+\beta}} + K_{10}\int_\Omega |\Delta w|^{\frac{k}{\alpha+\beta}} + K_{10},\ t \in (0, T_{\max}). \tag{3.54}$$

Invoking the Gagliardo-Nirenberg inequality (see [50, 51]) and Lemma 3.1, for some $C_5, C_6 > 0$, it is not difficult to get

$$\int_\Omega |\nabla w|^{\frac{2k}{\alpha+\beta}} = \|\nabla w\|^{\frac{2k}{\alpha+\beta}}_{L^{\frac{2k}{\alpha+\beta}}(\Omega)} \leq C_5\|\Delta w\|^{\frac{k}{\alpha+\beta}}_{L^{\frac{k}{\alpha+\beta}}(\Omega)}\|w\|^{\frac{k}{\alpha+\beta}}_{L^\infty(\Omega)} + C_5\|w\|^{\frac{2k}{\alpha+\beta}}_{L^\infty(\Omega)}$$
$$\leq C_6\int_\Omega |\Delta w|^{\frac{k}{\alpha+\beta}} + C_6,\ t \in (0, T_{\max}). \tag{3.55}$$

Substituting (3.55) into (3.54), we get

$$\frac{1}{2(\frac{k}{\alpha+\beta}-1)}\frac{d}{dt}\int_\Omega (|\nabla u|^{2(\frac{k}{\alpha+\beta}-1)} + |\nabla v|^{2(\frac{k}{\alpha+\beta}-1)}) + \int_\Omega (|\nabla u|^{2(\frac{k}{\alpha+\beta}-1)} + |\nabla v|^{2(\frac{k}{\alpha+\beta}-1)})$$
$$\leq C_7\int_\Omega |\Delta w|^{\frac{k}{\alpha+\beta}} + C_8, \tag{3.56}$$

with some $C_7, C_8 > 0$. Using Lemma 2.3, we deduce from (3.53) and (3.56) that

$$\int_\Omega (|\nabla u|^{2(\frac{k}{\alpha+\beta}-1)} + |\nabla v|^{2(\frac{k}{\alpha+\beta}-1)}) \leq C_9, \quad t \in (0, T_{\max}), \tag{3.57}$$

with some $C_9 > 0$. Thus, we can deduce (3.49).

Lemma 3.7. *Suppose that for any $k > \max\{\frac{(\alpha+\beta)(n+2)}{2}, 1\}$, there is $C > 0$ satisfying*

$$\int_t^{t+\delta}\int_\Omega (u^{\frac{\alpha k}{\alpha+\beta}} + v^{\frac{\beta k}{\alpha+\beta}}) \leq C, \quad t \in (0, T^*_{\max}), \tag{3.58}$$

with $\delta = \min\{1, \frac{T_{\max}}{2}\}$ and $T^*_{\max} = T_{\max} - \delta$, then $T_{\max} = \infty$.

Proof. Due to Lemmas 3.3 and 3.6, it is not difficult to find $\bar{k} = 2(\frac{k}{\alpha+\beta} - 1) > n$ and $C_1 > 0$ satisfying

$$\|u(\cdot, t)\|_{W^{1,\bar{k}}(\Omega)} + \|v(\cdot, t)\|_{W^{1,\bar{k}}(\Omega)} + \|w(\cdot, t)\|_{W^{1,\bar{k}}(\Omega)} \leq C_1, \quad t \in (0, T^*_{\max}). \tag{3.59}$$

Thus, based on Lemma 2.1, we know $T_{\max} = \infty$.

The proof of Theorem 1.1 Let $n \geq 2$ and $a_i, b_i, c_i, \alpha, \beta, \chi, \xi > 0, m, l > 1$ with $i = 1, 2$. We see that if $m > \max\{\frac{\alpha(n+2)}{2}, 1\}$ and $l > \max\{\frac{\beta(n+2)}{2}, 1\}$, Theorem 1.1 can be concluded from Lemma 3.7 and (3.3).

4. Conclusions and outlook

In this paper, we consider a predator-prey model involving nonlinear nutrient dissipation mechanisms and generalized logistic terms, and the sufficient condition for system (1.6) to have global solvability in the classical sense has been found. Compared to previous work, we use a method of a series of bootstrap-type arguments for some variational structures to obtain the global classical solvability of the system, overcoming the problems caused by nonlinear terms. The novelty of this paper lies in the fact that the existence result established here is more generalized depending only on the nonlinear power exponents and spatial dimensions.

From a purely mathematical perspective, there are also other interesting questions related to system (1.6) that are worth further exploration. For example, by adjusting parameters such as a_i, b_i, and c_i, it can exhibit richer dynamic behaviors, such as oscillation, stable equilibrium, and bifurcation, so as to adapt to different practical problems. We will consider these issues in our future work.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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