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Research article

SDD_2 tensors and B_2-tensors

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Abstract: Strong $\mathcal{H}$-tensors have many important applications in practical problems. In particular, strong $\mathcal{H}$-tensors play an important role in the positive qualitative determination of multivariate even-order homogeneous polynomials. Therefore, research in this field is of great theoretical and practical value. This paper focuses on introducing a novel class of tensors, termed SDD_2 tensors, which are derived from SDD_2 matrices and constitute a subclass of strong $\mathcal{H}$-tensors. Furthermore, we also investigate the relationships among SDD_2 tensors, strong $\mathcal{H}$-tensors, SDD_1 tensors and SDD tensors. Additionally, we extend the concept of SDD_2 tensors to B-tensors, thereby defining a new tensor class called B_2-tensors and analyzing their fundamental properties.

Keywords: $\mathcal{H}$-tensor; SDD_2 tensor; B_2-tensor; real symmetric tensor; even-order homogeneous polynomial

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1. Introduction

In 2005, Qi studied the eigenvalues of a real supersymmetric tensor [1], and this work gave us a more profound understanding of the tensors. Indeed, in mathematics, tensors are a generalization of matrices; a first-order tensor is a vector, and a second-order tensor is a matrix. Tensors play a crucial role in numerous scientific fields, including signal and image processing [2], continuum physics, high-order statistics [3], and magnetic resonance imaging [4]. As multilinear functions, tensors can express linear relationships among vectors, scalars, and other tensors. Recent research on tensors has primarily focused on several key areas, for example, establishing criteria for identifying strong $\mathcal{H}$-tensors [5]; generalizing $\mathcal{H}$-tensors to B-tensors using matrix theory [6]; analyzing the positive definiteness of $\mathcal{H}$-tensors [7]; investigating whether newly defined tensors retain the properties of $\mathcal{H}$-tensors; and deriving bounds for the infinity norm of tensors. Consequently, the structural properties, identification criteria, and iterative algorithms for strong $\mathcal{H}$-tensors have garnered substantial attention from researchers recently. In 2015, Song et al. discussed relationships among higher-order tensors, positive semi-definite tensors, and some other structured tensors. They demonstrate that every principal sub-tensor of such a structured tensor is still a structured tensor in the same class, with a lower dimension [8].

The positive definiteness of homogeneous polynomials plays a crucial role in numerous scientific fields, such as multivariate network realizability theory [9], a test for Lyapunov stability in multivariate filters [10], and polynomial problems [11]. And the $\mathcal{H}$-eigenvalues of tensors are widely used in data analysis, high-order Markov chains, and positive definiteness of even-order homogeneous polynomials [7, 12, 13]. Given the broad applications of even-order homogeneous polynomials in areas such as medical imaging and the stability study of non-linear autonomous systems via Lyapunov's direct method in automatic control [6, 7, 14, 15]. Determining whether an even-order homogeneous polynomial is positive definite has become increasingly significant. In this paper, we investigate whether SDD_2 tensors retain the properties of strong $\mathcal{H}$-tensors and explore their application to the positive definiteness of even-order homogeneous polynomials. The definitions of homogeneous polynomials and positive definiteness are provided below.

For positive integers n and m, $N = \{1, 2, \cdots, n\}$ and $\mathbb{C}$($resp.$ $\mathbb{R}$) denotes the set of all complex($resp.$ real) numbers. Let $\mathbb{C}^{n\times n}$($resp.$ $\mathbb{R}^{n\times n}$)$(n \geq 2)$ denotes the set of all n by n complex ($resp.$ real) matrices and let $\mathbb{C}^{[m,n]}$ ($resp.$ $\mathbb{R}^{[m,n]}$)$(m, n \geq 2)$ be the set of all complex ($resp.$ real) mth-order n-dimensional tensors. A tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ is called a complex($resp.$ real) mth-order n-dimensional tensor if $a_{i_1 i_2 \cdots i_m} \in \mathbb{C}$($resp.$ $\mathbb{R}$), where $i_j = 1, 2, \cdots, n$ for $j = 1, 2, \ldots, m$. A tensor $\mathcal{A}$ is called symmetric if its elements are invariant under any permutation of indices $\{i_1, i_2, \cdots, i_m\}$ [1]. An mth-degree homogeneous polynomial of n variables, $f(x)$, can be usually denoted as

$$f(x) \equiv \mathcal{A}x^m = \sum_{i_1, i_2, \ldots, i_m \in N} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m},$$

where $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ and $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$ is a symmetric tensor [7]. An mth-order n-dimensional tensor is denoted by $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}(m, n \geq 2)$, an n-dimensional vector is denoted by $x = (x_1, x_2, \ldots, x_n)^T$, and the ith of $\mathcal{A}x^{m-1}$ components are

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2, \cdots, i_m \in N} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m},$$

and

$$(x^{[m-1]})_i = x_i^{m-1}.$$

If there exists a λ such that the following homogeneous polynomial equation holds:

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

where $\mathcal{A}x^{m-1}$ and $\lambda x^{[m-1]}$ are vectors, and $\lambda \in \mathbb{C}$, $x = (x_1, x_2, \ldots, x_n)^T$ being a nonzero complex vector, then λ is referred to as an eigenvalue of $\mathcal{A}$, and x is its corresponding eigenvector [1, 16, 17]. Specifically, if λ, x, and all entries of $\mathcal{A}$ are constrained to the real field, then λ is termed an $\mathcal{H}$-eigenvalue of $\mathcal{A}$, and x is its corresponding $\mathcal{H}$-eigenvector [1]. If m is even, and

$$f(x) > 0, \; for\ all\ x \in \mathbb{R}^n, \; x \neq 0,$$

then we say that $f(x)$ is positive definite. The symmetric tensor $\mathcal{A}$ is called positive definite if $f(x)$ is positive definite [7].

Definition 1.1. *[18] Let* $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$*. If there is a positive vector* $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ *such that*

$$|a_{ii\cdots i}|x_i^{m-1} > \sum_{\substack{i_2,\cdots,i_m \in N \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| x_{i_2} \cdots x_{i_m}, \ \forall i \in N,$$

where $|a|$ *is the modulus of* $a \in \mathbb{C}$*, then* $\mathcal{A}$ *is called a strong* $\mathcal{H}$*-tensor.*

Theorem 1.1. *[19] Let* $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ *with* $a_{kk\cdots k} > 0$ *for all* $k \in N$ *and* m *be even. If* $\mathcal{A}$ *is a strong* $\mathcal{H}$*-tensor, then* $\mathcal{A}$ *is positive definite.*

Based on this theorem, to determine the positive definiteness of an even-order real symmetric tensor, one can first verify whether the given tensor is a strong $\mathcal{H}$-tensor. Numerous criteria for identifying strong $\mathcal{H}$-tensors have been extensively proposed in the literature; for example, using algorithmic criteria [20–22] and direct criteria to determine strong $\mathcal{H}$-tensor [23–27]. In the following sections, we will give the highlights of this article and present a new class of tensors, called SDD_2 tensors.

This paper is organized as follows: In Section 2, we introduce a new class of tensors, named SDD_2 tensors, which extend the concept of SDD_2 matrices. And we demonstrate that this new class of tensors is a subclass of strong $\mathcal{H}$-tensors. Furthermore, we use some numerical examples to illustrate these new results. In Section 3, we propose B_2-tensors inspired by SDD_2 tensors. Meanwhile, some properties of B_2-tensors are introduced. Finally, in Section 4, give a conclusion of this article.

2. SDD_2 tensors

In this section, we proposed a new class of tensors, which was inspired by the SDD_2 matrices, and named it SDD_2 tensors. First, let us begin by reviewing the concept of SDD_2 matrix. For the convenience of discussion, now some notations, definitions, lemmas, and theorems are given, which will be used in the sequel.

The calligraphic letters $\mathcal{A}, \mathcal{B}, \cdots$, represent tensors; the capital letters $A, B, \cdots$, denote matrices; the lowercase letters $x, y, \cdots$, refer to vectors. A tensor $\mathcal{I} = (\delta_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$ is called the unit tensor, where

$$\delta_{i_1 i_2 \cdots i_m} = \begin{cases} 1, & i_1 = i_2 = \cdots = i_m, \\ 0, & otherwise. \end{cases}$$

For a given matrix $M = (m_{ij}) \in \mathbb{C}^{n \times n}$, we denote

$$r_i(M) = \sum_{j \in N, j \neq i}^{n} |m_{ij}|,$$

$$N_1 = \{i \mid |m_{ii}| \leq r_i(M)\},$$

$$N_2 = \{i \mid |m_{ii}| > r_i(M)\}.$$

For a given tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$, we denote

$$r_i(\mathcal{A}) = \sum_{\substack{i_2 \cdots i_m \in N^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| = \sum_{i_2 \cdots i_m \in N^{m-1}} |a_{ii_2\cdots i_m}| - |a_{ii\cdots i}|,$$

$$\begin{aligned}
N_A &= N_A(\mathcal{A}) = \{i \in N : |a_{ii\cdots i}| \le r_i(\mathcal{A})\},\\
N_B &= N_B(\mathcal{A}) = \{i \in N : |a_{ii\cdots i}| > r_i(\mathcal{A})\},\\
S^{m-1} &= \{i_2 \cdots i_m : i_j \in S, j = 2, \ldots, m\},\ S \subseteq N,\\
N^{m-1} \backslash S^{m-1} &= \{i_2 i_3 \cdots i_m : i_2 i_3 \cdots i_m \in N^{m-1} \text{ and } i_2 i_3 \cdots i_m \notin S^{m-1}\},\\
N_C^{m-1} &= N^{m-1} \backslash (N_A^{m-1} \cup N_B^{m-1}),
\end{aligned}$$

where $r_i(\mathcal{A})$ denotes the weight of the off-diagonal entries in the ith row of the flattening $\mathcal{A}$. N_A is the set of indices where the modulus of the diagonal entry is less than or equal to the corresponding off-diagonal weight. Conversely, N_B is the set of indices where the modulus of the diagonal entry is greater than the corresponding off-diagonal weight. The set of S^{m-1} includes indices where i_2 to i_m belong to S and $S \subseteq N$. The set $N^{m-1} \backslash S^{m-1}$ refers to the difference set between N^{m-1} and S^{m-1}, where i_2 to i_m belong to N but not to S. And N_C^{m-1} denotes the difference set between N^{m-1} and $(N_A^{m-1} \cup N_B^{m-1})$, where i_2 to i_m partially belong to N_A and partially to N_B.

Definition 2.1. *[28] Given a matrix $M = \left(m_{ij}\right) \in \mathbb{C}^{n\times n}$ $(n \ge 2)$ is called an SDD_2 matrix, if*

$$|m_{ii}| > q_i(M), \forall i \in N_1(M),$$

where

$$q_i(M) = \sum_{j\in N_1\backslash\{i\}} \left|m_{ij}\right| + \sum_{j\in N_2\backslash\{i\}} \frac{p_j(M)}{\left|m_{jj}\right|} \left|m_{ij}\right|,$$

$$p_i(M) = \sum_{j\in N_1\backslash\{i\}} \left|m_{ij}\right| + \sum_{j\in N_2\backslash\{i\}} \frac{r_j(M)}{\left|m_{jj}\right|} \left|m_{ij}\right|.$$

Definition 2.2. *[1] Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$. $\mathcal{A}$ is called a diagonally dominant tensor if*

$$|a_{ii\cdots i}| \ge r_i(\mathcal{A}),\ \forall i \in N.$$

$\mathcal{A}$ is called a strictly diagonally dominant (SDD) tensor if all inequalities hold strictly.

Definition 2.3. *[29] Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$ and $X = diag(x_1, x_2, \ldots, x_n)$. If*

$$\mathcal{B} = (b_{i_1 i_2 \cdots i_m}) = \mathcal{A}X^{m-1},$$

where

$$b_{i_1 i_2 \cdots i_m} = a_{i_1 i_2 \cdots i_m} x_{i_2} \ldots x_{i_m},\ i_j \in N,\ j \in \{1, 2, \ldots, m\},$$

then $\mathcal{B}$ is referred to as the product of the tensor $\mathcal{A}$ and the matrix X.

Definition 2.4. *[6] A tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$ is called an SDD_1 tensor if*

$$|a_{i\cdots i}| > p_i(\mathcal{A}),\ i \in N_A,$$

where

$$p_i(\mathcal{A}) = \sum_{\substack{i_2\cdots i_m \in N_A^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| + \sum_{\substack{i_2\cdots i_m \in N_B^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,\cdots,i_m\}} \left\{\frac{r_j(\mathcal{A})}{|a_{jj\cdots j}|}\right\} |a_{ii_2\cdots i_m}| + \sum_{\substack{i_2\cdots i_m \in N_C^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|.$$

Through [6], it is established that SDD_1 matrices can be extended to SDD_1 tensors. Furthermore, we further attempt to generalize SDD_2 matrices to SDD_2 tensors. Specifically, we will demonstrate that SDD_2 tensors are a subclass of strong $\mathcal{H}$-tensors.

Definition 2.5. *A tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$ is called an SDD_2 tensor if*

$$|a_{i\cdots i}| > q_i(\mathcal{A}),\ i \in N_A,$$

where

$$q_i(\mathcal{A}) = \sum_{\substack{i_2\cdots i_m \in N_A^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| + \sum_{\substack{i_2\cdots i_m \in N_B^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,\cdots,i_m\}} \left\{ \frac{p_j(\mathcal{A})}{|a_{jj\cdots j}|} \right\} |a_{ii_2\cdots i_m}| + \sum_{\substack{i_2\cdots i_m \in N_C^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|,$$

and $p_i(\mathcal{A})$ is defined as the Definition 2.1.

Lemma 2.1. *[18] If $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$ is an SDD tensor, then $\mathcal{A}$ is a strong $\mathcal{H}$-tensor.*

Lemma 2.2. *[29] Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$. If there exists a positive diagonal matrix X such that $\mathcal{A}X^{m-1}$ is a strong $\mathcal{H}$-tensor, then $\mathcal{A}$ is a strong $\mathcal{H}$-tensor.*

In the following, we will give some properties of the SDD_2 tensor.

Theorem 2.1. *If a tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$ is an SDD_2 tensor and $N_A \neq \emptyset$, then we have $\sum_{\substack{i_2\cdots i_m \in N_B^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| \neq 0,\ i \in N_A$.*

Proof. For a tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$, if $\sum_{\substack{i_2\cdots i_m \in N_B^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| = 0,\ i \in N_A$, then there is $r_i(\mathcal{A}) = q_i(\mathcal{A})$. Since $\mathcal{A}$ is an SDD_2 tensor, by definition we have $|a_{i\cdots i}| > q_i(\mathcal{A}) = r_i(\mathcal{A})$ for all $i \in N_A$, it contradicts the definition of N_A. The proof is complete. □

Theorem 2.2. *A tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$ is an SDD_2 tensor if and only if $|a_{i\cdots i}| > q_i(\mathcal{A})$ for all $i \in N$.*

Proof. Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$ be an SDD_2 tensor. From Definition 2.5, we have $|a_{i\cdots i}| > q_i(\mathcal{A})$ for any $i \in N_A$. For any $i \in N_B$, from the definition of N_B and $q_i(\mathcal{A})$, we have $|a_{i\cdots i}| > r_i(\mathcal{A}) \geq q_i(\mathcal{A})$. Therefore, we obtain $|a_{i\cdots i}| > q_i(\mathcal{A})$ for all $i \in N$. □

Next, we will prove the SDD_2 tensor is a strong $\mathcal{H}$ tensor.

Theorem 2.3. *If a tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$ is an SDD_2 tensor, then $\mathcal{A}$ is a strong $\mathcal{H}$-tensor.*

Proof. Let a tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$ be an SDD_2 tensor; according to Theorems 2.1 and 2.2, $|a_{i\cdots i}| > q_i(\mathcal{A})$ for all $i \in N$. Hence, we have

$$|a_{i\cdots i}| - q_i(\mathcal{A}) > 0,\ \forall i \in N,$$

and

$$1 - \frac{q_i(\mathcal{A})}{|a_{ii\cdots i}|} > 0,\ \forall i \in N.$$

Then there exists a positive number $\varepsilon > 0$ such that

$$0 < \varepsilon < \min\left\{1 - \frac{q_i(\mathcal{A})}{|a_{ii\cdots i}|}, \frac{|a_{i\cdots i}| - q_i(\mathcal{A})}{\sum\limits_{\substack{i_2\cdots i_m \in N_B^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|}\right\}, \tag{2.1}$$

if $\sum\limits_{\substack{i_2\cdots i_m \in N_B^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| = 0,\ i \in N_B$, then the corresponding fraction is defined to be ∞. Next we construct a diagonal matrix $X = \text{diag}(x_1, x_2, \ldots, x_n)$, where

$$x_i = \begin{cases} 1, & i \in N_A, \\ \left(\frac{q_i(\mathcal{A})}{|a_{ii\cdots i}|} + \varepsilon\right)^{\frac{1}{m-1}}, & i \in N_B. \end{cases}$$

From inequality (2.1) we can obtain that $\frac{q_i(\mathcal{A})}{|a_{ii\cdots i}|} + \varepsilon < \frac{q_i(\mathcal{A})}{|a_{ii\cdots i}|} + (1 - \frac{q_i(\mathcal{A})}{|a_{ii\cdots i}|}) = 1$, so $x_i \neq +\infty$, which shows that X is a positive diagonal matrix. Let $\mathcal{B} = (b_{i_1 i_2\cdots i_m}) = \mathcal{A}X^{m-1}$; then we have $b_{i_1 i_2\cdots i_m} = a_{i_1 i_2\cdots i_m} x_{i_2} \cdots x_{i_m}$, for any $i_j \in N,\ j \in \{1, 2, \ldots, m\}$.

Next, we will prove that $\mathcal{B}$ is an SDD tensor.

$$\begin{aligned}
r_i(\mathcal{B}) &= \sum_{\substack{i_2\cdots i_m \in N_A^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |b_{ii_2\cdots i_m}| + \sum_{\substack{i_2\cdots i_m \in N_B^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |b_{ii_2\cdots i_m}| + \sum_{\substack{i_2\cdots i_m \in N_C^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |b_{ii_2\cdots i_m}| \\
&= \sum_{\substack{i_2\cdots i_m \in N_A^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| + \sum_{\substack{i_2\cdots i_m \in N_B^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| \left(\frac{q_{i_2}(\mathcal{A})}{|a_{i_2\cdots i_2}|} + \varepsilon\right)^{\frac{1}{m-1}} \cdots \left(\frac{q_{i_m}(\mathcal{A})}{|a_{i_m\cdots i_m}|} + \varepsilon\right)^{\frac{1}{m-1}} \\
&\quad + \sum_{\substack{i_2\cdots i_m \in N_C^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| x_{i_2} \cdots x_{i_m} \\
&\le \sum_{\substack{i_2\cdots i_m \in N_A^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| + \sum_{\substack{i_2\cdots i_m \in N_B^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| \left(\frac{q_{i_2}(\mathcal{A})}{|a_{i_2\cdots i_2}|} + \varepsilon\right)^{\frac{1}{m-1}} \cdots \left(\frac{q_{i_m}(\mathcal{A})}{|a_{i_m\cdots i_m}|} + \varepsilon\right)^{\frac{1}{m-1}} \\
&\quad + \sum_{\substack{i_2\cdots i_m \in N_C^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| \\
&\le \sum_{\substack{i_2\cdots i_m \in N_A^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| + \sum_{\substack{i_2\cdots i_m \in N_B^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| \left(\frac{p_{i_2}(\mathcal{A})}{|a_{i_2\cdots i_2}|} + \varepsilon\right)^{\frac{1}{m-1}} \cdots \left(\frac{p_{i_m}(\mathcal{A})}{|a_{i_m\cdots i_m}|} + \varepsilon\right)^{\frac{1}{m-1}} \\
&\quad + \sum_{\substack{i_2\cdots i_m \in N_C^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| \\
&\le \sum_{\substack{i_2\cdots i_m \in N_A^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| + \sum_{\substack{i_2\cdots i_m \in N_B^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,\cdots,i_m\}} \left\{\frac{p_j(\mathcal{A})}{|a_{j\cdots j}|} + \varepsilon\right\} |a_{ii_2\cdots i_m}| + \sum_{\substack{i_2\cdots i_m \in N_C^{m-1}\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|
\end{aligned}$$

$$=q_i(\mathcal{A}) + \varepsilon \sum_{\substack{i_2\cdots i_m \in N_B^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|.$$

For any $i \in N_A$, according to inequality (2.1) and Theorem 2.1, we have

$$r_i(\mathcal{B}) \le q_i(\mathcal{A}) + \varepsilon \sum_{\substack{i_2\cdots i_m \in N_B^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| < q_i(\mathcal{A}) + |a_{i\cdots i}| - q_i(\mathcal{A}) = |a_{ii\cdots i}| = |b_{ii\cdots i}|.$$

And for any $i \in N_B$, from definition of N_B, we have

$$r_i(\mathcal{B}) \le q_i(\mathcal{A}) + \varepsilon \sum_{\substack{i_2\cdots i_m \in N_B^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| < q_i(\mathcal{A}) + \varepsilon |a_{ii\cdots i}| = |b_{ii\cdots i}|.$$

Thus, we obtain $|b_{ii\cdots i}| > r_i(\mathcal{B})$ for any $i \in N$. This indicates that $\mathcal{B}$ is a strictly diagonally dominant (SDD) tensor, and by Lemma 2.1 we conclude that $\mathcal{B}$ is a strong $\mathcal{H}$-tensor. Furthermore, applying Lemma 2.2, it is straightforward to deduce that $\mathcal{A}$ is also a strong $\mathcal{H}$-tensor. The proof is completed. □

Remark 2.1. *From the Definitions 2.2, 2.4, and 2.5, it can be readily deduced that* $q_i(\mathcal{A}) \le p_i(\mathcal{A}) \le r_i(\mathcal{A})$*. Consequently,* SDD *tensors constitute a subclass of* SDD_1 *tensors, and* SDD_1 *tensors, in turn, form a subclass of* SDD_2 *tensors. Furthermore, from Theorem 2.3, we conclude that* SDD_2 *tensors are strong* $\mathcal{H}$*-tensor. This establishes the following inclusion relationship:*

$$\{SDD\text{-}tensors\} \subseteq \{SDD_1\text{-}tensors\} \subseteq \{SDD_2\text{-}tensors\} \subseteq \{strong\ \mathcal{H}\text{-}tensors\}.$$

Utilizing the following chart, we illustrate the relationships among these tensors.

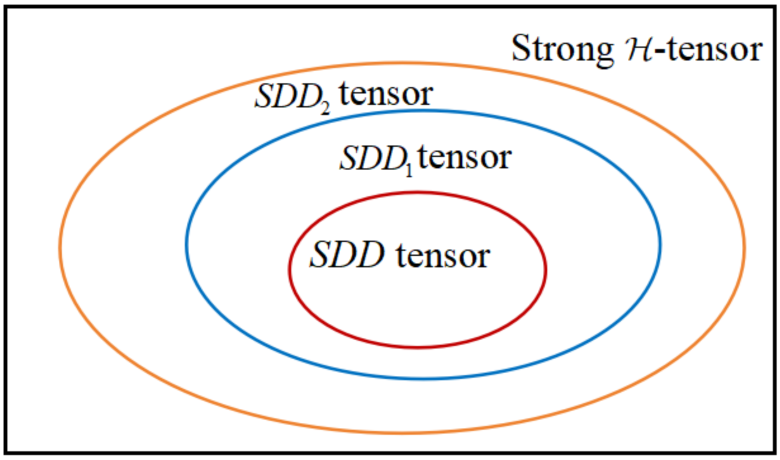

Figure 1. Relationships among some tensor classes.

Example 2.1. *Let us consider tensor* $\mathcal{A} = (a_{ijk}) = [A(1,:,:), A(2,:,:), A(3,:,:)] \in \mathbb{C}^{[3,3]}$*, where*

$$A(1,:,:) = \begin{pmatrix} 9 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 0 \end{pmatrix},\ A(2,:,:) = \begin{pmatrix} 1 & 1 & 0 \\ 0.5 & 10 & 0 \\ 0.5 & 0 & 1 \end{pmatrix},\ A(3,:,:) = \begin{pmatrix} 1 & 0 & 0.5 \\ 1 & 2 & 0 \\ 0 & 0.5 & 2 \end{pmatrix}.$$

Obviously,

$$|a_{111}| = 9,\ r_1(\mathcal{A}) = 5,\ |a_{222}| = 10,\ r_2(\mathcal{A}) = 4,\ |a_{333}| = 2 \text{ and } r_3(\mathcal{A}) = 5,$$

so $N_A = \{3\}$, $N_B = \{1, 2\}$. *By calculation, we obtain* $p_1(\mathcal{A}) = \frac{37}{9}$ *and* $p_2(\mathcal{A}) = \frac{26}{9}$, *then*

$$\max_{j\in N_B}\left\{\frac{p_j(\mathcal{A})}{|a_{jjj}|}\right\} = \frac{37}{81},$$

when $i \in N_A$, *we obtain*

$$\begin{aligned} q_3(\mathcal{A}) = &\sum_{\substack{i_2i_3\in N_A^2\\ \delta_{3i_2i_3}=0}} |a_{3i_2i_3}| + \sum_{i_2i_3\in N_B^2} \max_{j\in\{i_2,i_3\}}\left\{\frac{p_j(\mathcal{A})}{|a_{jjj}|}\right\}|a_{3i_2i_3}| + \sum_{i_2i_3\in N_C^2} |a_{3i_2i_3}| \\ =&\frac{37}{81}(1 + 0 + 1 + 2) + 1 = \frac{229}{81} > 2 = |a_{333}|. \end{aligned}$$

By Definition 2.5, $\mathcal{A}$ is not an SDD_1 tensor. However, there exists a positive diagonal matrix $D = diag(d_1, d_2, d_3)$, where $d_1 = 0.7^{\frac{1}{2}}, d_2 = 0.6^{\frac{1}{2}}, d_3 = 2.2^{\frac{1}{2}}$ such that $\mathcal{A}D$ is an SDD tensor, by Lemma 2.2, that tensor $\mathcal{A}$ is a strong H tensor.

Example 2.2. *Let us consider tensor* $\mathcal{A} = (a_{ijk}) = [A(1, :, :), A(2, :, :), A(3, :, :)] \in \mathbb{C}^{[3,3]}$, *where*

$$A(1,:,:) = \begin{pmatrix} 9 & 1 & 0\\ 0 & 1 & 2\\ 0 & 1 & 0 \end{pmatrix},\ A(2,:,:) = \begin{pmatrix} 1 & 1 & 0\\ 0.5 & 10 & 0\\ 0.5 & 0 & 1 \end{pmatrix},\ A(3,:,:) = \begin{pmatrix} 1 & 0 & 0.5\\ 0 & 1 & 0\\ 0 & 0.5 & 2 \end{pmatrix}.$$

Obviously,

$$|a_{111}| = 9,\ r_1(\mathcal{A}) = 5,\ |a_{222}| = 10,\ r_2(\mathcal{A}) = 4,\ |a_{333}| = 2 \text{ and } r_3(\mathcal{A}) = 3,$$

so $N_A = \{3\}$, $N_B = \{1, 2\}$. *By calculation, we obtain*

$$\max_{j\in N_B}\left\{\frac{r_j(\mathcal{A})}{|a_{jjj}|}\right\} = \frac{5}{9},$$

when $i \in N_A$, *we obtain*

$$\begin{aligned} p_3(\mathcal{A}) = &\sum_{\substack{i_2i_3\in N_A^2\\ \delta_{3i_2i_3}=0}} |a_{3i_2i_3}| + \sum_{i_2i_3\in N_B^2} \max_{j\in\{i_2,i_3\}}\left\{\frac{p_j(\mathcal{A})}{|a_{jjj}|}\right\}|a_{3i_2i_3}| + \sum_{i_2i_3\in N_C^2} |a_{3i_2i_3}| \\ =&\frac{5}{9}(1 + 0 + 0 + 1) + 1 = \frac{19}{9} > 2 = |a_{333}|. \end{aligned}$$

By Definition 2.4, we obtain that $\mathcal{A}$ *is not an* SDD_2 *tensor. Moreover, through computation we find* $p_1(\mathcal{A}) = \frac{37}{9}, p_2(\mathcal{A}) = \frac{26}{9}$, *then*

$$\max_{j\in N_B}\left\{\frac{p_j(\mathcal{A})}{|a_{jjj}|}\right\} = \frac{37}{81},$$

when $i \in N_A$, we obtain

$$q_3(\mathcal{A}) = \sum_{\substack{i_2i_3 \in N_A^2 \\ \delta_{3i_2i_3}=0}} |a_{3i_2i_3}| + \sum_{i_2i_3 \in N_B^2} \max_{j \in \{i_2,i_3\}} \left\{ \frac{p_j(\mathcal{A})}{|a_{jjj}|} \right\} |a_{3i_2i_3}| + \sum_{i_2i_3 \in N_C^2} |a_{3i_2i_3}|$$

$$= \frac{37}{81}(1+0+0+2) + 1 = \frac{155}{81} < 2 = |a_{333}|.$$

From Definition 2.5, we conclude that $\mathcal{A}$ is an SDD_2 tensor.

Next we give the application of the SDD_2 tensor from Theorems 1.1 and 2.3 as follows.

Theorem 2.4. *Let $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$ be an even-order symmetric tensor with $a_{kk\cdots k} > 0$ for all $k \in N$. If $\mathcal{A}$ is an SDD_2 tensor, then $\mathcal{A}$ is positive definite.*

We give an example to illustrate how the definition of an SDD_2 tensor can be applied to determine whether a given tensor is a strong $\mathcal{H}$-tensor.

Example 2.3. *Let us consider tensor $\mathcal{A} = (a_{ijk}) = [A(1,:,:), A(2,:,:), A(3,:,:)] \in \mathbb{C}^{[3,3]}$, where*

$$A(1,:,:) = \begin{pmatrix} 10 & 0.3 & 0 \\ 0 & 3 & 0 \\ 0.7 & 2 & 30 \end{pmatrix},\ A(2,:,:) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 4 \end{pmatrix},\ A(3,:,:) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 4 & 0 & 30 \end{pmatrix}.$$

Obviously,

$$|a_{111}| = 10,\ r_1(\mathcal{A}) = 36,\ |a_{222}| = 30,\ r_2(\mathcal{A}) = 7,\ |a_{333}| = 30 \text{ and } r_3(\mathcal{A}) = 9,$$

so $N_A = \{1\}$, $N_B = \{2,3\}$. By calculation, we obtain

$$\max_{j \in N_B} \left\{ \frac{r_j(\mathcal{A})}{|a_{jjj}|} \right\} = \frac{3}{10},$$

when $i \in N$, we obtain

$$p_1(\mathcal{A}) = \sum_{\substack{i_2i_3 \in N_A^2 \\ \delta_{1i_2i_3}=0}} |a_{1i_2i_3}| + \sum_{i_2i_3 \in N_B^2} \max_{j \in \{i_2,i_3\}} \left\{ \frac{r_j(\mathcal{A})}{|a_{jjj}|} \right\} |a_{1i_2i_3}| + \sum_{i_2i_3 \in N_C^2} |a_{1i_2i_3}|$$

$$= 0 + \frac{105}{10} + 1 = \frac{23}{2}.$$

$$p_2(\mathcal{A}) = \sum_{i_2i_3 \in N_A^2} |a_{2i_2i_3}| + \sum_{\substack{i_2i_3 \in N_B^2 \\ \delta_{2i_2i_3}=0}} \max_{j \in \{i_2,i_3\}} \left\{ \frac{r_j(\mathcal{A})}{|a_{jjj}|} \right\} |a_{2i_2i_3}| + \sum_{i_2i_3 \in N_C^2} |a_{2i_2i_3}|$$

$$= 3 + \frac{12}{10} + 0 = \frac{21}{5}.$$

$$p_3(\mathcal{A}) = \sum_{i_2i_3 \in N_A^2} |a_{3i_2i_3}| + \sum_{\substack{i_2i_3 \in N_B^2 \\ \delta_{3i_2i_3}=0}} \max_{j \in \{i_2,i_3\}} \left\{ \frac{r_j(\mathcal{A})}{|a_{jjj}|} \right\} |a_{3i_2i_3}| + \sum_{i_2i_3 \in N_C^2} |a_{3i_2i_3}|$$

$$=2+\frac{9}{10}+4=\frac{69}{10}.$$

Furthermore, we obtain

$$\max_{j\in N_B}\left\{\frac{p_j(\mathcal{A})}{|a_{jjj}|}\right\}=\frac{23}{100},$$

when $i\in N_A$, we obtain

$$\begin{aligned}q_1(\mathcal{A})=&\sum_{\substack{i_2i_3\in N_A^2\\ \delta_{1i_2i_3}=0}}|a_{1i_2i_3}|+\sum_{i_2i_3\in N_B^2}\max_{j\in\{i_2,i_3\}}\left\{\frac{p_j(\mathcal{A})}{|a_{jjj}|}\right\}|a_{1i_2i_3}|+\sum_{i_2i_3\in N_C^2}|a_{1i_2i_3}|\\ =&0+\frac{23}{100}(3+2+30)+1=\frac{181}{20}<10=|a_{111}|.\end{aligned}$$

Hence, $\mathcal{A}$ satisfies the conditions of the SDD_2 tensor. By Theorem 2.3, we can get that $\mathcal{A}$ is a strong $\mathcal{H}$-tensor.

Additionally, another example is provided to demonstrate the positive definiteness of an even-degree homogeneous polynomial.

Example 2.4. *Consider the following 4th-degree homogeneous polynomial*

$$f(x)=\mathcal{A}x^4=14x_1^4+12x_2^4+20x_3^4+19x_4^4-8x_1^3x_4+12x_1x_3^2x_4-12x_2x_3x_4^2+24x_1x_2x_3x_4,$$

where $x=(x_1,x_2,x_3,x_4)^T$. Then we can obtain a symmetric tensor $\mathcal{A}=(a_{ijkl})\in\mathbb{R}^{[4,4]}$, where

$$\begin{aligned}a_{1111}=14,a_{2222}=12,a_{3333}=20,a_{4444}=19,\\ a_{1114}=a_{1141}=a_{1411}=a_{4111}=-2,\\ a_{1334}=a_{1343}=a_{1433}=a_{4133}=a_{4313}=a_{4331}=1,\\ a_{3314}=a_{3341}=a_{3413}=a_{3143}=a_{3134}=a_{3431}=1,\\ a_{2344}=a_{3244}=a_{2443}=a_{3442}=a_{3424}=a_{2434}=-1,\\ a_{4423}=a_{4432}=a_{4234}=a_{4324}=a_{4342}=a_{4243}=-1,\\ a_{1234}=a_{1243}=a_{1324}=a_{1342}=a_{1423}=a_{1432}=1,\\ a_{2134}=a_{2143}=a_{2314}=a_{2341}=a_{2413}=a_{2431}=1,\\ a_{3124}=a_{3142}=a_{3214}=a_{3241}=a_{3412}=a_{3421}=1,\\ a_{4123}=a_{4132}=a_{4213}=a_{4231}=a_{4312}=a_{4321}=1,\end{aligned}$$

and others are zeros. Then,

$$|a_{1111}|=14,\ r_1(\mathcal{A})=15,\ |a_{2222}|=12,\ r_2(\mathcal{A})=9,$$

$$|a_{3333}|=20,\ r_3(\mathcal{A})=15,\ |a_{4444}|=19,\ r_4(\mathcal{A})=17,$$

hence $N_A=\{1\}$, $N_B=\{2,3,4\}$. By calculation, we obtain

$$\max_{j\in N_B}\left\{\frac{r_j(\mathcal{A})}{|a_{jjjj}|}\right\}=\frac{17}{19},$$

when $i \in N_B$, we obtain

$$
\begin{aligned}
p_2(\mathcal{A}) =& \sum_{i_2i_3i_4 \in N_A^3} |a_{2i_2i_3i_4}| + \sum_{\substack{i_2i_3i_4 \in N_B^3 \\ \delta_{2i_2i_3i_4}=0}} \max_{j\in\{i_2,i_3,i_4\}} \left\{\frac{r_j(\mathcal{A})}{|a_{jjjj}|}\right\} |a_{2i_2i_3i_4}| + \sum_{i_2i_3i_4 \in N_C^3} |a_{2i_2i_3i_4}| \\
=& 0 + \frac{51}{19} + 6 = \frac{165}{19}. \\
p_3(\mathcal{A}) =& \sum_{i_2i_3i_4 \in N_A^3} |a_{3i_2i_3i_4}| + \sum_{\substack{i_2i_3i_4 \in N_B^3 \\ \delta_{3i_2i_3i_4}=0}} \max_{j\in\{i_2,i_3,i_4\}} \left\{\frac{r_j(\mathcal{A})}{|a_{jjjj}|}\right\} |a_{3i_2i_3i_4}| + \sum_{i_2i_3i_4 \in N_C^3} |a_{3i_2i_3i_4}| \\
=& 0 + \frac{51}{19} + 12 = \frac{279}{19}. \\
p_4(\mathcal{A}) =& \sum_{i_2i_3i_4 \in N_A^3} |a_{4i_2i_3i_4}| + \sum_{\substack{i_2i_3i_4 \in N_B^3 \\ \delta_{4i_2i_3i_4}=0}} \max_{j\in\{i_2,i_3,i_4\}} \left\{\frac{r_j(\mathcal{A})}{|a_{jjjj}|}\right\} |a_{4i_2i_3i_4}| + \sum_{i_2i_3i_4 \in N_C^3} |a_{4i_2i_3i_4}| \\
=& 2 + \frac{102}{19} + 9 = \frac{311}{19}.
\end{aligned}
$$

Furthermore, we obtain

$$\max_{j\in N_B} \left\{\frac{p_j(\mathcal{A})}{|a_{jjjj}|}\right\} = \frac{311}{361},$$

when $i \in N_A$,

$$
\begin{aligned}
q_1(\mathcal{A}) =& \sum_{\substack{i_2i_3i_4 \in N_A^3 \\ \delta_{1i_2i_3i_4}=0}} |a_{1i_2i_3i_4}| + \sum_{i_2i_3i_4 \in N_B^3} \max_{j\in\{i_2,i_3,i_4\}} \left\{\frac{p_j(\mathcal{A})}{|a_{jjjj}|}\right\} |a_{1i_2i_3i_4}| + \sum_{i_2i_3i_4 \in N_C^3} |a_{1i_2i_3i_4}| \\
=& 0 + \frac{2799}{361} + 6 = \frac{4965}{361} < 14 = |a_{1111}|.
\end{aligned}
$$

Therefore, by the definition of an SDD_2 tensor, $\mathcal{A}$ is an SDD_2 tensor. According to Theorem 2.3, $\mathcal{A}$ also is a strong $\mathcal{H}$-tensor. Moreover, all its diagonal elements are positive. Furthermore, by applying Theorem 2.4, we conclude that $\mathcal{A}$ is positive definite, and consequently, $f(x)$ is positive definite.

3. B_2-tensor and its properties

In this section, we first introduce a new class of tensors, termed B_2-tensor, which is based on the SDD_2 tensor. This new class of tensors encompasses B-tensors and B_1-tensors as its subclass. Subsequently, we present several properties of B_2-tensors. For convenience, some notations, definitions, theorems, and lemmas are provided as follows:

Given a tensor $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$, for each row i, we denote

$$r_i^+(\mathcal{A}) = \max\{0, a_{ij_2\cdots j_m} : (j_2, \ldots, j_m) \neq (i, \ldots, i)\}. \tag{3.1}$$

Let $\mathcal{B}^+ = (b_{i_1i_2\cdots i_m})$ be the tensor defined as

$$b_{i_1i_2\cdots i_m} = a_{i_1i_2\cdots i_m} - r_{i_1}^+(\mathcal{A}), \tag{3.2}$$

clearly, $\mathcal{B}^+$ is a Z-tensor. A tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$ is called a Z-tensor if and only if $a_{i_1 i_2 \cdots i_m} \leq 0$ for all $(i_1, \ldots, i_m) \neq (i, \ldots, i)$ [15].

Definition 3.1. *[8] A tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ is called a B-tensor if and only if for all $i \in N$,*

$$\sum_{i_2,\ldots,i_m \in N} a_{ii_2 \cdots i_m} > 0,$$

and

$$\frac{1}{n^{m-1}} \left(\sum_{i_2,\ldots,i_m \in N} a_{ii_2 \cdots i_m} \right) > a_{ij_2 \cdots j_m}, \ \forall (j_2, \ldots, j_m) \neq (i, \ldots, i).$$

In this section, we reviewed the concept of B-tensor. Furthermore, in [8], it was also proved that a B-tensor can be characterized by the following equivalent definition.

Definition 3.2. *[8] A tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ is called a B-tensor if and only if for each $i \in N$,*

$$\sum_{i_2 \cdots i_m \in N^{m-1}} |a_{ii_2 \cdots i_m}| > n^{m-1} r_i^+(\mathcal{A}),$$

i.e.,

$$(a_{ii \cdots i} - r_i^+(\mathcal{A})) > \sum_{(i_2,\ldots,i_m) \neq (i,\ldots,i)} (r_i^+(\mathcal{A}) - a_{ii_2 \cdots i_m}) = r_i(\mathcal{B}^+).$$

Definition 3.3. *[6] A tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ is a B_1-tensor if for all $i \in N$,*

$$a_{i \cdots i} - r_i^+(\mathcal{A}) > p_i(\mathcal{B}^+).$$

Lemma 3.1. *[6] If a tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ is a B-tensor, then $\mathcal{A}$ is a B_1-tensor.*

Now, we give the definition of a B_2-tensor.

Definition 3.4. *A tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ is a B_2-tensor if for all $i \in N$,*

$$a_{i \cdots i} - r_i^+(\mathcal{A}) > q_i(\mathcal{B}^+).$$

Next, we introduce some useful properties of a B_2-tensors.

Proposition 3.1. *If a tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ is a B_1-tensor, then $\mathcal{A}$ is a B_2-tensor.*

Proof. If $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ is a B_1-tensor, and we have $r_i(\mathcal{B}^+) \geq p_i(\mathcal{B}^+) \geq q_i(\mathcal{B}^+)$, then by Definition 3.3,

$$a_{i \cdots i} - r_i^+(\mathcal{A}) > p_i(\mathcal{B}^+) \geq q_i(\mathcal{B}^+),$$

that is, $\mathcal{A}$ is a B_2-tensor. □

Remark 3.1. *From Lemma 3.1 and Proposition 3.1, it is evident that the B_2-tensors encompass the B_1-tensors, and the B_1-tensors encompass the B-tensors; that is,*

$$\{B\text{-}tensors\} \subseteq \{B_1\text{-}tensors\} \subseteq \{B_2\text{-}tensors\}.$$

Proposition 3.2. *Let tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ be a B_2-tensor; then $\mathcal{B}^+$ is an SDD_2 tensor.*

Proof. Since $b_{i_1i_2\cdots i_m} = a_{i_1i_2\cdots i_m} - r^+_{i_1}(\mathcal{A}) > q_{i_1}(\mathcal{B}^+)$, we have $\left|b_{i_1i_2\cdots i_m}\right| > q_{i_1}(\mathcal{B}^+)$, so $\mathcal{B}^+$ is an SDD_2 tensor. □

Proposition 3.3. *If tensor $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$ is a B_2-tensor, then $\mathcal{B}^+$ has positive diagonal elements.*

Proof. If tensor $\mathcal{A} = (a_{i_1i_2\cdots i_m})$ is a B_2-tensor, then it follows that $a_{i\cdots i} - r^+_i(\mathcal{A}) > q_i(\mathcal{B}^+) \geq 0$, which implies $a_{i\cdots i} - r^+_i(\mathcal{A}) > 0$. The proof is complete. □

From Proposition 3.3, we can easily obtain the following corollary.

Corollary 3.1. *If tensor $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$ is a B_2-tensor, then there must be $a_{i\cdots i} > r^+_i(\mathcal{A})$.*

Proposition 3.4. *A tensor $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$ is a B_2-tensor if and only if $\mathcal{B}^+$ is an SDD_2 tensor with positive diagonal entries.*

Proof. If the tensor $\mathcal{A} = (a_{i_1i_2\cdots i_m})$ is a B_2-tensor, then by Propositions 3.2 and 3.3, $\mathcal{B}^+$ is an SDD_2 tensor with positive diagonal entries. Conversely, if $\mathcal{B}^+$ is an SDD_2 tensor, then $|a_{i\cdots i} - r^+_i(\mathcal{A})| > q_i(\mathcal{B}^+)$. Since $\mathcal{B}^+$ has positive diagonal entries, i.e., $a_{i\cdots i} - r^+_i(\mathcal{A}) > 0$, it follows that $a_{i\cdots i} - r^+_i(\mathcal{A}) > q_i(\mathcal{B}^+)$. Therefore, $\mathcal{A}$ is B_2-tensor. □

Proposition 3.5. *Let $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$ be a Z-tensor with positive diagonal entries. Then $\mathcal{A}$ is a B_2-tensor if and only if $\mathcal{A}$ is an SDD_2 tensor.*

Proof. Since $\mathcal{A} = (a_{i_1i_2\cdots i_m})$ is a Z-tensor, we have $r^+_i(\mathcal{A}) = 0$ for all $i \in N$, and $\mathcal{B}^+ = \mathcal{A}$. Consequently, $\mathcal{A}$ is a B_2-tensor if and only if $a_{i\cdots i} - r^+_i(\mathcal{A}) > q_i(\mathcal{B}^+)$, which simplifies to $a_{i\cdots i} > q_i(\mathcal{A})$. Therefore, $\mathcal{A}$ is an SDD_2 tensor. Conversely, if $\mathcal{A}$ is an SDD_2 tensor, we have $|a_{i\cdots i}| > q_i(\mathcal{A})$. Since $\mathcal{A}$ has positive diagonal entries, it follows that $a_{i\cdots i} > q_i(\mathcal{A})$. Thus, $a_{i\cdots i} - r^+_i(\mathcal{A}) > q_i(\mathcal{B}^+)$ holds immediately, which implies that $\mathcal{A}$ is a B_2-tensor. □

The following corollary can be obtained from Proposition 3.5 and Theorem 2.3.

Corollary 3.2. *Let $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$ be a Z-tensor with positive diagonal entries. If $\mathcal{A}$ is a B_2-tensor, then $\mathcal{A}$ is a strong $\mathcal{H}$-tensor.*

Proposition 3.6. *$\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$ is a B_2-tensor if and only if $\mathcal{B}^+$ is a B_2-tensor.*

Proof. By Proposition 3.4, $\mathcal{A} = (a_{i_1i_2\cdots i_m})$ is a B_2-tensor if and only if $\mathcal{B}^+$ is an SDD_2 tensor with positive diagonal entries. Since $\mathcal{B}^+$ is a Z-tensor, according to Proposition 3.5, the conclusion follows immediately. □

Proposition 3.7. *If $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$ is a B_2-tensor, and $\mathcal{D} \in \mathbb{R}^{[m,n]}$ is a nonnegative diagonal tensor of the same order and dimension, then $\mathcal{A} + \mathcal{D}$ is a B_2-tensor.*

Proof. Let $\mathcal{D} = (d_{i_1i_2\cdots i_m})$, where

$$d_{ii_2\cdots i_m} = \begin{cases} d_i, & (i_2, \ldots, i_m) = (i, \ldots, i), \\ 0, & otherwise, \end{cases}$$

and $d_i \geq 0$. Let $C = \mathcal{A} + \mathcal{D}$, where

$$c_{ii_2\cdots i_m} = \begin{cases} a_{i\cdots i} + d_i, & (i_2, \ldots, i_m) = (i, \ldots, i), \\ a_{ii_2\cdots i_m}, & otherwise. \end{cases}$$

Then, $\mathcal{A}$ and C have the same nondiagonal elements, so that $r_i^+(\mathcal{A}) = r_i^+(C)$.

Next, let us prove $c_{i\cdots i} - r_i^+(C) > q_i(C^+)$ for all $i \in N$, where $C^+ = (v_{i_1 i_2\cdots i_m})$ with $v_{ii_2\cdots i_m} = c_{ii_2\cdots i_m} - r_i^+(C)$.

Since $\mathcal{A}$ is a B_2-tensor, for all $i \in N$, we have

$$c_{i\cdots i} - r_i^+(C) = a_{i\cdots i} + d_i - r_i^+(\mathcal{A}) \geq a_{i\cdots i} - r_i^+(\mathcal{A}) > q_i(\mathcal{B}^+) \geq 0. \tag{3.3}$$

Meanwhile, we have the following equation that holds:

$$r_i(C^+) = \sum_{\substack{i_2\cdots i_m \in N^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |v_{ii_2\cdots i_m}| = \sum_{\substack{i_2\cdots i_m \in N^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |c_{ii_2\cdots i_m} - r_i^+(C)| = \sum_{\substack{i_2\cdots i_m \in N^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m} - r_i^+(\mathcal{A})| = r_i(\mathcal{B}^+).$$

Hence, for any $i \in N_A(C^+)$, we have $|c_{i\cdots i} - r_i^+(C)| \leq r_i(C^+)$, i.e., $|a_{i\cdots i} + d_i - r_i^+(\mathcal{A})| \leq r_i(\mathcal{B}^+)$, we can immediately obtain that $|a_{i\cdots i} - r_i^+(\mathcal{A})| \leq r_i(\mathcal{B}^+)$. Therefore, $i \in N_A(\mathcal{B}^+)$. This indicates that $N_A(C^+) \subset N_A(\mathcal{B}^+)$; the same method can illustrate that $N_B(\mathcal{B}^+) \subset N_B(C^+)$.

For any $i \in N$,

$$\begin{aligned}
p_i(C^+) = & \sum_{\substack{i_2\cdots i_m \in N_A^{m-1}(C^+) \\ \delta_{ii_2\cdots i_m}=0}} |v_{ii_2\cdots i_m}| + \sum_{\substack{i_2\cdots i_m \in N_B^{m-1}(C^+) \\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,\cdots,i_m\}} \left\{ \frac{r_j(C^+)}{|v_{jj\cdots j}|} \right\} |v_{ii_2\cdots i_m}| \\
& + \sum_{\substack{i_2\cdots i_m \in N_C^{m-1}(C^+) \\ \delta_{ii_2\cdots i_m}=0}} |v_{ii_2\cdots i_m}| \\
= & \sum_{\substack{i_2\cdots i_m \in N_A^{m-1}(C^+) \\ \delta_{ii_2\cdots i_m}=0}} |c_{ii_2\cdots i_m} - r_i^+(C)| + \sum_{\substack{i_2\cdots i_m \in N_B^{m-1}(C^+) \\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,\cdots,i_m\}} \left\{ \frac{r_j(C^+)}{|c_{jj\cdots j} - r_j^+(C)|} \right\} |c_{ii_2\cdots i_m} - r_i^+(C)| \\
& + \sum_{\substack{i_2\cdots i_m \in N_C^{m-1}(C^+) \\ \delta_{ii_2\cdots i_m}=0}} |c_{ii_2\cdots i_m} - r_i^+(C)| \\
= & \sum_{\substack{i_2\cdots i_m \in N_A^{m-1}(C^+) \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m} - r_i^+(\mathcal{A})| + \sum_{\substack{i_2\cdots i_m \in N_B^{m-1}(C^+) \\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,\cdots,i_m\}} \left\{ \frac{r_j(\mathcal{B}^+)}{|a_{jj\cdots j} + d_j - r_j^+(\mathcal{A})|} \right\} \\
& |a_{ii_2\cdots i_m} - r_i^+(\mathcal{A})| + \sum_{\substack{i_2\cdots i_m \in N_C^{m-1}(C^+) \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m} - r_i^+(\mathcal{A})| \\
\leq & \sum_{\substack{i_2\cdots i_m \in N_A^{m-1}(\mathcal{B}^+) \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m} - r_i^+(\mathcal{A})| + \sum_{\substack{i_2\cdots i_m \in N_B^{m-1}(\mathcal{B}^+) \\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,\cdots,i_m\}} \left\{ \frac{r_j(\mathcal{B}^+)}{|a_{jj\cdots j} - r_j^+(\mathcal{A})|} \right\} \\
& |a_{ii_2\cdots i_m} - r_i^+(\mathcal{A})| + \sum_{\substack{i_2\cdots i_m \in N_C^{m-1}(\mathcal{B}^+) \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m} - r_i^+(\mathcal{A})| \\
= & p_i(\mathcal{B}^+).
\end{aligned}$$

$$
\begin{aligned}
q_i(C^+) &= \sum_{\substack{i_2\cdots i_m\in N_A^{m-1}(C^+)\\ \delta_{ii_2\cdots i_m}=0}} |v_{ii_2\cdots i_m}| + \sum_{\substack{i_2\cdots i_m\in N_B^{m-1}(C^+)\\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,\cdots,i_m\}}\left\{\frac{p_j(C^+)}{|v_{jj\cdots j}|}\right\}|v_{ii_2\cdots i_m}| \\
&\quad + \sum_{\substack{i_2\cdots i_m\in N_C^{m-1}(C^+)\\ \delta_{ii_2\cdots i_m}=0}} |v_{ii_2\cdots i_m}| \\
&= \sum_{\substack{i_2\cdots i_m\in N_A^{m-1}(C^+)\\ \delta_{ii_2\cdots i_m}=0}} |c_{ii_2\cdots i_m} - r_i^+(C)| + \sum_{\substack{i_2\cdots i_m\in N_B^{m-1}(C^+)\\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,\cdots,i_m\}}\left\{\frac{p_j(C^+)}{|c_{jj\cdots j} - r_j^+(C)|}\right\}|c_{ii_2\cdots i_m} - r_i^+(C)| \\
&\quad + \sum_{\substack{i_2\cdots i_m\in N_C^{m-1}(C^+)\\ \delta_{ii_2\cdots i_m}=0}} |c_{ii_2\cdots i_m} - r_i^+(C)| \\
&\le \sum_{\substack{i_2\cdots i_m\in N_A^{m-1}(C^+)\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m} - r_i^+(\mathcal{A})| + \sum_{\substack{i_2\cdots i_m\in N_B^{m-1}(C^+)\\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,\cdots,i_m\}}\left\{\frac{p_j(\mathcal{B}^+)}{|a_{jj\cdots j} + d_j - r_j^+(\mathcal{A})|}\right\} \\
&\quad |a_{ii_2\cdots i_m} - r_i^+(\mathcal{A})| + \sum_{\substack{i_2\cdots i_m\in N_C^{m-1}(C^+)\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m} - r_i^+(\mathcal{A})| \\
&\le \sum_{\substack{i_2\cdots i_m\in N_A^{m-1}(\mathcal{B}^+)\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m} - r_i^+(\mathcal{A})| + \sum_{\substack{i_2\cdots i_m\in N_B^{m-1}(\mathcal{B}^+)\\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,\cdots,i_m\}}\left\{\frac{p_j(\mathcal{B}^+)}{|a_{jj\cdots j} - r_j^+(\mathcal{A})|}\right\} \\
&\quad |a_{ii_2\cdots i_m} - r_i^+(\mathcal{A})| + \sum_{\substack{i_2\cdots i_m\in N_C^{m-1}(\mathcal{B}^+)\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m} - r_i^+(\mathcal{A})| \\
&= q_i(\mathcal{B}^+).
\end{aligned}
$$

Combining Eq (3.3), there is $c_{i\cdots i} - r_i^+(C) > q_i(C^+)$, and this proof is completed. □

Proposition 3.8. *If $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$ is a B_2-tensor, then we can write $\mathcal{A}$ as $\mathcal{A} = \mathcal{B} + C$, where $\mathcal{B}$ is a Z-tensor with positive diagonal entries, and C is a nonnegative tensor with $c_{ii_2\cdots i_m} = c_i,\ c_i \ge 0$, for any $i \in N$. In particular, if $\mathcal{A}$ is both a B_2-tensor and a Z-tensor, then C is a zero tensor.*

Proof. Suppose $\mathcal{B} = (b_{i_1i_2\cdots i_m})$, where $b_{ii_2\cdots i_m} = a_{ii_2\cdots i_m} - r_i^+(\mathcal{A})$. According to the definition of $r_i^+(\mathcal{A})$, it can be inferred that $b_{ii_2\cdots i_m} \le 0,\ (i_2, \ldots, i_m) \ne (i, \ldots i)$. Since $\mathcal{A}$ is a B_2-tensor, $b_{i\cdots i} = a_{i\cdots i} - r_i^+(\mathcal{A}) > q_i(\mathcal{B}) \ge 0$. Then $\mathcal{B}$ is a Z-tensor with positive diagonal entries.

Let $C = c_{ii_2\cdots i_m}$ with $c_{ii_2\cdots i_m} = r_i^+(\mathcal{A})$; obviously C is a nonnegative tensor with $c_i = r_i^+(\mathcal{A})$. In particular, if $\mathcal{A}$ is both a B_2-tensor and Z-tensor, we have $r_i^+(\mathcal{A}) = 0$, then C is a zero tensor. □

Proposition 3.9. *Let $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$ be a B_2-tensor, and let $C = (c_{i_1i_2\cdots i_m})$ be a nonnegative tensor of the form $c_{ii_2\cdots i_m} = c_i$; then $\mathcal{A} + C$ is a B_2-tensor.*

Proof. Let $\mathcal{P} = \mathcal{A} + C$, where $\mathcal{P} = (p_{i_1i_2\cdots i_m})$. By Proposition 3.6, we need to prove that $\mathcal{P}^+$ is a B_2-tensor. By definition, we can see that for all $i \in N$,

$$r_i^+(\mathcal{P}) = r_i^+(\mathcal{A}) + c_i.$$

Hence, for all $i, i_2, \ldots, i_m \in N$, we have

$$d_{ii_2\cdots i_m} = p_{ii_2\cdots i_m} - r_i^+(\mathcal{P}) = (a_{ii_2\cdots i_m} + c_i) - (r_i^+(\mathcal{A}) + c_i) = a_{ii_2\cdots i_m} - r_i^+(\mathcal{A}),$$

then we obtain that $\mathcal{P}^+ = \mathcal{B}^+$. Since $\mathcal{A}$ is a B_2-tensor, by Proposition 3.6, $\mathcal{B}^+$ is a B_2-tensor. The conclusion follows immediately. □

Proposition 3.10. *If $\mathcal{A} = (a_{i_1 i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$ is a B_2-tensor, then every principal sub-tensor of $\mathcal{A}$ is also a B_2-tensor.*

Proof. Let T be a nonempty subset of N with $|T| = r$, and let $C = \mathcal{A}_r^T \in \mathbb{R}^{[m,r]}$ be the principal sub-tensor of $\mathcal{A}$. Since $\mathcal{A}$ is a B_2-tensor, we have $a_{i\cdots i} - r_i^+(\mathcal{A}) > q_i(\mathcal{B}^+) \geq 0$. Consequently, $a_{i\cdots i} > r_i^+(\mathcal{A}) \geq r_i^+(C)$. Next, we prove that $a_{i\cdots i} - r_i^+(C) > q_i(C^+)$ for all $i \in T$. We have

$$\begin{aligned} r_i(\mathcal{B}^+) &= \sum_{\substack{i_2\cdots i_m \in N^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m} - r_i^+(\mathcal{A})| = \sum_{\substack{i_2\cdots i_m \in N^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} (r_i^+(\mathcal{A}) - a_{ii_2\cdots i_m}) \\ &\geq \sum_{\substack{i_2\cdots i_m \in T^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} (r_i^+(C) - a_{ii_2\cdots i_m}) = r_i(C^+). \end{aligned}$$

For any $i \in N_A(C^+)$, we have $0 \leq a_{i\cdots i} - r_i^+(C) \leq r_i(C^+) \leq r_i(\mathcal{B}^+)$, hence $0 \leq a_{i\cdots i} - r_i^+(\mathcal{A}) \leq r_i(\mathcal{B}^+)$. Therefore, $i \in N_A(\mathcal{B}^+)$. This indicates that $N_A(C^+) \subset N_A(\mathcal{B}^+)$. The same method can illustrate that $N_B(\mathcal{B}^+) \subset N_B(C^+)$.
For any $i \in T$,

$$\begin{aligned} p_i(\mathcal{B}^+) = &\sum_{\substack{i_2\cdots i_m \in N_A^{m-1}(\mathcal{B}^+) \\ \delta_{ii_2\cdots i_m}=0}} |b_{ii_2\cdots i_m}| + \sum_{\substack{i_2\cdots i_m \in N_B^{m-1}(\mathcal{B}^+) \\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,\cdots,i_m\}} \left\{ \frac{r_j(\mathcal{B}^+)}{|b_{jj\cdots j}|} \right\} |b_{ii_2\cdots i_m}| \\ &+ \sum_{\substack{i_2\cdots i_m \in N_C^{m-1}(\mathcal{B}^+) \\ \delta_{ii_2\cdots i_m}=0}} |b_{ii_2\cdots i_m}| \\ = &\sum_{\substack{i_2\cdots i_m \in N_A^{m-1}(\mathcal{B}^+) \\ \delta_{ii_2\cdots i_m}=0}} (r_i^+(\mathcal{A}) - a_{ii_2\cdots i_m}) + \sum_{\substack{i_2\cdots i_m \in N_B^{m-1}(\mathcal{B}^+) \\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,\cdots,i_m\}} \left\{ \frac{r_j(\mathcal{B}^+)}{a_{jj\cdots j} - r_j^+(\mathcal{A})} \right\} \\ &(r_i^+(\mathcal{A}) - a_{ii_2\cdots i_m}) + \sum_{\substack{i_2\cdots i_m \in N_C^{m-1}(\mathcal{A}^+) \\ \delta_{ii_2\cdots i_m}=0}} (r_i^+(\mathcal{A}) - a_{ii_2\cdots i_m}) \\ \geq &\sum_{\substack{i_2\cdots i_m \in N_A^{m-1}(C^+) \\ \delta_{ii_2\cdots i_m}=0}} (r_i^+(C) - a_{ii_2\cdots i_m}) + \sum_{\substack{i_2\cdots i_m \in N_B^{m-1}(C^+) \\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,\cdots,i_m\}} \left\{ \frac{r_j(C^+)}{a_{jj\cdots j} - r_j^+(\mathcal{A})} \right\} \\ &(r_i^+(C) - a_{ii_2\cdots i_m}) + \sum_{\substack{i_2\cdots i_m \in N_C^{m-1}(C^+) \\ \delta_{ii_2\cdots i_m}=0}} (r_i^+(C) - a_{ii_2\cdots i_m}) \\ = &p_i(C^+). \end{aligned}$$

So,

$$
\begin{aligned}
a_{i\cdots i} - r_i^+(\mathcal{C}) \geq & a_{i\cdots i} - r_i^+(\mathcal{A}) > q_i(\mathcal{B}^+) \\
= & \sum_{\substack{i_2\cdots i_m \in N_A^{m-1}(\mathcal{B}^+) \\ \delta_{ii_2\cdots i_m}=0}} |b_{ii_2\cdots i_m}| + \sum_{\substack{i_2\cdots i_m \in N_B^{m-1}(\mathcal{B}^+) \\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,\cdots,i_m\}} \left\{ \frac{p_j(\mathcal{B}^+)}{|b_{jj\cdots j}|} \right\} |b_{ii_2\cdots i_m}| \\
& + \sum_{\substack{i_2\cdots i_m \in N_C^{m-1}(\mathcal{B}^+) \\ \delta_{ii_2\cdots i_m}=0}} |b_{ii_2\cdots i_m}| \\
= & \sum_{\substack{i_2\cdots i_m \in N_A^{m-1}(\mathcal{B}^+) \\ \delta_{ii_2\cdots i_m}=0}} (r_i^+(\mathcal{A}) - a_{ii_2\cdots i_m}) + \sum_{\substack{i_2\cdots i_m \in N_B^{m-1}(\mathcal{B}^+) \\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,\cdots,i_m\}} \left\{ \frac{p_j(\mathcal{B}^+)}{a_{jj\cdots j} - r_j^+(\mathcal{A})} \right\} \\
& (r_i^+(\mathcal{A}) - a_{ii_2\cdots i_m}) + \sum_{\substack{i_2\cdots i_m \in N_C^{m-1}(\mathcal{A}^+) \\ \delta_{ii_2\cdots i_m}=0}} (r_i^+(\mathcal{A}) - a_{ii_2\cdots i_m}) \\
\geq & \sum_{\substack{i_2\cdots i_m \in N_A^{m-1}(\mathcal{C}^+) \\ \delta_{ii_2\cdots i_m}=0}} (r_i^+(\mathcal{C}) - a_{ii_2\cdots i_m}) + \sum_{\substack{i_2\cdots i_m \in N_B^{m-1}(\mathcal{C}^+) \\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,\cdots,i_m\}} \left\{ \frac{p_j(\mathcal{C}^+)}{a_{jj\cdots j} - r_j^+(\mathcal{A})} \right\} \\
& (r_i^+(\mathcal{C}) - a_{ii_2\cdots i_m}) + \sum_{\substack{i_2\cdots i_m \in N_C^{m-1}(\mathcal{C}^+) \\ \delta_{ii_2\cdots i_m}=0}} (r_i^+(\mathcal{C}) - a_{ii_2\cdots i_m}) \\
= & q_i(\mathcal{C}^+).
\end{aligned}
$$

In summary, we have $a_{i\cdots i} - r_i^+(\mathcal{C}) > q_i(\mathcal{C}^+)$, and this proof is completed. □

4. Conclusions

In this paper, we mainly introduce a novel class of tensors, which we named SDD_2 tensors, inspired by the SDD_1 tensors. We demonstrate that the SDD_2 tensor is a subclass of strong $\mathcal{H}$-tensor and give an application of the SDD_2 tensor to the determination of the positive definiteness of even-order real symmetric tensors. The validity of our results is supported by illustrated examples. Furthermore, we extend the concept of the SDD_2 tensor to the B-tensor, thereby introducing the B_2-tensor. Finally, we proposed several properties of the B_2-tensor. Regarding the error bounds for the linear complementarity problems based on strong $\mathcal{H}$-tensors. In the future, we can conduct similar research on SDD_2 tensors and B_2-tensors.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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