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Research article

Floquet theory for first-order delay equations and an application to height stabilization of a drone's flight

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Abstract: In this paper, we proposed a version of the Floquet theory for delay differential equations. We demonstrated that very natural assumptions for control in technical applications can lead us to a one-dimensional fundamental system. This approach allowed researchers to work with classical methods used in the case of ordinary differential equations. On this basis, new original unexpected results on the exponential stability were proposed. For example, in the equation $x'(t)+a(t)x(t-\tau(t)) = 0$, $t \in [0, \infty)$, we avoided the assumption on the smallness of the product $\sup_{t\in[0,\infty)} a(t) \sup_{t\in[0,\infty)} \tau(t) < 3/2$ for asymptotic stability. We obtained that in the case of ω-periodic coefficient and delay, the fact that the period ω was situated in a corresponding interval can lead to exponential stability. We then applied our new tests of stability to the stabilization of a drone's flight, where smallness of the noted above product could not be achieved from a technical point of view. For an equation with periodic coefficient and delay, we got a formula of the solution's representation on the semiaxis.

Keywords: Floquet theory; delay equation; exponential stability; drone flight

1. Introduction

Designing sampled-data observers and controllers has become an important topic in the last 20 years (see, for example, [1]). The idea of so-called sampled data was used in the book [2]. In the paper [3], it has been shown that the introduction of time-varying gains in a specific class of observers improves their exponential convergence properties in the presence of measurement delay. Currently, these properties are being investigated in the presence of measurement sampling. A long standing issue is how to enlarge the sampling intervals, see [4]. The paper [5] provides exponential stability results for two types of systems. The first class includes a family of nonlinear ordinary differential systems, while the second consists of semi-linear parabolic partial differential equations. A common feature of both types is that the systems include sampled-data states and a time-varying gain. Sufficient conditions for global exponential stability are formulated in terms of linear matrix inequalities based on Lyapunov–Krasovskiĭ functionals. The established stability results prove to be useful in designing exponentially convergent observers based on sampled-data measurements. It was demonstrated through examples that the introduction of time-varying gains is beneficial for enlargement of sampling intervals while preserving the stability of the system.

In our paper, we use the idea of sampled-data observers, but on corresponding time intervals only. This allows us to remain with a finite-dimensional fundamental system. In the cases of periodic coefficients and delay, we propose a simple version of the Floquet theory for first-order delay differential equations, which is completely analogous to the case of ordinary differential equations.

Floquet theory (originating in [6]) is an important part in the qualitative theory of differential equations. For equations with periodic coefficients, this theory allows us to get a way to represent solutions and helps essentially in stability analysis. The foundations of Floquet–Lyapunov theory for a system of ordinary differential equations were established in [7–11]. For equations with memory, Floquet theory was developed, for example, in the following works. Floquet theory for integrodifferential equations was developed in [12–14], where results on stability of these equations were also obtained. In the paper [15], the analytical theory of infinite determinants was proposed for integrodifferential Floquet theory. Floquet theory for difference equations was presented in [16]. It should be stressed that the first books on Floquet–Lyapunov theory for discrete time systems with periodic coefficients are [17, 18]. Floquet theory for dynamical systems on time scales was intensively developed (see the works [19–21]). A generalized Floquet theory for nonlinear systems was presented in [22]. For delay equations, the foundations of Floquet theory were proposed in [23–25]. Note the well-known books [26, 27], which are very closely related to our research. The idea of the semi-discretization method (see [26, Chapter 3]) presents an approach to the finite-dimensional representation of the monodromy operator. Discretization is made only in the "delay part" of the equations and not in their "ordinary part". The semi-discretization method is a sort of sampled data approach. It can be interpreted as a "way" to obtain a finite-dimensional fundamental system. The graph of the solution of the discretized retarded equation is a connection of graphs of the solutions of the ordinary equations forcing by constant righthand sides at intervals between discretization points. Stability conditions are determined by eigenvalues of the monodromy operators. In the standard approach, the monodromy operator in the case of delay differential equations was infinite-dimensional. Note also the book [27], which addresses the problems of stability analysis, stabilization, and robust fixed-order control of dynamical systems subject to delays. Within the eigenvalue-based framework, an overall solution is given to the stability analysis, stabilization, and robust control design problem, using both analytical methods and numerical algorithms, and it is applicable to a broad class of linear time-delay systems. The authors of [27] considered stabilization as well as the design of robust and optimal controllers.

In [28], computing the Floquet multipliers for functional differential equations was developed. In [29], unboundedness of all solutions to second-order delay equations on the basis of Floquet theory as well as estimates of distances between zeros was obtained. In [30], an analytical approach was proposed. It should be mentioned that an infinite-dimensional fundamental system does not allow obtaining a full analogue of Floquet theory (for example, the assertions about passing to systems with constant coefficients [24]). Note several papers on applications of Floquet theory to delay equations. In the paper [31], the authors discussed the properties of Floquet exponents for delay linear periodic systems with constant delays and a new cholera epidemic model with phage dynamics and seasonality was considered. Necessary and sufficient conditions of stability in the case of constant delays and triangular matrices of the coefficients were proposed. Generalized Floquet theory with application to dynamical systems with memory and Bloch's theorem for nonlocal potentials was proposed in [25].

The main goal is to propose a version of the Floquet theory for first-order delay differential equations in a form that the one-dimensional fundamental system is preserved. One-dimensional fundamental systems allow us not only to study stability but also to obtain formulas of solutions' representations. The standard approach through infinite-dimensional fundamental systems does not lead to suffciently simple formulas of solutions' presentations and, as far as we know, is used only for analysis of the Floquet multipliers and stability on this basis. One-dimensional fundamental systems allow us to avoid the standard assumption on smallness of the delay (see inequality (32) below, which comes from the classical books [24, 32], and in the general case continues to be very essential). However, these classical assumptions are not necessary, and we propose a way to avoid them through the use of the Floquet theory in our version. We not only consider variable delays, but also avoid the principal assumptions on smallness of a product of coeffcient and delay. In previous results, as far as we know, it was demonstrated that, in the corresponding cases, the constants estimating this product could not be increased. After our results, it is clear that conditions on the smallness could not only be improved but even avoided altogether.

We propose a one-dimensional version of Floquet theory for delay differential equations of the first order. We obtain Floquet formulas to represent solutions. Taking into account the fact that modern computer methods solve equations with simple delay (and these are the ones used in modern applications) on any finite interval, this gives us the opportunity to obtain solutions on the semi-axis. One of the results of the new possibilities that have opened up is that the tests of stability we obtained are significantly different from the existing ones. Researchers' attention has been focused on increasing the lengths of possible zones where the argument deviation ($t - \tau(t)$, where $\tau(t)$ is delay) is constant. We shift this attention to something else. Lengths of zones where the deviation is constant can be very small and constitute a very small part of the period of coefficients. It is only important that in this case, it remains possible to apply the version of the finite-dimensional Floquet theory we propose. Moreover, then only the value of the solution at the period point turns out to be important. Note that other mathematicians are trying to develop an idea that allows one to obtain a representation of solutions by somehow arriving at a finite-dimensional fundamental system. In this regard, note the recent work [33] and the references therein. In it, the initial function is expanded into a finite sum of some selected functions, and the solution becomes a linear combination of functions that are continuations of these functions. The finite-dimensionality of the fundamental system is preserved, but the one-dimensionality is not. Our results are based essentially on the fact that the fundamental system stays one-dimensional.

Our paper is constructed as follows. Technological motivations of our studies are presented in Section 2. A version of Floquet theory for delay differential equations is presented in Section 3. Tests of exponential stability based on Floquet theory are formulated in Section 4. Applications to stabilization of vertical motion of a drone's flight are proposed in Section 4. Proofs are written in Section 5. In Section 6, height regulations are considered. Discussion and open problems can be found in Section 7.

2. Height stabilization of a drone's flight

Our main motivation lies in the task of stabilizing the height of the drone's flight. Drone flight control tasks have recently become key to mathematical modeling in industrial and military aviation technology. The aerodynamic aspects of the flight of aircraft and similarly-shaped flying machines have been developed for quite some time and are included in the basic textbooks (see, for example, the book [34] from 1961).

With the development of technology, unmanned aerial vehicles (UAVs) have become a very important area of research around the world in recent decades (see, for example, the state of the art in the papers [35–38], where models of quadrotors are considered and many important references are given. Fixed-wing drones were also developed [39]). Modeling flight of such drone motion in the form of system of delay differential equations is described in the paper [40]. As examples of such use, we can note our projects in this area [41–43]. In this article, we will focus only on the problem of height stabilization for drone motion in the vertical plane. Let us now describe the process mathematically.

The flight of a drone is characterized by the following steady state parameters:

- α_0 is the angle of attack, i.e., the angle between the longitudinal axis of the drone and the projection of the drone's velocity on the symmetry plane of the drone,
- ϑ_0 is the pitch angle, i.e., the angle between the longitudinal drone axis and the horizontal plane,
- V_0 is the flight velocity tangent to the trajectory (with respect to air),
- H_0 is the height above mean sea level of the drone's flight,
- $(U_y)_0$ is the wind velocity along the y axis.

Let us consider a perturbation of these steady state parameters:

$$V(\bar{t}) = V_0 + \Delta V(\bar{t}), \quad \hat{\alpha}(\bar{t}) = \alpha_0 + \Delta\alpha(\bar{t}), \quad \hat{\vartheta}(\bar{t}) = \vartheta_0 + \Delta\vartheta(\bar{t}),$$
$$U_y(\bar{t}) = (U_y)_0 + \Delta U_y(\bar{t}), \quad H(\bar{t}) = H_0 + \Delta H(\bar{t}).$$

Their normalized values are usually taken as

$$\begin{cases} v(t) = \Delta V(\tau_a t)/V_0, \\ h(t) = \Delta H(\tau_a t)/V_0\tau_a, \\ \alpha(t) = \Delta\alpha(\tau_a t), \\ \vartheta(t) = \Delta\vartheta(\tau_a t), \\ v_y(t) = \Delta U_y(\tau_a t)/V_0, \end{cases}$$

where $t = \frac{\bar{t}}{\tau_a}$, $\tau_a = m/(\rho_0 V_0 S)$, m is the drone's mass, S is the area of the wings, $\rho_0 = \rho(0)(T_H(H_0)/T(0))^{1/(\gamma-1)}$, $T_H(H)$ is the temperature at flight height H, $T(0)$ and $\rho(0)$ are the temperature and air density at mean sea level, and γ is the adiabatic constant. See Figure 1. The notation used in Figure 1 is as follows:

- Y is the carrying force orthogonal to the flight velocity,
- V is the flight velocity tangent to the trajectory (with respect to air),
- X is the resistance force opposite to V,
- G is the gravitation force,

- P is the tractive force directed along the longitudinal drone axis,
- H is the height above mean sea level of the drone's flight,
- ϑ is the pitch angle, i.e., the angle between the longitudinal drone axis and the horizontal plane,
- θ is the tilting of the velocity about the horizontal plane,
- α is the angle of attack.

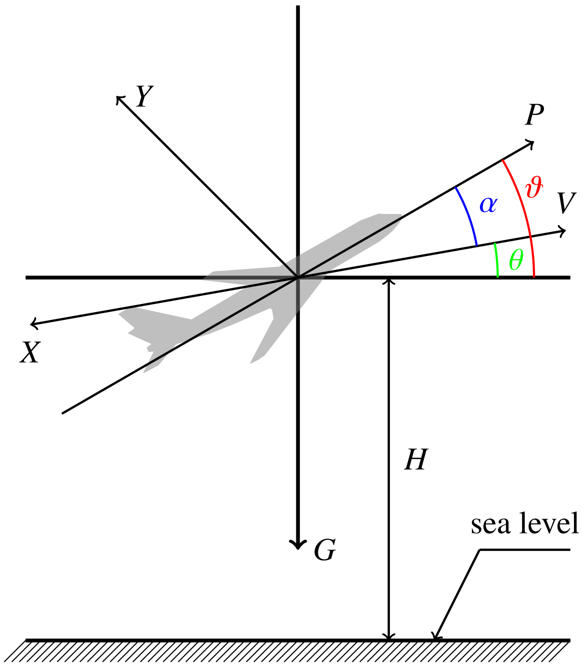

Figure 1. Drone longitudinal motion.

Our basic model consists of four equations for normalized perturbations of four drone motion parameters with respect to four steady state values [34, 40]:

$$\begin{cases} v'(t) + n_{11}v(t) + n_{12}\alpha(t) + n_{13}\vartheta(t) + n_{14}h(t) = n_p\delta_p(t) + f_1(t), \\ \alpha'(t) - n_{21}v(t) + n_{22}\alpha(t) - \vartheta'(t) - n_{23}\vartheta(t) + n_{24}h(t) = f_2(t), \\ n_{31}v(t) + n_0\alpha'(t) + n_{32}\alpha(t) + \vartheta''(t) + n_{33}\vartheta'(t) + n_{34}h(t) = -n_B\delta_B(t) + f_3(t), \\ h'(t) - n_{41}v(t) + n_{42}\alpha(t) - n_{42}\vartheta(t) = v_y(t). \end{cases} \tag{1}$$

In (1), we have the following controlling signals: $\delta_p(t)$ is the position of the drone central control knob, $\delta_B(t)$ is the deviation of the drone control elevator, $f_1(t), f_2(t), f_3(t)$ are random forces and random moments of forces, and $v_y(t)$ is the random fluctuation of the vertical component of wind velocities along axes y. The coefficients n_{ij}, n_B, n_p, n_0 are constant parameters obtained by linearization of basic nonlinear equations, and they can depend on parameters of steady state motion. The values of these constant parameters can be found in [40, Table 1].

Let us consider some simplified cases of the linear equations of motion (1) together with its stability analysis. We suppose that dependence on a flight height on $v(t)$, $\alpha(t)$, and $\vartheta(t)$ is negligible, i.e.,

$n_{14} = n_{24} = n_{34} = 0$. This leads us to the system

$$\begin{cases} v'(t) + n_{11}v(t) + n_{12}\alpha(t) + n_{13}\vartheta(t) = n_p\delta_p(t) + f_1(t), \\ \alpha'(t) - n_{21}v(t) + n_{22}\alpha(t) - \vartheta'(t) - n_{23}\vartheta(t) = f_2(t), \\ n_{31}v(t) + n_0\alpha'(t) + n_{32}\alpha(t) + \vartheta''(t) + n_{33}\vartheta'(t) = -n_B\delta_B(t) + f_3(t), \\ h'(t) - n_{41}v(t) + n_{42}\alpha(t) - n_{42}\vartheta(t) = v_y(t). \end{cases} \tag{2}$$

Let us start with analysis of system (2) for zero controlling signals and random external parameters

$$\delta_p(t) = \delta_B(t) = f_1(t) = f_2(t) = f_3(t) = v_y(t) = 0.$$

The first three equations for $\alpha(t), \vartheta(t), v(t)$ in the system (2) become independent from the fourth equation for $h(t)$. The characteristic equation for the system of the first three equations is

$$\Delta(p) = \begin{vmatrix} p + n_{11} & n_{12} & n_{13} \\ -n_{21} & p + n_{22} & -(p + n_{23}) \\ n_{31} & n_0 p + n_{32} & p^2 + n_{33}p \end{vmatrix} = 0,$$

which can be written in the form of a fourth-order polynomial $p^4 + c_1p^3 + c_2p^2 + c_3p + c_4 = 0$. The Routh–Hurwitz criterion for exponential stability is

$$\begin{cases} c_1 > 0, \quad c_2 > 0, \quad c_3 > 0, \quad c_4 > 0, \\ c_1c_2 - c_3 > 0, \\ c_1(c_2c_3 - c_1c_4) - c_3^2 > 0, \end{cases}$$

and we get the exponential stability of the horizontal motion.

Let us give some real examples of a characteristic equation for a jet airplane in [34]:

$$p^4 + 2.8p^3 + 4.45p^2 + 0.049p + 0.057 = 0.$$

The system described by these three first equations is exponentially stable, and its characteristic equation has the roots

$$p_{1,2} = -1.39853 \pm 1.57259\mathrm{i}, \quad p_{3,4} = -0.00147 \pm 0.11344\mathrm{i},$$

and the horizontal motion of the drone is exponentially stable. However, motion along the vertical direction is always not exponentially stable. We need a control for stabilization of a flight height.

Suppose that $v_y(t)$ changes at every small time interval Δt, has independent random values in these moments, has random normal (Gauss) distribution with mean value equal to zero, and standard deviation equal to σ_v. Then, $h(t)$ in these moments will also be randomly distributed with mean value equal to zero and standard deviation equal to $\sqrt{\langle h(t)^2 \rangle} \sim \sqrt{t}$ (here, $\langle \cdot \rangle$ means the value over random distribution). The dependence of $\alpha(t), \vartheta(t), v(t)$ on time is exponentially stable for these roots p_j ($j = 1, 2, 3, 4$). This exponential dependence is much faster than the square root dependence of $h(t)$ on time. So we suppose that $\alpha'(t) = \vartheta'(t) = v'(t) = 0$ for time scales of height changes.

We also need a feedback control of the autopilot for stabilization of a flight height. Let us suppose that our controlling parameters are equal to

$$\delta_p(t) = a_p h(g(t)), \quad \delta_B(t) = a_B h(g(t)), \quad g(t) = t - \tau(t),$$

where τ is time delay for a measurement of the normalized height h, and a_p, a_B are controlling constants that can be chosen by the user of the autopilot. It is natural to suppose that we get the value of $h(t)$ (in system (2)) for the autopilot with time delay $\tau(t) = t - g(t)$ from some measurement devices. The parameters δ_p and δ_B are control parameters (for example, steering angle) that can be chosen free (arbitrary) by the operator of the drone. So, we can choose also these parameters as linear functions of any variables that are known to the drone operator (measured by drone's sensors, for example, perturbation of height with measurement time delay with respect to desirable height $h(t-\tau(t))$). We try to choose the constant in this relation to get a small perturbation of h, i.e., to get a flight of the drone on stable height. It is not only realistic, but also the free and correct choice of the control parameter is the basis of control and "free will" of an operator in control. From system (2), we get

$$\begin{cases} n_{11}v + n_{12}\alpha + n_{13}\vartheta = n_p a_p h(g(t)), \\ -n_{21}v + n_{22}\alpha - n_{23}\vartheta = 0, \\ n_{31}v + n_{32}\alpha = -n_B a_B h(g(t)), \\ h'(t) - n_{41}v + n_{42}\alpha - n_{42}\vartheta = v_y(t). \end{cases}$$

Solving the first three equations for v, α, ϑ and then substituting the result into the fourth equation, we obtain

$$v_y(t) = h'(t) + ah(g(t)), \quad \text{where} \quad a = \begin{bmatrix} -n_{41} & n_{42} & -n_{42} \end{bmatrix} \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ -n_{21} & n_{22} & -n_{23} \\ n_{31} & n_{32} & 0 \end{bmatrix}^{-1} \begin{bmatrix} n_p a_p \\ 0 \\ -n_B a_B \end{bmatrix},$$

and denoting $x(t) = h(t)$, $f(t) = v_y(t)$, $g(t) = t - \tau(t)$, we arrive at the scalar equation

$$x'(t) + ax(t - \tau(t)) = f(t), \quad t \in [0, \infty). \tag{3}$$

Thus, we have to study stability of the scalar equation (3) with respect to the righthand side $f(t)$ to be sure that "small" errors in the righthand side do not imply "big" perturbations in the solution. According to the classical Bohl–Perron theorem for this case, the homogeneous equation

$$x'(t) + ax(t - \tau(t)) = 0, \quad t \in [0, \infty) \tag{4}$$

must be exponentially stable. Thus, by studying stability of (4), we can get conditions for exponential stability of the basic system (2).

Example 1. The equation

$$x'(t) + ax(t - \tau) = 0, \quad t \in [0, \infty) \tag{5}$$

with constant coefficient a and delay τ represents a classical object in the theory of delay differential equations [24, 32], and it is a particular case of (4). It is known [24, 32] that in the case of positive τ, the inequality $0 < a\tau < \frac{\pi}{2}$ is necessary and sufficient for exponential stability of (5). If $a\tau > \frac{\pi}{2}$, then all nontrivial solutions of (5) oscillate and amplitudes of these oscillating solutions can tend to infinity, and, of course, there is no stability with respect to the righthand side in this case. This situation is quite typical in applications. It takes time τ to receive, proceed, and send a signal to control the drone, which can be quite long. The sensitivity of the system does not allow the coefficient a to be reduced below a certain sensitivity threshold, i.e., the gain coefficient a cannot be, for instance, less than one when the control device works.

Suppose, for example, that two constraints hold: T1) the delay τ is not less than 10 and T2) the coefficient characterizing the minimum response of the control device cannot be less than 1. Building a mathematical model, we can now come to the particular case of (5) in the form

$$x'(t) + x(t - 10) = 0, \quad t \in [0, \infty),$$

which is unstable. It seems that this closes the possibility of stabilization by feedback with a delay. Is it possible to somehow stabilize this process, preserving only one control governing term and changing to some extent only the delay in the control action, i.e., to consider the equation

$$x'(t) + x(t - \tau(t)) = 0, \quad t \in [0, \infty), \tag{6}$$

while remaining within the framework of the completely natural technical constraints formulated above as T1 and T2? As far as we know, there is no complete solution to this issue. We propose in this paper a solution to this issue on the basis of Floquet theory for delay differential equations. Concerning the coefficient $a(t)$ in (4): In the paper [44], it was demonstrated that a corresponding transform reduces (4) with positive coefficient $a(t)$ to the equation

$$x'(t) + x(g(t)) = 0, \quad t \in [0, \infty), \tag{7}$$

in which the coefficient is equal to one.

Perturbations of the trajectory also can result in the crash of a drone (for example, because of collision with a mountain). The formula of the solution's representation obtained in this paper could solve this problem. It leads us to the conclusion that in real technical problems, together with analysis of stability, estimates of solutions have to be obtained. This results in constructing the Cauchy functions, or at least in obtaining their estimates. We deal with constructing the Cauchy functions in an upcoming paper.

3. Floquet theorem for first-order delay differential equations

In this paper, we consider the problem

$$x'(t) + \sum_{i=1}^{m} a_i(t)x(t - \tau_i(t)) = 0, \quad t \in [0, \infty) \tag{8}$$

$$x(\xi) = 0, \quad \xi < 0, \tag{9}$$

with essentially bounded measurable coefficients $a_i(t)$ and delays $\tau_i(t) \geq 0$ for $t \in [0, \infty)$.

Definition 2. We will say that (8) is *uniformly exponentially stable* if there exist $M > 0$ and $\gamma > 0$ such that the solution of problem (8) with initial condition

$$x(\xi) = \varphi(\xi) \quad \text{for} \quad \xi < t_0, \quad t_0 \geq 0,$$

where $\varphi : (-\infty, t_0)$ is a Borel-measurable bounded function, satisfies the estimate

$$|x(t)| \leq Me^{-\gamma(t-t_0)} \operatorname{ess\,sup} |\varphi(\xi)|, \quad t \geq t_0, \tag{10}$$

where the constants M and γ do not depend on t_0 and φ.

The solution of the nonhomogeneous equation

$$x'(t) + \sum_{i=1}^{m} a_i(t)x(t - \tau_i(t)) = f(t), \quad t \in [0, \infty)$$

with measurable essentially bounded f and the zero initial function (9) can be represented [45, 46] in the form

$$x(t) = \int_0^t C(t, s)f(s)\mathrm{d}s + C(t, 0)x(0), \quad t \in [0, \infty),$$

where $C(t, s)$ is called the Cauchy function (in other terminology it is known as the fundamental function). For fixed s as a function of the variable t, it satisfies the equation

$$\frac{\partial C(t, s)}{\partial t} + \sum_{i=1}^{m} a_i(t)C(t - \tau_i(t), s) = 0, \quad t \in [s, \infty)$$

with the initial condition $C(s, s) = 1$ and the initial function

$$C(t, s) = 0 \quad \text{for} \quad t < s.$$

Definition 3. We say that the Cauchy function $C(t, s)$ satisfies an *exponential estimate* if there exist constants $M > 0$ and $\gamma > 0$ such that

$$|C(t, s)| \leq Me^{-\gamma(t-s)} \quad \text{for} \quad 0 \leq s \leq t < \infty.$$

It is known [45, 46] that the facts of uniform exponential stability and the exponential estimate of the Cauchy function $C(t, s)$ coincide in the case of bounded delays.

The coefficients and delays in the frame of Floquet theory are assumed to be periodic with a corresponding period ω, i.e., the equalities

$$a_i(t + \omega) = a_i(t), \quad \tau_i(t + \omega) = \tau_i(t), \quad t \in [0, \infty) \tag{11}$$

are true. To obtain a version of Floquet theory, we assume also that

$$t - \tau_i(t) \geq 0, \quad t \in [0, \omega], \quad i = 1, \ldots, m. \tag{12}$$

We want to find solutions satisfying the equality

$$x(t + \omega) = \lambda x(t). \tag{13}$$

The following assertion can be considered as an analog of the Floquet theorem about the presentation of the solution for delay differential equation (8).

Theorem 4. *Let conditions* (11) *and* (12) *be fulfilled. Then, in the case of* $\lambda \neq 0$*, every solution* x *of* (8) *can be represented in the form*

$$x(t) = \Phi(t)\exp\left\{\frac{\ln|\lambda|}{\omega}t\right\}, \tag{14}$$

where the function Φ *is* ω*-periodic if* $\lambda > 0$ *and* 2ω*-periodic if* $\lambda < 0$*, and in the case of* $\lambda = 0$*, in the form*

$$x(t) = \begin{cases} y(t) & \text{if} \quad 0 \leq t \leq \omega, \\ 0 & \text{if} \quad t > \omega, \end{cases} \tag{15}$$

where y *is a solution of* (8) *on* $[0, \omega]$*. The function* Φ *in* (14) *can be defined by the formula*

$$\Phi(t) = \exp\left\{\frac{\ln|\lambda|}{\omega}(\omega k - t)\right\} y(t - k\omega) \quad \text{for} \quad t \in (k\omega, (k+1)\omega], \quad k = 0, 1, 2, \ldots.$$

Theorem 5. *Let conditions* (11) *and* (12) *be fulfilled. Then,* (8) *is uniformly exponentially stable, if and only if, the solution* y *of* (8) *on* $[0, \omega]$ *such that* $y(0) = 1$ *satisfies the inequality* $|y(\omega)| < 1$.

Remark 6. Note that the equality $\lambda = 0$ in (13) implies that the solution y satisfies the condition $y(\omega) = 0$. This case and the case of $\lambda < 0$ are impossible for the ordinary differential equation

$$x'(t) + a(t)x(t) = 0, \quad t \in [0, \infty),$$

for which $\lambda > 0$, since its general solution is

$$x(t) = x(0)\exp\left\{-\int_0^t a(s)\mathrm{d}s\right\} > 0 \quad \text{if} \quad x(0) > 0.$$

In the case of

$$\int_0^\omega a(s)\mathrm{d}s > 0,$$

it is clear that $x(\omega) < x(0)$, and this implies that $\lambda < 1$. The solution x satisfies $x(t) \to 0$ for $t \to \infty$. In the case of

$$\int_0^\omega a(s)\mathrm{d}s < 0, \tag{16}$$

the solution x satisfies $x(t) \to \infty$ for $t \to \infty$. In the case of

$$\int_0^\omega a(s)\mathrm{d}s = 0,$$

the solution is an ω-periodic function. In the case of the delay equation

$$x'(t) + a(t)x(t - \tau(t)) = 0, \quad t \in [0, \infty), \tag{17}$$

the behavior of solutions can be more complicated.

Example 7. Consider (17), where

$$a(t) = \begin{cases} -1 & \text{if} \quad 0 \le t < 0.5, \\ 1 & \text{if} \quad 0.5 \le t < 1, \end{cases}$$

$$t - \tau(t) = \begin{cases} 0 & \text{if} \quad 0 \le t < 0.5, \\ 0.5 & \text{if} \quad 0.5 \le t < 1, \end{cases}$$

$$a(t) = a(t-1) \quad \text{and} \quad \tau(t) = \tau(t-1) \quad \text{for} \quad t \ge 1.$$

Its solution satisfying the initial condition $x(0) = 1$ is

$$x(t) = \begin{cases} 1 + t & \text{if} \quad 0 \le t < 0.5, \\ \frac{3}{2} - \frac{3}{2}\left(t - \frac{1}{2}\right) & \text{if} \quad 0.5 \le t < 1, \\ \frac{3}{4} + \frac{3}{4}(t-1) & \text{if} \quad 1 \le t < 1.5, \\ \frac{9}{8} - \frac{9}{8}\left(t - \frac{3}{2}\right) & \text{if} \quad 1.5 \le t < 2, \\ \frac{9}{16} + \frac{9}{16}(t-2) & \text{if} \quad 2 \le t < 2.5, \\ \frac{27}{32} - \frac{27}{32}\left(t - \frac{5}{2}\right) & \text{if} \quad 2.5 \le t < 3, \\ \frac{27}{64} + \frac{27}{64}(t-3) & \text{if} \quad 3 \le t < 3.5, \\ \cdots, \end{cases} \tag{18}$$

and it is clear that $0 < x(t) < 1$ for $t \ge 3$.

Let us choose now $\omega > 1$ and consider the equation

$$y'(t) + a_\omega(t) y(t - \tau_\omega(t)) = 0, \quad t \in [0, \infty) \tag{19}$$

such that

$$a_\omega(t) = a(t) \quad \text{for} \quad t \in [0, \omega] \quad \text{and} \quad a_\omega(t) = a_\omega(t - \omega) \quad \text{for} \quad t \ge \omega,$$
$$\tau_\omega(t) = \tau(t) \quad \text{for} \quad t \in [0, \omega] \quad \text{and} \quad \tau_\omega(t) = \tau_\omega(t - \omega) \quad \text{for} \quad t \ge \omega$$

and find its solution satisfying the initial condition $y(0) = 1$. It is clear that the solution y of (19) coincides with the solution x of (17) on $[0, \omega]$. If we choose $\omega \in \left(\frac{4}{3}, \frac{11}{6}\right)$, then $y(\omega) > 1$ and $y(t) \to \infty$ as $t \to \infty$. If $\omega \in \left\{\frac{4}{3}, \frac{11}{6}\right\}$, then the solution y of (19) is periodic. If $\omega \in \left(1, \frac{4}{3}\right) \cup \left(\frac{11}{6}, \infty\right)$, then $y(\omega) < 1$, $y(t) \to 0$ as $t \to \infty$, and (19) is exponentially stable. Thus, we see that condition (16) can be fulfilled, but the solution y satisfies $y(t) \to 0$ as $t \to \infty$, and (19) is exponentially stable.

Remark 8. We see that the analysis of the uniform exponential stability is reduced to verifying the inequality $|y(\omega)| < 1$ for the value of the solution y at the point ω. For standard delays used in applications, we can compute the value $y(\omega)$. The conditions (11) and (12) look natural in applications. Our aim in stabilization by the control $u(t) = -a(t)x(t - \tau(t))$ with $a(t)$ and $\tau(t)$ satisfying conditions (11) and (12) is to obtain stability of (17).

Remark 9. Let us look on the period ω as a parameter. This parameter is used in conditions (11) and (12). Where can this parameter be taken on the semiaxis? Let us introduce the condition

(A) the delay in (17) is such that $t - \tau(t)$ is a nondecreasing function and $a(t) \gneqq 0$.

If the solution x of (17) oscillates on the semiaxis $t \geq 0$, then there exists an infinite number of intervals (α_n, β_n) such that $|x(t)| < 1$ for $t \in (\alpha_n, \beta_n)$. It is clear that in the case of

$$\omega \in \bigcup_{n=1,2,3,\ldots} (\alpha_n, \beta_n),$$

we achieve exponential stability of (17) satisfying conditions (A), (11), (12). Thus, the use of Floquet theory avoids the classical limitations of the form $0 < a\tau < \frac{\pi}{2}$ for (5) with constant coefficient a and delay τ [32] or $\int_{t-\tau(t)}^{t} a(s)\mathrm{d}s < \frac{3}{2}$, where $a(t) \gneqq 0$ for (17) with variable ones [47].

Remark 10. Consider (7) in which

$$g(t) = \frac{n}{10} \quad \text{for} \quad n \leq t < n+1, \quad n = 0, 1, 2, \ldots. \tag{20}$$

The delay in this case is $\tau(t) = t - g(t)$, and it tends to infinity as $t \to \infty$. The solution of (7) with $g(t)$ defined by (20) can be computed on every finite interval $[0, \omega]$ as $x(t) = x(n) - \int_n^t x\left(\frac{n}{10}\right)\mathrm{d}s$. This solution to (7) oscillates with amplitudes tending to infinity as $t \to \infty$. Let us construct now the ω-periodic delay as $\tau_\omega(t) = t - g(t)$ for $t \in [0, \omega)$ and $\tau_\omega(t) = \tau_\omega(t - \omega)$ for $t \geq \omega$. It is clear that (7) with $g(t)$ defined by (20) and the equation

$$x'(t) + x(t - \tau_\omega(t)) = 0, \quad t \in [0, \infty) \tag{21}$$

coincide on the interval $[0, \omega]$ for every $\omega > 0$. If the solution x starting with the initial condition $x(0) = 1$ satisfies the inequality $|x(\omega)| < 1$, then (21) is exponentially stable, according to Theorem 5. Existing tests of stability based on a "smallness" of delay cannot be applied here. Where can such values of ω be taken from? Remark 9 explains the existence of an infinite number of possible intervals for ω. The first two of them are $(0, 2.125)$ and $(19.5, 21.755)$. Thus a combination of sampled data, allowing us to solve equations on every finite interval, with Floquet representations opens absolutely new possibilities for stabilization.

4. Stability based on Floquet theory

In this section, we consider equations with constant coefficient a and "almost constant" delay $\tau(t)$. We explain in Remarks 12 and 13 why our assumption on the delay $\tau(t)$ appears natural from a technological point of view. We propose very unusual tests for exponential stability. In these tests, the classical (3/2) stability condition of A. D. Myshkis [32] and T. Yoneyama [47] does not appear at all. A condition on the period of ω of the delay is proposed instead. It should be noted that this condition does not assume a smallness of the period ω.

Let us take constant positive coefficient a and delay $\tau(t)$ which is constant on the interval $[\tau, \omega]$ and such that condition (10) is fulfilled on the interval $[0, \tau]$. The conditions of Theorem 5 are satisfied now. We propose the following assertion, which will be proven in Section 5 below on the basis of Lemmas 18 and 19.

Theorem 11. *Let $a(t) = a > 0$,*

$$\tau(t) = \begin{cases} t & \text{if} \quad 0 \leq t < \tau, \\ \tau & \text{if} \quad \tau \leq t < \omega \end{cases} \tag{22}$$

$\tau(t)$ be continued as a periodic function for all $t > \omega$, i.e., $\tau(t) = \tau(t-\omega)$, and

$$\left|\sum_{n=0}^{k} \frac{(-1)^n a^n}{n!}(\omega - (n-1)\tau)^n\right| < 1, \quad (k-1)\tau \le \omega \le k\tau, \quad k = 1, 2, 3, \ldots. \tag{23}$$

Then, (17) *is uniformly exponentially stable.*

Let us come back now to the equation described as a model in Example 1.

Remark 12. The condition (22) is fruitful under the following physical conditions: Suppose we have two measurement devices. The first one measures $x(t)$ in (17) with the time delay τ. We know from the previous considerations [24, 32, 45, 46] that, using only this measurement device for the autopilot's operation, results in instability of $x(t)$ for "big" time delay $a\tau > \frac{\pi}{2}$. So, we need to add the second measurement device that makes measurements with zero time delay, but only at discrete time moments with the time interval ω between these time moments. We use the second device measurement (for the autopilot) during time $[k\omega, k\omega + \tau]$ and the first device measurement during time $[k\omega + \tau, (k+1)\omega]$. By such a way, we can get conditions (22). Theorem 11 can give uniform exponential stability in this case.

Remark 13. In the case when we do not have the second device making measurements without delay, there is a simple technical possibility to arrive at (6) with ω-periodic delay satisfying (22), where $\tau = 10$. The Newton–Leibniz formula and substitution of the derivative from (6) lead to

$$|x(t)| = \left|x(t-10) + \int_{t-10}^{t} x'(s)\mathrm{d}s\right| = \left|x(t-10) - \int_{t-10}^{t} x(s-10)\mathrm{d}s\right| \tag{24}$$

which can be realized technically since we know the values of $x(s)$ for all $s \le t - 10$. Let us verify the inequality

$$|x(t)| = \left|x(t-10) - \int_{t-10}^{t} x(s-10)\mathrm{d}s\right| < |x(0)|.$$

If the solution x of (6) oscillates on the semiaxis $t \ge 0$, then there exist an infinite number of intervals (α_n, β_n) such that $|x(t)| < 1$ for $t \in (\alpha_n, \beta_n)$. It is clear that in the case of

$$\omega \in \bigcup_{n=1,2,3,\ldots} (\alpha_n, \beta_n),$$

we achieve the exponential stability of (6) satisfying condition (22).

Now let us choose ω from one of these intervals. Then,

$$x(\omega) = x(\omega - 10) - \int_{\omega-10}^{\omega} x(s-10)\mathrm{d}s. \tag{25}$$

Let us add to the process of building control a so-called "predictive" character. In constructing a feedback control, we do not have to wait for the moment ω in order to know the value of $x(\omega)$ and can use the value calculated by formula (25). We suppose that the values of $x(s)$ for $s \le \omega - 10$ are known when we calculate $x(t)$ for $t \ge \omega$, and this satisfies the constraints T1 and T2. This ω can be chosen as the period, and the delay can be

$$\tau(t) = \begin{cases} t - \omega(k-1) & \text{if} \quad \omega(k-1) \le t < \omega(k-1) + 10, \\ 10 & \text{if} \quad \omega(k-1) + 10 \le t \le \omega k, \end{cases} \tag{26}$$

which is ω-periodic and the condition (22) is fulfilled: It is clear that (26) and (22) with ω-periodic continuation of $\tau(t)$ coincide. Inequality (23) gives us such ω. The oscillation of the solution (see passage around formulas (24) and (25)) explains that such ω exists. In the case of nonoscillating solution, we have a fact that the solution is nondecreasing and $x(\omega) < x(0) = 1$, and inequality (23) is fulfilled for every ω.

Remark 14. We can find suitable intervals for ω such that condition (23) holds. In the case of the Eq (5) considered in Example 1, we construct (6), where $\tau(t)$ is defined by (22).

Corollary 15. *If $\tau(t)$ is defined by* (22) *with $\tau = 10$, then* (6) *is exponentially stable if*

$$\omega \in (30.106, 30.167) \cup (44.8064, 44.8146) \cup (59.4862, 59.4873) \\ \cup (74.16643, 74.16657) \cup (88.8454369, 88.8454563) \cup \cdots .$$

Proof. The proof follows from calculating the sum in (23) and verifying the inequality required in (23). □

We can take a finer division of intervals for ω to achieve more precise intervals within the union. For any N, we can choose a division such that the number of intervals M is greater than N.

Remark 16. (On optimizing the choice of ω for maximizing the rate of decay of the solution to zero at infinity). The lefthand side of inequality (23) is $|\lambda(\omega)|$. Formula (14) indicates the role of $\lambda(\omega)$ in the solution's representation.

Table 1. ω and decay rate values.

ω	30.106	30.112	30.125	30.135	30.137	30.153	30.167
λ	0.99657	0.8046	0.3877	0.00659	0.0014	−0.5156	−0.9699
$\frac{\ln(\lvert\lambda(\omega)\rvert)}{\omega}$	−0.00011	−0.00721	−0.0314	−0.0902	−0.217	−0.0219	−0.00214

Thus, we see that an appropriate choice of ω can increase the rate of convergence of the solution to zero when t tends to infinity. In Remark 12, we noted that the use of only one measurement device with large constant delay cannot achieve stability. In the corresponding case, the second measurement device, which makes measurements without any delay, but only in discrete time moments, could be added. We can increase ω by lengthening the time intervals between measurements. From a practical point of view: To make the next measurement by the second device not immediately after the previous measurement, but to introduce a corresponding break between two measurements of the second device. This simple step could, as our examples demonstrate, increase the speed of stabilization of the process $x(t)$.

Corollary 17. *Let $\tau(t)$ be defined by* (22)*, with $\tau = 100$. Then,* (6) *is exponentially stable if*

$$\omega \in (115.036, 115.177) \cup (238.23529, 238.23864) \cup (364.05305, 364.05309) \cup \cdots .$$

The optimal choice of ω, as we see from Table 1, is approximately at the center of the intervals for possible ω, in which the exponential stability is achieved, as noted above in the remarks. Using classical methods to approximate the roots of the polynomials describing the solutions, we can determine these ω with high accuracy. Selecting ω approximately equal to one of these roots will maximize the rate of the stabilization process.

5. Proofs

Proof of Theorem 4. The space of solutions to (8) under the condition $\tau_i(t) \leq t$ for $i = 1, \ldots, m$ is one-dimensional according to the general theory of functional differential equations [45,46,48]. Periodicity of the delay $\tau_i(t + \omega) = \tau_i(t)$ and the inequality $\tau_i(t) \leq t$ imply that $k\omega \leq t - \tau_i(t) \leq (k + 1)\omega$ for $t \in [k\omega, (k + 1)\omega]$ and for every $i = 1, \ldots, m$, $k = 0, 1, 2, \ldots$. This and periodicity of the coefficients $a_i(t + \omega) = a_i(t)$ for $i = 1, \ldots, m$ imply that the function $x_k(t) = y(t - k\omega)$, where y is the solution of Eq (8) on $[0, \omega]$, satisfies this equation on the interval $[k\omega, (k + 1)\omega]$ for every $k = 0, 1, 2, \ldots$.

From the equality $x(t + \omega) = \lambda x(t)$, it follows that

$$|x(k\omega)| = |\lambda x((k - 1)\omega)| = |\lambda^2||x((k - 2)\omega)| = \ldots = |\lambda^k||x(0)|,$$

and, consequently,

$$\begin{aligned} |x(t)| &= |\lambda|^k |x_0(t - k\omega)| = \exp\left\{\ln\left(|\lambda|^k\right)\right\} |y(t - k\omega)| \\ &= \exp\left\{\frac{\ln\left(|\lambda|^k\right)}{k\omega} \cdot k\omega\right\} |y(t - k\omega)| = \exp\left\{\frac{k \ln|\lambda|}{k\omega} \cdot k\omega\right\} |y(t - k\omega)| \\ &= \exp\left\{\frac{\ln|\lambda|}{\omega} \cdot k\omega\right\} |y(t - k\omega)| \quad \text{for} \quad t \in [k\omega, (k + 1)\omega]. \end{aligned}$$

To complete the proof, we have to obtain that $\Phi(t+\omega) = \Phi(t)$ in the case of $\lambda > 0$ and $\Phi(t+2\omega) = \Phi(t)$ in the case of $\lambda < 0$.

We have two representations of the solution x: $|x(t)| = \exp\left\{\frac{\ln|\lambda|}{\omega} \cdot k\omega\right\} |y(t-k\omega)|$ and (14). Comparing these representations, we can obtain the function

$$|\Phi(t)| = \exp\left\{\frac{\ln|\lambda|}{\omega}(\omega k - t)\right\} |y(t - k\omega)| \quad \text{for} \quad t \in [k\omega, (k + 1)\omega].$$

We can write

$$\begin{aligned} |\Phi(t + \omega)| &= \exp\left\{\frac{\ln|\lambda|}{\omega}(\omega k - (t + \omega))\right\} |y(t + \omega - k\omega)| \\ &= \exp\left\{\frac{\ln|\lambda|}{\omega}(\omega k - (t + \omega))\right\} |\lambda||y(t - k\omega)| \\ &= \exp\left\{\frac{\ln|\lambda|}{\omega}(\omega k - (t + \omega))\right\} \cdot \exp\left\{\frac{\ln|\lambda|}{\omega} \cdot \omega\right\} |y(t - k\omega)| \\ &= \exp\left\{\frac{\ln|\lambda|}{\omega}(\omega k - t)\right\} |y(t - k\omega)| = |\Phi(t)|. \end{aligned}$$

If $\lambda > 0$, then it follows from the equality $x(t + \omega) = \lambda x(t)$ that $\Phi(t + \omega) = \Phi(t)$. If $\lambda < 0$, then $\Phi(t + 2\omega) = \Phi(t)$. This completes the proof of the Floquet formula (14). □

Proof of Theorem 5. The conditions (11) and (12) imply, according to Theorem 4, the existence of positive constants α and N such that $|C(t,0)| \leq N\exp(-\alpha t)$ for $t \in [0,\infty)$. This implies $|C(t,n\omega)| \leq N\exp(-\alpha(t-n\omega))$ for $t \in [n\omega,\infty)$ and for every $n = 1,2,3,\ldots$. Conditions (11) and (12) imply that $C(t,s) = C(t,n\omega)C(n\omega,s)$ for $(n-1)\omega < s < n\omega$ and $t > n\omega$. It follows from the assumption that all $a_i(t)$ are essentially bounded functions (see the description of the coefficients a_i after (8)) that

$$\operatorname*{ess\,sup}_{(n-1)\omega\leq s\leq n\omega} |C(t,n\omega)| \equiv m < \infty.$$

Thus,

$$|C(t,s)| = |C(t,n\omega)C(n\omega,s)| \leq mN\exp(-\alpha(t-n\omega)).$$

This completes the proof. □

To prove Theorem 11, let us start with the following auxiliary assertion. Consider the equation

$$x'(t) + ax(t-\tau) = 0, \quad t \in [0,\infty) \tag{27}$$

with positive constant coefficients a and τ, the initial function

$$x(\xi) = 0, \quad \xi \in [-\tau,0), \tag{28}$$

and the initial condition

$$x(0) = 1. \tag{29}$$

Lemma 18. *The solution of problem* (27)–(29) *can be represented by the formula*

$$x(t) = \sum_{n=0}^{k} \frac{(-1)^n a^n}{n!}(t-n\tau)^n, \quad k\tau \leq t \leq (k+1)\tau, \quad k = 0,1,2,\ldots. \tag{30}$$

Proof. We obtain (30) by computing the solution step by step in each of the intervals $[k\tau,(k+1)\tau]$, $k = 0,1,2,\ldots$. On $[0,\tau]$, we have

$$x(t) \equiv 1 \quad \text{for} \quad t \in [0,\tau].$$

On $[\tau,2\tau]$, we have

$$x(t) = x(\tau) + \int_\tau^t x'(s)\mathrm{d}s = x(\tau) + \int_\tau^t (-a)\mathrm{d}s = 1 - a(t-\tau).$$

On $[2\tau,3\tau]$, we have

$$\begin{aligned} x(t) &= x(2\tau) + \int_{2\tau}^t x'(s)\mathrm{d}s = x(2\tau) - a\int_{2\tau}^t [1-a(s-2\tau)]\mathrm{d}s \\ &= 1 - a(t-\tau) + \frac{a^2}{2}(t-2\tau)^2. \end{aligned}$$

On $[3\tau,4\tau]$, we have

$$x(t) = x(3\tau) + \int_{3\tau}^t x'(s)\mathrm{d}s = x(3\tau) - a\int_{3\tau}^t \left[1 - a(s-2\tau) + \frac{a^2}{2}(s-3\tau)^2\right]\mathrm{d}s$$

$$= 1 - a(t-\tau) + \frac{a^2}{2}(t-2\tau)^2 - \frac{a^3}{3!}(t-3\tau)^3.$$

On $[4\tau, 5\tau]$, we have

$$\begin{aligned} x(t) &= x(4\tau) + \int_{4\tau}^{t} x'(s)\mathrm{d}s \\ &= x(4\tau) - a\int_{4\tau}^{t}\left[1 - a(s-2\tau) + \frac{a^2}{2}(s-3\tau)^2 - \frac{a^3}{3!}(s-4\tau)^3\right]\mathrm{d}s \\ &= 1 - a(t-\tau) + \frac{a^2}{2!}(t-2\tau)^2 - \frac{a^3}{3!}(t-3\tau)^3 + \frac{a^4}{4!}(t-4\tau)^4. \end{aligned}$$

By induction, we come to formula (30). This completes the proof. □

Lemma 19. *Let $a(t) = a > 0$, $\tau(t)$ be defined by the condition* (22) *and continued as an ω-periodic function for all $t > \omega$, i.e., $\tau(t) = \tau(t-\omega)$. Then, the solution of* (4) *satisfying initial condition* (29) *has the representation*

$$x(t) = \sum_{n=0}^{k} \frac{(-1)^n a^n}{n!}(t-(n-1)\tau)^n, \quad (k-1)\tau \le t \le k\tau, \quad k = 1, 2, 3, \ldots, \quad t \in [0, \omega]. \tag{31}$$

Proof. From the definition of the delay $\tau(t)$ by formula (22) with periodic continuation, it follows that the solutions z of (27)–(29) and x of (4), (29) satisfy the equality $x(t) = z(t+\omega)$, which holds for $t \in [0, \omega]$. Making shifting on τ in the left direction, we pass from formula (30) to (31). This completes the proof. □

Proof of Theorem 11. This follows now from Theorem 5 and Lemma 19. □

6. Calculations concerning height regulation of a drone's flight

We now calculate the coefficient a in the equation of height regulation of the drone in Eq (3). We substitute real data [40] into the system:

$$\begin{gathered} n_{11} = 0.024, \quad n_{12} = -0.11, \quad n_{13} = 0.2, \\ n_{21} = -0.4, \quad n_{22} = 2.4, \quad n_{23} = 0, \\ n_{31} = 0, \quad n_{32} = 38, \quad n_{33} = 2.45, \\ n_B = 49, \quad n_p = 0.022, \quad n_{41} = 0, \quad n_{42} = 1. \end{gathered}$$

Additionally, let a_B and a_p be arbitrary coefficients selected for their relevance to control and their generation by the user. To solve (3) and to find the coefficient a, we have

$$\begin{gathered} \alpha = \frac{-n_B a_B}{n_{32}} h(g(t)), \quad \nu = \frac{n_{22} n_B a_B}{n_{21} n_{32}} h(g(t)), \\ \vartheta = \left[n_p a_p + \frac{n_{12} n_B a_B}{n_{32}} - \frac{n_{11} n_{22} n_B a_B}{n_{21} n_{32}}\right] h(g(t)). \end{gathered}$$

Now, substituting this into $-n_{41}v + n_{42}\alpha - n_{42}\vartheta = ah(g(t))$, we obtain

$$a = \frac{-49 \cdot a_B}{38} - \left[0.022 \cdot a_p + \frac{-0.11 \cdot 49 \cdot a_B}{38} - \frac{0.024 \cdot 2.4 \cdot 49 \cdot a_B}{-0.4 \cdot 38}\right].$$

Taking, for example, $a_B = 1$ and $a_p = \frac{-775}{76}$ gives us $a = 1$, and we come to the stability zones described in Corollary 15.

7. Discussion and open problems

System (1), describing the drone's motion, consists of first and second-order equations, in which we add the control terms with delay. We considered simplified cases (see, for example, equalities $n_{14} = n_{24} = n_{34} = 0$). This allowed us to come to the analysis of the scalar equation of first order (4). Other ways of simplifying will lead us to second-order equations or to a corresponding system with delays. This raises a question about building Floquet theory for scalar delay equations of the order n and systems of delay equations.

All previous results assumed corresponding smallness of the product, for example (see [32]),

$$(\operatorname*{ess\,sup}_{t\geq 0} a(t))(\operatorname*{ess\,sup}_{t\geq 0} \tau(t)) < \frac{3}{2}, \tag{32}$$

which gives asymptotic stability of the (17). Note that even nonstrict inequality cannot be set instead of the strict one (see [32]). However, this condition can be very hard to reach for real technological problems, and this excludes the possibility of the use of results assuming smallness of the product (32). We demonstrate (see Example 1) the explanation of this situation.

Our approach opens new perspectives on the problem of stability for delay equations with periodic coefficients. We propose a new version of Floquet theory. Its main trait is the finite-dimensional character. This allows us to stay with a one-dimensional space of solutions.

Thus, we develop ideas of the sampled data [1–5] in the direction of Floquet theory. As a result, we have solution representation formulas (14) and (15). Note that modern computational capabilities (for example, Maple and MATLAB) allow us to obtain the solution on every finite interval $[0, \omega]$, which gives us the solution on the infinite interval thanks to the solution representation formulas. The question of exponential stability is reduced to computing $x(\omega)$. If $x(0) = 1$, then $x(\omega) = \lambda$. Thus, the inequality $|\lambda| < 1$ gives a necessary and sufficient condition for exponential stability. Therefore, we connect the idea of sampled data with stability through Floquet theory.

This idea can be developed for second-order equations and systems with delay. Based on Example 7, important developments can be achieved in the direction of equations of first order with oscillating coefficient. Using the idea of the W-transform, we can obtain sufficient tests of stability for equations with general delays. The way to achieve this is by constructing the Cauchy functions for the corresponding equations with sampled data.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflict of interest.

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