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Research article

Regularity results in grand variable exponent Morrey spaces and applications

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Abstract: The boundedness of commutators of Calderón–Zygmund operators in grand variable exponent Morrey spaces is established. The operators and spaces are defined on quasi-metric measure spaces with doubling measure. The obtained results are applied to study regularity properties of solutions of the second-order partial differential equations with discontinuous coefficients in the frame of grand variable exponent Morrey spaces.

Keywords: grand variable exponent Morrey spaces; commutators; Calderón–Zygmund operators; elliptic PDEs; regularity of solution

1. Introduction

In this paper the boundedness of commutators of Calderón–Zygmund operators in grand variable exponent Morrey spaces ($GVEMS\,s$ briefly) are studied under the condition that the variable exponents of the spaces satisfy the log–Hölder continuity condition. As a consequence of the general result we have, in particular, the boundedness of the commutator of the Calderón–Zygmund singular integral operator in these spaces defined on a bounded domain in $\mathbb{R}^n$. Based on the latter result, we study the regularity problem for elliptic partial differential equations (PDEs briefly).

One-weight estimates for commutators of sublinear operators in constant exponent weighted grand Morrey spaces ($CEWGMS\,s$ briefly) were investigated in [1] (see [2] for the unweighted case). The boundedness of operators of harmonic analysis via the Rubio de Francia's extrapolation in $CEWGMS$s with Muckenhoupt weights was studied in [3] and [4] (see also [5] for similar problems beyond the Muckenhoupt classes). The results of [3] and [4] were applied by the authors to study regularity of elliptic PDEs in the frame of $CEWGMS$s. We mention the paper [6] for the boundedness of commutators of sublinear operators in generalized Morrey spaces with constant exponent.

The study of function spaces with variable exponents is a very active area of research nowadays. A variable exponent Lebesgue space ($VELS$ briefly) $L^{p(\cdot)}$ is a special case of the space introduced by W. Orlicz in the 1930s and subsequently generalized by I. Musielak and W. Orlicz. These spaces are also called Nakano [7] spaces. For mapping properties of operators of harmonic analysis in $VELS$, we refer to the monographs [8, 9] and the survey [10].

Grand Lebesgue spaces $L^{p)}(\Omega)$ defined on bounded domains Ω were introduced in 1992 by T. Iwaniec and C. Sbordone [11] (see [12] for further generalization). In subsequent years, quite a number of problems of harmonic analysis and the theory of non-linear differential equations were studied in these spaces (see, e.g., the monograph [13] and references cited therein). Grand variable exponent Lebesgue spaces ($GVELS$ s briefly) were introduced in [14] (see also [15] for more precise spaces).

Morrey spaces describes regularity problems for solutions of elliptic PDEs more precisely than Lebesgue spaces. Classical Morrey spaces were introduced by C. B. Morrey [16] in 1938 and applied to the regularity problems of solutions of PDEs. Variable exponent Morrey spaces appeared in [17], while constant exponent grand Morrey spaces ($CEGMS$s briefly) were introduced and studied in [5] (see also [18] for further generalizations). Grand variable exponent Morrey spaces ($GVEMS$s briefly) were introduced and studied from different viewpoints in the papers [19–21]. For the progress in the directions of grand function spaces, see, e.g., [22].

Finally, we mention that some of the results of this paper were announced in [23].

2. Preliminaries

Let (X, d, μ) be a quasi-metric measure space ($QMMS$ briefly) with a quasi-metric d and measure μ, i.e., X is a topological space, μ is a complete measure on the σ-algebra defined on X, and d is a function (quasi-metric) $d : X \times X \to [0, \infty)$ that satisfies the following conditions:

(a) $d(x, y) = 0$ if and only if $x = y$;

(b) There is a constant $a_0 > 0$ such that $d(x, y) \le a_0 d(y, x)$ for all $x, y \in X$;

(c) There is a constant $a_1 > 0$ such that $d(x, y) \le a_1(d(x, z) + d(z, y))$ for all $x, y, z \in X$.

It is also assumed that all balls $B(x, r) := \{y \in X : d(x, y) < r\}$ with center x and radius r in X are measurable, $\mu\{x\} = 0$ for all $x \in X$, and that the class of continuous functions with compact supports is dense in the space of integrable functions on X ($L^1(X)$). We will assume that for all sufficiently small positive r, ρ, $0 < r < \rho$,

$$B(x, \rho) \setminus B(x, r) \neq \emptyset. \tag{2.1}$$

We say that a measure μ satisfies the doubling condition if there is a positive constant C_{dc} such that for all $x \in X$ and $r > 0$,

$$\mu B(x, 2r) \le C_{dc} \mu B(x, r).$$

We will deal with the $QMMS$ with doubling measure. Such a $QMMS$ is called a space of homogeneous type (SHT briefly).

A measure μ satisfies the reverse doubling condition if there are positive constants $\eta_1 > 1$ and $\eta_2 > 1$ such that for all $x \in X$ and sufficiently small $r > 0$,

$$\mu(B(x,\eta_1 r)) \geq \eta_2 \mu B(x,r).$$

In this case we write $\mu \in RD_{\eta_1,\eta_2}$.

There are many important examples of an SHT:

(a) Carleson (regular) curves on $\mathbb{C}$ with arc-length measure $d\nu$ and Euclidean distance on $\mathbb{C}$;

(b) Nilpotent Lie groups with Haar measure and homogeneous norm (homogeneous groups);

(c) Bounded domain Ω in $\mathbb{R}^n$ together with induced Lebesgue measure satisfying so called $\mathcal{A}$ condition, i.e., there is a positive constant C such that for all $x \in \Omega$ and $\rho \in (0, d_\Omega)$,

$$\mu(\widetilde{B}(x,\rho)) \geq C\rho^n, \tag{2.2}$$

where d_Ω is a diameter of Ω and $\widetilde{B}(x,\rho) := \Omega \cap B(x,\rho)$. In this case, balls are the sets $\widetilde{B}(x,\rho)$, $x \in \Omega$.

Notation:

By c and C we denote various absolute positive constants, which may have different values even in one and the same line;

f_B denotes the integral average of f, i.e., $f_B := \frac{1}{\mu B}\int_B f \,\mathrm{d}\mu$;

$p'(\cdot)$ stands for the conjugate exponent function defined by $1/p(\cdot) + 1/p'(\cdot) = 1$;

$\bar{a} := a_1\left(a_1\left(a_0 + 1\right) + 1\right)$ with the quasi-metric constants a_0 and a_1;

d_X denotes the diameter of X, i.e., $d_X := \sup\{d(x,y) : \; x, y \in X\}$;

by the symbol $D(X)$, we denote the class of bounded functions on X with compact supports;

we denote $aB := B(x, ar)$ for a ball $B := B(x,r)$, where a is a positive constant;

$B_k\left(x_0, r\right) := \left\{x \in X : d\left(x_0, x\right) < \eta^k r\right\}$ for a constant $\eta > 1$, where $k \in \mathbb{Z}$;

$A_k\left(x_0, r\right) := B_k\left(x_0, r\right) \backslash B_{k-1}\left(x_0, r\right), k \in \mathbb{Z}$, where x_0 is a point in X;

If $\mu(X) < \infty$, we will assume that m_0 is an integer depending on $r > 0$ such that the number $d_{x_0} := \sup_{x\in X} d\left(x_0, x\right)$ belongs to the interval $\left[\eta^{m_0} r, \eta^{m_0+1} r\right)$, where η is a certain positive number.

Throughout the paper we assume that $\mu(X) < \infty$.

2.1. Grand variable exponent Morrey spaces (GVEMS)

We denote by $P(X)$ the family of all real-valued μ-measurable functions $p(\cdot)$ on X such that

$$1 < p_- \leq p_+ < \infty,$$

where

$$p_- := p_-(X) := \inf_X p(x), \quad p_+ := p_+(X) := \sup_X p(x).$$

We say that a variable exponent belongs to the class $\mathcal{P}^{\log}(X)$ if there is a positive constant ℓ such that for all $x, y \in X$ with $d(x,y) \leq 1/2$,

$$|p(x) - p(y)| \leq \frac{\ell}{-\ln(d(x,y))}.$$

The variable exponent Lebesgue space, denoted by $L^{p(\cdot)}(X)$, is the linear space of all μ -measurable functions f on E for which

$$S_{p(\cdot)}(f) := \int_X |f(x)|^{p(x)} \mathrm{d}\mu < \infty.$$

The norm in $L^{p(\cdot)}(X)$ is defined as follows:

$$\|f\|_{L^{p(\cdot)}(X)} = \inf\left\{\eta > 0 : S_{p(\cdot)}(f/\eta) \le 1\right\}.$$

Let $\lambda(x)$ be a measurable function on X with values in $[0, 1]$. Denote by $L^{p(\cdot),\lambda(\cdot)}(X)$ the variable exponent Morrey space ($VEMS$ briefly), which is the class of measurable functions on X such that

$$\|f\|_{L^{p(\cdot),\lambda(\cdot)}(X)} = \sup_{x\in X, r\in(0,d_X)} \left\|(\mu B(x,r))^{-\lambda(x)} f\chi_{B(x,r)}\right\|_{L^{p(\cdot)}(X)} < \infty.$$

The $GVEMS$ denoted by $L^{p(\cdot),\lambda(\cdot),\theta}(X)$, where $\theta > 0$ and $\lambda(\cdot) \in [0, 1]$, are defined with respect to the following norm:

$$\|f\|_{L^{p(\cdot),\lambda(\cdot),\theta}(X)} = \sup_{0<\varepsilon<p_- -1} \sup_{x\in X, 0<r<d_X} \frac{\varepsilon^{\theta/(p(x)-\varepsilon)}}{\mu B(x,r)^{\lambda(x)}} \|f\|_{L^{p(\cdot)-\varepsilon}(B(x,r))}.$$

We are interested in the case when λ is constant. In particular, in this case, grand variable exponent Morrey space $L^{p(\cdot),\lambda,\theta}(X)$ is defined by the norm:

$$\|f\|_{L^{p(\cdot),\lambda(\cdot),\theta}(X)} = \sup_{0<\varepsilon<p_- -1} \sup_{x\in X, 0<r<d_X} \frac{\varepsilon^{\theta/(p(x)-\varepsilon)}}{\mu B(x,r)^{\lambda}} \|f\|_{L^{p(\cdot)-\varepsilon}(B(x,r))}.$$

2.2. Commutators of Calderón–Zygmund operators

The space of functions of bounded mean oscillation, denoted by $BMO(X)$, is the set of all real-valued locally integrable functions such that

$$\|f\|_{BMO(X)} = \sup_{\substack{x\in X\\ 0<r<d_X}} \frac{1}{\mu B(x,r)} \int_{B(x,r)} \left|f(y) - f_{B(x,r)}\right| \mathrm{d}\mu(y) < \infty.$$

It is well-known that $BMO(X)$ is a Banach space with respect to the norm $\|\cdot\|_{BMO(X)}$ when we regard the space $BMO(X)$ as the class of equivalent functions modulo additive constants.

An equivalent norm for $\|\cdot\|_{BMO(X)}$ is defined as

$$\|f\|_{BMO(X)} \sim \sup_{\substack{x\in X\\ 0<r<d_X}} \inf_{c\in R} \frac{1}{\mu B(x,r)} \int_{B(x,r)} |f(y) - c| \mathrm{d}\mu(y).$$

Let U be an operator defined on some subclass of $\mu-$ measurable functions and let b be a locally integrable function on X. We define the commutator $U_b f$ as

$$U_b f = bU(f) - U(bf).$$

Commutators are very useful when studying problems related with regularity of solutions of elliptic partial differential equations of the second order (see, e.g., [24]).

We are interested in commutators of Calderón–Zygmund operators defined on an SHT (see [25] for the boundedness results of such commutators in weighted classical Lebesgue spaces with Muckenhoupt weights).

Let K be the Calderón–Zygmund operator (see, e.g., [26])

$$Kf(x) = p.v. \int_X k(x,y) f(y) \mathrm{d}\mu(y),$$

where k is the Calderón–Zygmund kernel $k : X \times X \backslash \{(x,x) : x \in X\} \to \mathbf{R}$ satisfying the conditions:

(i) $|k(x,y)| \le \frac{C}{\mu B(x,d(x,y))}, \quad x, y \in X, \quad x \neq y;$

(ii) $|k(x_1,y) - k(x_2,y)| + |k(y,x_1) - k(y,x_2)| \le Cw\left(\frac{d(x_2,x_1)}{d(x_2,y)}\right) \frac{1}{\mu B(x_2,d(x_2,y))},$

for all x_1, x_2 and y with $d(x_2,y) \ge Cd(x_1,x_2)$, where w is a positive nondecreasing function on $(0,\infty)$ which satisfies the Δ_2 condition $w(2t) \le cw(t)$ $(t > 0)$ and the Dini condition $\int_0^1 w(t)/t \, \mathrm{d}t < \infty$. It is also assumed that k is such that Kf exists almost everywhere on X in the principal value sense for all $f \in L^2(X)$ and that K is bounded in $L^2(X)$.

For the Calderón–Zygmund operator K we will have the following assumptions on their commutators K_b:

$$|K_b f(x)| \le C \int_X \frac{|b(x) - b(y)||f(y)|}{\mu B(x, d(x,y))} \mathrm{d}\mu(y), \quad x \notin \operatorname{supp} f. \tag{2.3}$$

3. Boundedness result

Let us recall that by the symbol $D(X)$ is denoted the class of all bounded functions on X with compact supports.

To prove the main boundedness result, we need some auxiliary statements.

Lemma 3.1. *(see [27]) The following inequality holds for all $b \in BMO(X)$:*

$$\left|b_{B_k} - b_B\right| \le kA\|b\|_{BMO(X)}, \tag{3.1}$$

where $A := D_{dc}^{\log_2 \bar{a}+1}$ with the doubling constant D_{dc}.

Some relations between Lebesgue spaces with variable exponent and BMO spaces are given in [28]. The proof of [28] enables us to formulate it for an SHT:

Lemma 3.2. *For all $b \in BMO(X)$ and all $j, i \in \mathbb{Z}$ with $j > i$, we have that*

$$\sup_{B:ball} \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(X)}} \|(b - b_B)\chi_B\|_{L^{p(\cdot)}(X)} \le C_{p(\cdot)}\|b\|_{BMO(X)}$$

and

$$\left\|(b - b_{B_i})\chi_{B_j}\right\|_{L^{p(\cdot)}(X)} \le C_{p(\cdot)}(j-i)\|b\|_{BMO(X)} \left\|\chi_{B_j}\right\|_{L^{p(\cdot)}(X)},$$

where the constant $C_{p(\cdot)}$ depending on $p(\cdot)$ is such that

$$\sup_{0<\varepsilon<\delta} C_{p(\cdot)-\varepsilon} < \infty,$$

for some small positive constant δ.

For the next statement we refer, e.g., to [9] (P.9, Lemma 1.7), [8] (Lemma 3.3.1):

Lemma 3.3. *Let $p, q \in P(X)$ and $q(x) \leqslant p(x)$ almost everywhere, and $\frac{1}{r(x)} := \frac{1}{q(x)} - \frac{1}{p(x)}$. If $1 \in L^{r(\cdot)}(X)$, then*

$$\|f\|_{L^{q(\cdot)}} \leqslant 2^{1/q_-}\|1\|_{L^{r(\cdot)}}\|f\|_{L^{p(\cdot)}}.$$

The next auxiliary statement for constant exponents was proved in [1].

Lemma 3.4. *Let $1 < p_- \le p(x) \le p_+ < \infty, \theta > 0, 0 < \lambda < 1/p_+$. Let $p(\cdot) \in \mathcal{P}^{\log}(X)$. Suppose that σ is a positive constant less than $p_- - 1$. Let η be a positive constant $\eta = \max\{\eta_1, \overline{a}\}$. Then there is a positive constant $C \equiv C_{p(\cdot),BMO}$ such that for all $f \in L^{p(\cdot),\lambda}(X)$, all balls $B := B(x_0, r)$, all $\varepsilon \in (0, \sigma)$, and all sufficiently small $r > 0$, the inequality*

$$\begin{aligned}&\varepsilon^{\theta}\left(\int_{X\setminus\eta B}\frac{|f(y)|}{\mu B(x_0, d(x_0,y))}|b_B - b(y)|\,\mathrm{d}\mu(y)\right)^{p(x_0)-\varepsilon}(\mu(B))^{1-\lambda(p(x_0)-\varepsilon)}\\&\le C\|f\|^{p(x_0)-\varepsilon}_{L^{p(\cdot),\lambda,\theta}(X)}\end{aligned}$$

holds.

Proof. Observe that Hölder's inequality yields

$$\begin{aligned}&\varepsilon^{\theta}\left(\int_{X\setminus\eta B}\frac{|f(y)|}{\mu(x_0, d(x_0,y))}|b_B - b(y)|\,\mathrm{d}\mu(y)\right)^{p(x_0)-\varepsilon}(\mu(B))^{1-\lambda(p(x_0)-\varepsilon)}\\
&\le \varepsilon^{\theta}\left(\sum_{k=1}^{m_0}\int_{A_{k+1}}\frac{|f(y)|}{\mu(B(x_0, d(x_0,y))}|b_B - b(y)|\,\mathrm{d}\mu(y)\right)^{p(x_0)-\varepsilon}(\mu(B))^{1-\lambda(p(x_0)-\varepsilon)}\\
&\le \varepsilon^{\theta}\left(\sum_{k=1}^{m_0}\frac{1}{\mu(B_k)}\int_{A_{k+1}}|f(y)|\,|b_B - b(y)|\,\mathrm{d}\mu(y)\right)^{p(x_0)-\varepsilon}(\mu(B))^{1-\lambda(p(x_0)-\varepsilon)}\\
&\le C\varepsilon^{\theta}\left(\sum_{k=1}^{m_0}\frac{1}{\mu(B_k)}\left\|\chi_{A_k}f\right\|_{L^{p(\cdot)-\varepsilon}}\left\|\chi_{B_k}(b_B - b)\right\|_{L^{(p(\cdot)-\varepsilon)'}}\right)^{p(x_0)-\varepsilon}(\mu(B))^{1-\lambda(p(x_0)-\varepsilon)}\\
&\le C\left(\sum_{k=1}^{m_0}\frac{\varepsilon^{\frac{\theta}{p(x_0)-\varepsilon}}}{\mu(B_k)}\left\|\chi_{A_k}f\right\|_{L^{p(\cdot)-\varepsilon}}\left\|\chi_{B_k}(b_B - b)\right\|_{L^{(p(\cdot)-\varepsilon)'}}\right)^{p(x_0)-\varepsilon}(\mu(B))^{1-\lambda(p(x_0)-\varepsilon)}\\
&\le C\|f\|^{p(x_0)-\varepsilon}_{L^{p(\cdot),\lambda,\theta}(X)}\left(\sum_{k=1}^{m_0}\frac{1}{(\mu(B_k))^{1-\lambda}}\left\|\chi_{B_k}(b_B - b)\right\|_{L^{(p(\cdot)-\varepsilon)'}}\right)^{p(x_0)-\varepsilon}(\mu(B))^{1-\lambda(p(x_0)-\varepsilon)}.\end{aligned}$$

Further, it is easy to see that by adding and subtracting b_{B_k} and $b_{B_{k-1}}$ we find that

$$\begin{aligned}&\left\|(b_B - b)\chi_{B_k}\right\|_{L^{(p(\cdot)-\varepsilon)'}(X)}\\
&\le\left\{\left|b_B - b_{B_k}\right|\left\|\chi_{B_k}\right\|_{L^{(p(\cdot)-\varepsilon)'}(X)} + \left\|(b_{B_k} - b_{B_{k-1}})\chi_{B_k}\right\|_{L^{(p(\cdot)-\varepsilon)'}(X)} + \left\|(b_{B_{k-1}} - b)\chi_{B_k}\right\|_{L^{(p(\cdot)-\varepsilon)'}(X)}\right\}.\end{aligned}$$

This estimate, together with Lemmas 3.1 and 3.2 and condition $p(\cdot) \in \mathcal{P}^{\log}(X)$, gives

$$\left(\sum_{k=1}^{m_0}\frac{1}{(\mu(B_k))^{1-\lambda}}\left\|\chi_{B_k}(b_B-b)\right\|_{L^{(p(\cdot)-\varepsilon)'}}\right)^{p(x_0)-\varepsilon}(\mu(B))^{1-\lambda(p(x_0)-\varepsilon)}$$
$$\leq C\|b\|_{BMO(X)}\left(\sum_{k=1}^{\infty}\frac{k}{(\mu(B_k))^{1-\lambda}}\left\|\chi_{B_k}\right\|_{L^{(p(\cdot)-\varepsilon)'}}\right)^{p(x_0)-\varepsilon}(\mu(B))^{1-\lambda(p(x_0)-\varepsilon)}$$
$$\leq C_{p(\cdot),BMO}\left(\sum_{k=1}^{\infty}\frac{k}{(\mu(B_k))^{1-\lambda}}(\mu(B_k))^{1-\frac{1}{p(x_0)-\varepsilon}}\right)^{p(x_0)-\varepsilon}(\mu(B))^{1-\lambda(p(x_0)-\varepsilon)}$$
$$\leq C_{p(\cdot),BMO}\left(\sum_{k=1}^{\infty}\frac{k}{(\mu(B_k))^{\frac{1}{p(x_0)-\varepsilon}-\lambda}}(\mu(B))^{\frac{1}{p(x_0)-\varepsilon}-\lambda}\right)^{p(x_0)-\varepsilon}\leq C.$$

In the last inequality we used the fact that μ satisfies the reverse doubling condition $\mu\in RD_{\eta,\eta_2}$ which is guaranteed by the conditions that μ is doubling and (2.1) is satisfied (see, e.g., [29], P.11, Lemma 20 for the details).

Summarizing the estimates above, we obtain the desired result.

□

Proposition 3.1. *Let $p(\cdot)\in P(X)\cap\mathcal{P}^{\log}(X)$ and let λ be a constant such that $0<\lambda<1/p_+$. Let $b\in BMO(X)$. Then there is a positive constant C such that for all $f\in D(X)$,*

$$\|K_b f\|_{L^{p(\cdot),\lambda}(X)}\leq C\|f\|_{L^{p(\cdot),\lambda}(X)}$$

holds.

Proof. It is known that (see [26]) the Calderón–Zygmund operator K is bounded in Morrey space $L^{p(\cdot),\lambda}(X)$ if $p(\cdot)\in P(X)\cap\mathcal{P}^{\log}(X)$. Further, the boundedness of commutators of the Calderón–Zygmund operator in variable exponent Morrey spaces (even in more general spaces) was established in [30] in the case of the Euclidean spaces but the proof enables us to conclude that it is true for an SHT with finite measure under the condition that the exponent of the space satisfies the log–Hölder continuity condition. □

Now we formulate the main results of this note:

Theorem 3.1. *Let $p(\cdot)\in P(X)\cap\mathcal{P}^{\log}(X)$ and let θ and λ be constants such that $\theta>0$ and $0<\lambda<1/p_+$. Let $b\in BMO(X)$. Then there is a positive constant C independent of f such that*

$$\|K_b f\|_{L^{p(\cdot),\lambda,\theta}(X)}\leq C\|f\|_{L^{p(\cdot),\lambda,\theta}(X)}$$

holds for all $f\in D(X)$.

Proof. To prove the theorem, it is enough to prove that

$$\|K_b f\|_{\widetilde{L}^{p(\cdot),\lambda,\theta}(X)}\leq C\|f\|_{L^{p(\cdot),\lambda,\theta}(X)},\quad f\in D(X),$$

where the norm $\|\cdot\|_{\widetilde{L}^{p(\cdot),\lambda,\theta}(X)}$ is defined by the formula

$$\|g\|_{\widetilde{L}^{p(\cdot),\lambda,\theta}(X)}:=\sup_{0<\varepsilon\leq\sigma}\sup_{x,r}\frac{1}{(\mu(B(x,r)))^{\lambda}}\varepsilon^{\frac{\theta}{p(x)-\varepsilon}}\|g\|_{L^{p(\cdot)-\varepsilon}(B(x,r))}.$$

This follows from the following observation, which holds due to the Hölder inequality in $VELS$s:

$$\|K_b f\|_{L^{p(\cdot),\lambda,\theta}(X)} = \max\Big\{ \sup_{0<\varepsilon\le\sigma} \sup_{x,r} \frac{1}{(\mu(B(x,r)))^{\lambda}} \varepsilon^{\frac{\theta}{p(x)-\varepsilon}} \|K_b f\|_{L^{p(\cdot)-\varepsilon}(B(x,r))},$$
$$\sup_{\sigma<\varepsilon\le p_- -1} \sup_{x,r} \frac{1}{(\mu(B(x,r)))^{\lambda}} \varepsilon^{\frac{\theta}{p(x)-\varepsilon}} \|K_b f\|_{L^{p(\cdot)-\varepsilon}(B(x,r))}\Big\}$$
$$\le C_{p,\sigma,\theta} \sup_{0<\varepsilon\le\sigma} \sup_{x,r} \frac{1}{(\mu(B(x,r)))^{\lambda}} \varepsilon^{\frac{\theta}{p(x)-\varepsilon}} \|K_b f\|_{L^{p(\cdot)-\varepsilon}(B(x,r))} = C_{p,\sigma,\theta} \|K_b f\|_{\widetilde{L}^{p(\cdot),\lambda,\theta}(X)}.$$

Let $\eta := \max\{\bar{a}, \eta_1\}$, where η_1 is the reverse doubling constant. Let us take a ball $B := B(x_0, r)$. Using the representation $f = f_1 + f_2$, where $f_1 = f\chi_{\eta B}, f_2 = f - f_1$, where r is a sufficiently small positive number, we have

$$\frac{\varepsilon^{\frac{\theta}{p(x_0)-\varepsilon}}}{(\mu(B))^{\lambda}} \|K_b f\|_{L^{p(\cdot)-\varepsilon}(B)} \le \frac{\varepsilon^{\frac{\theta}{p(x_0)-\varepsilon}}}{(\mu(B))^{\lambda}} \|K_b f_1\|_{L^{p(\cdot)-\varepsilon}(B)} + \frac{\varepsilon^{\frac{\theta}{p(x_0)-\varepsilon}}}{(\mu(B))^{\lambda}} \|K_b f_2\|_{L^{p(\cdot)-\varepsilon}(B)} := I_1 + I_2$$

Observe that by Proposition 3.1 we have

$$I_1 \le \|K_b f_1\|_{\widetilde{L}^{p(\cdot),\lambda,\theta}(X)} \le C\|f\|_{L^{p(\cdot),\lambda,\theta}(X)}.$$

To estimate I_2, first we observe that if $x \in B$ and $y \notin \eta B$, then

$$\mu(B(x_0, d(x_0, y))) \le C\mu(B(x, d(x, y))),$$

with a positive constant C independent of x, x_0, y. Consequently, by condition (2.3), we find that

$$|K_b f_2(x)| \le C\left(\int_X \frac{|f_2(y)|\,|b(x)-b(y)|}{\mu(B(x,d(x,y))}\mathrm{d}\mu(y)\right)$$
$$\le C\left(\int_{X\setminus\eta B} \frac{|f(y)|}{\mu(B(x_0, d(x_0,y)))}\mathrm{d}\mu(y)\right)|b(x) - b_B|$$
$$+ C\left(\int_{X\setminus\eta B} \frac{|f(y)|\,|b(y)-b_B|}{\mu(B(x_0, d(x_0,y)))}\mathrm{d}\mu(y)\right),$$

with the positive constant C. Further, by the condition $p(\cdot) \in \mathcal{P}^{\log}(X)$, we find that

$$\frac{1}{(\mu(B))^{\lambda}}\, \varepsilon^{\frac{\theta}{p(x_0)-\varepsilon}} \|K_b f_2\|_{L^{p(\cdot)-\varepsilon}}$$
$$\le C\frac{\varepsilon^{\theta/(p(x_0)-\varepsilon)}}{(\mu(B))^{\lambda}}\left(\int_{X\setminus\eta B} \frac{|f(y)|}{\mu(B(x_0, d(x_0,y)))}\mathrm{d}\mu(y)\right)\|b(\cdot) - b_B\|_{L^{p(\cdot)-\varepsilon}}$$
$$+ C\frac{\varepsilon^{\theta/(p(x_0)-\varepsilon)}}{(\mu(B))^{\lambda}}\left(\int_{X\setminus\eta B} \frac{|f(y)|}{\mu(B(x_0, d(x_0,y)))}\,|b(y) - b_B|\,\mathrm{d}\mu(y)\right)(\mu(B))^{\frac{1}{p(x_0)-\varepsilon}}$$
$$:= I_{21} + I_{22}.$$

Observe that Lemma 3.4 yields that

$$I_{22} \le C\|f\|_{L^{p(\cdot),\lambda,\theta}(X)}.$$

Further, by the condition $p(\cdot) \in \mathcal{P}^{\log}(X)$, Lemmas 3.2 and 3.3, we have that

$$\begin{aligned}
&\frac{1}{\|\chi_B\|_{L^{p(\cdot)-\varepsilon}(X)}} \| b - b_B\|_{L^{p(\cdot)-\varepsilon}(B)} \\
\leq\ & C_{p(\cdot)} \frac{1}{(\mu(B))^{\frac{1}{p_-(B)-\varepsilon}}} (\mu(B))^{\frac{\varepsilon}{(p_-(B)-\varepsilon)p_-(B)}} \| b - b_B\|_{L^{p(\cdot)}(B)} \\
\leq\ & C_{p(\cdot)} (\mu(B))^{-1/p_-(B)} \| b - b_B\|_{L^{p(\cdot)}(B)} \leq \frac{C_{p(\cdot)}}{\|\chi_B\|_{L^{p(\cdot)}(X)}} \| b - b_B\|_{L^{p(\cdot)}(B)} \\
\leq\ & C_{p(\cdot)} \|b\|_{BMO(B)}
\end{aligned}$$

with the positive constant $C_{p(\cdot)}$ independent of B and ε. Consequently, this estimate together with the condition $p(\cdot) \in \mathcal{P}^{\log}(X)$ (which implies that $\mu(B)^{1/p_-(B)} \approx \mu(B)^{1/p_+(B)} \approx \mu(B)^{1/p(x_0)}$) and $\mu \in RD_{\eta,\eta_2}$ yield that

$$\begin{aligned}
I_{21} &\leq \frac{C\varepsilon^{\frac{\theta}{p(x_0)-\varepsilon}}}{(\mu(B))^{\lambda}} \left(\sum_{k=1}^{m_0} \frac{1}{\mu\left(B\left(x_0, \eta^k r\right)\right)} \int_{B_{k+1}} |f(y)| \mathrm{d}\mu(y) \right) \| b - b_B\|_{L^{p(\cdot)-\varepsilon}(B)} \\
&\leq \frac{C\varepsilon^{\theta/(p(x_0)-\varepsilon)}}{(\mu(B))^{\lambda}} \sum_{k=1}^{m_0} \frac{1}{\mu\left(B_{k+1}\right)\left(\mu\left(B_{k+1}\right)\right)^{\lambda}} \| f\|_{L^{p(\cdot)-\varepsilon}(B_{k+1})} \left(\mu\left(B_{k+1}\right)\right)^{\lambda} \\
&\times \left\|\chi_{B_{k+1}}\right\|_{L^{(p(\cdot)-\varepsilon)'}(X)} \| b - b_B\|_{L^{p(\cdot)-\varepsilon}(B)} \\
&\leq \frac{C}{(\mu(B))^{\lambda}} \|f\|_{L^{p(\cdot),\lambda,\theta}(X)} \| b - b_B\|_{L^{p(\cdot)-\varepsilon}(B)} \sum_{k=1}^{m_0} \left(\mu\left(B_{k+1}\right)\right)^{\lambda + \frac{1}{(p_-(B_{k+1})-\varepsilon)'} - 1} \\
&\leq \frac{C}{(\mu(B))^{\lambda}} \|f\|_{L^{p(\cdot),\lambda,\theta}(X)} \| b - b_B\|_{L^{p(\cdot)-\varepsilon}(B)} \sum_{k=1}^{m_0} \left(\mu\left(B_{k+1}\right)\right)^{\lambda - \frac{1}{p_-(B_{k+1})-\varepsilon}} \\
&\leq C_{p(\cdot),BMO} \|f\|_{L^{p(\cdot),\lambda,\theta}(X)} \sum_{k=1}^{m_0} \frac{(\mu(B))^{\frac{1}{p_+(B)-\varepsilon}-\lambda}}{\left(\mu\left(B_k\right)\right)^{\frac{1}{p_-(B_{k+1})-\varepsilon}-\lambda}} \\
&\leq C_{p(\cdot),BMO} \|f\|_{L^{p(\cdot),\lambda,\theta}(X)} \sum_{k=1}^{\infty} \frac{\eta_2^{k\left(\lambda - \frac{1}{p_- - \varepsilon}\right)} (\mu(B_k))^{\frac{1}{p_-(B_k)-\varepsilon}-\lambda}}{\left(\mu\left(B_k\right)\right)^{\frac{1}{p_-(B_k)-\varepsilon}-\lambda}} \\
&\leq C_{p(\cdot),BMO} \|f\|_{L^{p(\cdot),\lambda,\theta}(X)} \sum_{k=1}^{\infty} \eta_2^{k\left(\lambda - \frac{1}{p_- - \varepsilon}\right)} \\
&\leq C_{p(\cdot),BMO} \frac{\eta_2^{\lambda - \frac{1}{p_- - \varepsilon}}}{1 - \eta_2^{\lambda - \frac{1}{p_- - \varepsilon}}} \|f\|_{L^{p(\cdot),\lambda,\theta}(X)} \\
&\leq C_{p(\cdot),\sigma,BMO} \|f\|_{L^{p(\cdot),\lambda,\theta}(X)}.
\end{aligned}$$

Here we used the obvious fact:

$$\sup_{0<\varepsilon<\sigma} \frac{\eta_2^{\lambda - \frac{1}{p_- - \varepsilon}}}{1 - \eta_2^{\lambda - \frac{1}{p_- - \varepsilon}}} < \infty.$$

Summarizing these estimates we obtain the desired estimate.

□

Corollary 3.1. *Let Ω be a bounded domain in R^n such that condition* (2.2) *is satisfied. Let K_b be the commutator of the Calderón–Zygmund operator defined on Ω. Suppose that $p(\cdot) \in \mathcal{P}^{\log}(\Omega)$ and θ and λ are constants such that $\theta > 0, 0 < \lambda < 1/p_+$. Let $b \in BMO(\Omega, dx)$. Then there is a positive constant C independent of f such that*

$$\|K_b f\|_{L^{p(\cdot),\lambda,\theta}(\Omega)} \leq C\|f\|_{L^{p(\cdot),\lambda,\theta}(\Omega)}, \quad f \in D(\Omega).$$

4. Applications to *PDE*s

In the last thirty years a number of papers have been devoted to the study of local and global regularity properties of strong solutions to elliptic equations with discontinuous coefficients. To be more precise, let us consider the second-order equation

$$\mathcal{L}u \equiv \sum_{i,j=1}^{n} a_{ij}(x) D_{x_i x_j} u = f(x), \quad \text{for almost all } x \in \Omega, \tag{4.1}$$

where $\mathcal{L}$ is a uniformly elliptic operator over the bounded domain $\Omega \subset \mathbb{R}^n, n \geq 2$. We assume that a domain Ω satisfies $\mathcal{A}$ condition (see (2.2)). In this case Ω, with induced Lebesgue measure and Euclidean metrics is an SHT. Hence, the previous statements are valid for such domains.

Regularizing properties of $\mathcal{L}$ in Hölder spaces (i.e., $\mathcal{L}u \in C^{\alpha}(\bar{\Omega})$ implies $u \in C^{2+\alpha}(\bar{\Omega})\big)$ have been well studied in the case of Hölder continuous coefficients $a_{ij}(x)$. Also, unique classical solvability of the Dirichlet problem for (4.1) has been derived in this case (we refer to [31] and the references therein). In the case of uniformly continuous coefficients a_{ij}, an L^p-Schauder theory has been elaborated for the operator $\mathcal{L}$ (see [31, 32]). In particular, $\mathcal{L}u \in L^p(\Omega)$ always implies that the strong solution to (4.1) belongs to the Sobolev space $W^{2,p}(\Omega)$ for each $p \in (1, \infty)$. However, the situation becomes rather difficult if one tries to allow discontinuity at the principal coefficients of $\mathcal{L}$. In general, it is well-known (cf. [33]) that arbitrary discontinuity of a_{ij} implies that the L^p-theory of $\mathcal{L}$ and the strong solvability of the Dirichlet problem for (4.1) break down. A notable exception to that rule is the two-dimensional case ($\Omega \subset \mathbb{R}^2$). It was shown by G. Talenti that the sole condition on measurability and boundedness of the a_{ij} 's ensures isomorphic properties of $\mathcal{L}$ considered as a mapping from $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ into $L^2(\Omega)$. To handle the multidimensional case ($n \geq 3$) requires that additional properties on $a_{ij}(x)$ should be added to the uniform ellipticity in order to guarantee that $\mathcal{L}$ possesses the regularizing property in Sobolev functional scales. In particular, if $a_{ij}(x) \in W^{1,n}(\Omega)$ (cf. [30]), or if the difference between the largest and the smallest eigenvalues of $\left\{a_{ij}(x)\right\}$ is small enough (the Cordes condition), then $\mathcal{L}u \in L^2(\Omega)$ yields that $u \in W^{2,2}(\Omega)$, and these results can be extended to $W^{2,p}(\Omega)$ for $p \in (2-\varepsilon, 2+\varepsilon)$ with sufficiently small ε.

Later the Sarason class VMO of functions with vanishing mean oscillation was used in the study of local and global Sobolev regularity of the strong solutions to (4.1).

Next, we define the space BMO and then the smallest VMO class, where we consider coefficients a_{ij} and later that one where we consider the known term f.

In the sequel, let Ω be an open bounded set in R^n.

Definition 4.1. *Let $f \in L^1_{\text{loc}}(\Omega)$. We define the integral mean $f_{x,R}$ by*

$$f_{x,R} := \frac{1}{|\widetilde{B}(x,R)|} \int_{\widetilde{B}(x,R)} f(y) dy,$$

where, as before, $\widetilde{B}(x,R) = \Omega \cap B(x,R)$*,* $x \in \Omega$*, and* $|\widetilde{B}(x,R)|$ *is the Lebesgue measure of* $\widetilde{B}(x,R)$*. If we are not interested in specifying which the center is, we just use the notation* f_R*.*

We now give the definition of Bounded Mean Oscillation functions (BMO) that appeared at first in the note by F. John and L. Nirenberg [34].

Definition 4.2. *Let* $f \in L^1_{\mathrm{loc}}(\Omega)$*. We say that* f *belongs to* $BMO(\Omega)$ *if the seminorm* $\|f\|_*$ *is finite, where*

$$\|f\|_* := \sup_{|\widetilde{B}(x,R)|} \frac{1}{|\widetilde{B}(x,R)|} \int_{\widetilde{B}(x,R)} \left| f(y) - f_{x,R} \right| dy.$$

Next, we consider the definition of the space of VMO functions, given at first by D. Sarason [35].

Definition 4.3. *Let* $f \in BMO(\Omega)$ *and*

$$\eta(f,R) := \sup_{\rho \le R} \frac{1}{\left|B_\rho\right|} \int_{B_\rho} \left| f(y) - f_\rho \right| dy,$$

where $B\rho$ *ranges over the class of the balls of* $\mathbb{R}^n$ *of radius* ρ*. Further, a function* $f \in VMO(\Omega)$ *if* $\lim_{R\to 0} \eta(f,R) = 0$.

The Sarason class is then expressed as the subspace of the functions in the John-Nirenberg class whose BMO norm over a ball vanishes as the radius of the balls tends to zero. This property implies a number of good features of VMO functions not shared by general BMO functions; in particular, they can be approximated by smooth functions.

This class of functions was considered by many others. At first, we recall the paper by F. Chiarenza et al. [36], where the authors answer a question raised thirty years before by C. Miranda [37]. In his paper he considers a linear elliptic equation where the coefficients a_{ij} of the higher-order derivatives are in the class $W^{1,n}(\Omega)$ and asks whether the gradient of the solution is bounded if $p > n$. In [36], the authors suppose that $a_{ij} \in VMO$ and prove that Du is Hölder continuous for all $p \in (1, +\infty)$.

Also, it is possible to check that bounded uniformly functions are in VMO as well as functions of fractional Sobolev spaces $W^{\theta,\frac{n}{\theta}}, \theta \in (0,1)$.

The study of Sobolev regularity of strong solutions of (4.1) was initiated in 1991 with the pioneering work by F. Chiarenza et al. [24]. It was obtained that if $a_{ij}(x) \in VMO \cap L^\infty(\Omega)$ and $\mathcal{L}u \in L^p(\Omega)$, then $u \in W^{2,p}(\Omega)$ for each value of p in the range $(1,\infty)$. Moreover, well-posedness of the Dirichlet problem for (4.1) in $W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ was proved. As a consequence, Hölder continuity of the strong solution or of its gradient follows if the exponent p is sufficiently large.

Thanks to the fundamental accessibility of these two papers [36, 38], many other authors have used the VMO class to obtain regularity results for PDEs and systems with discontinuous coefficients.

Continuing the study of regularity of PDEs, we see that Hölder continuity can be inferred for small p if one has more information on $\mathcal{L}u$, such as its belonging to a suitable Morrey class $L^{p,\lambda}(\Omega)$.

Using these spaces a natural problem arises, namely, to study the regularizing properties of the operator $\mathcal{L}$ in Morrey spaces in the case of VMO principal coefficients. In [39], L. Caffarelli proved that each $W^{2,p}$-viscosity solution to (4.1) lies in $C^{1+\alpha}(\Omega)$ if $f(x)$ belongs to the Morrey space $M^{n,n\alpha}(\Omega)$ with $\alpha \in (0,1)$.

One of the main results of this note is to obtain local regularity, in grand Morrey spaces, for highest-order derivatives of solutions of elliptic non-divergence form with coefficients, which can be discontinuous.

We recall that, in the case of continuous coefficients of the above kind of equation, the results were obtained by S. Agmon et al. [32]. Later, discontinuous coefficients were considered by S. Campanato [40].

Then this paper can be regarded as a continuation of the study of L^p regularity of solutions of second-order elliptic *PDE*s to the maximum-order derivatives of the solutions to a certain class of linear elliptic equations in nondivergence form with discontinuous coefficients (see also [1] for related topics).

Let us consider the second-order differential operator

$$\mathcal{L} \equiv a_{ij}(x)D_{ij}, \quad D_{ij} \equiv \frac{\partial^2}{\partial x_i \partial x_j}.$$

Here we have adopted the usual summation convention on repeated indices. In the sequel, we need the following regularity and ellipticity assumptions on the coefficients of $\mathcal{L}, \forall i, j = 1, \dots, n$:

$$\begin{cases} a_{ij}(x) \in L^\infty(\Omega) \cap VMO, \\ a_{ij}(x) = a_{ji}(x), \quad \text{a.a. } x \in \Omega \\ \exists \kappa > 0 : \kappa^{-1}|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \kappa|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \text{ a.a. } x \in \Omega. \end{cases} \tag{4.2}$$

Set η_{ij} for the VMO-modulus of the function $a_{ij}(x)$ and let $\eta(r) = \left(\sum_{i,j=1}^n \eta_{ij}^2\right)^{1/2}$. We denote by $\Gamma(x, t)$ the normalized fundamental solution of $\mathcal{L}$, i.e.,

$$\Gamma(x, \xi) = \frac{1}{n(2-n)\omega_n \sqrt{\det\{a_{ij}(x)\}}} \left(\sum_{i,j=1}^n A_{ij}(x)\xi_i\xi_j\right)^{(2-n)/2}$$

for a.a. x and all $\xi \in \mathbb{R}^n \setminus \{0\}$, where $A_{ij}(x)$ stands for the entries of the inverse matrix of the matrix $\{a_{ij}(x)\}_{i,j=1,\dots,n}$, and ω_n is the measure of the unit ball in $\mathbb{R}^n$. We set also

$$\Gamma_i(x, \xi) = \frac{\partial}{\partial \xi_i}\Gamma(x, \xi), \quad \Gamma_{ij}(x, \xi) = \frac{\partial}{\partial \xi_i \partial \xi_j}\Gamma(x, \xi),$$
$$M = \max_{i,j=1,\dots,n} \max_{|\alpha| \le 2n} \left\| \frac{\partial^\alpha \Gamma_{ij}(x, \xi)}{\partial \xi^\alpha} \right\|_{L^\infty(\Omega \times \Sigma)}.$$

It is well known that $\Gamma_{ij}(x, \xi)$ are Calderón–Zygmund kernels in the ξ variable.

Corollary 3.1 enables us to formulate the next statement:

Theorem 4.1. *Let the coefficients of $\mathcal{L}$ satisfy* (4.2) *and $p(\cdot) \in \mathcal{P}^{\log}(\Omega), 0 < \lambda < 1/p_+, \theta > 0$. Let Ω be a domain satisfying $\mathcal{A}$ condition (see* (2.2)*). Then there exist positive constants $c = c(n, \kappa, p(\cdot), \lambda, \theta, M)$ and $\rho_0 = \rho_0(c, n)$ such that for every ball $B_\rho \subset\subset \Omega, \rho < \rho_0$ and every $u \in W_0^{2,p}\left(B_\rho\right)$ such that $D_{ij}u \in L^{p(\cdot),\lambda,\theta}\left(B_\rho\right)$, it holds true that*

$$\left\|D_{ij}u\right\|_{L^{p(\cdot),\lambda,\theta}\left(B_\rho\right)} \le c\|\mathcal{L}u\|_{L^{p(\cdot),\lambda,\theta}\left(B_\rho\right)}, \quad \forall i, j = 1, \dots, n.$$

5. Conclusions

The authors obtained regularity results for solutions of second-order *PDE*s having discontinuous coefficients in the framework of grand variable exponent Morrey spaces. In the future it will be possible to extend the obtained properties to other kinds of equations, making use of boundedness properties in grand variable exponent Morrey spaces that are proved in the present paper.

Use of AI tools declaration

The authors declare that Artificial Intelligence (AI) tools played no part in the creation of this article.

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Conflict of interest

Maria Alessandra Ragusa is an editorial board member for Electronic Research Archive and was not involved in the editorial review or the decision to publish this article. All authors declare that there are no competing interests.

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