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Research article

On three-dimensional system of rational difference equations with second-order

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Abstract: This work deals with the dynamical features of the system for three-dimensional difference equations

$$\begin{cases} u_{n+1} = \alpha + \frac{u_{n-1}^q}{v_n^q}, \\ v_{n+1} = \alpha + \frac{v_{n-1}^q}{w_n^q}, \qquad n = 0, 1, \cdots, \\ w_{n+1} = \alpha + \frac{w_{n-1}^q}{u_n^q}, \end{cases}$$

where the initial values $u_i, v_i, w_i \in (0, \infty), i \in \{-1, 0\}$, and the parameters $\alpha > 0, q \geq 1$. In detail, the local asymptotical stability of the positive equilibrium point, boundedness, persistence, and oscillation behavior of the positive solution for the systems are obtained. Furthermore, using Matlab software, we give some examples to show the validity of theoretic analysis.

Keywords: system of difference equation; equilibrium point; oscillation; stability; boundedness

1. Introduction

Difference equation, or discrete dynamical system, is a useful mathematical tool describing natural phenomena in economics, population dynamics, genetics, computer science, control theory, etc. [1–4]. It has been an important research topic that affects every aspect of mathematics and applied mathematics. In the past decades, there has been many results on the dynamical behavior of solutions such as asymptotic stability of equilibriums, oscillation, and periodic solutions for the nonlinear difference system (see [5–8]).

For instance, in 1998, Papaschinopoulos and Schinas [9] explored the following system.

$$y_{n+1} = B + \frac{z_n}{y_{n-s}}, \quad z_{n+1} = B + \frac{y_n}{z_{n-t}}, \quad n = 0, 1, \cdots,$$

here $B \in (0, \infty)$ and the initial values are $y_i \in (0, \infty), i \in \{-s, \cdots, 0\}, z_j \in (0, \infty), j \in \{-t, \cdots, 0\}$, $s, t \in Z^+$(positive integer).

In 2000, Papaschinopoulos and Schinas [10] considered the following two-dimensional system.

$$u_{n+1} = \delta + \frac{u_{n-1}}{v_n}, \quad v_{n+1} = \delta + \frac{v_{n-1}}{u_n}, \quad n \in N,$$

here $\delta > 0$, and the initial values are $u_i \in (0, \infty), v_i \in (0, \infty), i \in \{-1, 0\}$.

In 2012, Zhang et al. [11] discussed the qualitative feature for a rational difference system.

$$u_{n+1} = \frac{u_{n-2}}{\delta + v_{n-2}v_{n-1}v_n}, \quad v_{n+1} = \frac{v_{n-2}}{\gamma + u_{n-2}u_{n-1}u_n}, \quad n \in N,$$

where the initial values $u_i \in (0, \infty), v_i \in (0, \infty), i \in \{-2, -1, 0\}$, and the parameters $\delta, \gamma \in (0, \infty)$.

In 2015, Bao [12] considered boundedness, oscillation, and the local stability of the system.

$$u_{n+1} = \delta + \frac{u_{n-1}^m}{v_n^m}, \quad v_{n+1} = \delta + \frac{v_{n-1}^m}{u_n^m}, \quad n = 0, 1, \cdots,$$

where the parameters $\delta \in (0, \infty), m \geq 1$, and the initial values $u_i \in (0, \infty), v_i \in (0, \infty), i \in \{-1, 0\}$.

In 2018, Inci and Yüksel [13] investigated the boundedness, persistence, periodicity, and the global asymptotic features of the three-dimensional system

$$\alpha_{n+1} = \eta + \frac{\alpha_{n-1}}{\gamma_n}, \quad \beta_{n+1} = \eta + \frac{\beta_{n-1}}{\gamma_n}, \quad \gamma_{n+1} = \eta + \frac{\gamma_{n-1}}{\beta_n}, \quad n \in N,$$

where $\eta \in (0, \infty)$ and the initial values $\alpha_i, \beta_i, \gamma_i \in (0, \infty), i \in \{-1, 0\}$.

In 2023, Abdul et al. [14] considered the persistence, boundedness, and local and global features of a second-order system.

$$\begin{cases} u_{n+1} = a + bu_{n-1} + cu_{n-1}e^{-v_n}, \\ v_{n+1} = \alpha + \beta v_{n-1} + \gamma v_{n-1}e^{-w_n}, \quad n \in N, \\ w_{n+1} = \eta + \delta w_{n-1} + \vartheta w_{n-1}e^{-u_n}, \end{cases}$$

where the parameters $a, b, c, \alpha, \beta, \gamma, \eta, \delta, \vartheta$ are positive real numbers, and the initial values $u_i, v_i, w_i \in (0, \infty), i \in \{-1, 0\}$. Readers can refer to other works [15, 16].

Inspired by the above-mentioned publications, in this paper, we will extend the system ([12, 13]) to a three-dimensional system

$$\begin{cases} u_{n+1} = \alpha + \frac{u_{n-1}^q}{v_n^q}, \\ v_{n+1} = \alpha + \frac{v_{n-1}^q}{w_n^q}, \quad n = 0, 1, \cdots, \\ w_{n+1} = \alpha + \frac{w_{n-1}^q}{u_n^q}, \end{cases} \tag{1.1}$$

where the initial values $u_i, v_i, w_i \in (0, \infty), i \in \{-1, 0\}$, and the parameters $q \in [1, \infty), \alpha \in (0, \infty)$.

The main aim of this article is to discuss the boundedness, persistence, and asymptotic features of positive solutions for (1.1). The organization of this article is as follows. Section 2 gives preliminaries and some definitions. In Section 3, we explore theoretical results, including the boundedness, persistence, oscillation, and asymptotic features of system (1.1). Section 4 presents some examples to show the validity of theoretic findings. In Section 5, we draw a general conclusion and give our future works.

2. Preliminaries

In this section, firstly, we give some definitions and a basic theorem that are applied in the next section.

Consider the following abstract discrete dynamical system

$$\begin{cases} u_{n+1} = \phi(u_n, u_{n-1}, v_n, v_{n-1}, w_n, w_{n-1}), \\ v_{n+1} = \psi(u_n, u_{n-1}, v_n, v_{n-1}, w_n, w_{n-1}), \quad n \in N, \\ w_{n+1} = h(u_n, u_{n-1}, v_n, v_{n-1}, w_n, w_{n-1}), \end{cases} \tag{2.1}$$

where $\phi : I_1^2 \times I_2^2 \times I_3^2 \to I_1, \psi : I_1^2 \times I_2^2 \times I_3^2 \to I_2$ and $h : I_1^2 \times I_2^2 \times I_3^2 \to I_3$ are C^1-continuous and differentiable functions, and $I_i (i = 1, 2, 3)$ are some real intervals. It is well known that the solution $\{(u_n, v_n, w_n)\}$ of (2.1) is ascertained by the initial values $(u_i, v_i, w_i) \in I_1 \times I_2 \times I_3$ for $i \in \{0, -1\}$.

Definition 2.1. A point $(\bar{u}, \bar{v}, \bar{w})$ is said to be an equilibrium of system (2.1) provided that

$$\begin{cases} \bar{u} = \phi(\bar{u}, \bar{u}, \bar{v}, \bar{v}, \bar{w}, \bar{w}), \\ \bar{v} = \psi(\bar{u}, \bar{u}, \bar{v}, \bar{v}, \bar{w}, \bar{w}), \\ \bar{w} = h(\bar{u}, \bar{u}, \bar{v}, \bar{v}, \bar{w}, \bar{w}). \end{cases}$$

Also, let $F = (\phi, \psi, h)$, then $(\bar{u}, \bar{v}, \bar{w})$ is said to be a fixed point of F.

Definition 2.2. [3] Suppose that $(\bar{u}, \bar{v}, \bar{w})$ is an equilibrium of system (2.1) and $I_i, i \in \{1, 2, 3\}$, is a real interval.

(i) $(\bar{u}, \bar{v}, \bar{w})$ is stable, provided that $\forall \varepsilon > 0$, $\exists \gamma > 0$, for any initial value $(u_i, v_i, w_i) \in I_1 \times I_2 \times I_3, i \in \{-1, 0\}$, with

$$\sum_{i=-1}^{0} | u_i - \bar{u} | < \gamma, \ \sum_{i=-1}^{0} | v_i - \bar{v} | < \gamma, \ \sum_{i=-1}^{0} | w_i - \bar{w} | < \gamma,$$

deducing $| u_n - \bar{u} | < \varepsilon, | v_n - \bar{v} | < \varepsilon, | w_n - \bar{w} | < \varepsilon$ as $n \geq 1$.

(ii) $(\bar{u}, \bar{v}, \bar{w})$ is LAS (locally asymptotically stable), provided that it is stable, and also $\exists \sigma > 0$ satisfying

$$\sum_{i=-1}^{0} | u_i - \bar{u} | < \sigma, \ \sum_{i=-1}^{0} | v_i - \bar{v} | < \sigma, \ \sum_{i=-1}^{0} | w_i - \bar{w} | < \sigma,$$

and $(u_n, v_n, w_n) \to (\bar{u}, \bar{v}, \bar{w})$ as $n \to \infty$.
(iii) $(\bar{u}, \bar{v}, \bar{w})$ is a global attractor if $(u_n, v_n, w_n) \to (\bar{u}, \bar{v}, \bar{w})$ as $n \to \infty$.
(iv) $(\bar{u}, \bar{v}, \bar{w})$ is GAS (globally asymptotically stable), provided that it is stable and a global attractor.
(v) $(\bar{u}, \bar{v}, \bar{w})$ is unstable, provided that it is not stable.

Let $\overline{U} = (\bar{u}, \bar{v}, \bar{w})$ be an equilibrium of $F = (\phi, \psi, h)$, here $\phi, \psi, h \in C^1$. The associated linearized system of (2.1) at the equilibrium $\overline{U}$ is written as

$$U_{n+1} = F(U_n) = GU_n,$$

where $U_n = (u_n, u_{n-1}, v_n, v_{n-1}, w_n, w_{n-1})^T$, and G is a Jacobian matrix of (2.1) at the equilibrium $\overline{U} = (\bar{u}, \bar{v}, \bar{w})$.

Theorem 2.1. [3] Let

$$U_{n+1} = F(U_n) = GU_n$$

be a linear system of (2.1) at the equilibrium $\overline{U}$.
(i) If all the characteristic values of the Jacobian matrix G at the equilibrium $\overline{U}$ lies inside the open unit disk, i.e., $|\lambda| < 1$, then $\overline{U}$ is LAS.
(ii) If at least one of the characteristic values of the Jacobian matrix G at the equilibrium $\overline{U}$ lies outside the open unit disk, i.e., $|\lambda| > 1$, then $\overline{U}$ is unstable.

Definition 2.3. A positive solution $\{(u_n, v_n, w_n)\}_{n=-1}^{\infty}$ of system (2.1) is persistent and bounded, provided that $\exists V > 0, W > 0$ satisfying

$$V \le u_n \le W, \quad V \le v_n \le W, \quad V \le w_n \le W, \quad n \ge -1.$$

3. Main results

In this section, we will explore qualitative features of the positive solution of system (1.1).

Theorem 3.1. *Consider system (1.1): the following assertions hold true.*
(i) There is a positive equilibrium $(\bar{u}, \bar{v}, \bar{w}) = (\alpha + 1, \alpha + 1, \alpha + 1)$ *for system (1.1).*
(ii) The positive equilibrium $(\alpha + 1, \alpha + 1, \alpha + 1)$ *of system (1.1) is LAS provided that* $\alpha > 2q - 1$.
(iii) The positive equilibrium $(\alpha + 1, \alpha + 1, \alpha + 1)$ *of system (1.1) is unstable provided that* $0 < \alpha < 2q - 1$.

Proof. (i) Suppose that u, v, w are positive numbers satisfying

$$\begin{cases} u = \alpha + \frac{u^q}{v^q}, \\ v = \alpha + \frac{v^q}{w^q}, \\ w = \alpha + \frac{w^q}{u^q}. \end{cases} \tag{3.1}$$

From this, one has the positive equilibrium $(\bar{u}, \bar{v}, \bar{w}) = (\alpha + 1, \alpha + 1, \alpha + 1)$.

(ii) The linearized equation of system (1.1) at the positive equilibrium $(\alpha + 1, \alpha + 1, \alpha + 1)$ is

$$U_{n+1} = GU_n, \tag{3.2}$$

in which

$$U_n = \begin{pmatrix} u_n \\ u_{n-1} \\ v_n \\ v_{n-1} \\ w_n \\ w_{n-1} \end{pmatrix}, \quad G = \begin{pmatrix} 0 & \frac{q}{\alpha+1} & -\frac{q}{\alpha+1} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{q}{\alpha+1} & -\frac{q}{\alpha+1} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{q}{\alpha+1} & 0 & 0 & 0 & 0 & \frac{q}{\alpha+1} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let $\nu_i, i \in \{1, \cdots, 6\}$ be the characteristic value of matrix G, and $L = \operatorname{diag}(l_1, \cdots, l_6)$ is a diagonal matrix, where $l_1 = l_3 = l_5 = 1, l_k = 1 - k\varepsilon, k = 2, 4, 6$, and

$$0 < \varepsilon < \frac{1}{6}\left(1 - \frac{q}{\alpha + 1 - q}\right). \tag{3.3}$$

Since L is invertible, calculating LGL^{-1}, we obtain

$$LGL^{-1} = \begin{pmatrix} 0 & \frac{ql_1l_2^{-1}}{\alpha+1} & -\frac{ql_1l_3^{-1}}{\alpha+1} & 0 & 0 & 0 \\ l_2l_1^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{ql_3l_4^{-1}}{\alpha+1} & -\frac{ql_3l_5^{-1}}{\alpha+1} & 0 \\ 0 & 0 & l_4l_3^{-1} & 0 & 0 & 0 \\ -\frac{ql_5l_1^{-1}}{\alpha+1} & 0 & 0 & 0 & 0 & \frac{ql_5l_6^{-1}}{\alpha+1} \\ 0 & 0 & 0 & 0 & l_6l_5^{-1} & 0 \end{pmatrix}.$$

It is clear that

$$l_2l_1^{-1} < 1, \; l_4l_3^{-1} < 1, \;\; l_6l_5^{-1} < 1. \tag{3.4}$$

Furthermore, from (3.3), one has that

$$\frac{ql_1l_2^{-1}}{\alpha+1} + \frac{ql_1l_3^{-1}}{\alpha+1} = \frac{q}{\alpha+1}\left(1 + \frac{1}{l_2}\right) = \frac{q}{\alpha+1}\left(1 + \frac{1}{1-2\varepsilon}\right) < 1,$$

$$\frac{ql_3l_4^{-1}}{\alpha+1} + \frac{ql_3l_5^{-1}}{\alpha+1} = \frac{q}{\alpha+1}\left(1 + \frac{1}{l_4}\right) = \frac{q}{\alpha+1}\left(1 + \frac{1}{1-4\varepsilon}\right) < 1,$$

$$\frac{ql_5l_1^{-1}}{\alpha+1} + \frac{ql_5l_6^{-1}}{\alpha+1} = \frac{q}{\alpha+1}\left(1 + \frac{1}{l_6}\right) = \frac{q}{\alpha+1}\left(1 + \frac{1}{1-6\varepsilon}\right) < 1.$$

Since G has the same eigenvalues as LGL^{-1}, one has

$$\begin{aligned} \max_{1\le i\le 6} |\nu_i| &\le \|LGL^{-1}\| \\ &= \max\left\{l_2l_1^{-1}, l_4l_3^{-1}, l_6l_5^{-1}, \frac{q}{\alpha+1}\left(1+\frac{1}{l_2}\right),\right. \\ &\qquad \left.\frac{q}{\alpha+1}\left(1+\frac{1}{l_4}\right), \frac{q}{\alpha+1}\left(1+\frac{1}{l_6}\right)\right\} \\ &< 1. \end{aligned}$$

It implies that the fixed point $(\alpha + 1, \alpha + 1, \alpha + 1)$ of system (1.1) is LAS.

(iii) It is true from the proof of (ii).

Remark 3.1. In [12], Bao studied the dynamical behaviours of two-dimensional difference equations. In [13], Inci Okumu and Yüksel Soykan investigated the global asymptotic stability of three-dimensional difference equations. Compared with some literatures [12, 13], the results obtained in this paper are an extension of existing results.

Theorem 3.2. *Let $\alpha \in (0, 1)$, and suppose that $\{(u_n, v_n, w_n)\}$ is a positive solution of system (1.1). Then the following assertions hold true.*
(i) If

$$\Theta_{-1} \in (0, 1), \quad \Theta_0 \in \left(\frac{1}{(1-\alpha)^{1/q}}, +\infty\right), \; \Theta \in \{u, v, w\}, \tag{3.5}$$

then

$$\lim_{n\to\infty} \Theta_{2n} = \infty, \quad \lim_{n\to\infty} \Theta_{2n+1} = \alpha, \quad \Theta \in \{u, v, w\}.$$

(ii) If

$$\Theta_{-1} \in \left(\frac{1}{(1-\alpha)^{1/q}}, +\infty\right), \quad \Theta_0 \in (0, 1), \quad \Theta \in \{u, v, w\}, \tag{3.6}$$

then

$$\lim_{n\to\infty} \Theta_{2n} = \alpha, \quad \lim_{n\to\infty} \Theta_{2n+1} = \infty, \quad \Theta \in \{u, v, w\}.$$

Proof. (i) Since $\alpha \in (0, 1)$, so $(1-\alpha)^2 < 1$, and $1/(1-\alpha) > 1 + \alpha$. It implies

$$u_0^q > 1 + \alpha, \; v_0^q > 1 + \alpha, \;\; w_0^q > 1 + \alpha. \tag{3.7}$$

From (3.5) and (1.1), it has

$$\begin{cases} \alpha < u_1 = \alpha + \frac{u_{-1}^q}{v_0^q} \le \alpha + \frac{1}{v_0^q} \le 1, \\ \alpha < v_1 = \alpha + \frac{v_{-1}^q}{w_0^q} \le \alpha + \frac{1}{w_0^q} \le 1, \\ \alpha < w_1 = \alpha + \frac{w_{-1}^q}{u_0^q} \le \alpha + \frac{1}{u_0^q} \le 1. \end{cases} \tag{3.8}$$

Thus

$$(u_1, v_1, w_1) \in (\alpha, 1] \times (\alpha, 1] \times (\alpha, 1]. \tag{3.9}$$

Similarly, it has

$$\begin{cases} u_2 = \alpha + \frac{u_0^q}{v_1^q} \ge \alpha + u_0^q, \\ v_2 = \alpha + \frac{v_0^q}{w_1^q} \ge \alpha + v_0^q, \\ w_2 = \alpha + \frac{w_0^q}{u_1^q} \ge \alpha + w_0^q, \end{cases} \tag{3.10}$$

and

$$
\begin{cases}
\alpha < u_3 = \alpha + \frac{u_1^q}{v_2^q} \leq \alpha + \frac{1}{v_2^q} \leq \alpha + \frac{1}{(\alpha+v_0^q)^q} \leq \alpha + \frac{1}{\alpha+v_0^q} \leq \alpha + 1 - \alpha = 1, \\
\alpha < v_3 = \alpha + \frac{v_1^q}{w_2^q} \leq \alpha + \frac{1}{w_2^q} \leq \alpha + \frac{1}{(\alpha+w_0^q)^q} \leq \alpha + \frac{1}{\alpha+w_0^q} \leq \alpha + 1 - \alpha = 1, \\
\alpha < w_3 = \alpha + \frac{w_1^q}{u_2^q} \leq \alpha + \frac{1}{u_2^p} \leq \alpha + \frac{1}{(\alpha+u_0^q)^q} \leq \alpha + \frac{1}{\alpha+u_0^q} \leq \alpha + 1 - \alpha = 1.
\end{cases} \tag{3.11}
$$

So

$$
(u_3, v_3, w_3) \in (\alpha, 1] \times (\alpha, 1] \times (\alpha, 1]. \tag{3.12}
$$

Also similarly, it has

$$
\begin{cases}
u_4 = \alpha + \frac{u_2^q}{v_3^q} \geq \alpha + (\alpha + u_0^q)^q \geq \alpha + (\alpha + u_0^q) = 2\alpha + u_0^q, \\
v_4 = \alpha + \frac{v_2^q}{w_3^q} \geq \alpha + (\alpha + v_0^q)^q \geq \alpha + (\alpha + v_0^q) = 2\alpha + v_0^q, \\
w_4 = \alpha + \frac{w_2^q}{u_3^q} \geq \alpha + (\alpha + w_0^q)^q \geq \alpha + (\alpha + w_0^q) = 2\alpha + w_0^q.
\end{cases} \tag{3.13}
$$

By mathematical induction, one has

$$
\begin{cases}
(u_{2n}, v_{2n}, w_{2n}) \in \left[n\alpha + u_0^q, +\infty\right) \times \left[n\alpha + v_0^q, +\infty\right) \times \left[n\alpha + w_0^q, +\infty\right), \\
(u_{2n+1}, v_{2n+1}, w_{2n+1}) \in (\alpha, 1].
\end{cases} \tag{3.14}
$$

Therefore, we have

$$
\lim_{n\to\infty} u_{2n} = \infty, \ \lim_{n\to\infty} v_{2n} = \infty, \ \lim_{n\to\infty} w_{2n} = \infty,
$$

and

$$
\begin{cases}
\lim_{n\to\infty} u_{2n+1} = \alpha + \lim_{n\to\infty} \frac{u_{2n-1}^q}{v_{2n}^q} = \alpha, \\
\lim_{n\to\infty} v_{2n+1} = \alpha + \lim_{n\to\infty} \frac{v_{2n-1}^q}{w_{2n}^q} = \alpha, \\
\lim_{n\to\infty} w_{2n+1} = \alpha + \lim_{n\to\infty} \frac{w_{2n-1}^q}{u_{2n}^q} = \alpha.
\end{cases}
$$

(ii) The proof of (ii) is similar, so we omit it.

Theorem 3.3. *Suppose that* $\{(u_n, v_n, w_n)\}$ *is a positive solution of system (1.1), if one of the following conditions is true, for* $S \geq 0$,

$$
u_{S-1} < \alpha + 1 \leq u_S, \ \ v_{S-1} < \alpha + 1 \leq v_S, \ \ w_{S-1} < \alpha + 1 \leq w_S, \tag{3.15}
$$

or

$$
u_{S-1} > \alpha + 1 \geq u_S, \ \ v_{S-1} > \alpha + 1 \geq v_S, \ \ w_{S-1} < \alpha + 1 \geq z_S, \tag{3.16}
$$

then all the coordinate components of the solution $\{(u_n, v_n, w_n)\}$ *are oscillatory.*

Proof. If condition (3.15) is true, then

$$\begin{cases} u_{S+1} = \alpha + \frac{u_{S-1}^q}{v_S^q} < \alpha + 1, \\ v_{S+1} = \alpha + \frac{v_{S-1}^q}{w_S^q} < \alpha + 1, \\ w_{S+1} = \alpha + \frac{w_{S-1}^q}{u_S^q} < \alpha + 1, \end{cases} \tag{3.17}$$

and

$$\begin{cases} u_{S+2} = \alpha + \frac{u_S^q}{v_{S+1}^q} > \alpha + 1, \\ v_{S+2} = \alpha + \frac{v_S^q}{w_{S+1}^q} > \alpha + 1, \\ w_{S+2} = \alpha + \frac{w_S^q}{u_{S+1}^q} > \alpha + 1. \end{cases} \tag{3.18}$$

It follows from (3.17) and (3.18) that

$$u_{S+1} < \alpha + 1 \le u_{S+2}, \;\; v_{S+1} < \alpha + 1 \le v_{S+2}, \;\; w_{S+1} < \alpha + 1 \le w_{S+2}. \tag{3.19}$$

If condition (3.16) holds true, then

$$\begin{cases} u_{S+1} = \alpha + \frac{u_{S-1}^q}{v_S^q} > \alpha + 1, \\ v_{S+1} = \alpha + \frac{v_{S-1}^q}{w_S^q} > \alpha + 1, \\ w_{S+1} = \alpha + \frac{w_{S-1}^q}{u_S^q} > \alpha + 1, \end{cases} \tag{3.20}$$

and

$$\begin{cases} u_{S+2} = \alpha + \frac{u_S^q}{v_{S+1}^q} < \alpha + 1, \\ v_{S+2} = \alpha + \frac{v_S^q}{w_{S+1}^q} < \alpha + 1, \\ w_{S+2} = \alpha + \frac{w_S^q}{u_{S+1}^q} < \alpha + 1. \end{cases} \tag{3.21}$$

It follows from (3.20) and (3.21) that

$$u_{S+1} > \alpha + 1 > u_{S+2}, \;\; v_{S+1} > \alpha + 1 > v_{S+2}, \;\; w_{S+1} > \alpha + 1 > w_{S+2}. \tag{3.22}$$

This implies that all the coordinate components of the solution $\{(u_n, v_n, w_n)\}$ are oscillatory.

Theorem 3.4. *Suppose that $\{(u_n, v_n, w_n)\}$ is a positive solution of system (1.1), if one of the following conditions holds true*

$$w_0 > v_0 > u_0 > w_{-1} > v_{-1} > u_{-1} > \alpha + 1 \tag{3.23}$$

or

$$u_{-1} > v_{-1} > w_{-1} > u_0 > v_0 > w_0 > \alpha + 1, \tag{3.24}$$

then all the coordinate components of the solution $\{(u_n, v_n, w_n)\}$ *are oscillatory.*

Proof. If condition (3.23) holds true, then we have

$$u_1 = \alpha + \frac{u_{-1}^q}{v_0^q} < \alpha + 1, \quad v_1 = \alpha + \frac{v_{-1}^q}{w_0^q} < \alpha + 1, \quad w_1 = \alpha + \frac{w_{-1}^q}{w_0^q} < \alpha + 1. \tag{3.25}$$

It follows from (3.25) that

$$u_2 = \alpha + \frac{u_0^q}{v_1^q} > \alpha + 1, \quad v_2 = \alpha + \frac{v_0^q}{w_1^q} > \alpha + 1, \quad w_2 = \alpha + \frac{w_0^q}{u_1^q} > \alpha + 1. \tag{3.26}$$

By induction, suppose that for $n = k$,

$$u_{2k-1} < \alpha + 1, v_{2k-1} < \alpha + 1, w_{2k-1} < \alpha + 1, \tag{3.27}$$

and

$$u_{2k} > \alpha + 1, v_{2k} > \alpha + 1, w_{2k} > \alpha + 1. \tag{3.28}$$

Then, for $n = k + 1$, one has

$$\begin{cases} u_{2(k+1)-1} = \alpha + \frac{u_{2k-1}^q}{v_{2k}^q} < \alpha + 1, \\ \\ v_{2(k+1)-1} = u + \frac{v_{2k-1}^q}{w_{2k}^q} < \alpha + 1, \\ \\ w_{2(k+1)-1} = \alpha + \frac{w_{2k-1}^q}{u_{2k}^q} < \alpha + 1, \end{cases} \tag{3.29}$$

and

$$\begin{cases} u_{2(k+1)} = u + \frac{u_{2k}^q}{v_{2k+1}^q} > \alpha + 1, \\ v_{2(k+1)} = \alpha + \frac{v_{2k}^q}{w_{2k+1}^q} > \alpha + 1, \\ \\ w_{2(k+1)} = \alpha + \frac{w_{2k}^q}{u_{2k+1}^q} > \alpha + 1. \end{cases} \tag{3.30}$$

It follows from (3.29) and (3.30) that all the coordinate components of the solution $\{(u_n, v_n, w_n)\}$ are oscillatory.

If Condition (3.24) holds true, the proof is similar to the above. So it is omitted.

Theorem 3.5. *Every positive solution* (u_n, v_n, w_n) *of system (1.1) is bounded and persists if* $\alpha^q > 1$.

Proof. From system (1.1), it has, for $n \geq 1$,

$$u_n \geq \alpha, \quad v_n \geq \alpha, \quad w_n \geq \alpha. \tag{3.31}$$

On the other hand, we have

$$u_n \quad = \quad \alpha + \frac{u_{n-2}^q}{v_{n-1}^q} \leq \alpha + \frac{u_{n-2}^q}{\alpha^p} \leq \alpha + \frac{\alpha}{\alpha^q} + \frac{1}{\alpha^{2q}} u_{n-4}^q$$

$$\begin{aligned}
&\leq \alpha + \frac{\alpha}{\alpha^q} + \frac{\alpha}{\alpha^{2q}} + \frac{1}{\alpha^{3q}} u_{n-6}^q \\
&\leq \alpha + \frac{\alpha}{\alpha^q} + \frac{\alpha}{\alpha^{2q}} + \cdots + \frac{\alpha}{\alpha^{(k-1)q}} + \frac{1}{\alpha^{kq}} u_{n-2k}^q \\
&= \frac{\alpha^{q+1}\left[1 - \left(\frac{1}{\alpha^q}\right)^k\right]}{\alpha^q - 1} + \frac{1}{\alpha^{kq}} u_{n-2k}^q \\
&\leq \begin{cases} \frac{\alpha^{q+1}}{\alpha^q - 1} + u_0^q, & n = 2k, \\ \frac{\alpha^{q+1}}{\alpha^q - 1} + u_{-1}^q, & n = 2k-1. \end{cases}
\end{aligned} \tag{3.32}$$

Similarly, we have

$$\alpha \leq v_n \leq \begin{cases} \frac{\alpha^{q+1}}{\alpha^q - 1} + v_0^q, & n = 2k, \\ \frac{\alpha^{q+1}}{\alpha^q - 1} + v_{-1}^q, & n = 2k-1, \end{cases} \tag{3.33}$$

and

$$\alpha \leq w_n \leq \begin{cases} \frac{\alpha^{q+1}}{\alpha^q - 1} + w_0^q, & n = 2k, \\ \frac{\alpha^{q+1}}{\alpha^q - 1} + w_{-1}^q, & n = 2k-1. \end{cases} \tag{3.34}$$

Let

$$M = \frac{\alpha^{q+1}}{\alpha^q - 1} + \max\{u_0, u_{-1}, v_0, v_{-1}, w_0, w_{-1}\}. \tag{3.35}$$

So, from (3.31)–(3.35), one has

$$\alpha \leq u_n, v_n, w_n \leq M, \quad n \in N^+.$$

The proof of this theorem is completed.

4. Numerical examples

To show the validity of theoretic results, using Matlab software, we give some numerical examples in this section.

Example 4.1 Consider system (1.1): suppose that the initial values are $u_{-1} = 0.6, v_{-1} = 1.2, w_{-1} = 0.9, u_0 = 2.6, v_0 = 2.7, w_0 = 2.9$, and the parameters are $\alpha = 1.5, q = 1.1$, it is clear that $\alpha > 2q - 1$. Then the positive equilibrium $(2.5, 2.5, 2.5)$ of system (1.1) is LAS (see Figure 1).

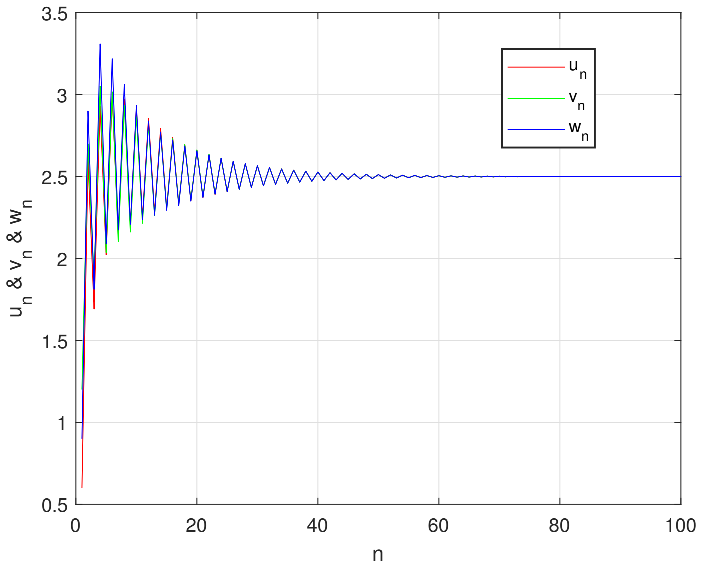

Figure 1. The stability of positive equilibrium $(2.5, 2.5, 2.5)$ for system (1.1) under $\alpha > 2q - 1$.

Example 4.2 Consider system (1.1): suppose that the initial values $u_{-1} = 0.6, v_{-1} = 1.2, w_{-1} = 0.9, u_0 = 2.6, v_0 = 2.7, w_0 = 2.9$, and the parameters are $\alpha = 1.5, q = 1.28$. It is clear that $0 < \alpha < 2q - 1$. Then the positive equilibrium $(2.5, 2.5, 2.5)$ is unstable (see Figure 2).

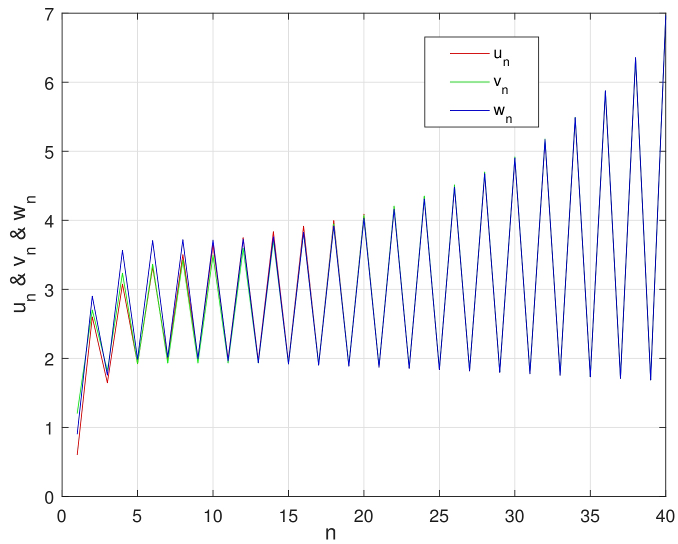

Figure 2. The instability of positive equilibrium $(2.5, 2.5, 2.5)$ for system (1.1) under $0 < \alpha < 2q - 1$.

Example 4.3 Consider system (1.1): suppose that the initial values $u_{-1} = 1.2, v_{-1} = 1.5, w_{-1} = 1.7, u_0 = 1.9, v_0 = 2.2, w_0 = 2.5$ and the parameters $\alpha = 0.8, q = 1$. It is clear that the conditions of Theorem 3.3 are satisfied. So all the coordinate components of the positive solution $\{(u_n, v_n, w_n)\}$ are oscillatory (see Figure 3).

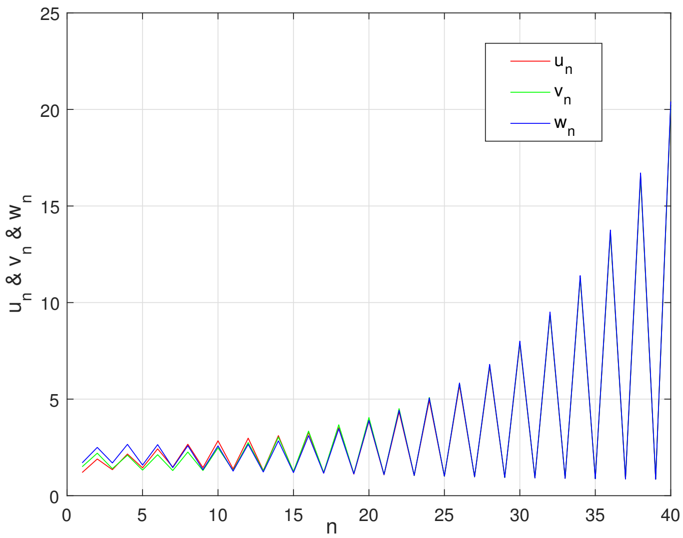

Figure 3. The oscillatory solution of system (1.1) with $q = 1$.

Example 4.4 Consider system (1.1): suppose that the initial values $u_{-1} = 3.6, v_{-1} = 3.7, w_{-1} = 3.8, u_0 = 4.0, v_0 = 4.3, w_0 = 4.5$ and the parameters $\alpha = 2.5, q = 1.8$. It is clear that the conditions of Theorem 3.4 are satisfied. So all the coordinate components of the positive solution $\{(u_n, v_n, w_n)\}$ are oscillatory (see Figure 4).

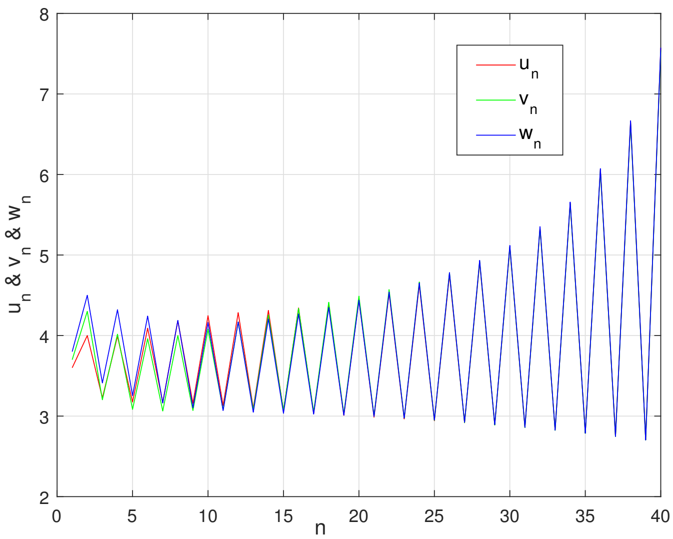

Figure 4. The oscillatory solution of system (1.1) with $q = 1.8$.

5. Conclusions and future works

In this work, we utilize a linearized equation at the equilibrium point to study the local dynamics of a three-dimensional discrete system. The qualitative features of the systems are the main results of this article. Some sufficient conditions guarantee the asymptotic feature of system (1.1). The theoretic findings are stated below.
(i) The positive equilibrium $(\alpha + 1, \alpha + 1, \alpha + 1)$ of the system (1.1) is LAS if the parameters satisfy $\alpha > 2q - 1$. The positive solution is unstable if the parameters satisfy $0 < \alpha < 2q - 1$.
(ii) The positive solution is oscillatory under different initial conditions. Moreover, the positive solutions are bounded and persistent under certain parametric and initial conditions.
Open problem: To find a sufficient condition, the positive equilibrium $(\alpha + 1, \alpha + 1, \alpha + 1)$ is GAS.

In the future, we will continue to explore high-order difference system

$$u_{n+1} = \alpha + \frac{u_{n-m}^q}{v_n^q}, v_{n+1} = \alpha + \frac{v_{n-m}^q}{w_n^q}, w_{n+1} = \alpha + \frac{w_{n-m}^q}{u_n^q}, n = 0, 1, \cdots .$$

where $\alpha \in (0, \infty), q \in [1, +\infty)$ and $u_i, v_i, w_i \in (0, \infty), i \in \{0, -1, \cdots, -m\}$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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