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Research article

Sufficient conditions for exact bifurcation curves in Minkowski curvature problems and their applications

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Abstract: In this paper, we continued and extended the work in [1–3] by establishing sufficient conditions to determine the exact shape of the bifurcation curve of positive solutions for the Minkowski curvature problem

$$\begin{cases} -\left(u'/\sqrt{1-u'^2}\right)' = \lambda f(u), & \text{in } (-L, L), \\ u(-L) = u(L) = 0, \end{cases}$$

where $\lambda, L > 0$ and $f \in C^2(0,\infty)$. Notice that we allow $f(0^+) = -\infty$. To illustrate the applicability of these results, we presented some examples, such as the diffusive logistic equation with a Holling type-II functional response.

Keywords: positive solution; bifurcation curve; Minkowski curvature problem; diffusive logistic equation

1. Introduction

In this paper, we establish sufficient conditions to determine the exact shape of the bifurcation curve of positive solutions for the Minkowski curvature problem

$$\begin{cases} -\left(u'/\sqrt{1-u'^2}\right)' = \lambda f(u), & \text{in } (-L, L), \\ u(-L) = u(L) = 0, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a bifurcation parameter, $L > 0$ is an evolution parameter, $f \in C^2(0,\infty)$, $f(0^+) < \infty$ and the following conditions (H_1) and (H_2) hold:

(H_1) There exist $0 \le \beta_0 < \beta \le \infty$ such that one of the following four cases holds:
(H_{1a}): $\beta_0 = 0, \beta < \infty$ and $(\beta - u)\, f(u) > 0$ for $u > 0$ and $u \neq \beta$;

(H_{1b}): $\beta_0 > 0, \beta < \infty$ and $f(u) < 0$ on $(0,\beta_0) \cup (\beta,\infty)$, and $f(u) > 0$ on (β_0,β) ;

(H_{1c}): $\beta_0 = 0, \beta = \infty$ and $f(u) > 0$ for $u > 0$;

(H_{1d}): $\beta_0 > 0, \beta = \infty$ and $(\beta_0 - u) f(u) < 0$ for $u > 0$ and $u \neq \beta_0$;

(H_2) Let $F(u) \equiv \int_0^u f(t)dt$. Then $F : [0,\infty) \longrightarrow \mathbb{R}$ is continuous and differentiable for $u > 0$, and $F(\beta^-) > 0$.

Notice that we allow $f(0^+) = -\infty$, and that the condition (H_2) naturally holds if either (H_{1a}) or (H_{1c}) is satisfied. By (H_1)–(H_2), there exists $\zeta \in [0,\beta)$ such that

$$(\zeta - u) F(u) < 0 \quad \text{for } u > 0 \text{ and } u \neq \zeta, \tag{1.2}$$

see Figure 1. Moreover, $\zeta = 0$ if $\beta_0 = 0$, and $\zeta \in (0,\beta)$ if $\beta_0 > 0$.

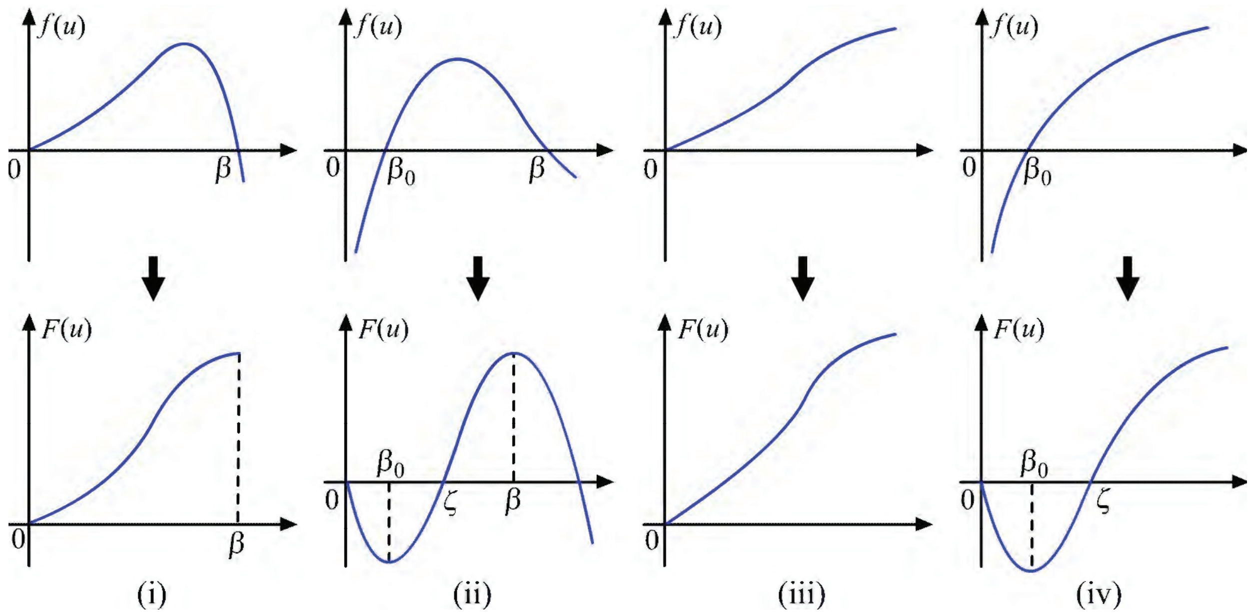

Figure 1. Graphs of f and F on $(0,\beta)$. (i) $\beta_0 = 0$ and $\beta < \infty$. (ii) $\beta_0 > 0$ and $\beta < \infty$. (iii) $\beta_0 = 0$ and $\beta = \infty$. (iv) $\beta_0 > 0$ and $\beta = \infty$.

Define the bifurcation curve of (1.1) on the $(\lambda, \|u\|_\infty)$-plane as follows:

$$S_L \equiv \Big\{(\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \in C^2(-L,L) \cap C[-L,L] \text{ is a positive solution of (1.1)}\Big\} \quad \text{for } L > 0. \tag{1.3}$$

It is well known that studying the exact shape of the bifurcation curve S_L of (1.1) is equivalent to studying the exact multiplicity of positive solutions of problem (1.1). Therefore, many researchers have devoted significant efforts to studying the shapes of bifurcation curves (cf. [1–9]). In particular, [6–8] demonstrated that the corresponding bifurcation curves may be monotone increasing or ⊂-shaped; [4,5,9] showed that they may be monotone increasing or S-shaped; and [1–3] analyzed the possible forms of the bifurcation curves. In addition, [10, 11] also discussed the shapes of bifurcation curves, although their focus was on semilinear problems.

Next, we introduce the studies in [1–3]. If either (H_{1a}) or (H_{1c}) holds, we define

$$\eta \equiv \lim_{u \to 0^+} \frac{f(u)}{u}.$$

Clearly, $\eta \geq 0$. Then the following seven possibilities (C_1)–(C_7) arise:

(C_1) $\eta = 0$.
(C_2) $\eta = \infty$.
(C_3) $\eta \in (0, \infty)$ and $f''(0^+) \in (0, \infty]$.
(C_4) $\eta \in (0, \infty)$ and $f''(0^+) \in [-\infty, 0)$.
(C_5) $\eta \in (0, \infty)$, $f''(0^+) = 0$ and $f^{(3)}(0^+) = \infty$ (if $f^{(3)}(u)$ exists).
(C_6) $\eta \in (0, \infty)$, $f''(0^+) = 0$ and $f^{(3)}(0^+) \in [-\infty, 0]$ (if $f^{(3)}(u)$ exists).
(C_7) $\eta \in (0, \infty)$, $f''(0^+) = 0$ and $f^{(3)}(0^+) \in (0, \infty)$ (if $f^{(3)}(u)$ exists).

References [1,2] provide the classification of the bifurcation curve S_L for the Minkowski curvature problem (1.1) under the condition (H_{1a}) or (H_{1c}), see Theorem 1.1.

Theorem 1.1 ([1, Theorem 2.1] and [2, Theorem 2.1]). *Consider (1.1). Assume that (H_{1a}) or (H_{1c}) holds. Then the bifurcation curve S_L of (1.1) is continuous on the $(\lambda, \|u_\lambda\|_\infty)$-plane, starts from the point $(\kappa_L, 0)$, and goes to $\left(\infty, m_{L,\beta}\right)$ for $L > 0$ where*

$$\kappa_L \equiv \begin{cases} \infty & \text{if } \eta = 0, \\ \frac{\pi^2}{4\eta L^2} & \text{if } \eta \in (0, \infty), \\ 0 & \text{if } \eta = \infty, \end{cases} \quad \text{and} \quad m_{L,\beta} \equiv \min\{L, \beta\}. \tag{1.4}$$

Furthermore,

(i) if one of (C_1), (C_3) and (C_5) holds, then S_L is $\subset$-like shaped for all $L > 0$.
(ii) if one of (C_2), (C_4) and (C_6) holds, then S_L is either monotone increasing or S-like shaped for $L > 0$.
(iii) if (C_7) holds, then S_L is $\subset$-like shaped for $L > \mathring{L}$, and is either monotone increasing or S-like shaped for $\mathring{L} > L > 0$ where

$$\mathring{L} \equiv \pi \sqrt{\frac{3\eta}{2 f^{(3)}(0^+)}}. \tag{1.5}$$

On the other hand, if (H_{1d}) holds, we define the following conditions (D_1)–(D_4):

(D_1) $\lim_{u \to 0^+} f(u)/u \in (-\infty, 0]$.
(D_2) $\lim_{u \to 0^+} f(u)/u^{r_1} \in [-\infty, 0)$ for some $r_1 \in (0, 1)$.
(D_3) $\lim_{u \to 0^+} u^{\frac{1}{3}} f(u) \in (-\infty, 0]$.
(D_4) $\lim_{u \to 0^+} u^{r_2} f(u) \in [-\infty, 0)$ for some $r_2 \in (\frac{1}{3}, 1)$.

Reference [3] provides the classification of the bifurcation curve S_L for the Minkowski curvature problem (1.1) under the condition (H_{1d}), see Theorem 1.2.

Theorem 1.2 ([3, Theorems 2.1 and 2.2]). *Consider (1.1). Assume that (H_{1d}) holds. Let*

$$G \equiv \int_0^{\zeta} \frac{-\zeta f(\zeta) - 2F(u) + uf(u)}{[-F(u)]^{3/2}} du, \tag{1.6}$$

where ζ is defined in (1.2). Then the following statements (i)–(iii) hold:

(i) The bifurcation curve S_L does not exist for $L \leq \zeta$.
(ii) Assume that (D_1) holds. For $L > \zeta$, the bifurcation curve S_L is $\subset$-like shaped, starts from (∞, ζ) and goes to (∞, L).
(iii) Assume that (D_2) holds. For $L > \zeta$, there exists $\bar{\kappa}_L \in (0, \infty)$ such that the bifurcation curve S_L starts from (κ_L, ζ) and goes to (∞, L), with a shape that can be monotone increasing, $\subset$-like shaped, or S-like shaped. Furthermore,

(a) if (D_3) holds, then the bifurcation curve S_L is $\subset$-like shaped for $L > \zeta$.
(b) if (D_4) holds, $G \geq 0$ and $3f(u) + uf'(u) > 0$ for $\beta_0 < u \leq \zeta$, then the bifurcation curve S_L is either monotone increasing or S-like shaped for $L > \zeta$.
(c) if (D_4) holds, $G < 0$ and $3f(u) + uf'(u) > 0$ for $\beta_0 < u \leq \zeta$, then there exists $\tilde{L} > \zeta$ such that the bifurcation curve S_L is $\subset$-like shaped for $L > \tilde{L}$, and is either monotone increasing or S-like shaped for $\tilde{L} \geq L > \zeta$.

The shape of the bifurcation curve obtained from Theorems 1.1 and 1.2 is clearly not precise enough. Therefore, references [1–3] have investigated the exact shape of the bifurcation curve S_L for the Minkowski curvature problem (1.1). These results are summarized and presented in Theorem 1.3 below.

Theorem 1.3. *Consider (1.1). Then the following statements (i)–(iii) hold:*

(i) If (H_{1a}) holds and $[f(u)/u]' < 0$ on $(0, \beta)$, then the bifurcation curve S_L is monotone increasing for $L > 0$.
(ii) If (H_{1c}) holds and $f''(u) < 0$ on $(0, \infty)$, then the bifurcation curve S_L is monotone increasing for $L > 0$.
(iii) If (H_{1d}) holds and $f''(u) < 0$ on $(0, \infty)$, then

(a) if (D_3) holds, then the bifurcation curve S_L is $\subset$-shaped for $L > \zeta$;
(b) if (D_4) holds and $G \geq 0$, then the bifurcation curve S_L is monotone increasing for $L > \zeta$;
(c) if (D_4) holds and $G < 0$, there exists $\tilde{L} > \zeta$ such that S_L is $\subset$-shaped for $L > \tilde{L}$ and monotone increasing for $\tilde{L} \geq L > \zeta$.

From Theorems 1.1–1.3, we find that references [1–3] lack discussion on condition (H_{1b}) and provide only limited results regarding the exact shape of the bifurcation curve. Therefore, we extend the work in [1–3] by establishing sufficient conditions to rigorously characterize its structure. These results can then be applied to various examples, the most notable of which is the diffusive logistic equation with predation, modeled by a Holling type-II functional response:

$$\begin{cases} -\left(u'/\sqrt{1-u'^2}\right)' = \lambda u\left(k - u - \frac{1}{1+mu}\right), \text{ in } (-L, L), \\ u(-L) = u(L) = 0, \end{cases} \tag{1.7}$$

where $\lambda, L, k, m > 0$. The problem (1.7) models predator-prey dynamics with a Holling type-II functional response, frequently used in biological modeling (cf. [10]).

In recent years, there has been an increasing amount of intensive research on one-dimensional Minkowski-curvature problems, particularly those involving indefinite weights or super-exponential nonlinearities. Equations with indefinite weights arise naturally in spacelike hypersurface geometry in Lorentz-Minkowski space. Such equations also serve as models for physical or biological systems in spatially heterogeneous environments. He and Miao [5] studied the Minkowski-curvature Dirichlet problem with indefinite weight $a(x)$:

$$\begin{cases} -\left(u'/\sqrt{1-u'^2}\right)' = \lambda a(x) f_1(u), & x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.8}$$

where $\lambda > 0$, $f_1 \in C[0,\infty)$, $f_1(0) = 0$, $f_1(u) > 0$ for all $u > 0$, and $a \in C[0,1]$ satisfies that $a(x) > 0$ on (x_1, x_2), and $a(x) < 0$ on $[0,1]\backslash[x_1, x_2]$ for some $x_1, x_2 \in [0,1]$. Under suitable assumptions, the authors proved the existence of an S-shaped connected component in the set of positive solutions, which reflects the existence and multiplicity of positive solutions of (1.8) with respect to the parameter λ.

Boscaggin et al. [12] considered the Minkowski-curvature equation:

$$\begin{cases} -\left(u'/\sqrt{1-u'^2}\right)' + a(x) f_2(u), & x \in (0,T), \\ \mathfrak{B}(u) = 0, \end{cases} \tag{1.9}$$

where $a : \mathbb{R} \to \mathbb{R}$ is a sign-changing T-periodic function, $f_2 : [0,\infty) \to [0,\infty)$ is a continuous function vanishing only at $u = 0$, and the boundary operator $\mathfrak{B} : C^1[0,T] \to \mathbb{R}^2$ is either of periodic or Neumann type. Under suitable conditions, the authors used topological degree theory to prove the existence of positive solutions of (1.9).

Ye et al. [13] investigated global bifurcation of one-signed radial solutions for Minkowski-curvature equations with indefinite weight and non-differentiable nonlinearities:

$$\begin{cases} -\mathrm{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) = \lambda a(x) u + f_3(x, y, \lambda), & \text{in } \Omega, \\ \frac{\partial y}{\partial \nu} = 0 \text{ on } \partial B_{R_1}, \ y = 0 \text{ on } \partial B_{R_2}, \end{cases} \tag{1.10}$$

where $\Omega = \{x \in \mathbb{R}^n : R_1 \le |x| \le R_2\}$ is an annular domain, $\lambda \neq 0$, $a \in L^\infty(\Omega)$, and $f_3 : \bar{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and radially symmetric with respect to x. Under suitable conditions, the authors used global bifurcation methods to prove the existence of unbounded continua of one-signed solutions for this problem (1.10) which bifurcate from the line of trival solution. Furthermore, they investigated the asymptotic behavior of the one-signed radial solution as $\lambda \to \infty$.

The paper is organized as follows. Section 2 presents the main results. Section 3 discusses three applications. Section 4 provides several lemmas necessary for proving the main results, while Section 5 contains the proofs of the main results and their applications.

2. Main results

In this section, we present the main results. Recall the numbers ζ, $m_{L,\beta}$ and G defined in (1.2), (1.4) and (1.6), respectively.

Theorem 2.1. *Consider (1.1). Assume that (H_{1b}) holds. Then the following statements (i)–(iii) hold:*

(i) The bifurcation curve S_L does not exist for $L \leq \zeta$.

(ii) Assume that (D_1) holds. For $L > \zeta$, the bifurcation curve S_L is $\subset$-like shaped, starts from (∞, ζ) and goes to $\left(\infty, m_{L,\beta}\right)$.

(iii) Assume that (D_2) holds. For $L > \zeta$, there exists $\bar{\kappa}_L \in (0, \infty)$ such that the bifurcation curve S_L starts from (κ_L, ζ) and goes to $\left(\infty, m_{L,\beta}\right)$, with a shape that can be monotone increasing, $\subset$-like shaped, or S-like shaped. Furthermore, assume that $f''(u) < 0$ on $(0, \infty)$. Then

(a) if (D_3) holds, then the bifurcation curve S_L is $\subset$-shaped for $L > \zeta$;

(b) if (D_4) holds and $G \geq 0$, then the bifurcation curve S_L is monotone increasing for $L > \zeta$;

(c) if (D_4) holds and $G < 0$, there exists $\tilde{L} > \zeta$ such that S_L is $\subset$-shaped for $L > \tilde{L}$ and monotone increasing for $\tilde{L} \geq L > \zeta$.

Let

$$g(u) \equiv \frac{f(u)}{u}, \quad N(u) \equiv \frac{uf(u) - u^2 f'(u)}{uf(u) - 2F(u)} \quad \text{and} \quad W(u) \equiv \frac{uf'(u)}{f(u)}. \tag{2.1}$$

Then we define the following conditions (E_1)–(E_4):

(E_1) There exists $\bar{\beta} \in (0, \beta)$ such that $g'(u) > 0$ on $\left(0, \bar{\beta}\right)$, $g'(\bar{\beta}) = 0$, and $g'(u) < 0$ on $\left(\bar{\beta}, \beta\right)$.

(E_2) $N(0^+) \geq -3$ and $N'(u) \geq 0$ for $u \in \Lambda$ where $\Lambda \equiv \{u \in (0, \beta) : uf(u) \neq 2F(u)\}$.

(E_3) $W'(u) \leq 0$ for $0 < u < \beta$ and $u \neq \beta_0$.

(E_4) f is convex-concave on $(0, \beta)$.

Theorem 2.2. *Consider (1.1). Assume that one of the following conditions (i)–(iii) holds:*

(i) (E_1) and (E_2) hold.

(ii) (E_1) and (E_3) hold, and one of the following (a)–(c) holds:

(a) $\beta_0 = 0$;

(b) $\beta_0 > 0$, $W(0^+) \leq 1$ and $\zeta < \bar{\beta}$; and

(c) $\beta_0 > 0$, $W(0^+) \leq W(\beta^-)$ and $W(\beta^-) > -1/3$.

(iii) (E_4) holds, $\beta_0 > 0$ and $W(\zeta) \leq 2$.

Then the bifurcation curve S_L is either monotone increasing or $\subset$-shaped for $L > \zeta$.

Remark 2.3. *When Theorem 2.2 is combined with Theorems 1.1 and 1.2, we can obtain a more comprehensive understanding of the bifurcation curve, including its precise shape, starting point, and endpoint.*

Theorem 2.4 (see Figure 2)**.** *Consider (1.7). Let*

$$\bar{\kappa}_L \equiv \begin{cases} \frac{\pi^2}{4(k-1)L^2} & \text{if } k > 1, \\ \infty & \text{if } k = 1, \end{cases} \quad \text{and} \quad \beta_{k,m} \equiv \frac{\sqrt{(km+1)^2 - 4m} + km - 1}{2m}. \tag{2.2}$$

Then the following statements (i) and (ii) hold.

(i) *If* $m > 1$, *there exists* $k_m \in (\frac{2\sqrt{m-1}}{m}, 1)$ *such that the following statements (a)–(c) hold.*

(a) *If* $k \geq 1$, *then the bifurcation curve* S_L *is* $\subset$*-shaped, starts from* $(\bar{\kappa}_L, 0)$ *and goes to* $(\infty, m_{L,\beta_{k,m}})$ *for* $L > 0$.

(b) *If* $k_m < k < 1$, *then the bifurcation curve* S_L *does not exist for* $0 < L \leq \zeta$, *and is* $\subset$*-shaped, starts from* (∞, ζ) *and goes to* $(\infty, m_{L,\beta_{k,m}})$ *for* $L > \zeta$ *where* $\zeta \in (0, \beta_{k,m})$ *satisfies*

$$6\ln(m\zeta + 1) - 2m^2\zeta^3 + 3km^2\zeta^2 - 6m\zeta = 0. \tag{2.3}$$

(c) *If* $0 < k \leq k_m$, *then the bifurcation curve* S_L *does not exist for* $L > 0$.

(ii) *If* $0 < m \leq 1$, *then the following statements (d) and (e) hold.*

(d) *If* $k > 1$, *then the bifurcation curve* S_L *is monotone increasing, starts from* $(\bar{\kappa}_L, 0)$ *and goes to* $(\infty, m_{L,\beta_{k,m}})$ *for* $L > 0$.

(e) *If* $k \leq 1$, *then the bifurcation curve* S_L *does not exist for* $L > 0$.

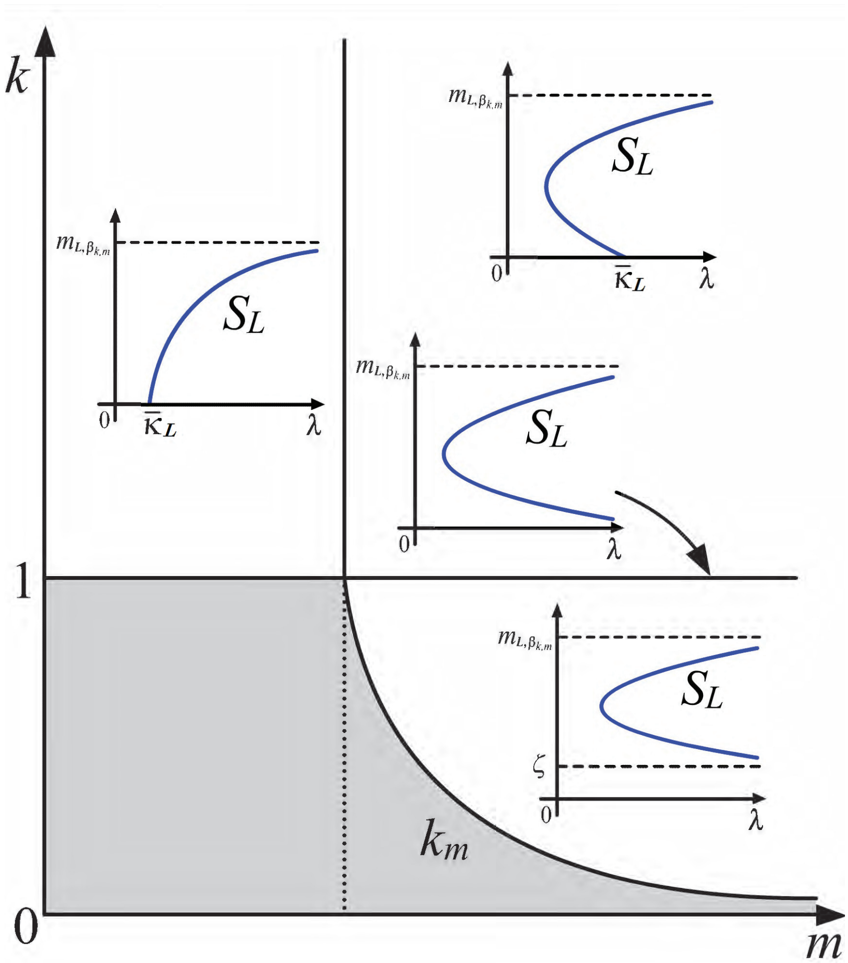

Figure 2. Graphs of the bifurcation curve S_L of (1.7).

3. Other examples

Besides the diffusive logistic equation (1.7), this section provides two additional examples to illustrate our results.

Example 3.1. *Consider (1.1) with*

$$f(u) = -u^3 + \frac{3(a+b-c)}{4}u^2 + \frac{ac+bc-ab}{2}u - \frac{abc}{4}, \quad b > a > 0 \text{ and } c \geq 0. \tag{3.1}$$

Let β be the root of $f(u) = 0$. First, we present the following conclusions:

(i) *If $c = 0$, the bifurcation curve S_L is $\subset$-shaped, starts from (∞, a) and goes to $\left(\infty, m_{L,\beta}\right)$ for $L > a$, see Figure 3(i); and*

(ii) *if $c > 0$ and $b - c \leq 2a$, there exists $\hat{\kappa}_L \in (0, \infty)$ such that the bifurcation curve S_L is $\subset$-shaped, starts from $(\hat{\kappa}_L, a)$ and goes to $\left(\infty, m_{L,\beta}\right)$ for $L > a$, see Figure 3(ii).*

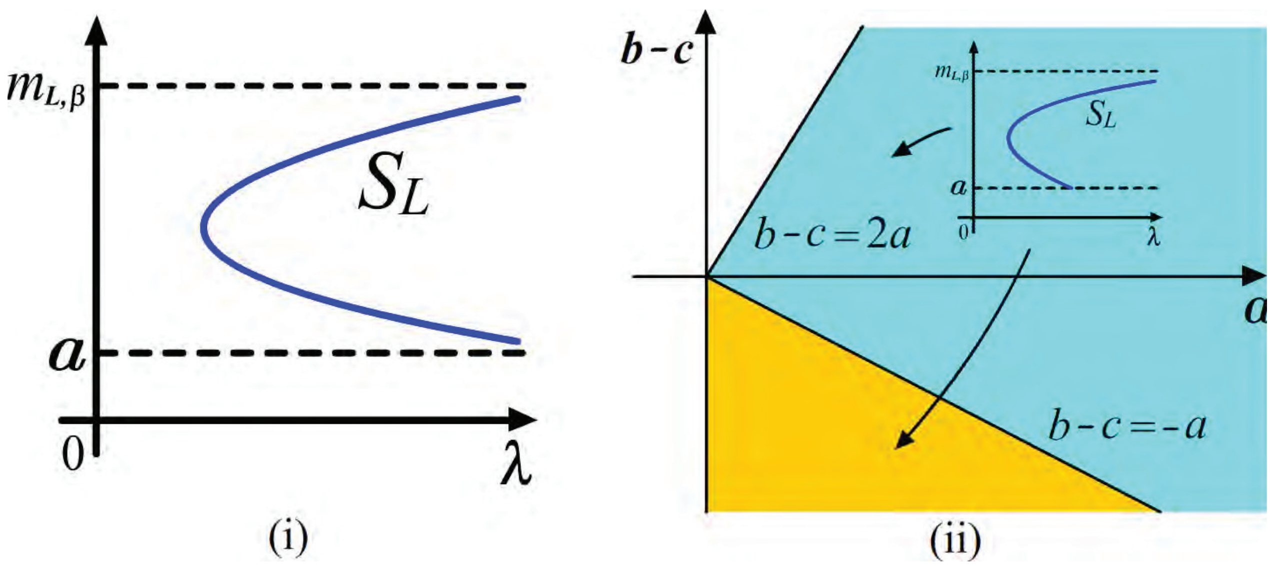

Figure 3. Graphs of the bifurcation curve S_L of (1.1) with (3.1). (i) $c = 0$. (ii) $c > 0$.

Indeed, since

$$F(u) = -\frac{u}{4}(u+c)(u-a)(u-b), \tag{3.2}$$

we see that (H_1) and (H_2) hold. Moreover, $\beta_0 > 0$, $\zeta = a$ and $\beta < \infty$. We compute

$$\lim_{u \to 0^+} \frac{f(u)}{u} = \frac{-ab}{2} \in (-\infty, 0) \ \text{ if } c = 0$$

and

$$\lim_{u \to 0^+} \frac{f(u)}{\sqrt{u}} = -\infty \ \text{ and } \ \lim_{u \to 0^+} u^{\frac{1}{3}} f(u) = 0 \ \text{ if } c > 0.$$

Then

$$\begin{cases} (D_1) \text{ holds} & \text{if } c = 0, \\ (D_2) \text{ and } (D_3) \text{ hold} & \text{if } c > 0. \end{cases} \tag{3.3}$$

In addition, we observe that

$$W(\zeta) = W(a) = \frac{2\left(3a^2 - 2(b-c)a - bc\right)}{(a+c)(a-b)} \le 2 \text{ if and only if } 2a - b + c \ge 0. \tag{3.4}$$

Next, we consider four cases:

Case 1. *Assume that* $c = 0$ *and* $2a - b > 0$. *Since* $f''(u) = -6u + 3(a+b)/2$, *we see that* (E_4) *holds. By (3.4), and Theorems 1.2 and 2.2, the bifurcation curve* S_L *is* $\subset$*-shaped, starts from* (∞, a) *and goes to* $\left(\infty, m_{L,\beta}\right)$ *for* $L > a$.

Case 2. *Assume that* $c = 0$ *and* $2a - b \le 0$. *Observe that*

$$g'(u) = -2u + \frac{3(a+b)}{4}\begin{cases} > 0 & \text{for } 0 < u < \bar{\beta}, \\ = 0 & \text{for } u = \bar{\beta} = \frac{3(a+b)}{8}, \\ < 0 & \text{for } u > \bar{\beta}, \end{cases}$$

and

$$\begin{aligned} W'(u) &= \frac{12(a+b)u^2}{\left[4u^3 - 3u^2(a+b) + 2abu\right]^2}\left[-u^2 + \frac{8ab}{3(a+b)}u - \frac{ab}{2}\right] \\ &\le \frac{12(a+b)u^2}{\left[4u^3 - 3u^2(a+b) + 2abu\right]^2}\left[-\left(\frac{4ab}{3(a+b)}\right)^2 + \frac{8ab}{3(a+b)}\frac{4ab}{3(a+b)} - \frac{ab}{2}\right] \\ &= \frac{-2abu^2}{3(a+b)\left[4u^3 - 3u^2(a+b) + 2abu\right]^2}\left[9a^2 + (9b - 14a)b\right] \\ &< 0 \quad (\text{since } b \ge 2a). \end{aligned}$$

So (E_1) *and* (E_3) *hold. Since* $2a - b \le 0$, *we see that* $g'(\zeta) = (3b - 5a)/4 > 0$. *It follows that* $\zeta < \bar{\beta}$. *Since* $W(0^+) = 1$, *and by Theorems 1.2 and 2.2, the bifurcation curve* S_L *is* $\subset$*-shaped, starts from* (∞, a) *and goes to* $\left(\infty, m_{L,\beta}\right)$ *for* $L > a$.

Case 3. *Assume that* $c > 0$ *and* $b - c \le -a$. *Since* $f''(u) = -6u + 3(a+b-c)/2 < 0$ *for* $u > 0$, *and by Theorems 1.2 and 2.1, there exists* $\hat{\kappa}_L \in (0, \infty)$ *such that the bifurcation curve* S_L *is* $\subset$*-shaped, starts from* $(\hat{\kappa}_L, a)$ *and goes to* $\left(\infty, m_{L,\beta}\right)$ *for* $L > a$.

Case 4. *Assume that* $c > 0$ *and* $-a < b - c \le 2a$. *Clearly, we have* $a + b - c > 0$ *and* $2a - b + c \ge 0$. *Then* (E_4) *holds and* $W(\zeta) \le 2$ *by (3.4). So by Theorems 1.2 and 2.2, there exists* $\hat{\kappa}_L \in (0, \infty)$ *such that the bifurcation curve* S_L *is* $\subset$*-shaped, starts from* $(\hat{\kappa}_L, a)$ *and goes to* $\left(\infty, m_{L,\beta}\right)$ *for* $L > a$.

Remark 3.2. *For any cubic polynomial* f *satisfying* (H_{1b}) *and* (H_2), *the corresponding function* F *has a similar form to (3.2), differing only by a multiplicative coefficient. Although the results in Example 3.1 are not yet fully complete and require further investigation, they are sufficient to determine the shape of the corresponding bifurcation curve for most cubic polynomials. For example, consider the case where*

$$f(u) = -\frac{1}{2}(u-1)\left(2u^2 - 4u - 3\right).$$

In this case, we find that $a = 2$, $b = 3$, *and* $c = 1$. *Clearly,* $\zeta = 2$ *and* $\beta = 1 + \frac{1}{2}\sqrt{10}$. *Since* $c = 1 > 0$ *and* $b - c = 1 \le 6 = 2a$, *there exists* $\hat{\kappa}_L \in (0, \infty)$ *such that the bifurcation curve* S_L *is* $\subset$*-shaped, starts from* $(\bar{\kappa}_L, 2)$ *and goes to* $(\infty, m_{L,1+\frac{1}{2}\sqrt{10}})$ *for* $L > 2$.

Next, we provide an example where $f(0^+) = -\infty$, and the corresponding bifurcation curve S_L may be monotone increasing.

Example 3.3. *Consider (1.1) with*

$$f(u) = \frac{1}{u^p} - \frac{1}{u^q},\ \ 0 \le p < q < 1. \tag{3.5}$$

Let

$$\bar{G} \equiv \int_0^{\zeta} \frac{-\frac{p+1}{1-p}u^{1-p} + \frac{q+1}{1-q}u^{1-q} - \left(\frac{q-p}{1-q}\right)\zeta^{1-q}}{\left(-\frac{1}{1-p}u^{1-p} + \frac{1}{1-q}u^{1-q}\right)^{3/2}}du,$$

where

$$\zeta = \left(\frac{1-p}{1-q}\right)^{\frac{1}{q-p}}.$$

Then the following statements (i)–(iv) hold:

(i) There exists $\check{\kappa}_L \in (0,\infty)$ such that the bifurcation curve S_L starts from $(\check{\kappa}_L, \zeta)$ and goes to (∞, L) for $L > \zeta$.
(ii) If $0 < q \le \frac{1}{3}$, then S_L is $\subset$-shaped for $L > \zeta$.
(iii) If $\frac{1}{3} < q < 1$ and $\bar{G} \ge 0$, then S_L is monotone increasing for $L > \zeta$.
(iv) If $\frac{1}{3} < q < 1$ and $\bar{G} < 0$, then there exists $\tilde{L} > \zeta$ such that S_L is $\subset$-shaped for $L > \tilde{L}$ and monotone increasing for $\tilde{L} \ge L > \zeta$.

Indeed, (H_1) and (H_2) hold, $\beta_0 = 1$ and $\beta = \infty$. We compute

$$g'(u) = \frac{p+1}{u^{q+2}}\left(\frac{1+q}{1+p} - u^{q-p}\right)\begin{cases} > 0 & \text{for } 0 < u < \bar{\beta}, \\ = 0 & \text{for } u = \bar{\beta} = \left(\frac{1+p}{1+q}\right)^{\frac{1}{q-p}}, \\ < 0 & \text{for } u > \bar{\beta}. \end{cases}$$

So (E_1) holds. We compute

$$f'(u) = -pu^{-p-1} + qu^{-q-1} \ \text{ and } \ F(u) = \frac{u^{1-q}}{1-p}\left(u^{q-p} - \frac{1-p}{1-q}\right).$$

Clearly, $F(\zeta) = 0$ and we have

$$N(u) = \frac{uf(u) - u^2 f'(u)}{uf(u) - 2F(u)} = \frac{(1+p)\,u^{q-p} - 1 - q}{\frac{p+1}{p-1}u^{q-p} - \frac{q+1}{q-1}}.$$

It follows that $N(0) = q - 1 > -3$, and

$$N'(u) = \frac{u^{q-p-1}\,(q+1)\,(p+1)\,(p-q)^2}{\left[\frac{p+1}{p-1}u^{q-p} - \frac{q+1}{q-1}\right]^2 (q-1)\,(p-1)} > 0 \ \text{for } u \ne \left[\frac{(q+1)\,(p-1)}{(p+1)\,(q-1)}\right]^{\frac{1}{q-p}}.$$

So (E_2) holds. In addition, we compute

$$\lim_{u\to 0^+}\frac{f(u)}{\sqrt{u}}=-\infty \text{ and } \lim_{u\to 0^+}u^{\frac{1}{3}}f(u)=\lim_{u\to 0^+}u^{\frac{1}{3}-q}\left(u^{q-p}-1\right)\begin{cases}0 & \text{if } 0<q<\frac{1}{3},\\ -1 & \text{if } q=\frac{1}{3},\\ -\infty & \text{if } \frac{1}{3}<q<1.\end{cases}$$

It follows that

$$\begin{cases}(D_2) \text{ and } (D_3) \text{ hold} & \text{if } 0<q\leq\frac{1}{3},\\ (D_2) \text{ and } (D_4) \text{ hold} & \text{if } \frac{1}{3}<q<1.\end{cases} \tag{3.6}$$

By Theorem 1.2(iii), statement (i) holds. Next, we consider three cases.

Case 1. *Assume that $0<q\leq\frac{1}{3}$. By (3.6), and Theorems 1.2(iii) and 2.2, then statement (ii) holds.*

Case 2. *Assume that $\frac{1}{3}<q<1$ and $\bar{G}\geq 0$. We compute*

$$\begin{aligned}3f(u)+uf'(u) &= u^{-q}\left[(3-p)u^{q-p}+q-3\right]>u^{-q}\left[(3-p)\beta_0^{q-p}+q-3\right]\\ &= u^{-q}(q-p)>0\end{aligned} \tag{3.7}$$

for $\beta_0=1<u\leq\zeta$. By (1.6), then $G=\bar{G}>0$. By (3.7) and Theorems 1.2(iii) and 2.2, statement (iii) holds.

Case 3. *Assume that $\frac{1}{3}<q<1$ and $\bar{G}<0$. By (3.7) and Theorems 1.2(iii) and 2.2, statement (iv) holds.*

4. Lemmas

Since $F'(u)=f(u)$, and by (H_1), we see that

$$B(\alpha,u)\equiv F(\alpha)-F(u)>0 \text{ for } 0<u<\alpha \text{ and } \zeta<\alpha<\beta. \tag{4.1}$$

We define the time-map formula for (1.1) by

$$T_\lambda(\alpha)\equiv\int_0^\alpha\frac{\lambda B(\alpha,u)+1}{\sqrt{\{\lambda B(\alpha,u)+1\}^2-1}}du \text{ for } \zeta<\alpha<\beta \text{ and } \lambda>0, \tag{4.2}$$

where ζ is defined by (1.2), cf. [6, p.127]. Observe that positive solutions $u_\lambda\in C^2(-L,L)\cap C[-L,L]$ for (1.1) correspond to

$$\|u_\lambda\|_\infty=\alpha \text{ and } T_\lambda(\alpha)=L.$$

So by the definition of S_L in (1.3), we have

$$S_L=\{(\lambda,\alpha):T_\lambda(\alpha)=L \text{ for some } \alpha\in(\zeta,\beta) \text{ and } \lambda>0\} \text{ for } L>0. \tag{4.3}$$

Thus, it is important to understand fundamental properties of the time-map $T_\lambda(\alpha)$ on (ζ,β) in order to study the shape of the bifurcation curve S_L of (1.1) for any fixed $L>0$. Note that it can be proved that $T_\lambda(\alpha)$ is a twice continuously differentiable function of $\alpha\in(\zeta,\beta)$ and $\lambda>0$. The proofs are easy but tedious and hence we omit them.

By (4.2), we compute

$$T'_\lambda(\alpha)=\frac{1}{\alpha}\int_0^\alpha\frac{\lambda^3B^3+3\lambda^2B^2+\lambda(2B-A)}{(\lambda^2B^2+2\lambda B)^{3/2}}du \tag{4.4}$$

and

$$T''_\lambda(\alpha) = \frac{1}{\alpha^2}\int_0^\alpha \frac{\left(3A^2B - B^2C - 2AB^2\right)\lambda^3 + \left(3A^2 - 4AB - 2BC\right)\lambda^2}{[\lambda^2B^2 + 2\lambda B]^{5/2}}du, \tag{4.5}$$

where $A(\alpha, u) \equiv \alpha f(\alpha) - uf(u)$ and $C(\alpha, u) \equiv \alpha^2 f'(\alpha) - u^2 f'(u)$, cf. [3, (18) and (24)]. In addition, by (4.4) and (4.5), we have

$$\alpha T''_\lambda(\alpha) + hT'_\lambda(\alpha) = \frac{1}{\alpha}\int_0^r \frac{hB^5\lambda^5 + 5hB^4\lambda^4 + \bar{H}_h(\alpha, u)\lambda^3 + H_h(\alpha, u)\lambda^2}{[\lambda^2B^2 + 2\lambda B]^{5/2}}du, \tag{4.6}$$

where

$$H_h(\alpha, u) \equiv 4hB^2 - 2(h + 2)AB - 2BC + 3A^2 \tag{4.7}$$

and

$$\bar{H}_h(\alpha, u) \equiv 8hB^2 - (h + 2)AB - BC + 3A^2. \tag{4.8}$$

Lemma 4.1. *Consider (1.1). Then*

$$\lim_{u\to 0^+} uf(u) = \lim_{u\to 0^+} u^2 f'(u) = 0 \;\; and \;\; \lim_{u\to 0^+} uf'(u) \geq 0.$$

Proof. We divide this proof into the following three steps.

Step 1. We prove that $\lim_{u\to 0^+} uf(u) = 0$. Suppose $\lim_{u\to 0^+} uf(u) \neq 0$. We consider two cases:

Case 1. Assume that $\lim_{u\to 0^+} uf(u) < 0$. There exist $\delta_1, \rho_1 > 0$ such that $uf(u) < -\rho_1$ for $0 < u \leq \delta_1$. Then

$$F(\delta_1) = \int_0^{\delta_1} f(t)dt < -\rho_1 \int_0^{\delta_1} \frac{1}{t}dt = -\infty,$$

which is a contradiction by (H_2).

Case 2. Assume that $\lim_{u\to 0^+} uf(u) > 0$. There exist $\delta_2, \rho_2 > 0$ such that $uf(u) > \rho_2$ for $0 < u \leq \delta_2$. Then

$$F(\delta_2) = \int_0^{\delta_2} f(t)dt > \rho_2 \int_0^{\delta_2} \frac{1}{t}dt = \infty,$$

which is a contradiction by (H_2).

Thus, by Cases 1 and 2, we obtain that $\lim_{u\to 0^+} uf(u) = 0$.

Step 2. We prove that $\lim_{u\to 0^+} u^2 f'(u) = 0$. Suppose $\lim_{u\to 0^+} u^2 f'(u) \neq 0$. We consider two cases.

Case 1. Assume that $\lim_{u\to 0^+} u^2 f'(u) < 0$. There exist $\delta_3, \rho_3 > 0$ such that $u^2 f'(u) < -\rho_3$ for $0 < u \leq \delta_3$. Then

$$f(\delta_3) - f(0^+) = \int_0^{\delta_3} f'(u)du < -\rho_3 \int_0^{\delta_3} \frac{1}{u^2}dt = -\infty,$$

which implies that $f(0^+) = \infty$. This is a contradiction.

Case 2. Assume that $\lim_{u\to 0^+} u^2 f'(u) > 0$. There exist $\delta_4, \rho_4 > 0$ such that $u^2 f'(u) > \rho_4$ for $0 < u \leq \delta_4$. Then

$$f(\delta_4) - f(0^+) = \int_0^{\delta_4} f'(u)du > \rho_4 \int_0^{\delta_4} \frac{1}{u^2}dt = \infty,$$

which implies that $f(0^+) = -\infty$. So by Step 1 and L'Hôpital's rule, we see that

$$0 = \lim_{u\to0^+} uf(u) = \lim_{u\to0^+} \frac{f(u)}{\frac{1}{u}} = -\lim_{u\to0^+} u^2 f'(u) < 0,$$

which is a contradiction.

Thus, by Cases 1 and 2, we obtain that $\lim_{u\to0^+} u^2 f'(u) = 0$.

Step 3. We prove that $\lim_{u\to0^+} uf'(u) \geq 0$. Suppose $\lim_{u\to0^+} uf'(u) < 0$. There exist $\delta_5, \rho_5 > 0$ such that $uf'(u) < -\rho_5$ for $0 < u \leq \delta_5$. Then

$$f(\delta_5) - f(0^+) = \int_0^{\delta_5} f'(u)du < -\rho_5 \int_0^{\delta_5} \frac{1}{u} du = -\infty,$$

which implies that $f(0^+) = \infty$. This is a contradiction.

The proof is complete. □

Lemma 4.2 (see Figure 4)**.** *Consider (1.1). Assume that* (E_1) *holds. Let* $\theta(u) \equiv 2F(u) - uf(u)$. *Then the following statements (i)–(iii) hold:*

(i) $\theta'(u) < 0$ *on* $\left(0, \bar{\beta}\right)$, $\theta'(\bar{\beta}) = 0$ *and* $\theta'(u) > 0$ *on* $\left(\bar{\beta}, \beta\right)$.

(ii) If $\theta(\beta^-) > 0$, *there exists* $\tau \in (\zeta, \beta)$ *such that* $\theta(u) < 0$ *on* $(0, \tau)$, $\theta(\tau) = 0$ *and* $\theta(u) > 0$ *on* (τ, β). *Moreover,* $T'_\lambda(\alpha) > 0$ *for* $\tau \leq \alpha < \beta$ *and* $\lambda > 0$.

(iii) If $\theta(\beta^-) \leq 0$, *then* $\theta(u) < 0$ *for* $0 < u < \beta$.

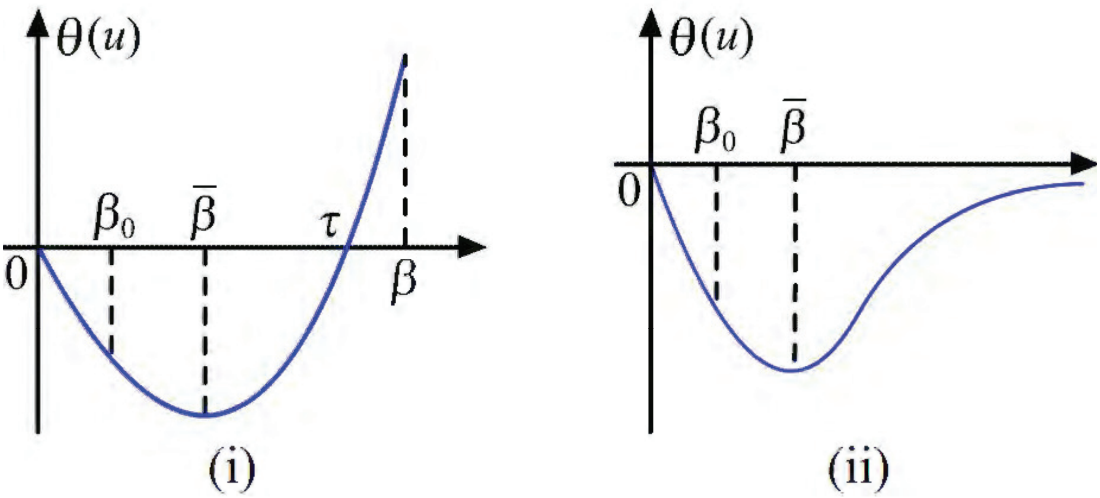

Figure 4. Graphs of θ on $(0, \beta)$. (i) $\theta(\beta^-) > 0$. (ii) $\theta(\beta^-) \leq 0$.

Proof. (I) Since

$$g'(u) = \frac{uf'(u) - f(u)}{u^2},$$

and by (E_1), we see that

$$\theta'(u) = f(u) - uf'(u) = -u^2 g'(u) \begin{cases} < 0 & \text{for } 0 < u < \bar{\beta}, \\ = 0 & \text{for } u = \bar{\beta}, \\ > 0 & \text{for } \bar{\beta} < u < \beta, \end{cases} \tag{4.9}$$

which implies that statement (i) holds.

(II) Assume that $\theta(\beta^-) > 0$. By Lemma 4.1, then $\theta(0^+) = 0$. So by (4.9), there exists $\tau \in (\bar{\beta}, \beta)$ such that $\theta(u) < 0$ on $(0, \tau)$, $\theta(\tau) = 0$, and $\theta(u) > 0$ on (τ, β). Since $\bar{\beta} < \tau < \beta$ and $\theta(\tau) = 0$, we observe that $2F(\tau) = \tau f(\tau) > 0$. It follows that $\tau > \zeta$. In addition, since

$$2B(\alpha, u) - A(\alpha, u) = \theta(\alpha) - \theta(u) > 0 \quad \text{for } 0 < u < \alpha \text{ and } \tau \le \alpha < \beta,$$

and by (4.1) and (4.4), we see that $T'_\lambda(\alpha) > 0$ for $\tau \le \alpha < \beta$ and $\lambda > 0$. Statement (ii) holds.

(III) Assume that $\theta(\beta^-) \le 0$. Since $\theta(0^+) = 0$, and by (4.9), we see that $\theta(u) < 0$ for $0 < u < \beta$. Statement (iii) holds. The proof is complete. □

Lemma 4.3. *Consider (1.1). Fix $\alpha \in (\zeta, \beta)$, then $\partial T_\lambda(\alpha)/\partial\lambda < 0$ for $\lambda > 0$.*

Proof. By (4.1) and (4.2), we compute that

$$\frac{\partial}{\partial\lambda} T_\lambda(\alpha) = \int_0^\eta \frac{-B(\alpha, u)}{\left[\lambda^2 B^2(\alpha, u) + 2\lambda B(\alpha, u)\right]^{3/2}} du < 0 \quad \text{for } \zeta < \alpha < \beta \text{ and } \lambda > 0,$$

cf. [3, p. 295]. The proof is complete. □

Lemma 4.4. *Consider (1.1). Assume that (E_1) holds. Let*

$$\omega \equiv \begin{cases} \beta & \text{if } \theta(\beta^-) \le 0, \\ \tau & \text{if } \theta(\beta^-) > 0, \end{cases} \tag{4.10}$$

where τ is defined in Lemma 4.2. Then the following statements (i) and (ii) hold.

(i) If (E_2) holds, then $\alpha T''_\lambda(\alpha) + [3 + N(\alpha)]\, T'_\lambda(\alpha) > 0$ for $\zeta < \alpha < \omega$ and $\lambda > 0$.
(ii) If (E_3) holds and $\zeta \le \bar{\beta}$, then $\alpha T''_\lambda(\alpha) + [3 + N(\alpha)]\, T'_\lambda(\alpha) > 0$ for $\bar{\beta} < \alpha < \omega$ and $\lambda > 0$.

Proof. (I) Assume that (E_2) holds. Since $\theta(u) = 2F(u) - uf(u)$, and by Lemma 4.2, we obtain

$$\Lambda = \{u \in (0, \beta) : \theta(u) \ne 0\} = (0, \omega).$$

By (E_2), then

$$-3 \le N(0^+) \le N(u) \le N(\alpha) \quad \text{for } 0 < u < \alpha < \omega. \tag{4.11}$$

Since

$$\begin{aligned} A(\alpha, u) - C(\alpha, u) &= \alpha f(\alpha) - uf(u) - \alpha^2 f'(\alpha) + u^2 f'(u) \\ &= \alpha \left[f(\alpha) - \alpha f'(\alpha)\right] - u \left[f(u) - uf'(u)\right] \\ &= \alpha\theta'(\alpha) - u\theta'(u), \end{aligned}$$

and by (4.1), (4.7), (4.8), (4.11) and Lemma 4.2, we observe that

$$\begin{aligned} 2\bar{H}_{3+N(\alpha)}(\alpha, u) &= H_{3+N(\alpha)}(\alpha, u) + 3A^2(\alpha, u) + 12\left[3 + N(\alpha)\right] B^2(\alpha, u) \\ &\ge H_{3+N(\alpha)}(\alpha, u) \\ &= 4\left[N(\alpha) + 3\right] B^2 - 2(\left[N(\alpha) + 5\right] AB - 2BC + 3A^2 \end{aligned}$$

$$
\begin{aligned}
&= 3\,[2B(\alpha,u)-A(\alpha,u)]^2 \\
&\quad +\ 2B(\alpha,u)\,\{A(\alpha,u)-C(\alpha,u)+N(\alpha)\,[2B(\alpha,u)-A(\alpha,u)]\} \\
&\geq 2B(\alpha,u)\,\{A(\alpha,u)-C(\alpha,u)+N(\alpha)\,[2B(\alpha,u)-A(\alpha,u)]\} \\
&= 2B(\alpha,u)\,\{\alpha\theta'(\alpha)-u\theta'(u)+N(\alpha)\,[\theta(\alpha)-\theta(u)]\} \\
&= 2B(\alpha,u)\theta(u)\left[-N(\alpha)\frac{\theta(\alpha)}{\theta(u)}+N(u)+N(\alpha)\left(\frac{\theta(\alpha)}{\theta(u)}-1\right)\right] \\
&= 2B(\alpha,u)\,[-\theta(u)]\,[N(\alpha)-N(u)]\geq 0 \qquad (4.12)
\end{aligned}
$$

for $0<u<\alpha<\omega$. By (4.1), (4.6) and (4.12), statement (i) holds.

(II) Assume that (E_3) holds. Since $\zeta\leq\bar{\beta}$, and by the similar argument in [7, Proof of Theorem 2.1], we obtain that

$$N'(u)\geq 0\ \ \text{for}\ \bar{\beta}<u<\omega. \qquad (4.13)$$

By Lemma 4.2, we see that

$$N(\alpha)>0\ \ \text{for}\ \bar{\beta}<\alpha<\omega. \qquad (4.14)$$

Let $\phi(u)=u\theta'(u)+N(\alpha)\theta(u)$. If $\bar{\beta}<\alpha<\omega$, by (4.13), (4.14) and Lemma 4.2, then we observe that

$$
\begin{cases}
\phi(u)<0\ \ \text{for}\ 0<u<\bar{\beta}, \\
\phi(\bar{\beta})=N(\alpha)\theta(\bar{\beta})<0, \\
\phi(u)=\theta(u)\,[N(\alpha)-N(u)]<0\ \ \text{for}\ \bar{\beta}<u<\alpha, \\
\phi(\alpha)=0,
\end{cases}
$$

see Figure 5. It follows that

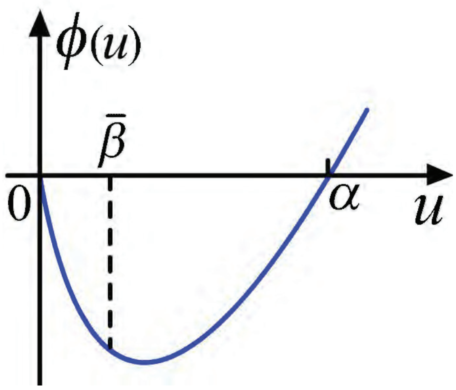

Figure 5. The graph of $\phi(u)$.

$$A(\alpha,u)-C(\alpha,u)+N(\alpha)\,[2B(\alpha,u)-A(\alpha,u)]=\phi(\alpha)-\phi(u)>0 \qquad (4.15)$$

for $0<u<\alpha$ and $\bar{\beta}<\alpha<\omega$. By (4.15) and the similar argument in (4.12), we obtain that

$$
\begin{aligned}
2\bar{H}_{3+N(\alpha)}(\alpha,u) &\geq H_{3+N(\alpha)}(\alpha,u) \\
&\geq 2B(\alpha,u)\,\{A(\alpha,u)-C(\alpha,u)+N(\alpha)\,[2B(\alpha,u)-A(\alpha,u)]\} \\
&> 0
\end{aligned}
$$

for $0 < u < \alpha$ and $\bar{\beta} < \alpha < \omega$. Thus

$$\bar{H}_{3+N(\alpha)}(\alpha, u) > 0 \text{ and } H_{3+N(\alpha)}(\alpha, u) > 0 \text{ for } 0 < u < \alpha \text{ and } \bar{\beta} < \alpha < \omega.$$

Then by (4.1) and (4.6), statement (ii) holds. The proof is complete. □

Lemma 4.5. *Consider (1.1). Assume that (E_1) and (E_3) hold, and either ($\beta_0 = 0$) or ($\beta_0 > 0$, $W(0^+) \leq 1$ and $\zeta < \bar{\beta}$). Then $T''_\lambda(\alpha) > 0$ for $\zeta < \alpha \leq \bar{\beta}$ and $\lambda > 0$.*

Proof. Notice that if $\beta_0 = 0$, by (H_1), we observe that $\zeta = 0 < \bar{\beta}$. Thus, regardless of whether $\beta_0 = 0$ or $\beta_0 > 0$, we can assume that $\zeta < \bar{\beta}$. Let $\alpha \in (\zeta, \bar{\beta}]$ be given. We divide this proof into the following four steps.

Step 1. We prove that $f(u)[W(\alpha) - W(u)] \leq 0$ for $0 < u < \alpha$ and $u \neq \beta_0$. We consider two cases.

Case 1. Assume that $\beta_0 = 0$. By (H_1), then $f(u) > 0$ and $W(u)$ is continuous on $(0, \alpha)$. So by (E_3), then $f(u)[W(\alpha) - W(u)] \leq 0$ for $0 < u < \alpha$.

Case 2. Assume that $\beta_0 > 0$ and $W(0^+) \leq 1$. By Lemma 4.2, then $0 = \theta'(\bar{\beta}) = f(\bar{\beta}) - \bar{\beta} f'(\bar{\beta})$. It implies that $W(\bar{\beta}) = 1$. Since $W(u)$ is continuous on $\left(\beta_0, \bar{\beta}\right)$, and by ($E_3$), we see that

$$W(u) \geq W(\alpha) \geq W(\bar{\beta}) = 1 \text{ for } \beta_0 < u < \alpha. \tag{4.16}$$

Since $f(u) > 0$ on (β_0, β), and by (4.16), we see that

$$f(u)[W(\alpha) - W(u)] \leq 0 \text{ for } \beta_0 < u < \alpha. \tag{4.17}$$

In addition, since $W(u)$ is continuous on $(0, \beta_0)$, and by (E_3) and (4.16), we see that

$$W(u) \leq W(0^+) \leq 1 \leq W(\alpha) \text{ for } 0 < u < \beta_0. \tag{4.18}$$

Since $f(u) < 0$ on $(0, \beta_0)$, and by (4.18), we see that

$$f(u)[W(\alpha) - W(u)] \leq 0 \text{ for } 0 < u < \beta_0. \tag{4.19}$$

By (4.17) and (4.19), then $f(u)[W(\alpha) - W(u)] \leq 0$ for $0 < u < \alpha$.

Step 2. We prove that

$$\frac{A(\alpha, u)}{B(\alpha, u)} \geq W(\alpha) + 1 > 0 \text{ for } 0 < u < \alpha. \tag{4.20}$$

Let $K(u) \equiv A(\alpha, u) - [W(\alpha) + 1] B(\alpha, u)$. By Step 1, we see that

$$K'(u) = f(u)[W(\alpha) - W(u)] \leq 0 \text{ for } 0 < u < \alpha \text{ and } u \neq \beta_0. \tag{4.21}$$

Since $K(u)$ is continuous on $(0, \alpha]$, and by (4.21), we see that $K(u) \geq K(\alpha) = 0$ for $0 < u < \alpha$. Then (4.20) holds by (4.1) and (4.16).

Step 3. We prove that

$$\frac{C(\alpha, u)}{A(\alpha, u)} \leq W(\alpha) \text{ for } 0 < u < \alpha. \tag{4.22}$$

By (4.1) and (4.20), then $A(\alpha,u) > 0$ for $0 < u < \alpha$. Then by Step 1, we see that

$$\frac{C(\alpha,u)}{A(\alpha,u)} - W(\alpha) = \frac{uf(u)\,[W(\alpha) - W(u)]}{A(\alpha,u)} \leq 0 \text{ for } 0 < u < \alpha \text{ and } u \neq \beta_0.$$

By continuity of $\frac{C(\alpha,u)}{A(u,\alpha)}$ on $(0,\beta)$, then (4.22) holds.

Step 4. We prove Lemma 4.5. By (4.1), (4.7), (4.8), (4.16), (4.20) and (4.22), we observe that

$$\begin{aligned}
2\bar{H}_0(\alpha,u) &= H_0(\alpha,u) + 3A^2(\alpha,u)\\
&\geq H_0(\alpha,u)\\
&= -4A(\alpha,u)B(\alpha,u) - 2B(\alpha,u)C(\alpha,u) + 3A^2(\alpha,u)\\
&= B^2(\alpha,u)\left[-4\frac{A(\alpha,u)}{B(\alpha,u)} - 2\frac{C(\alpha,u)}{B(\alpha,u)} + 3\left(\frac{A(\alpha,u)}{B(\alpha,u)}\right)^2\right]\\
&\geq B^2(\alpha,u)\left[-4\frac{A(\alpha,u)}{B(\alpha,u)} - 2W(\alpha)\frac{A(\alpha,u)}{B(\alpha,u)} + 3\left(\frac{A(\alpha,u)}{B(\alpha,u)}\right)^2\right]\\
&= A(\alpha,u)B(\alpha,u)\left[3\frac{A(\alpha,u)}{B(\alpha,u)} - 2W(\alpha) - 4\right]\\
&\geq A(\alpha,u)B(\alpha,u)\,[3\,(W(\alpha)+1) - 2W(\alpha) - 4]\\
&= A(\alpha,u)B(\alpha,u)\,[W(\alpha) - 1] \geq 0
\end{aligned}$$

for $0 < u < \alpha$. So by (4.1) and (4.6), $T''_\lambda(\alpha) > 0$ for $\zeta < \alpha < \bar{\beta}$ and $\lambda > 0$.

The proof is complete. □

Lemma 4.6. *Consider (1.1) with $\beta_0 > 0$. Assume that (E_1) and (E_3) hold, and that $W(0^+) \leq W(\beta^-)$ and $W(\beta^-) > -1/3$. Then $\alpha T''_\lambda(\alpha) + \left[\frac{W(\alpha)+1}{2}\right] T'_\lambda(\alpha) > 0$ for $\zeta < \alpha < \beta$ and $\lambda > 0$.*

Proof. Let $\alpha \in (\zeta,\beta)$ be given. We divide this proof into the following two steps.

Step 1. We prove that $f(u)\,[W(\alpha) - W(u)] \leq 0$ for $0 < u < \alpha$ and $u \neq \beta_0$. If $\alpha \leq \bar{\beta}$, by a similar argument in the proof of Lemma 4.5, we obtain that

$$f(u)\,[W(\alpha) - W(u)] \leq 0 \text{ for } 0 < u < \alpha \text{ and } u \neq \beta_0.$$

If $\bar{\beta} < \alpha < \omega$, by a similar argument used in the proofs of (4.16) and (4.18), we observe that

$$W(u) \geq W(\alpha) \text{ for } \beta_0 < u < \alpha$$

and

$$W(u) \leq W(0^+) \leq W(\beta^-) \leq W(\alpha) \text{ for } 0 < u < \beta_0.$$

Thus, $f(u)\,[W(\alpha) - W(u)] \leq 0$ for $0 < u < \alpha$ and $u \neq \beta_0$.

Step 2. We prove Lemma 4.6. By a similar argument in the proof of Lemma 4.5, we see that

$$\frac{A(\alpha,u)}{B(\alpha,u)} \geq W(\alpha) + 1 \geq W(\beta^-) + 1 > \frac{2}{3} \text{ for } 0 < u < \alpha. \tag{4.23}$$

In addition, by the same argument as in Step 3 of the proof of Lemma 4.5, we see that

$$\frac{C(\alpha,u)}{A(\alpha,u)} \leq W(\alpha) \quad \text{for } 0 < u < \alpha. \tag{4.24}$$

By (4.1), (4.7), (4.8), (4.23) and (4.24), we observe that

$$\begin{aligned}
& 2\bar{H}_{\frac{W(\alpha)+1}{2}}(\alpha,u) \\
= \;& H_{\frac{W(\alpha)+1}{2}}(\alpha,u) + 6\left[W(\alpha)+1\right]B^2(\alpha,u) + 3A^2(\alpha,u) \\
> \;& H_{\frac{W(\alpha)+1}{2}}(\alpha,u) \\
= \;& B^2(\alpha,u)\left[2W(\alpha)+2-\left[W(\alpha)+5\right]\frac{A(\alpha,u)}{B(\alpha,u)} - 2\frac{C(\alpha,u)}{B(\alpha,u)} + 3\left(\frac{A(\alpha,u)}{B(\alpha,u)}\right)^2\right] \\
\geq \;& B^2(\alpha,u)\left[2W(\alpha)+2-\left[W(\alpha)+5\right]\frac{A(\alpha,u)}{B(\alpha,u)} - 2W(\alpha)\frac{A(\alpha,u)}{B(\alpha,u)} + 3\left(\frac{A(\alpha,u)}{B(\alpha,u)}\right)^2\right] \\
= \;& 3B^2(\alpha,u)\left(\frac{A(\alpha,u)}{B(\alpha,u)} - \frac{2}{3}\right)\left[\frac{A(\alpha,u)}{B(\alpha,u)} - (W(\alpha)+1)\right] > 0
\end{aligned}$$

for $0 < u < \alpha$. So by (4.1) and (4.6), $\alpha T''_\lambda(\alpha) + \left[\frac{W(\alpha)+1}{2}\right] T'_\lambda(\alpha) > 0$ for $\zeta < \alpha < \beta$ and $\lambda > 0$. The proof is complete. □

Lemma 4.7. *Consider (1.1). Assume that* (E_4) *holds,* $\beta_0 > 0$ *and* $W(\zeta) \leq 2$*. Then* $\alpha T''_\lambda(\alpha) + 2T'_\lambda(\alpha) > 0$ *for* $\zeta < \alpha < \beta$ *and* $\lambda > 0$*.*

Proof. Let $Q(u) \equiv u\theta'(u) - \theta(u)$. By ($E_4$), there exists $\gamma \in (0,\beta)$ such that

$$Q'(u) = u\theta''(u) = -u^2 f''(u) \begin{cases} < 0 & \text{for } 0 < u < \gamma, \\ = 0 & \text{for } u = \gamma, \\ > 0 & \text{for } \gamma < u < \beta. \end{cases} \tag{4.25}$$

Since $\beta_0 > 0$, we see that $\zeta > 0$ and $f(\zeta) > 0$. Since $Q(u) = 2uf(u) - 2F(u) - u^2 f'(u)$ and $W(\zeta) \leq 2$, and by Lemma 4.1, we further see that

$$Q(0^+) = 0 \quad \text{and} \quad Q(\zeta) = 2\zeta f(\zeta) - \zeta^2 f'(\zeta) = \zeta f(\zeta)\left[2 - W(\zeta)\right] \geq 0 \tag{4.26}$$

By (4.1), (4.7), (4.8), (4.25) and (4.26), we see that

$$\begin{aligned}
2\bar{H}_2(\alpha,u) &= H_2(\alpha,u) + 3\left[A^2(\alpha,u) + 8B^2(\alpha,u)\right] \\
&> H_2(\alpha,u) \\
&= 3\left[2B(\alpha,u) - A(\alpha,u)\right]^2 + 2B(\alpha,u)\left[2A(\alpha,u) - 2B(\alpha,u) - C(\alpha,u)\right] \\
&> 2B(\alpha,u)\left[Q(\alpha) - Q(u)\right] \\
&> 0.
\end{aligned}$$

So by (4.1) and (4.6), $\alpha T''_\lambda(\alpha) + 2T'_\lambda(\alpha) > 0$ for $\zeta < \alpha < \beta$ and $\lambda > 0$. The proof is complete. □

Lemma 4.8. *Consider (1.1). Assume that the hypotheses of Theorem 2.2 hold, and that $T_\lambda(\alpha)$ has a critical number $\bar{\alpha}_\lambda$ on (ζ,β) for $\lambda > 0$. Then*

$$T'_\lambda(\alpha)\begin{cases} < 0 & \text{for } \zeta < \alpha < \bar{\alpha}_\lambda, \\ = 0 & \text{for } \alpha = \bar{\alpha}_\lambda, \\ > 0 & \text{for } \bar{\alpha}_\lambda < \alpha < \beta. \end{cases} \tag{4.27}$$

Furthermore, $T_\lambda(\bar{\alpha}_\lambda)$ is a strictly decreasing and continuous function with respect to $\lambda > 0$.

Proof. Recall the number ω defined in (4.10). If (E_1) holds, and by Lemma 4.2(ii), then $\zeta < \bar{\alpha}_\lambda < \omega$. Next, we divide the proof into the following four steps.

Step 1. We prove Lemma 4.8 if (E_1) and (E_2) hold. Since $T'_\lambda(\bar{\alpha}_\lambda) = 0$ for $\lambda > 0$, and by Lemma 4.4(i), we see that

$$T''_\lambda(\bar{\alpha}_\lambda) = T''_\lambda(\bar{\alpha}_\lambda) + \frac{3 + N(\bar{\alpha}_\lambda)}{\bar{\alpha}_\lambda} T'_\lambda(\bar{\alpha}_\lambda) > 0 \text{ for } \lambda > 0.$$

It follows that $T_\lambda(\alpha)$ has exactly one critical number $\bar{\alpha}_\lambda$, a local minimum, on (ζ,β). So (4.27) holds. In addition, since $T''_\lambda(\bar{\alpha}_\lambda) > 0$ for $\lambda > 0$, and by the implicit function theorem, $\bar{\alpha}_\lambda$ is a continuously differentiable function for $\lambda > 0$. Then $T_\lambda(\bar{\alpha}_\lambda)$ is also continuously differentiable for $\lambda > 0$. Since $T'_\lambda(\bar{\alpha}_\lambda) = 0$, and by Lemma 4.3, we observe that

$$\frac{\partial}{\partial\lambda} T_\lambda(\bar{\alpha}_\lambda) = T'_\lambda(\bar{\alpha}_\lambda)\frac{\partial\bar{\alpha}_\lambda}{\partial\lambda} + \left[\frac{\partial}{\partial\lambda} T_\lambda(\alpha)\right]_{\alpha=\bar{\alpha}_\lambda} = \left[\frac{\partial}{\partial\lambda} T_\lambda(\alpha)\right]_{\alpha=\bar{\alpha}_\lambda} < 0 \text{ for } \lambda > 0.$$

Thus Lemma 4.8 holds.

Step 2. We prove Lemma 4.8 if (E_1) and (E_3) hold, and either $(\beta_0 = 0)$ or $(\beta_0 > 0,\ W(0^+) \le 1$ and $\zeta < \bar{\beta})$. If $\beta_0 = 0$, then $\zeta = 0 < \bar{\beta}$. Thus, we can assume that $\zeta < \bar{\beta}$. By Lemmas 4.4(ii) and 4.5, then

$$T''_\lambda(\bar{\alpha}_\lambda) > 0 \text{ if } \zeta < \bar{\alpha}_\lambda \le \bar{\beta}$$

and

$$T''_\lambda(\bar{\alpha}_\lambda) = T''_\lambda(\bar{\alpha}_\lambda) + \frac{3 + N(\bar{\alpha}_\lambda)}{\bar{\alpha}_\lambda} T'_\lambda(\bar{\alpha}_\lambda) > 0 \text{ if } \bar{\beta} < \bar{\alpha}_\lambda < \omega$$

for $\lambda > 0$. So $T_\lambda(\alpha)$ has exactly one critical number $\bar{\alpha}_\lambda$, a local minimum, on (ζ,β). It implies that (4.27) holds. Then by the similar argument in Step 1, $T_\lambda(\bar{\alpha}_\lambda)$ is a strictly decreasing and continuous function with respect to $\lambda > 0$. Thus Lemma 4.8 holds.

Step 3. We prove Lemma 4.8 if (E_1) and (E_3) hold, $\beta_0 > 0$, $W(0^+) \le W(\beta^-)$ and $W(\beta^-) > -1/3$. By Lemma 4.6, then

$$T''_\lambda(\bar{\alpha}_\lambda) = T''_\lambda(\bar{\alpha}_\lambda) + \left[\frac{W(\bar{\alpha}_\lambda) + 1}{2\bar{\alpha}_\lambda}\right] T'_\lambda(\bar{\alpha}_\lambda) > 0 \text{ for } \lambda > 0.$$

By the similar argument in Step 1, Lemma 4.8 holds.

Step 4. We prove Lemma 4.8 if (E_4) holds, $\beta_0 > 0$ and $W(\zeta) \le 2$. By Lemma 4.7, we see that

$$T''_\lambda(\bar{\alpha}_\lambda) = T''_\lambda(\bar{\alpha}_\lambda) + \frac{2}{\bar{\alpha}_\lambda} T'_\lambda(\bar{\alpha}_\lambda) > 0 \text{ for } \lambda > 0.$$

By the similar argument in Step 1, Lemma 4.8 holds.

The proof is complete. □

By [2, Lemma 4.6] and [3, Lemma 4.6], we have the following lemma.

Lemma 4.9. *Consider (1.1) with fixed $L > 0$. Then the following statements (i)–(ii) hold:*

(i) There exists a positive function $\lambda_L(\alpha) \in C^1(\zeta, m_{L,\beta})$ such that $T_{\lambda_L(\alpha)}(\alpha) = L$. Moreover, the bifurcation curve $S_L = \left\{(\lambda_L(\alpha), \alpha) : \alpha \in (\zeta, m_{L,\beta})\right\}$ is continuous on the $(\lambda, \|u\|_\infty)$-plane.
(ii) $sgn(\lambda_L'(\alpha)) = sgn(T'_{\lambda_L(\alpha)}(\alpha))$ for $\alpha \in (\zeta, m_{L,\beta})$ where $sgn(u)$ is the signum function.

Lemma 4.10. *Consider (1.1) with*

$$f(u) = u\left(k - u - \frac{1}{1 + mu}\right).$$

Then the following statements (i) and (ii) hold.

(i) Assume that $m > 1$. Then there exists $k_m \in (\frac{2\sqrt{m}-1}{m}, 1)$ such that
   (a) if $k > k_m$, then (H_1)–(H_2), (E_1) and (E_2) hold; and
   (b) if $0 < k \le k_m$, then $F(u) \le 0$ on $(0, \infty)$.

(ii) Assume that $0 < m \le 1$. Then
   (c) if $k > 1$, then (H_1)–(H_2) hold, and $g'(u) < 0$ on $(0, \infty)$; and
   (d) if $k \le 1$, then $F(u) \le 0$ on $(0, \infty)$.

Proof. Clearly, we have

$$g(u) = \frac{f(u)}{u} = k - u - \frac{1}{1 + mu}.$$

(I) Assume that $m > 1$. The proof of statement (i) is divided into the following three steps.

Step 1. We prove that (H_1) and (E_1) hold for $k > \frac{2\sqrt{m}-1}{m}$. Let $\bar{\beta}_m \equiv \left(\sqrt{m} - 1\right)/m$ and $\beta_{k,m}$ be defined by (2.2). It is easy to compute that

$$g'(u) = \frac{-m^2u^2 - 2mu + m - 1}{(mu + 1)^2} \begin{cases} > 0 & \text{for } 0 < u < \bar{\beta}_m, \\ = 0 & \text{for } u = \bar{\beta}_m, \\ < 0 & \text{for } u > \bar{\beta}_m, \end{cases} \tag{4.28}$$

$$g(\beta_{k,m}) = 0 \text{ and } g(\bar{\beta}_m) = k - \frac{2\sqrt{m} - 1}{m} \begin{cases} \le 0 & \text{if } 0 < k \le \frac{2\sqrt{m}-1}{m}, \\ > 0 & \text{if } k > \frac{2\sqrt{m}-1}{m}. \end{cases} \tag{4.29}$$

We observe that

$$\begin{aligned} \beta_{k,m} - \bar{\beta}_m &= \frac{\sqrt{(km + 1)^2 - 4m} + km - 2\sqrt{m} + 1}{2m} \\ &> \frac{\sqrt{\left[\left(\frac{2\sqrt{m}-1}{m}\right)m + 1\right]^2 - 4m} + \left(\frac{2\sqrt{m}-1}{m}\right)m - 2\sqrt{m} + 1}{2m} \\ &= 0 \text{ for } k > \frac{2\sqrt{m} - 1}{m}. \end{aligned} \tag{4.30}$$

Since $f(u) = ug(u)$, and by (4.28)–(4.30), we see that (E_1) holds and either (H_{1a}) or (H_{1b}) holds if $k > \frac{2\sqrt{m}-1}{m}$.

Step 2. We prove that there exists $k_m \in (\frac{2\sqrt{m}-1}{m}, 1)$ such that (H_2) holds if $k > k_m$ and statement (i)(b) holds. We compute

$$F(u) = \frac{6\ln(mu+1) - 2m^2u^3 + 3km^2u^2 - 6mu}{6m^2}. \tag{4.31}$$

Since $f(\beta_{k,m}) = 0$ for $k \geq \frac{2\sqrt{m}-1}{m}$, and by (4.31), we observe that

$$\frac{\partial}{\partial k}F(\beta_{k,m}) = \left[\frac{\partial}{\partial k}F(u)\right]_{u=\beta_{k,m}} = \frac{\beta_{k,m}^2}{2} > 0 \ \text{ for } k \geq \frac{2\sqrt{m}-1}{m}. \tag{4.32}$$

Since

$$\frac{\partial}{\partial m}\left[3\ln(m) - \frac{\left(\sqrt{m}-1\right)\left(2m+5\sqrt{m}-1\right)}{m}\right] = \frac{\left(1-\sqrt{m}\right)^3}{m^2} < 0 \ \text{ for } m > 1,$$

we observe that

$$\begin{aligned}[F(\beta_{k,m})]_{k=\frac{2\sqrt{m}-1}{m}} &= \frac{3\ln(m) - \frac{(\sqrt{m}-1)(2m+5\sqrt{m}-1)}{m}}{6m^2} \\ &< \frac{\lim\limits_{m\to1^+}\left[3\ln(m) - \frac{(\sqrt{m}-1)(2m+5\sqrt{m}-1)}{m}\right]}{6m^2} = 0 \ \text{ for } m > 1. \end{aligned} \tag{4.33}$$

Since

$$\frac{\partial}{\partial m}\left[6\ln(m) + (m-1)\,\frac{m^2-5m-2}{m}\right] = \frac{2(m-1)^3}{m^2} > 0 \ \text{ for } m > 1,$$

we observe that

$$\begin{aligned}[F(\beta_{k,m})]_{k=1} &= \frac{6\ln(m) + (m-1)\,\frac{m^2-5m-2}{m}}{6m^2} \\ &> \frac{\lim\limits_{m\to1^+}\left[6\ln(m) + (m-1)\,\frac{m^2-5m-2}{m}\right]}{6m^2} = 0 \ \text{ for } m > 1. \end{aligned} \tag{4.34}$$

Clearly, $1 > \frac{2\sqrt{m}-1}{m}$ for $m > 1$. So by (4.32)–(4.34), there exists $k_m \in (\frac{2\sqrt{m}-1}{m}, 1)$ such that

$$F(\beta_{k,m}) \leq 0 \ \text{ for } \frac{2\sqrt{m}-1}{m} < k \leq k_m, \ \text{ and } F(\beta_{k,m}) > 0 \ \text{ for } k > k_m. \tag{4.35}$$

By (4.28) and (4.29), then $f(u) = ug(u) \leq ug(\bar{\beta}_m) \leq 0$ for $u > 0$ and $0 < k \leq \frac{2\sqrt{m}-1}{m}$. So we see that

$$F(u) \leq 0 \ \text{ for } u > 0 \text{ and } 0 < k \leq \frac{2\sqrt{m}-1}{m}. \tag{4.36}$$

Since $F(u) < F(\beta_{k,m})$ for $0 < u < \beta_{k,m}$, and by (4.35) and (4.36), we see that (H_2) holds if $k > k_m$ and statement (i)(b) holds.

Step 3. We prove statement (i). By Steps 1 and 2, it is sufficient to prove that (E_2) holds if $k > k_m$. Assume that $k > k_m$. Since $\theta(u) = 2F(u) - uf(u)$, we see that

$$N'(u) = -\frac{\theta(u)\theta'(u) + u\theta(u)\theta''(u) - u\theta'(u)\theta'(u)}{\theta^2(u)} = \frac{2(mu+1)u^2}{m^2\,(mu+1)^4}V_1(u), \tag{4.37}$$

where

$$\begin{aligned} V_1(u) \;\equiv\; & V_2(u)\ln(mu+1) \\ & + \frac{mu\left[4m^4u^4 + 37m^3u^3 + 78m^2u^2 + 3m\,(21-5m)\,u - 18\,(m-1)\right]}{6(mu+1)} \end{aligned}$$

and

$$V_2(u) \equiv -3m^3u^3 - 9m^2u^2 + m\,(m-9)\,u + 3\,(m-1)\,.$$

Since $V_2''(u) = -18m^3u - 18m^2 < 0$ for $u > 0$, we see that $V_2(u)$ is either strictly decreasing, or strictly increasing and then strictly decreasing on $(0,\beta_{k,m})$. Since $V_2(0) = 3\,(m-1) > 0$ and $V_2(\bar{\beta}_m) = -2m\left(\sqrt{m}-1\right) < 0$, there exists $\beta_1 \in (0,\bar{\beta}_m)$ such that

$$V_2(u)\begin{cases} > 0 & \text{for } 0 < u < \beta_1, \\ = 0 & \text{for } u = \beta_1, \\ < 0 & \text{for } \beta_1 < u < \beta_{k,m}. \end{cases} \tag{4.38}$$

We compute

$$\frac{d}{du}\frac{V_1(u)}{V_2(u)} = \frac{(2m^3u^3 + 3m^2u^2 + m - 1)m^4u^3}{(mu+1)^2V_2^2(u)}\bar{V}(u) \;\text{ for } u \neq \beta_1, \tag{4.39}$$

where $\bar{V}(u) \equiv -m^2u^2 - 2mu + m - 1$. We further compute $\bar{V}(0) = m - 1 > 0$ and $\bar{V}(\bar{\beta}_m) = 0$. Since $\bar{V}(u)$ is a quadratic polynomial of u, we observe that

$$\bar{V}(u)\begin{cases} > 0 & \text{for } 0 < u < \bar{\beta}_m, \\ = 0 & \text{for } u = \bar{\beta}_m, \\ < 0 & \text{for } \bar{\beta}_m < u < \beta_{k,m}. \end{cases}$$

So by (4.39),

$$\frac{d}{du}\frac{V_1(u)}{V_2(u)}\begin{cases} > 0 & \text{for } 0 < u < \bar{\beta}_m \text{ and } u \neq \beta_1, \\ = 0 & \text{for } u = \bar{\beta}_m, \\ < 0 & \text{for } \bar{\beta}_m < u < \beta_{k,m}. \end{cases} \tag{4.40}$$

By (4.40), we observe that

$$\frac{V_1(u)}{V_2(u)} > \frac{V_1(0)}{V_2(0)} = 0 \;\text{ for } 0 < u < \beta_1.$$

So by (4.38), $V_1(u) \geq 0$ for $0 < u \leq \beta_1$, from which it follows that by (4.37), $N'(u) \geq 0$ for $0 < u \leq \beta_1$. In addition, by (4.40), we observe that

$$\frac{V_1(u)}{V_2(u)} \leq \frac{V_1(\bar{\beta}_m)}{V_2(\bar{\beta}_m)} = \frac{\ln m}{2} - \frac{\sqrt{m}-1}{6m}\left(2m + 5\sqrt{m} - 1\right) < 0 \;\text{ for } \beta_1 < u < \beta_{k,m},$$

because

$$\frac{d}{dm}\frac{V_1(\bar{\beta}_m)}{V_2(\bar{\beta}_m)} = -\frac{\left(\sqrt{m}-1\right)^3}{6m^2} < 0 \text{ and } \left[\frac{V_1(\bar{\beta}_m)}{V_2(\bar{\beta}_m)}\right]_{m=1} = 0.$$

So by (4.38), $V_1(u) > 0$ for $\beta_1 < u < \beta_{k,m}$, from which it follows that by (4.37), $N'(u) > 0$ for $\beta_1 < u < \beta_{k,m}$.

Finally, we compute that

$$\theta'(u) = \frac{u^2\left(m^2u^2 + 2mu - m + 1\right)}{(mu+1)^2} \text{ and } \theta''(u) = \frac{2m^3u^4 + 6m^2u^3 + 6mu^2 - (2m-2)\,u}{(mu+1)^3}.$$

Then by L'Hôpital's rule, we further compute

$$\begin{aligned} N(0^+) &= \lim_{u\to 0^+}\frac{u\theta'(u)}{-\theta(u)} = \lim_{u\to 0^+}\frac{\theta'(u)+u\theta''(u)}{-\theta'(u)} \\ &= \lim_{u\to 0^+}\frac{-3m^3u^3 - 9m^2u^2 + m\,(m-9)\,u + 3\,(m-1)}{(mu+1)\,(-m+m^2u^2+2mu+1)} = -3. \end{aligned}$$

Thus (E_2) holds.

(II) Assume that $0 < m \le 1$ and $k > 1$. It follows that

$$g'(u) = \frac{-m^2u^2 - 2mu + m - 1}{(mu+1)^2} < 0 \text{ for } u > 0. \tag{4.41}$$

Since $g(0^+) = k - 1 > 0$ and $g(\beta_{k,m}) = 0$, and by (4.41), we observe that

$$f(u) = ug(u)\begin{cases} > 0 & \text{for } 0 < u < \beta_{k,m}, \\ = 0 & \text{for } u = \beta_{k,m}, \\ < 0 & \text{for } u > \beta_{k,m}. \end{cases}$$

It follows that $F(\beta_{k,m}) > 0$. So (H_1) and (H_2) hold. Then Lemma 4.10(ii)(c) holds.

Finally, we assume that $0 < m \le 1$ and $k \le 1$. By (4.41), then $g(u) < g(0^+) = k - 1 \le 0$ for $u > 0$. So $F(u) < 0$ for $u > 0$. Then Lemma 4.10(ii)(d) holds. The proof is complete. □

5. Proofs of main results

Proof of Theorem 2.1. Referring to the proof in [2], we deduce that the bifurcation curve S_L approaches $\left(\infty, m_{L,\beta}\right)$ for $L > \zeta$. The remaining results follow from the proofs of Theorems 1.2 and 1.3(iii); see [3]. Thus, we omit them here. □

Proof of Theorem 2.2. We divide this proof into the following three steps.

Step 1. We prove that $\lambda_L(\alpha)$ has at most one critical number on $(\zeta, m_{L,\beta})$. Assume that $\lambda_L(\alpha)$ has two distinct critical points α_1 and α_2 on $(\zeta, m_{L,\beta})$. Let $\lambda_1 = \lambda_L(\alpha_1)$ and $\lambda_2 = \lambda_L(\alpha_2)$. By Lemma 4.9,

$$T_{\lambda_1}(\alpha_1) = T_{\lambda_2}(\alpha_2) = L \text{ and } T'_{\lambda_1}(\alpha_1) = T'_{\lambda_2}(\alpha_2) = 0. \tag{5.1}$$

So by Lemma 4.8,

$$T'_{\lambda_1}(\alpha)\begin{cases} <0 & \text{for } \zeta<\alpha<\alpha_1, \\ =0 & \text{for } \alpha=\alpha_1, \\ >0 & \text{for } \alpha_1<\alpha<\beta, \end{cases} \quad \text{and} \quad T'_{\lambda_2}(\alpha)\begin{cases} <0 & \text{for } \zeta<\alpha<\alpha_2, \\ =0 & \text{for } \alpha=\alpha_2, \\ >0 & \text{for } \alpha_2<\alpha<\beta. \end{cases} \tag{5.2}$$

Next, we consider three cases.

Case 1. If $\lambda_1 < \lambda_2$, by (5.1), (5.2) and Lemma 4.3, then

$$L = T_{\lambda_1}(\alpha_1) > T_{\lambda_2}(\alpha_1) \geq T_{\lambda_2}(\alpha_2) = L,$$

which is a contradiction.

Case 2. If $\lambda_1 > \lambda_2$, by (5.1), (5.2) and Lemma 4.3, then

$$L = T_{\lambda_2}(\alpha_2) > T_{\lambda_1}(\alpha_2) \geq T_{\lambda_1}(\alpha_1) = L,$$

which is a contradiction.

Case 3. If $\lambda_1 = \lambda_2$, by (5.1) and Lemma 4.8, then $\alpha_1 = \alpha_2$, which is a contradiction.

So by Cases 1–3, $\lambda_L(\alpha)$ has at most one critical number on $(\zeta, m_{L,\beta})$.

Step 2. We prove that if $\lambda_L(\alpha)$ has a critical number $\bar{\alpha}$ on $(\zeta, m_{L,\beta})$, then $\lambda_L(\bar{\alpha})$ is a local minimum on $(\zeta, m_{L,\beta})$. Assume that $\lambda_L(\alpha)$ has a critical number $\bar{\alpha}$ on $(\zeta, m_{L,\beta})$. Let $\bar{\lambda} = \lambda_L(\bar{\alpha})$. By Lemma 4.9, then $T_{\bar{\lambda}}(\bar{\alpha}) = L$ and $T'_{\bar{\lambda}}(\bar{\alpha}) = 0$. So by Lemma 4.8, then $\bar{\alpha} = \bar{\alpha}_{\bar{\lambda}}$. Suppose $\lambda_L(\bar{\alpha}) > \lambda_L(\alpha_3)$ for some $\alpha_3 \in (\zeta, m_{L,\beta})$. Let $\lambda_3 = \lambda_L(\alpha_3)$. Since $\bar{\lambda} > \lambda_3$, and by Lemmas 4.3 and 4.9, we observe that

$$T_{\bar{\lambda}}(\alpha_3) < T_{\lambda_3}(\alpha_3) = L = T_{\bar{\lambda}}(\bar{\alpha}) = T_{\bar{\lambda}}(\bar{\alpha}_{\bar{\lambda}}),$$

which is a contradiction by Lemma 4.8. So $\lambda_L(\bar{\alpha}) \leq \lambda_L(\alpha)$ for $\zeta < \alpha < m_{L,\beta}$. It follows that $\lambda_L(\bar{\alpha})$ is a local minimum on $(\zeta, m_{L,\beta})$.

Step 3. We prove Theorem 2.2. We consider two cases.

Case 1. Assume that $\lambda_L(\alpha)$ has no critical numbers. Then $\lambda'_L(\alpha) > 0$ on $(\zeta, m_{L,\beta})$, or $\lambda'_L(\alpha) < 0$ on $(\zeta, m_{L,\beta})$. Suppose that $\lambda'_L(\alpha) < 0$ on $(\zeta, m_{L,\beta})$. Let $\alpha_4 \in (\zeta, m_{L,\beta})$. Since

$$\lim_{\lambda\to\infty} T_\lambda(\alpha_4) = \int_0^{\alpha_4} 1du = \alpha_4 < m_{L,\beta},$$

there exists $\lambda_4 > 0$ such that $T_{\lambda_4}(\alpha_4) < m_{L,\beta} < L$. By [2, (4.11)] and [3, Lemma 4.2], we see that

$$\lim_{\alpha\to\infty} T_{\lambda_4}(\alpha) \geq \lim_{\alpha\to\infty} \alpha = \infty \quad \text{if } \beta = \infty$$

and

$$\lim_{\alpha\to\beta^-} T_{\lambda_4}(\alpha) = \infty \quad \text{if } \beta < \infty.$$

So there exists $\alpha_5 \in (\alpha_4, \beta)$ such that $T_{\lambda_4}(\alpha_5) = L$ and $T'_{\lambda_4}(\alpha_5) \geq 0$. Then by Lemma 4.9, we see that $\lambda'_L(\alpha_5) \geq 0$. This is a contradiction. Thus $\lambda'_L(\alpha) > 0$ on $(\zeta, m_{L,\beta})$.

Case 2. Assume that $\lambda_L(\alpha)$ has a critical number. By Steps 1 and 2 and Lemma 4.9, the bifurcation curve S_L has exactly one turning point where this curve turns to the right for $L > \zeta$. Thus S_L is $\subset$-shaped for $L > \zeta$.

Thus by Cases 1 and 2, the bifurcation curve S_L is either monotone increasing or $\subset$-shaped for $L > \zeta$. The proof is complete. □

Proof of Theorem 2.4. For the problem (1.7), we see that

$$f(u) = u\left(k - u - \frac{1}{1+mu}\right) \text{ and } g(u) = k - u - \frac{1}{1+mu}.$$

Since $g(0^+) = k - 1$, we observe that $\beta_0 > 0$ if $0 < k < 1$, and $\beta_0 = 0$ if $k \geq 1$.

(I) Assume that $m > 1$. We consider three cases.

Case 1. Assume that $k \geq 1$. By Lemmas 4.10(i)(a) and 4.10, then (H_1), (H_2), (E_1) and (E_2) hold. So condition (i) in Theorem 2.2 holds. In this case, $\beta_0 = 0$. We compute and find that

$$\lim_{u\to 0^+} \frac{f(u)}{u} = k - 1 \begin{cases} = 0 & \text{if } k = 1, \\ > 0 & \text{if } k > 1, \end{cases} \quad \text{and} \quad \lim_{u\to 0^+} f''(u) = 2(m-1) > 0,$$

which implies that (C_1) holds if $k = 1$, and (C_3) holds if $k > 1$. So by Theorems 1.1 and 2.2, the bifurcation curve S_L of (1.7) is ⊂-shaped, starts from $(\bar{\kappa}_L, 0)$ and goes to $(\infty, m_{L,\beta_{k,m}})$ for $L > 0$. So Theorem 2.4(i)(a) holds.

Case 2. Assume that $k_m < k < 1$. By Lemmas 4.10(i)(a) and 4.10, then (H_1), (H_2), (E_1) and (E_2) hold. So condition (i) in Theorem 2.2 holds. In this case, $\beta_0 > 0$. By (4.31), then

$$F(\zeta) = \frac{6\ln(m\zeta + 1) - 2m^2\zeta^3 + 3km^2\zeta^2 - 6m\zeta}{6m^2} = 0,$$

from which it follows that (2.3) holds. We compute and find that

$$\lim_{u\to 0^+} \frac{f(u)}{u} = k - 1 \in (-\infty, 0),$$

which implies that (D_1) holds. So by Theorems 1.2, 2.1 and 2.2, the bifurcation curve S_L does not exist for $0 < L \leq \zeta$, and is ⊂-shaped, starts from (∞, ζ) and goes to $(\infty, m_{L,\beta_{k,m}})$ for $L > \zeta$. So Theorem 2.4(i)(b) holds.

Case 3. Assume that $0 < k \leq \tilde{k}_m$. By Lemma 4.10(i)(c), then $F(u) \leq 0$ for $u > 0$. It follows that $T_\lambda(\alpha)$ does not exist. So the bifurcation curve S_L does not exists for $L > 0$. Theorem 2.4(i)(c) holds.

(II) Assume that $0 < m \leq 1$. We consider two cases.

Case 1. Assume that $k > 1$. In this case, $\beta_0 = 0$. By Lemma 4.10(ii)(c) and Theorem 1.3(i), the bifurcation curve S_L is monotone increasing for $L > 0$. Since

$$\lim_{u\to 0^+} \frac{f(u)}{u} = k - 1 > 0,$$

and by Theorem 1.1, the bifurcation curve S_L starts from $(\bar{\kappa}_L, 0)$ and goes to $(\infty, m_{L,\beta_{k,m}})$ for $L > 0$. So Theorem 2.4(ii)(d) holds.

Case 2. Assume that $k \leq 1$. By Lemma 4.10(ii)(d), then $F(u) \leq 0$ for $u > 0$. It follows that the bifurcation curve S_L does not exists for $L > 0$. So Theorem 2.4(ii)(e) holds.

The proof is complete. □

Use of AI tools declaration

The authors declare that they have used generative AI tools (specifically ChatGPT by OpenAI) to assist with language editing and improving clarity. The final content was reviewed and approved by the authors, who take full responsibility for the integrity and accuracy of the manuscript.

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Conflict of interest

The authors declare there is no conflicts of interest.

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