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Research article

The semiclassical limit of the Kastler–Kalau–Walze-type theorem

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Abstract: In physics, the semiclassical limit principle asserts that as Planck's constant $\hbar \to 0$, quantum states reduce to classical configurations. We extend this framework to the noncommutative residue by applying the semiclassical limit to the spectral geometry. By introducing the coefficient ε, we establish a proof of the Kastler–Kalau–Walze-type theorem for the perturbations of the Dirac operator on four-dimensional compact manifolds with (without) boundary. As $\varepsilon \to 0$, we demonstrate the emergence of a semiclassical limit, thereby providing the classical formulation of the theorem. This result elucidates the interplay between quantum corrections and classical geometric invariants in the presence of boundary conditions.

Keywords: the semiclassical limit; the noncommutative residue; the perturbations of the Dirac operator; the Kastler–Kalau–Walze-type theorem

1. Introduction

The noncommutative residue, also known as great important study subject in noncommutative geometry, has been extensively studied in [1, 2]. In [3], Connes employed the noncommutative residue to derive a four-dimensional conformal Polyakov action analogue and demonstrated that the noncommutative residue on a compact manifold M coincides with the Dixmier's trace for pseudodifferential operators of order$-\dim M$ in [4]. Moreover, Connes claimed the noncommutative residue of the square of the inverse of the Dirac operator was proportional to the Einstein–Hilbert action. Kastler, Kalau, and Walze proved this conclusion respectively in [5, 6], which is called the Kastler–Kalau–Walze theorem. Afterwards, Ackermann proved that the noncommutative residue of the square of the inverse of the Dirac operator $\mathrm{Wres}(D^{-2})$ in turn is essentially the second coefficient of the heat kernel expansion of D^2 in [7], which enriches the results on noncommutative residues on manifolds without boundary.

Furthermore, Wang uses $\widetilde{\mathrm{Wres}}[(\pi^+D^{-1})^2]$ instead of $\mathrm{Wres}(D^{-2})$ to generalize the results from manifolds without boundary to manifolds with boundary in [8,9], and proved the Kastler–Kalau–Walze-type theorem for the Dirac operator and the signature operator on lower-dimensional manifolds with boundary [10]. Here $\widetilde{\mathrm{Wres}}$ denotes the noncommutative residue for manifolds with boundary, and π^+D^{-1} is an element in Boutet de Monvel's algebra (see (3.1) in Section 3.1). In [10, 11], Wang computed $\widetilde{\mathrm{Wres}}[\pi^+D^{-1} \circ \pi^+D^{-1}]$ and $\widetilde{\mathrm{Wres}}[\pi^+D^{-2} \circ \pi^+D^{-2}]$ for symmetric operators, where the boundary term vanished in these cases. However, when computing $\widetilde{\mathrm{Wres}}[\pi^+D^{-1} \circ \pi^+D^{-3}]$, a nonvanishing boundary term emerged [12], leading Wang to provide a theoretical interpretation of gravitational action on the boundary. In other words, this work effectively established a framework for investigating the Kastler–Kalau–Walze-type theorem on manifolds with boundary.

Subsequent studies [13–18] explored various perturbations of the Dirac operator by zero-order differential operators. In [15], Wang extended the Kastler–Kalau–Walze-type theorem for perturbations of Dirac operators on compact manifolds (with or without boundary) and proposed two distinct operator-theoretic interpretations of boundary gravitational action. Further developments by Wang, Wang, and Yang [17] ocused on 4-dimensional compact manifolds with boundary, where they derived two operator-theoretic explanations for gravitational action and proved a Kastler–Kalau–Walze-type theorem for nonminimal operators on complex manifolds. Additionally, in [16], Wang, Wang, and Wu introduced novel spectral functionals, which extended traditional spectral functionals to noncommutative realm with torsion and connected them to the noncommutative residue for manifolds with boundary.

The semiclassical limit not only connects quantum and classical physics theoretically but also provides important research tools and application value in the field of mathematics. In physics, the semiclassical limit refers to the transitional regime between quantum mechanics and classical mechanics. When the characteristic action $\bar{S}$ of a system is much larger than Planck's constant $\hbar$, quantum effects gradually diminish, and the system's behavior approaches that of classical mechanics. In mathematics, this is often achieved by taking the limit where Planck's constant $\hbar \to 0$.

There are many studies on the semiclassical limit of the spectral geometry. B*ä*r and Pf*ä*ffle studied semiclassical approximations for the heat kernel of a general self-adjoint Laplace-type operator within a geometric framework in [19]. Later, Ludewig [20] examined the semiclassical asymptotic expansion of the heat kernel arising from Witten's perturbation of the de Rham complex by a given function. By employing the stationary phase method, Ludewig derived a time-dependent integral formula, ultimately leading to a proof of the Poincar*é*-Hopf theorem. Meanwhile, Savale [21] analyzed the remainder term in the semiclassical limit formula (introduced in [22]) for the eta invariant on a metric contact manifold. Specifically, Savale demonstrated that this remainder term is governed by the volumes of recurrence sets of the Reeb flow. Obviously, the noncommutative residues as a part of the spectral geometry; thus, in order to extend the study of the semiclassical limit of the spectral geometry, motivated by [19–21] and Theorem 3.12 in [23], we introduce the semiclassical limit into the noncommutative residue. Based on the research of [24], we prove the semiclassical limit of the Kastler–Kalau–Walze-type theorem for the perturbations of the Dirac operator on 4-dimensional compact oriented spin manifolds with (without) boundary by taking the limit $\varepsilon \to 0$. For a fixed $\varepsilon > 0$, we may consider the Kastler–Kalau–Walze-type theorem as a theorem in the quantum state. And when $\varepsilon \to 0$, we give the classical state of the Kastler–Kalau–Walze-type theorem.

This paper is organized as follows: By using $\mathrm{Wres}(P) := \int_{S^*M} \mathrm{tr}(\sigma_{-n}^{P})(x,\xi)$, Section 2 gives semiclassical limits of the noncommutative residues of three cases for the perturbations of the Dirac operator on 4-dimensional manifolds without boundary. Moreover, we give the semiclassical limit of the Kastler–Kalau–Walze-type theorem about the perturbation of the Dirac operator on 4-dimensional manifolds with boundary in Section 3.

2. The semiclassical limits of the noncommutative residue on 4-dimensional manifolds without boundary

In this section, we study the semiclassical limits of the noncommutative residues on 4-dimensional manifolds without boundary in three different cases.

Firstly, we recall the main facts regarding the Dirac operator D. Let M be a 4-dimensional compact oriented spin manifold with Riemannian metric g, and let ∇ denote the Levi–Civita connection associated with g. Then the Dirac operator D can be expressed locally in terms of an orthonormal frame e_i (with corresponding dual coframe θ^k) of the frame bundle of M [5]:

$$\begin{aligned}
&D = i\gamma^i\widetilde{\nabla}_i = i\gamma^i(e_i + \sigma_i);\\
&\sigma_i(x) = \frac{1}{4}\gamma_{ij,k}(x)\gamma^i\gamma^k = \frac{1}{8}\gamma_{ij,k}(x)[\gamma^j\gamma^k - \gamma^k\gamma^j],\\
&\gamma_{ij,k} = -\gamma_{ik,j} = \frac{1}{2}[c_{ij,k} + c_{ki,j} + c_{kj,i}], \quad i, j, k = 1, \cdots, 4;\\
&c_{ij}^k = \theta^k([e_i.e_j]),
\end{aligned}$$

where the $\gamma_{ij,k}$ represents the Levi–Civita connection ∇ with spin connection $\widetilde{\nabla}$, the γ^i denote constant self-adjoint Dirac matrices, which satisfy $\gamma^i\gamma^j + \gamma^j\gamma^i = -2\delta^{ij}$.

Using local coordinates x^μ that induce the alternative vierbein $\partial_\mu = S_\mu^i(x)e_i$ (with dual vierbein dx^μ), $\gamma^i e_i = \gamma^\mu\partial_\mu$ is obtained, where the γ^μ are now x-dependent Dirac matrices, which satisfy $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2g^{\mu\nu}$ (we use Latin sub-(super-) scripts for the basic e_i and Greek sub-(super-) scripts for the basis ∂_μ, the type of sub-(super-) scripts specifying the type of Dirac matrices). Then the Dirac operator in the Greek basis is expressed by

$$\begin{aligned}
&D = i\gamma^\mu\widetilde{\nabla}_\mu = i\gamma^\mu(e_\mu + \sigma_\mu);\\
&\sigma_\mu(x) = S_\mu^i(x)\sigma_i.
\end{aligned}$$

Consider a pseudodifferential operator P that acts on sections of a vector bundle over a compact Riemannian manifold M. In [5], the noncommutative residues of P is defined by

$$\mathrm{Wres}(P) := \int_M \int_{\|\xi\|=1} \mathrm{tr}\left[\sigma_{-n}(P)\right](x,\xi)\sigma(\xi)dx, \tag{2.1}$$

where $\xi \in S^{n-1}$*and* tr denotes shorthand for trace.

Next, by (2.1), to obtain the semiclassical limit of the noncommutative residues on manifolds without boundary, we consider the following three different cases. From the point of view of the following three different cases, we give the classical state of the noncommutative residue on manifolds without boundary.

$$(1) \quad \lim_{\varepsilon\to 0}\varepsilon^3\mathrm{Wres}(\varepsilon D^2 + \lambda_1 D + \lambda_2)^{-1};$$

(2) $\lim_{\varepsilon\to 0}\varepsilon^3\mathrm{Wres}(\varepsilon D^2+\lambda_1 c(X)D+\lambda_2)^{-1}$;

(3) $\lim_{\varepsilon\to 0}\varepsilon^3\mathrm{Wres}(\varepsilon D^2+\lambda_1\nabla_X^{S(TM)}+\lambda_2)^{-1}$,

where λ_1,λ_2 are $C^\infty(M)$ functions.

2.1. *The first case:* $\lim_{\varepsilon\to 0}\varepsilon^3\mathrm{Wres}(\varepsilon D^2+\lambda_1 D+\lambda_2)^{-1}$

In this subsection, we want to compute $\lim_{\varepsilon\to 0}\varepsilon^3\mathrm{Wres}(\varepsilon D^2+\lambda_1 D+\lambda_2)^{-1}$, by $\varepsilon^3\mathrm{Wres}(\varepsilon D^2+\lambda_1 D+\lambda_2)^{-1}=\varepsilon^2\mathrm{Wres}\Big(D^2+\frac{\lambda_1}{\varepsilon}D+\frac{\lambda_2}{\varepsilon}\Big)^{-1}$, we need to compute $\mathrm{Wres}\Big(D^2+\frac{\lambda_1}{\varepsilon}D+\frac{\lambda_2}{\varepsilon}\Big)^{-1}$.

Set $A=D^2+\frac{\lambda_1}{\varepsilon}D+\frac{\lambda_2}{\varepsilon}$, we utilize the composition of pseudodifferential operators to express the symbol of the operator. Simplify the abbreviation of the principal symbol: $\xi=\sum_j\xi_j dx_j$, $\partial_\xi^\alpha=\partial^\alpha/\partial\xi_\alpha$, $\partial_\alpha^x=\partial_\alpha/\partial x^\alpha$, then the following identity holds:

$$\sigma^{PQ}(x,\xi)=\sum_\alpha\frac{(-i)^\alpha}{\alpha!}\partial_\xi^\alpha\sigma^P(x,\xi)\cdot\partial_\alpha^x\sigma^Q(x,\xi).\tag{2.2}$$

Firstly, we compute the total symbol $\sigma(x,\xi)$ of A, which is given by the sum of terms A_k of order k $(k=0,1,2)$:

$$A=A_2+A_1+A_0.$$

Then, we have

$$\begin{aligned}
&\sigma_2^A(x,\xi)=|\xi|^2;\\
&\sigma_1^A(x,\xi)=i(\Gamma^\mu-2\sigma^\mu)\xi_\mu+\frac{i\lambda_1}{\varepsilon}c(\xi);\\
&\sigma_0^A(x,\xi)=-(\partial^x\sigma_\mu+\sigma^\mu\sigma_\mu-\Gamma^\mu\sigma_\mu)+\frac{1}{4}s+\frac{i\lambda_1}{\varepsilon}\gamma^\mu\sigma_\mu+\frac{\lambda_2}{\varepsilon}.
\end{aligned}\tag{2.3}$$

Next, we compute A^{-1} from order -4 to order -2 using the above results; that is, $\sigma_{-k}^{A^{-1}}, k=2,3,4$. The full symbol σ of A is expressed in terms of decreasing order:

$$\sigma^{A^{-1}}=\sigma_{-2}^{A^{-1}}+\sigma_{-3}^{A^{-1}}+\sigma_{-4}^{A^{-1}}+terms\ of\ order\leq -5.$$

Using (2.2), the negative order of the symbol of A^{-1} yields:

$$\begin{aligned}
\sigma_{-2}^{A^{-1}}&=(\sigma_2^A)^{-1};\\
\sigma_{-3}^{A^{-1}}&=-\sigma_{-2}^{A^{-1}}[\sigma_1^A\sigma_{-2}^{A^{-1}}-i\partial_\xi^\mu\sigma_2^A\partial_\mu^x\sigma_{-2}^{A^{-1}}];\\
\sigma_{-4}^{A^{-1}}&=-\sigma_{-2}^{A^{-1}}[\sigma_1^A\sigma_{-3}^{A^{-1}}+\sigma_0^A\sigma_{-2}^{A^{-1}}-i\partial_\xi^\mu\sigma_1^A\partial_\mu^x\sigma_{-2}^{A^{-1}}-i\partial_\xi^\mu\sigma_2^A\partial_\mu^x\sigma_{-3}^{A^{-1}}].
\end{aligned}$$

Moreover, by (2.3), the following result is obtained.

$$\sigma_{-2}^{A^{-1}}=|\xi|^{-2};$$

$$
\begin{aligned}
\sigma_{-3}^{A^{-1}} &= -|\xi|^{-2}[(i(\Gamma^{\mu}-2\sigma^{\mu})\xi_{\mu}+\frac{i\lambda_1}{\varepsilon}c(\xi))|\xi|^{-2}-i\partial_{\xi}^{\mu}(|\xi|^{2})\partial_{\mu}^{x}(|\xi|^{-2})];\\
\sigma_{-4}^{A^{-1}} &= -|\xi|^{-6}\xi_{\mu}\xi_{\nu}(\Gamma^{\mu}-2\sigma^{\mu})(\Gamma^{\nu}-2\sigma^{\nu})-2|\xi|^{-8}\xi^{\mu}\xi_{\alpha}\xi_{\beta}(\Gamma^{\nu}-2\sigma^{\nu})\partial_{\mu}^{x}g^{\alpha\beta}+|\xi|^{-4}(\partial^{x\mu}\sigma_{\mu}+\sigma^{\mu}\sigma_{\mu}-\Gamma^{\mu}\sigma_{\mu})\\
&\quad-\frac{1}{4}|\xi|^{-4}s-2i|\xi|^{-2}\xi^{\mu}\cdot\partial_{\mu}^{x}\sigma_{-3}^{A^{-1}}+|\xi|^{-6}\xi_{\alpha}\xi_{\beta}(\Gamma^{\mu}-2\sigma^{\mu})\partial_{\mu}^{x}g^{\alpha\beta}-|\xi|^{-6}\xi_{\alpha}\xi_{\beta}g^{\mu\nu}\partial_{\mu\nu}^{x}g^{\alpha\beta}+2|\xi|^{-8}\xi_{\alpha}\xi_{\beta}\xi_{\gamma}\xi_{\delta}g^{\mu\nu}\\
&\quad\partial_{\mu}^{x}g^{\alpha\beta}\partial_{\nu}^{x}g^{\gamma\delta}-|\xi|^{-6}\frac{\lambda_1}{\varepsilon}c(\xi)(\Gamma^{\mu}-2\sigma^{\mu})\xi_{\mu}-|\xi|^{-6}(\Gamma^{\mu}-2\sigma^{\mu})\xi_{\mu}\frac{\lambda_1}{\varepsilon}c(\xi)-|\xi|^{-4}\frac{1}{\varepsilon}(i\lambda_1\gamma^{\mu}\sigma_{\mu}+\lambda_2)\\
&\quad+2|\xi|^{-8}\frac{\lambda_1}{\varepsilon}c(\xi)\xi^{\mu}\xi_{\alpha}\xi_{\beta}\partial_{\mu}^{x}g^{\alpha\beta}+|\xi|^{-4}\frac{\lambda_1^{2}}{\varepsilon^{2}}-|\xi|^{-4}\partial_{\xi}^{\mu}[\frac{\lambda_1}{\varepsilon}c(\xi)]\xi_{\alpha}\xi_{\beta}\partial_{\mu}^{x}g^{\alpha\beta}.
\end{aligned}
$$

Regrouping the terms and inserting

$$
\begin{aligned}
\partial_{\mu}^{x}\sigma_{-3}^{A^{-1}} &= 2i|\xi|^{-6}\xi_{\nu}\xi_{\alpha}\xi_{\beta}(\Gamma^{\nu}-2\sigma^{\nu})\partial_{\mu}^{x}g^{\alpha\beta}-i|\xi|^{-4}\xi_{\nu}\partial_{\mu}^{x}(\Gamma^{\nu}-2\sigma^{\nu})+6i|\xi|^{-8}\xi_{\nu}\xi_{\alpha}\xi_{\beta}\xi_{\gamma}\xi_{\delta}\partial_{\mu}^{x}g^{\alpha\beta}\partial_{\nu}^{x}g^{\gamma\delta}\\
&\quad-2i|\xi|^{-6}\xi_{\alpha}\xi_{\gamma}\xi_{\delta}\partial_{\mu}^{x}g^{\nu\alpha}\partial_{\nu}^{x}g^{\gamma\delta}-2i|\xi|^{-6}\xi^{\nu}\xi_{\gamma}\xi_{\delta}\partial_{\mu\nu}^{x}g^{\gamma\delta}-i\partial_{\mu}^{x}[|\xi|^{-4}\frac{\lambda_1}{\varepsilon}c(\xi)].
\end{aligned}
$$

We obtain for $\sigma_{-4}^{A^{-1}}$ the sum of terms:

$$
\begin{aligned}
N_1 &= -|\xi|^{-6}\xi_{\mu}\xi_{\nu}\Gamma^{\mu}\Gamma^{\nu}+|\xi|^{-4}[g_{\mu\nu}-|\xi|^{-4}\xi_{\mu\nu}][\sigma^{\mu}\sigma^{\nu}-\Gamma^{\nu}\sigma^{\nu}];\\
N_2 &= |\xi|^{-4}\partial^{x\mu}\sigma_{\mu}-\frac{1}{4}|\xi|^{-4}s;\\
N_3 &= -6|\xi|^{-8}\xi^{\mu}\xi_{\nu}\xi_{\alpha}\xi_{\beta}(\Gamma^{\nu}-2\sigma^{\nu})\partial_{\mu}^{x}g^{\alpha\beta};\\
N_4 &= 2|\xi|^{-6}\xi^{\mu}\xi_{\nu}\partial_{\mu}^{x}(\Gamma^{\nu}-2\sigma^{\nu});\\
N_5 &= -12|\xi|^{-10}\xi^{\mu}\xi^{\nu}\xi_{\alpha}\xi_{\beta}\xi_{\gamma}\xi_{\delta}\partial_{\mu}^{x}g^{\alpha\beta}\partial_{\nu}^{x}g^{\gamma\delta};\\
N_6 &= 4|\xi|^{-8}\xi^{\mu}\xi_{\alpha}\xi_{\gamma}\xi_{\delta}\partial_{\mu}^{x}g^{\nu\alpha}\partial_{\nu}^{x}g^{\gamma\delta};\\
N_7 &= |\xi|^{-6}\xi_{\alpha}\xi_{\beta}(\Gamma^{\mu}-2\sigma^{\mu})\partial_{\mu}^{x}g^{\alpha\beta};\\
N_8 &= 4|\xi|^{-8}\xi^{\mu}\xi^{\nu}\xi_{\gamma}\xi_{\delta}\partial_{\mu\nu}^{x}g^{\gamma\delta};\\
N_9 &= -|\xi|^{-6}\xi_{\alpha}\xi_{\beta}g^{\mu\nu}\partial_{\mu\nu}^{x}g^{\alpha\beta};\\
N_{10} &= 2|\xi|^{-8}\xi_{\alpha}\xi_{\beta}\xi_{\gamma}\xi_{\delta}g^{\mu\nu}\partial_{\mu}^{x}g^{\alpha\beta}\partial_{\nu}^{x}g^{\gamma\delta},
\end{aligned}
$$

and

$$
\begin{aligned}
&M_1 = -|\xi|^{-6}\frac{\lambda_1}{\varepsilon}c(\xi)(\Gamma^{\mu}-2\sigma^{\nu})\xi_{\mu};\quad M_2 = -|\xi|^{-6}(\Gamma^{\mu}-2\sigma^{\nu})\xi_{\mu}\frac{\lambda_1}{\varepsilon}c(\xi);\\
&M_3 = 2|\xi|^{-8}\frac{\lambda_1}{\varepsilon}c(\xi)\xi^{\mu}\xi_{\alpha}\xi_{\beta}\partial_{\mu}^{x}g^{\alpha\beta};\quad M_4 = |\xi|^{-4}\frac{\lambda_1^{2}}{\varepsilon^{2}};\quad M_5 = -|\xi|^{-4}\frac{1}{\varepsilon}(\lambda_1 i\gamma^{\mu}\sigma_{\mu}+\lambda_2);\\
&M_6 = -|\xi|^{-4}\frac{\lambda_1}{\varepsilon}\partial_{\xi}^{\mu}[c(\xi)]\xi_{\alpha}\xi_{\beta}\partial_{\mu}^{x}g^{\alpha\beta};\quad M_7 = -2|\xi|^{-2}\xi_{\mu}\partial_{x}^{\mu}[|\xi|^{-4}\frac{\lambda_1}{\varepsilon}c(\xi)].
\end{aligned}
$$

Let s denote the scalar curvature, from [5], we obtain

$$
\int_{|\xi|=1}\mathrm{tr}[\sum_{i=1}^{10}N_i]\sigma(\xi) = -\frac{s}{12}\mathrm{tr}[\mathtt{id}]. \tag{2.4}
$$

The next step involves computing $\int_{|\xi|=1} \mathrm{tr}[\sum_{i=1}^{7} M_i]\sigma(\xi)$.

(**1**) :

In normal coordinates, using the facts: $\Gamma^{\mu}_{\alpha\beta}(x_0) = \sigma_\mu(x_0) = 0$, $\partial^x_\mu g^{\alpha\beta}(x_0) = 0$, the results of the terms M_1, M_2, M_3, and M_6 disappear.

(**2**) :

$$\int_{|\xi|=1} \mathrm{tr}(M_4)(x_0)\sigma(\xi) = \frac{\lambda_1^2}{\varepsilon^2} Vol_{S^3}\mathrm{tr}[\mathtt{id}] = \frac{2\lambda_1^2}{\varepsilon^2}\pi^2\mathrm{tr}[\mathtt{id}],$$

and

$$\int_{|\xi|=1} \mathrm{tr}(M_5)(x_0)\sigma(\xi) = -\frac{\lambda_2}{\varepsilon} Vol_{S^3}\mathrm{tr}[\mathtt{id}] = -\frac{2\lambda_2}{\varepsilon}\pi^2\mathrm{tr}[\mathtt{id}].$$

(**3**) :

By $\partial^x_\mu[|\xi|^{-4}c(\xi)] = -2|\xi|^{-6}\partial^x_\mu(|\xi|^2)c(\xi) + |\xi|^{-4}\partial^x_\mu[c(\xi)]$, $\partial^x_\mu(|\xi|^2)(x_0) = 0$ and $\partial^x_\mu[c(\xi)] = 0$, we have

$$\int_{|\xi|=1} \mathrm{tr}(M_7)(x_0)\sigma(\xi) = 0.$$

Therefore, when $n = 4$, $\mathrm{tr}_{S(TM)}[\mathtt{id}] = 4$ and by (2.1), this implies

$$\mathrm{Wres}\Big(D^2 + \frac{\lambda_1}{\varepsilon}D + \frac{\lambda_2}{\varepsilon}\Big)^{-1} = 4\int_M \Big(\frac{2\lambda_1^2}{\varepsilon^2}\pi^2 - \frac{2\lambda_2}{\varepsilon}\pi^2 + \frac{1}{12}s\Big)d\mathrm{Vol_M}.$$

Further, we obtain the semiclassical limit of the above result. That is the following theorem.

Theorem 2.1. *If M is a 4-dimensional compact oriented spin manifolds without boundary, then we derive the semiclassical limit of the noncommutative residue about $\varepsilon D^2 + \lambda_1 D + \lambda_2$*

$$\lim_{\varepsilon\to 0}\varepsilon^3\mathrm{Wres}(\varepsilon D^2 + \lambda_1 D + \lambda_2)^{-1} = 8\int_M \lambda_1^2\pi^2 d\mathrm{Vol_M}.$$

Corollary 2.2. *If M is a 4-dimensional compact oriented spin manifolds without boundary, then when $\lambda_1 = \sqrt{\varepsilon}$, we obtain the following equality:*

$$\lim_{\varepsilon\to 0}\varepsilon^2\mathrm{Wres}(\varepsilon D^2 + \sqrt{\varepsilon}D + \lambda_2)^{-1} = 8\int_M (1-\lambda_2)\pi^2 d\mathrm{Vol_M}.$$

2.2. The second case: $\lim_{\varepsilon\to 0}\varepsilon^3\mathrm{Wres}(\varepsilon D^2 + \lambda_1 c(X)D + \lambda_2)^{-1}$

Let $c(X)$ denote a Clifford action on M, where $X = \sum_{\alpha=1}^{n} a_\alpha e_\alpha = \sum_{j=1}^{n} X_j\partial_j$ is a vector field. Then we can set $B = D^2 + \frac{\lambda_1}{\varepsilon}c(X)D + \frac{\lambda_2}{\varepsilon}$, the next step is to compute the total symbol $\sigma(x,\xi)$ of B; the sum of terms B_k of order k ($k = 0, 1, 2$) is given by:

$$B = B_2 + B_1 + B_0.$$

By (2.2), we have

$$\sigma^B_2(x,\xi) = |\xi|^2;$$

$$\sigma_1^B(x,\xi) = i(\Gamma^\mu - 2\sigma^\mu)\xi_\mu + \frac{i\lambda_1}{\varepsilon}c(X)c(\xi);$$
$$\sigma_0^B(x,\xi) = -(\partial^x\sigma_\mu + \sigma^\mu\sigma_\mu - \Gamma^\mu\sigma_\mu) + \frac{1}{4}s + \frac{i\lambda_1}{\varepsilon}c(X)\gamma^\mu\sigma_\mu + \frac{\lambda_2}{\varepsilon}. \tag{2.5}$$

Next, we compute B^{-1} from order -4 to order -2 using the above results; that is, we compute $\sigma_{-k}^{B^{-1}}, k = 2, 3, 4$. the full symbol σ of B is expressed into terms of decreasing order:

$$\sigma^{B^{-1}} = \sigma_{-2}^{B^{-1}} + \sigma_{-3}^{B^{-1}} + \sigma_{-4}^{B^{-1}} + terms\ of\ order \leq -5.$$

Using (2.2), the negative order of the symbol of B^{-1} yields:

$$\sigma_{-2}^{B^{-1}} = (\sigma_2^B)^{-1};$$
$$\sigma_{-3}^{B^{-1}} = -\sigma_{-2}^{B^{-1}}[\sigma_1^B\sigma_{-2}^{B^{-1}} - i\partial_\xi^\mu\sigma_2^B\partial_\mu^x\sigma_{-2}^{B^{-1}}];$$
$$\sigma_{-4}^{B^{-1}} = -\sigma_{-2}^{B^{-1}}[\sigma_1^B\sigma_{-3}^{B^{-1}} + \sigma_0^B\sigma_{-2}^{B^{-1}} - i\partial_\xi^\mu\sigma_1^B\partial_\mu^x\sigma_{-2}^{B^{-1}} - i\partial_\xi^\mu\sigma_2^B\partial_\mu^x\sigma_{-3}^{B^{-1}}].$$

Then by (2.5), it follows that

$$\sigma_{-2}^{B^{-1}} = |\xi|^{-2};$$
$$\sigma_{-3}^{B^{-1}} = -|\xi|^{-2}[(i(\Gamma^\mu - 2\sigma^\mu)\xi_\mu + \frac{i\lambda_1}{\varepsilon}c(X)c(\xi))|\xi|^{-2} - i\partial_\xi^\mu(|\xi|^2)\partial_\mu^x(|\xi|^{-2})];$$
$$\begin{aligned}\sigma_{-4}^{B^{-1}} = &-|\xi|^{-6}\xi_\mu\xi_\nu(\Gamma^\mu - 2\sigma^\mu)(\Gamma^\nu - 2\sigma^\nu) - 2|\xi|^{-8}\xi^\mu\xi_\alpha\xi_\beta(\Gamma^\nu - 2\sigma^\nu)\partial_\mu^x g^{\alpha\beta} + |\xi|^{-4}(\partial^{x\mu}\sigma_\mu + \sigma^\mu\sigma_\mu - \Gamma^\mu\sigma_\mu)\\ &- \frac{1}{4}|\xi|^{-4}s - 2i|\xi|^{-2}\xi^\mu\cdot\partial_\mu^x\sigma_{-3} + |\xi|^{-6}\xi_\alpha\xi_\beta(\Gamma^\mu - 2\sigma^\mu)\partial_\mu^x g^{\alpha\beta} - |\xi|^{-6}\xi_\alpha\xi_\beta g^{\mu\nu}\partial_{\mu\nu}^x g^{\alpha\beta} + 2|\xi|^{-8}\xi_\alpha\xi_\beta\xi_\gamma\xi_\delta\\ &g^{\mu\nu}\partial_\mu^x g^{\alpha\beta}\partial_\nu^x g^{\gamma\delta} - |\xi|^{-6}\frac{\lambda_1}{\varepsilon}c(X)c(\xi)(\Gamma^\mu - 2\sigma^\mu)\xi_\mu - |\xi|^{-6}(\Gamma^\mu - 2\sigma^\mu)\xi_\mu\frac{\lambda_1}{\varepsilon}c(X)c(\xi) - |\xi|^{-4}\frac{1}{\varepsilon}(i\lambda_1c(X)\gamma^\mu\\ &\sigma_\mu + \lambda_2) + 2|\xi|^{-8}\frac{\lambda_1}{\varepsilon}c(X)c(\xi)\xi^\mu\xi_\alpha\xi_\beta\partial_\mu^x g^{\alpha\beta} - |\xi|^{-6}\frac{\lambda_1^2}{\varepsilon^2}[c(X)c(\xi)]^2 - |\xi|^{-4}\partial_\xi^\mu[\frac{\lambda_1}{\varepsilon}c(X)c(\xi)]\xi_\alpha\xi_\beta\partial_\mu^x g^{\alpha\beta}.\end{aligned}$$

Regrouping the terms and inserting

$$\begin{aligned}\partial_\mu^x\sigma_{-3}^{B^{-1}} = &\ 2i|\xi|^{-6}\xi_\nu\xi_\alpha\xi_\beta(\Gamma^\nu - 2\sigma^\nu)\partial_\mu^x g^{\alpha\beta} - i|\xi|^{-4}\xi_\nu\partial_\mu^x(\Gamma^\nu - 2\sigma^\nu) + 6i|\xi|^{-8}\xi_\nu\xi_\alpha\xi_\beta\xi_\gamma\xi_\delta\partial_\mu^x g^{\alpha\beta}\partial_\nu^x g^{\gamma\delta}\\ &- 2i|\xi|^{-6}\xi_\alpha\xi_\gamma\xi_\delta\partial_\mu^x g^{\nu\alpha}\partial_\nu^x g^{\gamma\delta} - 2i|\xi|^{-6}\xi^\nu\xi_\gamma\xi_\delta\partial_{\mu\nu}^x g^{\gamma\delta} - i\partial_\mu^x[|\xi|^{-4}\frac{\lambda_1}{\varepsilon}c(X)c(\xi)].\end{aligned}$$

Then $\sigma_{-4}^{B^{-1}}$ includes the sum of terms: $N_1 - N_{10}$ and $R_1 - R_7$:

$$R_1 = -|\xi|^{-6}\frac{\lambda_1}{\varepsilon}c(X)c(\xi)(\Gamma^\mu - 2\sigma^\nu)\xi_\mu; \quad R_2 = -|\xi|^{-6}(\Gamma^\mu - 2\sigma^\nu)\xi_\mu\frac{\lambda_1}{\varepsilon}c(X)c(\xi);$$
$$R_3 = 2|\xi|^{-8}\frac{\lambda_1}{\varepsilon}c(X)c(\xi)\xi^\mu\xi_\alpha\xi_\beta\partial_\mu^x g^{\alpha\beta}; \quad R_4 = -|\xi|^{-6}\frac{\lambda_1^2}{\varepsilon^2}c(X)c(\xi)c(X)c(\xi);$$
$$R_5 = -|\xi|^{-4}\frac{1}{\varepsilon}(\lambda_1 ic(X)\gamma^\mu\sigma_\mu + \lambda_2); \quad R_6 = -|\xi|^{-4}\frac{\lambda_1}{\varepsilon}\partial_\xi^\mu[c(X)c(\xi)]\xi_\alpha\xi_\beta\partial_\mu^x g^{\alpha\beta};$$
$$R_7 = -2|\xi|^{-2}\xi_\mu\partial_x^\mu[|\xi|^{-4}\frac{\lambda_1}{\varepsilon}c(X)c(\xi)].$$

Then, similarly, we compute $\int_{|\xi|=1}\mathrm{tr}[\sum_{i=1}^{7}R_i]\sigma(\xi)$.

(1) :

In normal coordinates, using the facts, we have: $\Gamma^{\mu}_{\alpha\beta}(x_0)=\sigma_{\mu}(x_0)=0$, $\partial^{x}_{\mu}g^{\alpha\beta}(x_0)=0$, the results of the terms R_1, R_2, R_3, and R_6 disappear.

(2) :

$$\mathrm{tr}[c(X)c(\xi)c(X)c(\xi)]_{|\xi|=1}=-2\xi(X)\mathrm{tr}[c(X)c(\xi)]_{|\xi|=1}-|X|^2\mathrm{tr}[\mathtt{id}],$$

and

$$-2\xi(X)\mathrm{tr}[c(X)c(\xi)]_{|\xi|=1}=4\xi(X)^2\mathrm{tr}[\mathtt{id}]+2\xi(X)\mathrm{tr}[c(\xi)c(X)]_{|\xi|=1}. \tag{2.6}$$

Then by $\int_{|\xi|=1}\xi(X)^2\sigma(\xi)=-\frac{1}{2}|X|^2\pi^2\mathrm{tr}[\mathtt{id}]$, we have

$$\int_{|\xi|=1}\mathrm{tr}(R_4)(x_0)\sigma(\xi)=\frac{\lambda_1^2}{\varepsilon^2}|X|^2\pi^2\mathrm{tr}[\mathtt{id}].$$

(3) :

$$\int_{|\xi|=1}\mathrm{tr}(R_5)(x_0)\sigma(\xi)=-\frac{2\lambda_2}{\varepsilon}\pi^2\mathrm{tr}[\mathtt{id}].$$

(4) :

By $\partial^{\mu}_{x}[c(X)c(\xi)](x_0)=c(X)\partial^{\mu}_{x}[c(\xi)]+\partial^{\mu}_{x}[c(X)]c(\xi)=\sum_{j=1}^{n}\partial^{\mu}_{x}(X_j)c(e_j)c(\xi)(x_0)$, we have

$$\mathrm{tr}(R_7)(x_0)=2\xi^{\mu}\xi^{k}\frac{\lambda_1}{\varepsilon}\sum_{k}^{n-1}\partial^{\mu}_{x}(X_k)\mathrm{tr}[\mathtt{id}]. \tag{2.7}$$

Then

$$\begin{aligned}\int_{|\xi|=1}\mathrm{tr}(R_7)(x_0)\sigma(\xi)&=\frac{\lambda_1}{2\varepsilon}\sum_{k}\partial_{x_k}(X_k)Vol_{S^3}\mathrm{tr}[\mathtt{id}]\\&=\frac{\lambda_1}{2\varepsilon}div_M(X)Vol_{S^3}\mathrm{tr}[\mathtt{id}]\\&=\frac{\lambda_1}{\varepsilon}div_M(X)\pi^2\mathrm{tr}[\mathtt{id}],\end{aligned}$$

where div_M denotes divergence of M.

Thus by (2.1), we obtain the following result:

$$\mathrm{Wres}(D^2+\frac{\lambda_1}{\varepsilon}c(X)D+\frac{\lambda_2}{\varepsilon})^{-1}=4\int_M\Big(\frac{\lambda_1^2}{\varepsilon^2}|X|^2\pi^2-\frac{2\lambda_2}{\varepsilon}\pi^2+\frac{\lambda_1}{\varepsilon}div_M(X)\pi^2+\frac{1}{12}s\Big)d\mathrm{Vol_M}.$$

Further, we obtain the following theorem.

Theorem 2.3. *If M is a 4-dimensional compact oriented spin manifolds without boundary, then we derive the semiclassical limit of the noncommutative residue about $\varepsilon D^2+\lambda_1 c(X)D+\lambda_2$*

$$\lim_{\varepsilon\to 0}\varepsilon^3\mathrm{Wres}(\varepsilon D^2+\lambda_1 c(X)D+\lambda_2)^{-1}=4\int_M\lambda_1^2|X|^2\pi^2 d\mathrm{Vol_M}.$$

Corollary 2.4. *If M is a 4-dimensional compact oriented spin manifolds without boundary, then when $\lambda_1 = \sqrt{\varepsilon}$, the following equality holds:*

$$\lim_{\varepsilon\to 0}\varepsilon^2\mathrm{Wres}(\varepsilon D^2+\sqrt{\varepsilon}c(X)D+\lambda_2)^{-1}=4\int_M(|X|^2-2\lambda_2)\pi^2 d\mathrm{Vol_M}.$$

2.3. The third case: $\lim_{\varepsilon\to 0}\varepsilon^3\mathrm{Wres}(\varepsilon D^2+\lambda_1\nabla_X^{S(TM)}+\lambda_2)^{-1}$

Define $\nabla_X^{S(TM)} := X+\frac{1}{4}\sum_{ij}\langle\nabla_X^L e_i,e_j\rangle c(e_i)c(e_j)$, which is a spin connection. And let $g^{ij}=g(dx_i,dx_j)$ and $\nabla^L_{\partial_i}\partial_j=\sum_k\Gamma_{ij}^k\partial_k$, we denote that

$$\sigma_i=-\frac{1}{4}\sum_{s,t}\omega_{s,t}(e_i)c(e_i)c(e_s)c(e_t);\quad \xi^j=g^{ij}\xi_i;\quad \Gamma^k=g^{ij}\Gamma_{ij}^k;\quad \sigma^j=g^{ij}\sigma_i.$$

Set $C=D^2+\frac{\lambda_1}{\varepsilon}\nabla_X^{S(TM)}+\frac{\lambda_2}{\varepsilon}$, $E(X)=\frac{1}{4}\sum_{ij}\langle\nabla_X^L e_i,e_j\rangle c(e_i)c(e_j)$. The next step is to compute the total symbol $\sigma(x,\xi)$ of C^{-1} from order -4 to order -2, with C the following sum of terms C_k of order k:

$$C=C_2+C_1+C_0.$$

Then, we have

$$\begin{aligned}
&\sigma_2^C(x,\xi)=|\xi|^2;\\
&\sigma_1^C(x,\xi)=i(\Gamma^\mu-2\sigma^\mu)\xi_\mu+\frac{i\lambda_1}{\varepsilon}\sum_{j=1}^n X_j\xi_j;\\
&\sigma_0^C(x,\xi)=-(\partial^x\sigma_\mu+\sigma^\mu\sigma_\mu-\Gamma^\mu\sigma_\mu)+\frac{1}{4}s+\frac{i\lambda_1}{\varepsilon}E(X)+\frac{\lambda_2}{\varepsilon}.
\end{aligned}$$

Further, by (2.2), we obtain

$$\begin{aligned}
\sigma_{-2}^{C^{-1}}&=|\xi|^{-2};\\
\sigma_{-3}^{C^{-1}}&=-|\xi|^{-2}[(i(\Gamma^\mu-2\sigma^\mu)\xi_\mu+\frac{i\lambda_1}{\varepsilon}\sum_{j=1}^n X_j\xi_j)|\xi|^{-2}-i\partial_\xi^\mu(|\xi|^2)\partial_\mu^x(|\xi|^{-2})];\\
\sigma_{-4}^{C^{-1}}&=-|\xi|^{-6}\xi_\mu\xi_\nu(\Gamma^\mu-2\sigma^\mu)(\Gamma^\nu-2\sigma^\nu)-2|\xi|^{-8}\xi^\mu\xi_\alpha\xi_\beta(\Gamma^\nu-2\sigma^\nu)\partial_\mu^x g^{\alpha\beta}+|\xi|^{-4}(\partial^{x\mu}\sigma_\mu+\sigma^\mu\sigma_\mu-\Gamma^\mu\sigma_\mu)\\
&\quad-\frac{1}{4}|\xi|^{-4}s-2i|\xi|^{-2}\xi^\mu\cdot\partial_\mu^x\sigma_{-3}+|\xi|^{-6}\xi_\alpha\xi_\beta(\Gamma^\mu-2\sigma^\mu)\partial_\mu^x g^{\alpha\beta}-|\xi|^{-6}\xi_\alpha\xi_\beta g^{\mu\nu}\partial_{\mu\nu}^x g^{\alpha\beta}+2|\xi|^{-8}\xi_\alpha\xi_\beta\xi_\gamma\xi_\delta g^{\mu\nu}\\
&\quad\partial_\mu^x g^{\alpha\beta}\partial_\nu^x g^{\gamma\delta}-|\xi|^{-6}\frac{\lambda_1}{\varepsilon}\sum_{j=1}^n X_j\xi_j(\Gamma^\mu-2\sigma^\mu)\xi_\mu-|\xi|^{-6}(\Gamma^\mu-2\sigma^\mu)\xi_\mu\frac{\lambda_1}{\varepsilon}\sum_{j=1}^n X_j\xi_j-|\xi|^{-4}\frac{1}{\varepsilon}(i\lambda_1E(X)+\lambda_2)\\
&\quad+2|\xi|^{-8}\frac{\lambda_1}{\varepsilon}\sum_{j=1}^n X_j\xi_j\xi^\mu\xi_\alpha\xi_\beta\partial_\mu^x g^{\alpha\beta}-|\xi|^{-6}\frac{\lambda_1^2}{\varepsilon^2}\sum_{j=1}^n X_j\xi_j\sum_{k=1}^n X_k\xi_k-|\xi|^{-4}\partial_\xi^\mu[\frac{\lambda_1}{\varepsilon}\sum_{j=1}^n X_j\xi_j]\xi_\alpha\xi_\beta\partial_\mu^x g^{\alpha\beta}.
\end{aligned}$$

Regrouping the terms and inserting

$$\partial_\mu^x\sigma_{-3}^{C^{-1}}=2i|\xi|^{-6}\xi_\nu\xi_\alpha\xi_\beta(\Gamma^\nu-2\sigma^\nu)\partial_\mu^x g^{\alpha\beta}-i|\xi|^{-4}\xi_\nu\partial_\mu^x(\Gamma^\nu-2\sigma^\nu)+6i|\xi|^{-8}\xi_\nu\xi_\alpha\xi_\beta\xi_\gamma\xi_\delta\partial_\mu^x g^{\alpha\beta}\partial_\nu^x g^{\gamma\delta}$$

$$- 2i|\xi|^{-6}\xi_\alpha\xi_\gamma\xi_\delta\partial^x_\mu g^{\nu\alpha}\partial^x_\nu g^{\gamma\delta} - 2i|\xi|^{-6}\xi^\nu\xi_\gamma\xi_\delta\partial^x_{\mu\nu}g^{\gamma\delta} - i\partial^x_\mu[|\xi|^{-4}\frac{\lambda_1}{\varepsilon}\sum_{j=1}^n X_j\xi_j].$$

We obtain for $\sigma^{C^{-1}}_{-4}$ the sum of terms: $N_1 - N_{10}$ and $T_1 - T_7$:

$$T_1 = -|\xi|^{-6}\frac{\lambda_1}{\varepsilon}\sum_{j=1}^n X_j\xi_j(\Gamma^\mu - 2\sigma^\nu)\xi_\mu; \quad T_2 = -|\xi|^{-6}(\Gamma^\mu - 2\sigma^\nu)\xi_\mu\frac{\lambda_1}{\varepsilon}\sum_{j=1}^n X_j\xi_j;$$

$$T_3 = 2|\xi|^{-8}\frac{\lambda_1}{\varepsilon}\sum_{j=1}^n X_j\xi_j\xi^\mu\xi_\alpha\xi_\beta\partial^x_\mu g^{\alpha\beta}; \quad T_4 = -|\xi|^{-6}\frac{\lambda_1^2}{\varepsilon^2}\sum_{j=1}^n X_j\xi_j\sum_{k=1}^n X_k\xi_k;$$

$$T_5 = -|\xi|^{-4}\frac{1}{\varepsilon}(\lambda_1 c(X)E(X) + \lambda_2); \quad T_6 = -|\xi|^{-4}\frac{\lambda_1}{\varepsilon}\partial^\mu_\xi[\sum_{j=1}^n X_j\xi_j]\xi_\alpha\xi_\beta\partial^x_\mu g^{\alpha\beta};$$

$$T_7 = -2|\xi|^{-2}\xi_\mu\partial^\mu_\xi[|\xi|^{-4}\frac{\lambda_1}{\varepsilon}\sum_{j=1}^n X_j\xi_j].$$

Then, we proceed to compute $\int_{|\xi|=1}\mathrm{tr}[\sum_{i=1}^7 T_i]\sigma(\xi)$.

(**1**) :

In normal coordinates, using the facts: $\Gamma^\mu_{\alpha\beta}(x_0) = \sigma_\mu(x_0) = 0$, $\partial^x_\mu g^{\alpha\beta}(x_0) = 0$, the results of the terms T_1, T_2, T_3, and T_6 disappear.

(**2**) :

By $\int_{|\xi|=1}\xi_j\xi_k\sigma(\xi) = \frac{1}{4}Vol_{S^3}\delta_{jk} = \frac{1}{2}\pi^2\delta_{jk}$, we have

$$\int_{|\xi|=1}\mathrm{tr}(T_4)(x_0)\sigma(\xi) = -\frac{\lambda_1^2}{2\varepsilon^2}|X|^2\pi^2\mathrm{tr}[\mathtt{id}].$$

(**3**) :

$$\int_{|\xi|=1}\mathrm{tr}(T_5)(x_0)\sigma(\xi) = -\frac{2\lambda_2}{\varepsilon}\pi^2\mathrm{tr}[\mathtt{id}].$$

(**4**) :

Similar to (2.7), we have

$$\int_{|\xi|=1}\mathrm{tr}(T_7)(x_0)\sigma(\xi) = -\frac{\lambda_1}{\varepsilon}div_M(X)\pi^2\mathrm{tr}[\mathtt{id}].$$

Thus, we obtain the following result:

$$\mathrm{Wres}(D^2 + \frac{\lambda_1}{\varepsilon}\nabla^{S(TM)}_X + \frac{\lambda_2}{\varepsilon})^{-1} = 4\int_M\Big(- \frac{\lambda_1^2}{2\varepsilon^2}|X|^2\pi^2 - \frac{2\lambda_2}{\varepsilon}\pi^2 - \frac{\lambda_1}{\varepsilon}div_M(X)\pi^2 + \frac{1}{12}s\Big)d\mathrm{Vol_M}.$$

Building on these preliminaries, we obtain:

Theorem 2.5. *If M is a 4-dimensional compact oriented spin manifolds without boundary, then we obtain the semiclassical limit of the noncommutative residue about $\varepsilon D^2+\lambda_1\nabla_X^{S(TM)}+\lambda_2$*

$$\lim_{\varepsilon\to 0}\varepsilon^3\mathrm{Wres}(\varepsilon D^2+\lambda_1\nabla_X^{S(TM)}+\lambda_2)^{-1}=4\int_M-\frac{\lambda_1^2}{2}|X|^2\pi^2d\mathrm{Vol_M}.$$

Corollary 2.6. *If M is a 4-dimensional compact oriented spin manifolds without boundary, then when $\lambda_1=\sqrt{\varepsilon}$, we obtain the following equality:*

$$\lim_{\varepsilon\to 0}\varepsilon^2\mathrm{Wres}(\varepsilon D^2+\sqrt{\varepsilon}\nabla_X^{S(TM)}+\lambda_2)^{-1}=4\int_M\Big(-\frac{1}{2}|X|^2-2\lambda_2\Big)\pi^2d\mathrm{Vol_M}.$$

3. The semiclassical limit of the Kastler–Kalau–Walze-type theorem on 4-dimensional manifolds with boundary

In this section, we study the semiclassical limit of the Kastler–Kalau–Walze-type theorem for the perturbation of the Dirac operator on 4-dimensional manifolds with boundary, that is, to compute $\lim_{\varepsilon\to 0}\varepsilon^4\widetilde{\mathrm{Wres}}[\pi^+(\varepsilon D+c(X))^{-1}\circ\pi^+(\varepsilon D+c(Z))^{-1}]$.

3.1. Boutet de Monvel's calculus

In this subsection, we recall some fundamental concepts and key formulas about Boutet de Monvel's calculus, along with the definition of the noncommutative residue for manifolds with boundary. These preliminaries will be essential for our subsequent analysis. For a more comprehensive treatment of these topics, we refer readers to Section 2 in [10].

Denote by π^+ (resp. π^-) the projection on H^+ (resp. H^-). Let $\widetilde{H}$ = {rational functions having no poles on the real axis}. Then for $h\in\tilde{H}$,

$$\pi^+h(\xi_0)=\frac{1}{2\pi i}\lim_{u\to 0^-}\int_{\Gamma^+}\frac{h(\xi)}{\xi_0+iu-\xi}d\xi, \tag{3.1}$$

where Γ^+ is a Jordan closed curve included in $\mathrm{Im}(\xi)>0$ surrounding all the singularities of h in the upper half-plane and $\xi_0\in\mathbf{R}$. Similarly, we define π' on $\tilde{H}$,

$$\pi'h=\frac{1}{2\pi}\int_{\Gamma^+}h(\xi)d\xi. \tag{3.2}$$

So $\pi'(H^-)=0$.

For $h\in H\bigcap L^1(\mathbf{R})$,

$$\pi'h=\frac{1}{2\pi}\int_{\mathbf{R}}h(v)dv,$$

and for $h\in H^+\bigcap L^1(\mathbf{R})$, $\pi'h=0$.

Let G,T be, respectively, the singular Green operator and the trace operator of order m and type d. Let K be a potential operator and S be a classical pseudodifferential operator of order m along the boundary. An operator of order $m\in\mathbf{Z}$ and type d is a matrix

$$\widetilde{A}=\begin{pmatrix}\pi^+P+G & K\\ T & S\end{pmatrix}:\begin{matrix}C^\infty(M,E_1)\\ \bigoplus\\ C^\infty(\partial M,F_1)\end{matrix}\longrightarrow\begin{matrix}C^\infty(M,E_2)\\ \bigoplus\\ C^\infty(\partial M,F_2)\end{matrix},$$

where M is a manifold with boundary ∂M and E_1, E_2 (resp. F_1, F_2) are vector bundles over M (resp. ∂M). Here, $P : C_0^\infty(\Omega, \overline{E_1}) \to C^\infty(\Omega, \overline{E_2})$ is a classical pseudodifferential operator of order m on Ω, where Ω is a collar neighborhood of M and $\overline{E_i}|M = E_i$ $(i = 1, 2)$. P has an extension: $\mathcal{E}'(\Omega, \overline{E_1}) \to \mathcal{D}'(\Omega, \overline{E_2})$, where $\mathcal{E}'(\Omega, \overline{E_1})$ $(\mathcal{D}'(\Omega, \overline{E_2}))$ is the dual space of $C^\infty(\Omega, \overline{E_1})$ $(C_0^\infty(\Omega, \overline{E_2}))$. Let $e^+ : C^\infty(M, E_1) \to \mathcal{E}'(\Omega, \overline{E_1})$ denotes extension by zero from M to Ω, and $r^+ : \mathcal{D}'(\Omega, \overline{E_2}) \to \mathcal{D}'(\Omega, E_2)$ denotes the restriction from Ω to X; then define

$$\pi^+ P = r^+ P e^+ : C^\infty(M, E_1) \to \mathcal{D}'(\Omega, E_2).$$

In addition, P is supposed to have the transmission property; this means that, for all j, k, α, the homogeneous component p_j of order j in the asymptotic expansion of the symbol p of P in local coordinates near the boundary satisfies

$$\partial^k_{x_n} \partial^\alpha_{\xi'} p_j(x', 0, 0, +1) = (-1)^{j-|\alpha|} \partial^k_{x_n} \partial^\alpha_{\xi'} p_j(x', 0, 0, -1),$$

then $\pi^+ P : C^\infty(M, E_1) \to C^\infty(M, E_2)$ by Theorem 4 in [25] page 139.

Denote by $\mathcal{B}$ the Boutet de Monvel's algebra. We recall that the main theorem is in [10, 26].

Theorem 3.1. [26] **(Fedosov-Golse-Leichtnam-Schrohe)** *Let M and ∂M be connected,* $\dim M = n \geq 3$, *and let S (resp. S') be the unit sphere about ξ (resp. ξ') and $\sigma(\xi)$ (resp. $\sigma(\xi')$) be the corresponding canonical $n-1$ (resp. $(n-2)$) volume form. Set $\widetilde{A} = \begin{pmatrix} \pi^+ P + G & K \\ T & S \end{pmatrix} \in \mathcal{B}$, and denote by p, b and s the local symbols of P, G, and S, respectively. Define:*

$$\begin{aligned}\widetilde{\mathrm{Wres}}(\widetilde{A}) = & \int_X \int_{\mathbf{S}} \mathrm{tr}_E \left[p_{-n}(x, \xi) \right] \sigma(\xi) dx \\ & + 2\pi \int_{\partial X} \int_{\mathbf{S}'} \left\{ \mathrm{tr}_E \left[(\mathrm{tr} b_{-n})(x', \xi') \right] + \mathrm{tr}_F \left[s_{1-n}(x', \xi') \right] \right\} \sigma(\xi') dx',\end{aligned}$$

where $\widetilde{\mathrm{Wres}}$ denotes the noncommutative residue of an operator in the Boutet de Monvel's algebra, and

$$S = \{(\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n | \sum_{i,j=1}^n g^{ij} \xi_i \xi_j = 1\},$$

in the normal coordinate,

$$S(x_0) = \{(\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n | \sum_{i=1}^n \xi_i^2 = 1\}.$$

Then a) $\widetilde{\mathrm{Wres}}([\widetilde{A}, B]) = 0$, for any $\widetilde{A}, B \in \mathcal{B}$; b) It is the unique continuous trace on $\mathcal{B}/\mathcal{B}^{-\infty}$.

Definition 3.2. [10] *Lower-dimensional volumes of spin manifolds with boundary are defined by*

$$\mathrm{Vol}_n^{(p_1, p_2)} M := \widetilde{\mathrm{Wres}}[\pi^+ D^{-p_1} \circ \pi^+ D^{-p_2}],$$

and

$$\widetilde{\mathrm{Wres}}[\pi^+ D^{-p_1} \circ \pi^+ D^{-p_2}] = \int_M \int_{|\xi|=1} \mathrm{tr}_{\wedge^* T^* M \bigotimes \mathbb{C}}[\sigma_{-n}(D^{-p_1-p_2})] \sigma(\xi) dx + \int_{\partial M} \Phi, \tag{3.3}$$

where

$$\Phi = \int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{j,k=0}^{\infty}\sum \frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!}\times \mathrm{tr}_{\wedge^*T^*M\bigotimes\mathbb{C}}[\partial_{x_n}^j\partial_{\xi'}^{\alpha}\partial_{\xi_n}^k\sigma_r^+(D^{-p_1})(x',0,\xi',\xi_n)$$
$$\times\partial_{x'}^{\alpha}\partial_{\xi_n}^{j+1}\partial_{x_n}^k\sigma_l(D^{-p_2})(x',0,\xi',\xi_n)]d\xi_n\sigma(\xi')dx', \tag{3.4}$$

and the sum is taken over $r+l-k-|\alpha|-j-1=-n,\ \ r\leq -p_1,\ \ l\leq -p_2$.

3.2. *The interior term of* $\lim_{\varepsilon\to 0}\varepsilon^4\widetilde{\mathrm{Wres}}[\pi^+(\varepsilon D+c(X))^{-1}\circ\pi^+(\varepsilon D+c(Z))^{-1}]$

By $\varepsilon^4\widetilde{\mathrm{Wres}}[\pi^+(\varepsilon D+c(X))^{-1}\circ\pi^+(\varepsilon D+c(Z))^{-1}]=\varepsilon^2\widetilde{\mathrm{Wres}}\Big[\pi^+\Big(D+\frac{c(X)}{\varepsilon}\Big)^{-1}\circ\pi^+\Big(D+\frac{c(Z)}{\varepsilon}\Big)^{-1}\Big]$ and (3.3), we first compute

$$\widetilde{\mathrm{Wres}}\Big[\pi^+\Big(D+\frac{c(X)}{\varepsilon}\Big)^{-1}\circ\pi^+\Big(D+\frac{c(Z)}{\varepsilon}\Big)^{-1}\Big]$$
$$=\int_M\int_{|\xi|=1}\mathrm{tr}_{S(TM)\bigotimes\mathbb{C}}\Big[\sigma_{-4}\Big(D^2+\frac{c(Z)D}{\varepsilon}+\frac{Dc(X)}{\varepsilon}+\frac{c(Z)c(X)}{\varepsilon^2}\Big)^{-1}\Big]\sigma(\xi)dx+\int_{\partial M}\Phi, \tag{3.5}$$

where

$$\Phi = \int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{j,k=0}^{\infty}\sum \frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!}\times \mathrm{tr}_{S(TM)\bigotimes\mathbb{C}}\Big[\partial_{x_n}^j\partial_{\xi'}^{\alpha}\partial_{\xi_n}^k\sigma_r^+\Big(D+\frac{c(X)}{\varepsilon}\Big)^{-1}(x',0,\xi',\xi_n)$$
$$\times\partial_{x'}^{\alpha}\partial_{\xi_n}^{j+1}\partial_{x_n}^k\sigma_l\Big(D+\frac{c(Z)}{\varepsilon}\Big)^{-1}(x',0,\xi',\xi_n)\Big]d\xi_n\sigma(\xi')dx', \tag{3.6}$$

and the sum is taken over $r+l-k-j-|\alpha|=-3,\ \ r\leq -1,\ \ l\leq -1.$

Since $[\sigma_{-n}(D^{-p_1-p_2})]|_M$ has the same expression as $\sigma_{-n}(D^{-p_1-p_2})$ in the case of manifolds without boundary, so locally we can compute the interior term by [5,6,10,27].

Set $V=D^2+\frac{c(Z)D}{\varepsilon}+\frac{Dc(X)}{\varepsilon}+\frac{c(Z)c(X)}{\varepsilon^2}$, where $Z=\sum_{\alpha=1}^n a_\alpha e_\alpha=\sum_{j=1}^n Z_j\partial_j$ is a vector field. The next step is to compute the total symbol $\sigma(x,\xi)$ of V^{-1} from order -4 to order -2, with V the following sum of terms V_k of order k:

$$\sigma_2^V(x,\xi)=|\xi|^2;$$
$$\sigma_1^V(x,\xi)=i(\Gamma^\mu-2\sigma^\mu)\xi_\mu+\frac{i}{\varepsilon}c(Z)c(\xi)+\frac{i}{\varepsilon}c(\xi)c(X);$$
$$\sigma_0^V(x,\xi)=-(\partial^x\sigma_\mu+\sigma^\mu\sigma_\mu-\Gamma^\mu\sigma_\mu)+\frac{1}{4}s+\frac{i}{\varepsilon}c(Z)\gamma^\mu\sigma_\mu+\frac{i}{\varepsilon}\gamma^\mu\sigma_\mu c(X)+\frac{c(Z)c(X)}{\varepsilon^2}.$$

By (2.2) and the composition formula of pseudodifferential operators, $\sigma_{-4}^{V^{-1}}$ is obtained , which include the sum of terms N_1-N_{10} and F_1-F_7:

$$F_1=-|\xi|^{-6}\frac{1}{\varepsilon}[c(Z)c(\xi)+c(\xi)c(X)](\Gamma^\mu-2\sigma^\nu)\xi_\mu;\quad F_2=-|\xi|^{-6}(\Gamma^\mu-2\sigma^\nu)\xi_\mu\frac{1}{\varepsilon}[c(Z)c(\xi)+c(\xi)c(X)];$$
$$F_3=2|\xi|^{-8}\frac{1}{\varepsilon}[c(Z)c(\xi)+c(\xi)c(X)]\xi^\mu\xi_\alpha\xi_\beta\partial_\mu^x g^{\alpha\beta};\quad F_4=-|\xi|^{-6}f\frac{1}{\varepsilon^2}[c(Z)c(\xi)+c(\xi)c(X)]^2;$$

$F_5 = -|\xi|^{-4}[\frac{i}{\varepsilon}c(Z)\gamma^\mu\sigma_\mu + \frac{i}{\varepsilon}\gamma^\mu\sigma_\mu c(X) + \frac{c(Z)c(X)}{\varepsilon^2}]$; $F_6 = -|\xi|^{-4}\frac{1}{\varepsilon}\partial^\mu_\xi[c(Z)c(\xi) + c(\xi)c(X)]\xi_\alpha\xi_\beta\partial^x_\mu g^{\alpha\beta}$;
$F_7 = 2|\xi|^{-2}\xi_\mu\partial^\mu_\xi[|\xi|^{-4}\frac{1}{\varepsilon}[c(Z)c(\xi) + c(\xi)c(X)]]$.

Next, we proceed to compute $\int_{|\xi|=1}\mathrm{tr}[\sum_{i=1}^7 F_i]\sigma(\xi)$.

(**1**) : In normal coordinates, using the facts: $\Gamma^\mu_{\alpha\beta}(x_0) = \sigma_\mu(x_0) = 0$, $\partial^x_\mu g^{\alpha\beta}(x_0) = 0$, the terms F_1, F_2, F_3, and F_6 disappear.

(**2**) :

$$\begin{aligned}&\mathrm{tr}[c(Z)c(\xi) + c(\xi)c(X)]^2_{|\xi|=1}\\&= \mathrm{tr}[c(Z)c(\xi)c(Z)c(\xi)] + \mathrm{tr}[c(Z)c(\xi)c(\xi)c(X)] + \mathrm{tr}[c(\xi)c(X)c(Z)c(\xi)] + \mathrm{tr}[c(\xi)c(X)c(\xi)c(X)].\end{aligned}$$

By (2.6), we have

$$\int_{|\xi|=1} = \mathrm{tr}[c(Z)c(\xi)c(Z)c(\xi)]\sigma(\xi) = |Z|^2\pi^2\mathrm{tr}[\mathtt{id}],$$

$$\int_{|\xi|=1} = \mathrm{tr}[c(\xi)c(X)c(\xi)c(X)]\sigma(\xi) = |X|^2\pi^2\mathrm{tr}[\mathtt{id}],$$

$$\int_{|\xi|=1}\Big(\mathrm{tr}[c(Z)c(\xi)c(\xi)c(X)] + \mathrm{tr}[c(\xi)c(X)c(Z)c(\xi)]\Big)\sigma(\xi) = 4g(X,Z)\pi^2\mathrm{tr}[\mathtt{id}].$$

Then

$$\int_{|\xi|=1}\mathrm{tr}(F_4)(x_0)\sigma(\xi) = \frac{1}{\varepsilon^2}\Big(|Z|^2 + |X|^2 + 4g(X,Z)\Big)\pi^2\mathrm{tr}[\mathtt{id}].$$

(**3**) :

$$\int_{|\xi|=1}\mathrm{tr}(F_5)(x_0)\sigma(\xi) = \frac{2}{\varepsilon^2}g(X,Z)\pi^2\mathrm{tr}[\mathtt{id}].$$

(**4**) :

By (2.7), we have

$$\int_{|\xi|=1}\mathrm{tr}(F_7)(x_0)\sigma(\xi) = -\frac{1}{\varepsilon}[div_M(X) + div_M(Z)]\pi^2\mathrm{tr}[\mathtt{id}].$$

Therefore, we obtain the following result

$$\begin{aligned}&\mathrm{Wres}\Big(D^2 + \frac{c(Z)D}{\varepsilon} + \frac{Dc(X)}{\varepsilon} + \frac{c(Z)c(X)}{\varepsilon^2}\Big)^{-1}\\&= 4\int_M\Big(\frac{1}{\varepsilon^2}|X|^2\pi^2 + \frac{1}{\varepsilon^2}|Z|^2\pi^2 + \frac{6}{\varepsilon^2}g(X,Z)\pi^2 - \frac{1}{\varepsilon}div_M(X)\pi^2 - \frac{1}{\varepsilon}div_M(Z)\pi^2 + \frac{1}{12}s\Big)d\mathrm{Vol_M}.\end{aligned}$$

Further, above observations yields the following theorem

Theorem 3.3. *If M is a 4-dimensional compact oriented spin manifolds without boundary, then we derive the following equality:*

$$\lim_{\varepsilon\to 0}\varepsilon^4\mathrm{Wres}[\pi^+(\varepsilon D + c(X))^{-1}\circ\pi^+(\varepsilon D + c(Z))^{-1}] = 4\int_M\Big(|X|^2 + |Z|^2 + 6g(X,Z)\Big)\pi^2 d\mathrm{Vol_M}.$$

3.3. The boundary term of $\lim_{\varepsilon\to 0}\varepsilon^4\widetilde{\mathrm{Wres}}[\pi^+(\varepsilon D+c(X))^{-1}\circ\pi^+(\varepsilon D+c(Z))^{-1}]$

In this subsection, we proceed to calculate the boundary term: $\int_{\partial M}\Phi$. From [10], some symbols associated with these operators can be expressed.

Lemma 3.4. *The positive order symbol of* $D+\frac{c(Z)}{\varepsilon}$ *holds:*

$$\begin{aligned}
\sigma_1\Big(D+\frac{c(Z)}{\varepsilon}\Big)&=\sigma_1\Big(D+\frac{c(X)}{\varepsilon}\Big)=\sigma_1(D)=ic(\xi);\\
\sigma_0\Big(D+\frac{c(Z)}{\varepsilon}\Big)&=\sigma_0(D)+\frac{c(Z)}{\varepsilon}=-\frac{1}{4}\sum_{i,s,t}\omega_{s,t}(e_i)c(e_i)c(e_s)c(e_t)+\frac{c(Z)}{\varepsilon};\\
\sigma_0\Big(D+\frac{c(X)}{\varepsilon}\Big)&=\sigma_0(D)+\frac{c(X)}{\varepsilon}=-\frac{1}{4}\sum_{i,s,t}\omega_{s,t}(e_i)c(e_i)c(e_s)c(e_t)+\frac{c(X)}{\varepsilon}.
\end{aligned}$$

Then, utilizing the composition formula of pseudodifferential operators, we arrive at the following lemma.

Lemma 3.5. *The negative order symbol of* $\Big(D+\frac{c(Z)}{\varepsilon}\Big)^{-1}$ *holds:*

$$\begin{aligned}
\sigma_{-1}\Big(D+\frac{c(Z)}{\varepsilon}\Big)^{-1}&=\sigma_{-1}\Big(D+\frac{c(X)}{\varepsilon}\Big)^{-1}=\frac{ic(\xi)}{|\xi|^2};\\
\sigma_{-2}\Big(D+\frac{c(Z)}{\varepsilon}\Big)^{-1}&=\frac{c(\xi)\sigma_0\Big(D+\frac{c(Z)}{\varepsilon}\Big)c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}\sum_j c(dx_j)\Big[\partial_{x_j}(c(\xi))|\xi|^2-c(\xi)\partial_{x_j}(|\xi|^2)\Big];\\
\sigma_{-2}\Big(D+\frac{c(X)}{\varepsilon}\Big)^{-1}&=\frac{c(\xi)\sigma_0\Big(D+\frac{c(X)}{\varepsilon}\Big)c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}\sum_j c(dx_j)\Big[\partial_{x_j}(c(\xi))|\xi|^2-c(\xi)\partial_{x_j}(|\xi|^2)\Big].
\end{aligned}$$

By computations, we obtain the semiclassical limit of the Kastler–Kalau–Walze-type theorem.

Theorem 3.6. *Let* M *be a* 4*-dimensional oriented compact manifold with boundary* ∂M*, then*

$$\lim_{\varepsilon\to 0}\varepsilon^4\widetilde{\mathrm{Wres}}[\pi^+(\varepsilon D+c(X))^{-1}\circ\pi^+(\varepsilon D+c(Z))^{-1}]=4\int_M\Big(|X|^2+|Z|^2+6g(X,Z)\Big)\pi^2d\mathrm{Vol_M}.$$

In particular, as the semiclassical limit is taken, the boundary term goes to zero.

Proof. For $n=4$, the summation condition $r+l-k-j-|\alpha|=-3,\ \ r\leq -1,\ \ l\leq -1$, it leads to the following five cases:

case a) When $r=-1,\ l=-1,\ k=j=0,\ |\alpha|=1$.

By (3.6), we obtain

$$\Phi_1=-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{tr}\Big[\partial^\alpha_{\xi'}\pi^+_{\xi_n}\sigma_{-1}\Big(D+\frac{c(X)}{\varepsilon}\Big)^{-1}\times\partial^\alpha_{x'}\partial_{\xi_n}\sigma_{-1}\Big(D+\frac{c(Z)}{\varepsilon}\Big)^{-1}\Big](x_0)d\xi_n\sigma(\xi')dx'.$$

For $i<n$, we obtain

$$\partial_{x_i}\left(\frac{ic(\xi)}{|\xi|^2}\right)(x_0)=\frac{i\partial_{x_i}[c(\xi)](x_0)}{|\xi|^2}-\frac{ic(\xi)\partial_{x_i}(|\xi|^2)(x_0)}{|\xi|^4}=0,$$

so $\Phi_1 = 0$.
case b) When $r = -1,\ l = -1,\ k = |\alpha| = 0,\ j = 1$.
From (3.6), we obtain

$$\Phi_2 = -\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty} \mathrm{tr}\Big[\partial_{x_n}\pi^+_{\xi_n}\sigma_{-1}\Big(D+\frac{c(X)}{\varepsilon}\Big)^{-1}\times\partial^2_{\xi_n}\sigma_{-1}\Big(D+\frac{c(Z)}{\varepsilon}\Big)^{-1}\Big](x_0)d\xi_n\sigma(\xi')dx'.$$

Applying Lemma 3.5 yields

$$\partial^2_{\xi_n}\sigma_{-1}\Big(D+\frac{c(Z)}{\varepsilon}\Big)^{-1}(x_0) = i\Big(-\frac{6\xi_n c(dx_n)+2c(\xi')}{|\xi|^4}+\frac{8\xi_n^2 c(\xi)}{|\xi|^6}\Big);$$

$$\partial_{x_n}\sigma_{-1}\Big(D+\frac{c(X)}{\varepsilon}\Big)^{-1}(x_0) = \frac{i\partial_{x_n}c(\xi')(x_0)}{|\xi|^2}-\frac{ic(\xi)|\xi'|^2h'(0)}{|\xi|^4}.$$

Using the Clifford algebra relations and the trace property $\mathrm{tr}ab = \mathrm{tr}ba$, we obtain:

$$\mathrm{tr}[c(\xi')c(dx_n)] = 0;\ \ \mathrm{tr}[c(dx_n)^2] = -4;\ \ \mathrm{tr}[c(\xi')^2](x_0)|_{|\xi'|=1} = -4;$$
$$\mathrm{tr}[\partial_{x_n}c(\xi')c(dx_n)] = 0;\ \ \mathrm{tr}[\partial_{x_n}c(\xi')c(\xi')](x_0)|_{|\xi'|=1} = -2h'(0).$$

Then, we obtain

$$\begin{aligned}\Phi_2 &= -\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{ih'(0)(\xi_n-i)^2}{(\xi_n-i)^4(\xi_n+i)^3}d\xi_n\sigma(\xi')dx'\\ &= -ih'(0)\Omega_3\int_{\Gamma^+}\frac{1}{(\xi_n-i)^2(\xi_n+i)^3}d\xi_n dx'\\ &= -ih'(0)\Omega_3 2\pi i\Big[\frac{1}{(\xi_n+i)^3}\Big]^{(1)}\Big|_{\xi_n=i}dx'\\ &= -\frac{3}{8}\pi h'(0)\Omega_3 dx',\end{aligned}$$

where Ω_3 is the canonical volume of S^2.
case c) When $r = -1,\ l = -1,\ j = |\alpha| = 0,\ k = 1$.
From (3.6), we obtain

$$\Phi_3 = -\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty} \mathrm{tr}\Big[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}\Big(D+\frac{c(X)}{\varepsilon}\Big)^{-1}\times\partial_{\xi_n}\partial_{x_n}\sigma_{-1}\Big(D+\frac{c(Z)}{\varepsilon}\Big)^{-1}\Big](x_0)d\xi_n\sigma(\xi')dx'.$$

Applying Lemma 3.5 yields

$$\partial_{\xi_n}\partial_{x_n}\sigma_{-1}\Big(D+\frac{c(Z)}{\varepsilon}\Big)^{-1}(x_0)|_{|\xi'|=1} = -ih'(0)\left[\frac{c(dx_n)}{|\xi|^4}-4\xi_n\frac{c(\xi')+\xi_n c(dx_n)}{|\xi|^6}\right]-\frac{2\xi_n i\partial_{x_n}c(\xi')(x_0)}{|\xi|^4};$$

$$\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}\Big(D+\frac{c(X)}{\varepsilon}\Big)^{-1}(x_0)|_{|\xi'|=1} = -\frac{c(\xi')+ic(dx_n)}{2(\xi_n-i)^2}.$$

Similar to case b), we obtain

$$\mathrm{tr}\left\{\frac{c(\xi')+ic(dx_n)}{2(\xi_n-i)^2}\times ih'(0)\left[\frac{c(dx_n)}{|\xi|^4}-4\xi_n\frac{c(\xi')+\xi_n c(dx_n)}{|\xi|^6}\right]\right\}=2h'(0)\frac{i-3\xi_n}{(\xi_n-i)^4(\xi_n+i)^3}$$

and

$$\mathrm{tr}\left[\frac{c(\xi')+ic(dx_n)}{2(\xi_n-i)^2}\times\frac{2\xi_n i\partial_{x_n}c(\xi')(x_0)}{|\xi|^4}\right]=\frac{-2ih'(0)\xi_n}{(\xi_n-i)^4(\xi_n+i)^2}.$$

Thus, we obtain

$$\begin{aligned}\Phi_3&=-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{h'(0)(i-3\xi_n)}{(\xi_n-i)^4(\xi_n+i)^3}d\xi_n\sigma(\xi')dx'-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{h'(0)i\xi_n}{(\xi_n-i)^4(\xi_n+i)^2}d\xi_n\sigma(\xi')dx'\\&=-h'(0)\Omega_3\frac{2\pi i}{3!}\left[\frac{(i-3\xi_n)}{(\xi_n+i)^3}\right]^{(3)}\bigg|_{\xi_n=i}dx'+h'(0)\Omega_3\frac{2\pi i}{3!}\left[\frac{i\xi_n}{(\xi_n+i)^2}\right]^{(3)}\bigg|_{\xi_n=i}dx'\\&=\frac{3}{8}\pi h'(0)\Omega_3dx'.\end{aligned}$$

case d) When $r=-2,\ l=-1,\ k=j=|\alpha|=0$.
From (3.6), we obtain

$$\Phi_4=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}\Big[\pi^+_{\xi_n}\sigma_{-2}\Big(D+\frac{c(X)}{\varepsilon}\Big)^{-1}\times\partial_{\xi_n}\sigma_{-1}\Big(D+\frac{c(Z)}{\varepsilon}\Big)^{-1}\Big](x_0)d\xi_n\sigma(\xi')dx'.$$

Denote

$$Q(x_0)=-\frac{1}{4}\sum_{s,t,i}\omega_{s,t}(e_i)(x_0)c(e_i)c(e_s)c(e_t).$$

Then applying Lemma 3.5 yields

$$\begin{aligned}\pi^+_{\xi_n}\sigma_{-2}\Big(D+\frac{c(X)}{\varepsilon}\Big)^{-1}\Big|_{|\xi'|=1}&=\pi^+_{\xi_n}\Big[\frac{c(\xi)Q(x_0)c(\xi)}{(1+\xi_n^2)^2}\Big]+\pi^+_{\xi_n}\Big[\frac{c(\xi)c(X)c(\xi)}{\varepsilon(1+\xi_n^2)^2}\Big]\\&\quad+\pi^+_{\xi_n}\Big[\frac{c(\xi)c(dx_n)\partial_{x_n}[c(\xi')](x_0)}{(1+\xi_n^2)^2}-h'(0)\frac{c(\xi)c(dx_n)c(\xi)}{(1+\xi_n^2)^3}\Big]\\&:=E_1-E_2+E_3,\end{aligned}$$

where

$$\begin{aligned}E_1=&\frac{-1}{4(\xi_n-i)^2}[(2+i\xi_n)c(\xi')Q_0^2(x_0)c(\xi')+i\xi_nc(dx_n)Q_0^2(x_0)c(dx_n)\\&+(2+i\xi_n)c(\xi')c(dx_n)\partial_{x_n}c(\xi')+ic(dx_n)Q_0^2(x_0)c(\xi')+ic(\xi')Q_0^2(x_0)c(dx_n)-i\partial_{x_n}c(\xi')],\end{aligned}\tag{3.7}$$

$$E_2=\frac{h'(0)}{2}\left[\frac{c(dx_n)}{4i(\xi_n-i)}+\frac{c(dx_n)-ic(\xi')}{8(\xi_n-i)^2}+\frac{3\xi_n-7i}{8(\xi_n-i)^3}[ic(\xi')-c(dx_n)]\right],\tag{3.8}$$

and

$$E_3=\frac{2+i\xi_n}{4\varepsilon(\xi_n-i)^2}c(\xi')c(X)c(\xi')+\frac{i}{4\varepsilon(\xi_n-i)^2}c(\xi')c(X)c(dx_n)+\frac{i}{4\varepsilon(\xi_n-i)^2}c(dx_n)c(X)c(\xi')$$
$$+\frac{i\xi_n}{4\varepsilon(\xi_n-i)^2}c(dx_n)c(X)c(dx_n).$$

Since

$$\partial_{\xi_n}\sigma_{-1}\Big(D+\frac{c(Z)}{\varepsilon}\Big)^{-1}=i\left[\frac{c(dx_n)}{1+\xi_n^2}-\frac{2\xi_nc(\xi')+2\xi_n^2c(dx_n)}{(1+\xi_n^2)^2}\right]. \tag{3.9}$$

Using the Clifford algebra relations and the trace property $\mathrm{tr}ab=\mathrm{tr}ba$, we obtain:

$$\mathrm{tr}[c(\xi')c(X)c(\xi')c(dx_n)]=-4X_n;\ \ \mathrm{tr}[c(\xi')c(X)c(\xi')c(\xi')]=4g(X,\xi');$$
$$\mathrm{tr}[c(dx_n)c(X)c(dx_n)c(dx_n)]=4X_n;\ \ \mathrm{tr}[c(dx_n)c(X)c(\xi')c(\xi')c(dx_n)]=4g(X,\xi').$$

By (3.7) and (3.9), we have

$$\mathrm{tr}\Big[C_1\times\partial_{\xi_n}\sigma_{-1}\Big(D+\frac{c(Z)}{\varepsilon}\Big)^{-1}\Big]\Big|_{|\xi'|=1}=\frac{3ih'(0)}{2(\xi_n-i)^2(1+\xi_n^2)^2}+h'(0)\frac{\xi_n^2-i\xi_n-2}{2(\xi_n-i)(1+\xi_n^2)^2},$$

By (3.8) and (3.9), we have

$$\mathrm{tr}\Big[C_2\times\partial_{\xi_n}\sigma_{-1}\Big(D+\frac{c(Z)}{\varepsilon}\Big)^{-1}\Big]\Big|_{|\xi'|=1}=2ih'(0)\frac{-i\xi_n^2-\xi_n+4i}{4(\xi_n-i)^3(\xi_n+i)^2},$$

and

$$\mathrm{tr}\Big[C_3\times\partial_{\xi_n}\sigma_{-1}\Big(D+\frac{c(Z)}{\varepsilon}\Big)^{-1}\Big]\Big|_{|\xi'|=1}=\frac{2i}{\varepsilon(\xi_n-i)^3(\xi_n+i)}X_n-4\frac{\xi_n+i\xi_n^2}{\varepsilon(\xi_n-i)^4(\xi_n+i)^2}X_n$$
$$+\frac{2}{\varepsilon(\xi_n-i)^3(\xi_n+i)}g(X,\xi')+\frac{4i\xi_n-\xi_n^2}{\varepsilon(\xi_n-i)^4(\xi_n+i)^2}g(X,\xi').$$

When $i<n$, $\int_{|\xi'|=1}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2d+1}}\sigma(\xi')=0$, so $g(X,\xi')$ has no contribution for computing case d). Thus, we obtain

$$-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}\Big[(E_1-E_2)\times\partial_{\xi_n}\sigma_{-1}\Big(D+\frac{c(X)}{\varepsilon}\Big)^{-1}\Big](x_0)d\xi_n\sigma(\xi')dx'$$
$$=\Omega_3\int_{\Gamma^+}\frac{3h'(0)(\xi_n-i)+ih'(0)}{2(\xi_n-i)^3(\xi_n+i)^2}d\xi_ndx'$$
$$=\frac{9}{8}\pi h'(0)\Omega_3dx'.$$

$$-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}\Big[E_3\times\partial_{\xi_n}\sigma_{-1}\Big(D+\frac{c(X)}{\varepsilon}\Big)^{-1}\Big](x_0)d\xi_n\sigma(\xi')dx'$$
$$=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{2i}{\varepsilon(\xi_n-i)^3(\xi_n+i)}X_nd\xi_n\sigma(\xi')dx'-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}-4\frac{\xi_n+i\xi_n^2}{\varepsilon(\xi_n-i)^4(\xi_n+i)^2}X_nd\xi_n\sigma(\xi')dx'$$

$$= \Omega_3 X_n \frac{1}{\varepsilon}\int_{\Gamma^+}\frac{2}{(\xi_n-i)^3(\xi_n+i)}d\xi_n dx' + 4if\Omega_3 X_n\frac{1}{\varepsilon}\int_{\Gamma^+}\frac{\xi_n+i\xi_n^2}{(\xi_n-i)^4(\xi_n+i)^2}d\xi_n dx'$$
$$= \Omega_3 X_n\frac{2\pi i}{2!\varepsilon}\Big[\frac{2}{(\xi_n+i)}\Big]^{(2)}\Big|_{\xi_n=i}dx' + 4if\Omega_3 X_n\frac{2\pi i}{3!\varepsilon}\Big[\frac{\xi_n+i\xi_n^2}{(\xi_n+i)^2}\Big]^{(3)}\Big|_{\xi_n=i}dx'$$
$$= -\frac{1}{\varepsilon}X_n\pi\Omega_3 dx'.$$

Thus

$$\Phi_4 = \Big(\frac{9}{8}h'(0) - \frac{1}{\varepsilon}X_n\Big)\pi\Omega_3 dx'.$$

case e) When$r=-1,\ l=-2,\ k=j=|\alpha|=0.$
From (3.6), we obtain

$$\Phi_5 = -i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}\Big[\pi^+_{\xi_n}\sigma_{-1}\Big(D+\frac{c(X)}{\varepsilon}\Big)^{-1}\times\partial_{\xi_n}\sigma_{-2}\Big(D+\frac{c(Z)}{\varepsilon}\Big)^{-1}\Big](x_0)d\xi_n\sigma(\xi')dx'.$$

Applying Lemma 3.5 yields

$$\pi^+_{\xi_n}\sigma_{-1}\Big(D+\frac{c(X)}{\varepsilon}\Big)^{-1}\Big|_{|\xi'|=1} = \frac{c(\xi')+ic(dx_n)}{2(\xi_n-i)}. \tag{3.10}$$

Since

$$\sigma_{-2}\Big(D+\frac{c(Z)}{\varepsilon}\Big)^{-1}(x_0)$$
$$= \frac{c(\xi)\sigma_0\Big(D+\frac{c(Z)}{\varepsilon}\Big)(x_0)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6}c(dx_n)\Big[\partial_{x_n}[c(\xi')](x_0)|\xi|^2 - c(\xi)h'(0)|\xi|^2_{\partial_M}\Big].$$

Further

$$\partial_{\xi_n}\sigma_{-2}\Big(D+\frac{c(Z)}{\varepsilon}\Big)^{-1}(x_0)\Big|_{|\xi'|=1}$$
$$= \partial_{\xi_n}\Big\{\frac{c(\xi)\Big(Q(x_0)+\frac{c(Z)}{\varepsilon}\Big)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6}c(dx_n)[\partial_{x_n}[c(\xi')](x_0)|\xi|^2 - c(\xi)h'(0)]\Big\}$$
$$= \partial_{\xi_n}\Big\{\frac{[c(\xi)Q(x_0)]c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6}c(dx_n)[\partial_{x_n}[c(\xi')](x_0)|\xi|^2 - c(\xi)h'(0)]\Big\} + \partial_{\xi_n}\Big(\frac{c(\xi)\frac{c(Z)}{\varepsilon}c(\xi)}{|\xi|^4}\Big).$$

By computations, we have

$$\partial_{\xi_n}\Big(\frac{c(\xi)\frac{c(Z)}{\varepsilon}c(\xi)}{|\xi|^4}\Big) = -\frac{4\xi_n}{\varepsilon(1+\xi_n^2)^3}c(\xi')c(Z)c(\xi') + \Big(\frac{1}{\varepsilon(1+\xi_n^2)^2} - \frac{4\xi_n^2}{\varepsilon(1+\xi_n^2)^3}\Big)\Big(c(\xi')c(Z)c(dx_n)$$
$$+ c(dx_n)c(Z)c(\xi')\Big) + \Big(\frac{2\xi_n}{\varepsilon(1+\xi_n^2)^2} - \frac{4\xi_n^3}{\varepsilon(1+\xi_n^2)^3}\Big)c(dx_n)c(Z)c(dx_n). \tag{3.11}$$

We denote

$$q_{-2}^{1}=\frac{c(\xi)Q(x_0)c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}c(dx_n)[\partial_{x_n}[c(\xi')](x_0)|\xi|^2-c(\xi)h'(0)],$$

then

$$\begin{aligned}\partial_{\xi_n}(q_{-2}^{1})=&\frac{1}{(1+\xi_n^2)^3}\Big[(2\xi_n-2\xi_n^3)c(dx_n)Q(x_0)c(dx_n)+(1-3\xi_n^2)c(dx_n)Q(x_0)c(\xi')\\&+(1-3\xi_n^2)c(\xi')Q(x_0)c(dx_n)-4\xi_nc(\xi')Q(x_0)c(\xi')+(3\xi_n^2-1)\partial_{x_n}c(\xi')\\&-4\xi_nc(\xi')c(dx_n)\partial_{x_n}c(\xi')+2h'(0)c(\xi')+2h'(0)\xi_nc(dx_n)\Big]+6\xi_nh'(0)\frac{c(\xi)c(dx_n)c(\xi)}{(1+\xi_n^2)^4}.\end{aligned}\tag{3.12}$$

By (3.10) and (3.12), we have

$$\mathrm{tr}\Big[\pi_{\xi_n}^{+}\sigma_{-1}\Big(D+\frac{c(X)}{\varepsilon}\Big)^{-1}\times\partial_{\xi_n}(q_{-2}^{1})\Big](x_0)=\frac{3h'(0)(i\xi_n^2+\xi_n-2i)}{(\xi-i)^3(\xi+i)^3}+\frac{12h'(0)i\xi_n}{(\xi-i)^3(\xi+i)^4}.$$

Then

$$-i\Omega_3\int_{\Gamma_+}\Big[\frac{3h'(0)(i\xi_n^2+\xi_n-2i)}{(\xi_n-i)^3(\xi_n+i)^3}+\frac{12h'(0)i\xi_n}{(\xi_n-i)^3(\xi_n+i)^4}\Big]d\xi_ndx'=-\frac{9}{8}\pi h'(0)\Omega_3dx'.$$

Then, using the Clifford algebra relations and the trace property $\mathrm{tr}ab=\mathrm{tr}ba$, we obtain:

$$\begin{aligned}&\mathrm{tr}[c(\xi')c(Z)c(\xi')c(dx_n)]=-4Z_n;\ \ \mathrm{tr}[c(\xi')c(Z)c(\xi')c(\xi')]=4g(Z,\xi');\\&\mathrm{tr}[c(dx_n)c(Z)c(dx_n)c(dx_n)]=4Z_n;\ \ \mathrm{tr}[c(dx_n)c(Z)c(\xi')c(\xi')c(dx_n)]=4g(Z,\xi').\end{aligned}$$

By (3.10) and (3.11), we have

$$\begin{aligned}&\mathrm{tr}\Big[\pi_{\xi_n}^{+}\sigma_{-1}\Big(D+\frac{c(X)}{\varepsilon}\Big)^{-1}\times\partial_{\xi_n}\Big(\frac{c(\xi)c(Z)c(\xi)}{\varepsilon|\xi|^4}\Big)\Big](x_0)\\&=4\frac{1-3\xi_n^2+3i\xi_n-i\xi_n^3}{\varepsilon(\xi_n-i)^4(\xi_n+i)^3}Z_n+4\frac{i(1-3\xi_n^2)-3\xi_n+\xi_n^3}{\varepsilon(\xi_n-i)^4(\xi_n+i)^3}g(Z,\xi').\end{aligned}$$

When $i<n$, $\int_{|\xi'|=1}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2d+1}}\sigma(\xi')=0$ and $g(Z,\xi')$ has no contribution for computing case e), we have

$$\begin{aligned}&-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}\Big[\pi_{\xi_n}^{+}\sigma_{-1}\Big(D+\frac{c(X)}{\varepsilon}\Big)^{-1}\times\partial_{\xi_n}\Big(\frac{c(\xi)c(Z)c(\xi)}{\varepsilon|\xi|^4}\Big)\Big](x_0)d\xi_n\sigma(\xi')dx'\\&=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}4\frac{1-3\xi_n^2+3i\xi_n-i\xi_n^3}{\varepsilon(\xi_n-i)^4(\xi_n+i)^3}Z_nd\xi_n\sigma(\xi')dx'\\&=-4i\Omega_3Z_n\frac{1}{\varepsilon}\int_{\Gamma^+}\frac{1-3\xi_n^2+3i\xi_n-i\xi_n^3}{(\xi_n-i)^4(\xi_n+i)^3}d\xi_ndx'\\&=-4i\Omega_3Z_n\frac{2\pi i}{3!\varepsilon}\Big[\frac{1-3\xi_n^2+3i\xi_n-i\xi_n^3}{(\xi_n+i)^3}\Big]^{(3)}\Big|_{\xi_n=i}dx'\\&=\frac{1}{\varepsilon}Z_n\pi\Omega_3dx'.\end{aligned}$$

Therefore

$$\Phi_5 = \Big(-\frac{9}{8}h'(0) + \frac{1}{\varepsilon}Z_n \Big)\pi\Omega_3 dx'.$$

Now Φ can be expressed as the sum of the **case a)–case e)**,

$$\Phi = \sum_{i=1}^{5} \Phi_i = \frac{1}{\varepsilon}(Z_n - X_n)\pi\Omega_3 dx'.$$

Finally, we obtain

$$\lim_{\varepsilon\to 0} \varepsilon^2 \int_{|\xi'|=1} \Phi = \lim_{\varepsilon\to 0} \varepsilon^2 \int_{|\xi'|=1} \frac{1}{\varepsilon}(Z_n - X_n)\pi\Omega_3 d\mathrm{Vol}_{\mathrm{M}} = 0.$$

By Theorem 3.3, Theorem 3.6 holds. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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