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Research article

A Liouville-type theorem of a weighted semilinear parabolic equation on weighted manifolds with boundary

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Abstract: We derive a Liouville-type theorem for positive ancient solutions to a weighted semilinear parabolic equation with a Dirichlet boundary condition on complete noncompact weighted manifolds with a compact boundary. This result can be viewed as an extension of Dung et al.'s work on a linear heat equation.

Keywords: Liouville-type theorem; ancient solution; semilinear parabolic equation; weighted manifold; gradient estimate

1. Introduction

Let $(\hat{M}^n, g, e^{-h}dv)$ be a *weighted manifold*, which is in fact an n-dimensional complete Riemannian manifold endowed with a weighted measure $e^{-h}dv$, where h is a smooth function on $\hat{M}^n$ and dv is the volume element of the metric g. The associated weighted Laplacian is defined as $\triangle_h := \triangle - \nabla h \cdot \nabla$, where $\triangle$ and ∇, respectively, denote the rough Laplacian and the Levi-Civita connection. The Bakry-Émery Ricci curvature is given by $\mathrm{Ric}_h := \mathrm{Ric} + \mathrm{Hess}h$, where Ric and Hess, respectively, are the Ricci curvature of $\hat{M}^n$ and the Hessian operator with respect to g (see [1]).

Recently, many authors have devoted themselves to studying Liouville-type theorems of parabolic equations on weighted manifolds with or without a boundary, and there have been plenty of results obtained (see [2–4] and the references therein). For example, Wu [5] proved elliptic gradient estimates for positive solutions to the linear heat equation

$$u_t = \triangle_h u \tag{1.1}$$

on $(\hat{M}^n, g, e^{-h}dv)$ without a boundary and obtained Liouville theorems for positive ancient solutions (i.e., solutions defined in all space and negative time) satisfying some growth restriction near infinity.

Abolarinwa [6] showed Souplet-Zhang gradient estimates for positive solutions to the weighted semilinear parabolic equation

$$u_t = \triangle_h u + q(x,t)u^\alpha \tag{1.2}$$

on weighted manifolds without a boundary, where $\alpha \in \mathbb{R}$ and the function $q(x,t)$ is C^1 in x and C^0 in t. In particular, he obtained the following Liouville-type theorem.

Theorem A. *Let $(\hat{M}^n, g, e^{-h}dv)$ be a complete noncompact weighted manifold (without a boundary) with* $\mathrm{Ric}_h \geq 0$. *Assume that $q(x,t) = q(x) \neq 0$, that is, it is time-independent and satisfies $\|q^+\|_{L^\infty(B_\rho(x_0))} = o(\rho^{-(\alpha-1)})$ and $\|\nabla q\|_{L^\infty(B_\rho(x_0))} = o(\rho^{-(\alpha-1)})$ as $\rho \to \infty$. If u is a positive ancient solution to Eq* (1.2) *satisfying $u(x,t) = o(r^{\frac{1}{2}}(x) + |t|^{\frac{1}{4}})$ as $r(x) \to \infty$ and $t \to -\infty$, then u is a constant, where $q^+ =$* $\max\{q(x), 0\}$, *$B_\rho(x_0)$ is a geodesic ball with the center at x_0 and a radius ρ.*

For further generalization of the result above, see also [7] and a recent paper [8]. Inspired by the works of Kunikawa and Sakurai [9], Dung et al. [10] gave elliptic gradient estimates for positive solutions to Eq (1.1) on weighted manifolds with a compact boundary. They also showed the following result.

Theorem B. *Let $(\hat{M}^n, g, e^{-h}dv)$ be a complete noncompact weighted manifold with the compact boundary $\partial\hat{M}$,* $\mathrm{Ric}_h \geq 0$ *and* $\mathrm{H}_h \geq 0$. *Assume that u is a positive ancient solution to Eq* (1.1) *with a Dirichlet boundary condition (i.e., the solution is constant on the boundary). If $u_\eta \geq 0$, $u_t \leq 0$ over $\partial\hat{M} \times (-\infty, 0]$, and $u(x,t) = e^{o(r_{\partial\hat{M}}(x)+|t|)}$ (as $r_{\partial\hat{M}}(x) \to \infty$ and $t \to -\infty$, $o(r_{\partial\hat{M}}(x) + |t|)$ is an infinitely small quantity), then u must be constant.*

Here and below, H_h stands for the weighted mean curvature on $\partial\hat{M}$ and is defined as $\mathrm{H}_h := \mathrm{H} - \nabla h \cdot \eta$, where H denotes the mean curvature of the boundary $\partial\hat{M}$, η is the outer unit normal vector to $\partial\hat{M}$, u_η stands for the derivative of u along the direction η, and $r_{\partial\hat{M}}(x)$ is the distance function from the boundary.

In this paper, on weighted manifolds with a compact boundary, we study the Liouville property of positive ancient solutions to Eq (1.2). On the basis of Souplet-Zhang gradient estimates for positive solutions to Eq (1.2) with a Dirichlet boundary condition, we obtain the following Liouville-type theorem, in the spirit of Theorem 3.3 of Souplet and Zhang in [11].

Main theorem. *Let $(\hat{M}^n, g, e^{-h}dv)$ be a complete noncompact weighted manifold with the compact boundary $\partial\hat{M}$,* $\mathrm{Ric}_h \geq 0$ *and* $\mathrm{H}_h \geq 0$. *Assume that $q(x,t) = q(x) \neq 0$ satisfies $\|q^+\|_{L^\infty(B_\rho(\partial\hat{M}))} = o(\rho^{-(\alpha-1)})$ and $\|\nabla q\|_{L^\infty(B_\rho(\partial\hat{M}))} = o(\rho^{-(\alpha-1)})$ as $\rho \to \infty$. Let u be a positive ancient solution to Eq* (1.2) *with a Dirichlet boundary condition. If $u_\eta \geq 0$, $u_t \leq qu^\alpha$ over $\partial\hat{M} \times (-\infty, 0]$ and $u(x,t) = e^{o(r_{\partial\hat{M}}(x)+|t|)}$ (as $r_{\partial\hat{M}}(x) \to \infty$ and $t \to -\infty$, $o(r_{\partial\hat{M}}(x) + |t|)$ is an infinitely small quantity), then u must be constant, where $B_\rho(\partial\hat{M}) := \{x \in \hat{M}^n | d(x, \partial\hat{M}) < \rho\}$.*

Remark. *If $q(x,t) \equiv 0$, Eq* (1.2) *reduces to Eq* (1.1), *and hence our result generalizes the corresponding result of Dung et al. in [10]. When $q(x,t) \equiv -1$, $\alpha = 1$, and $\hat{M} = (-\infty, 0]$, $h = x$. It can be checked that $u = e^{x-t}$ is a positive ancient solution to Eq* (1.2), *where $u_\eta \geq 0$, $u_t \leq qu^\alpha$ over $\partial\hat{M} \times (-\infty, 0]$ and its growth rate is $e^{|x|+|t|}$. The example shows that our growth condition is necessary and sharp in both the spatial and time directions. Hence, it is better than the condition $u(x,t) = o(r^{\frac{1}{2}}(x) + |t|^{\frac{1}{4}})$ (as $r(x) \to \infty$ and $t \to -\infty$) used in [6].*

2. Basic lemmas

In this section, we present some definitions and results. On a weighted manifold $(\hat{M}^n, g, e^{-h}dv)$ with the compact boundary $\partial\hat{M}$, the distance function from the boundary is given by

$$r(x) := r_{\partial\hat{M}}(x) = d(x, \partial\hat{M}), \quad x \in \hat{M}^n.$$

This is a smooth function outside of the cut locus for the boundary (see [12]). We introduce the weighted Laplacian comparison theorem for the distance function on weighted manifolds with a boundary.

Lemma 2.1. [13] *Let $(\hat{M}^n, g, e^{-h}dv)$ be an n-dimensional weighted manifold with the compact boundary $\partial\hat{M}$. If $\mathrm{Ric}_h \geq -(n-1)K$ and $\mathrm{H}_h \geq -L$ for some non-negative constants K and L, then*

$$\triangle_h r(x) \leq (n-1)K\rho + L \tag{2.1}$$

for all $x \in B_\rho(\partial\hat{M})$.

We give the following useful derivative equality, which is called the Reilly's formula.

Lemma 2.2. [14] *Let φ be a smooth function on a weighted manifold $(\hat{M}^n, g, e^{-h}dv)$ with the compact boundary $\partial\hat{M}$. Then*

$$\begin{aligned}\frac{1}{2}(|\nabla\varphi|^2)_\eta =&\varphi_\eta[\triangle_h\varphi - \triangle_{\partial\hat{M},h}(\varphi|_{\partial\hat{M}}) - \varphi_\eta \mathrm{H}_h] + g_{\partial\hat{M}}(\nabla_{\partial\hat{M}}(\varphi|_{\partial\hat{M}}), \nabla_{\partial\hat{M}}\varphi_\eta)\\ &- \Pi(\nabla_{\partial\hat{M}}(\varphi|_{\partial\hat{M}}), \nabla_{\partial\hat{M}}(\varphi|_{\partial\hat{M}})),\end{aligned} \tag{2.2}$$

where Π is the second fundamental form of $\partial\hat{M}$.

Next, we introduce a smooth cut-off function originally developed by Li-Yau. It is very useful in the proof of elliptic gradient estimates.

Lemma 2.3. [15] Let $(\hat{M}^n, g, e^{-h}dv)$ be an n-dimensional weighted manifold with the compact boundary $\partial\hat{M}$. A smooth cut-off function $\psi = \psi(x,t)$ supported in $Q_{\rho,T}(\partial\hat{M}) := B_\rho(\partial\hat{M}) \times [-T, 0]$ exists such that

(i) $\psi = \psi(r_{\partial\hat{M}}(x), t) \equiv \psi(r,t)$; $\psi(r,t) = 1$ in $Q_{\rho/2,T/2}(\partial\hat{M})$, $0 \leq \psi \leq 1$;

(ii) ψ is decreasing as a radial function in the spatial variables, and $\psi_r = 0$ in $Q_{\rho/2,T}(\partial\hat{M})$;

(iii) $|\psi_t| \leq \frac{C\psi^{1/2}}{T}$, $|\psi_r| \leq \frac{C_\epsilon\psi^\epsilon}{\rho}$ and $|\psi_{rr}| \leq \frac{C_\epsilon\psi^\epsilon}{\rho^2}$, where $C > 0$ is a universal constant and $C_\epsilon > 0$ is a constant depending only on $0 < \epsilon < 1$.

According to Souplet and Zhang's idea in [16], by introducing the auxiliary function $\sqrt{1 + \log(N/u)}$ instead of $\log(u/N)$ used in [6], we prove a derivative inequality, which plays an important role in the proof of Proposition 3.1.

Lemma 2.4. *Let u be a smooth solution to Eq* (1.2) *and $0 < u \leq N$ for some constant N. Let $v = \sqrt{\log(P/u)}$, where $P = Ne$ and $\omega = |\nabla v|^2$. We then have*

$$\begin{aligned}\triangle_h\omega - \omega_t \geq& 2(v^{-2} + 2)\omega^2 - [2(\alpha-1) + v^{-2}]qu^{\alpha-1}\omega + 2\mathrm{Ric}_h(\nabla v, \nabla v)\\ &+ 2(2v - v^{-1})\langle\nabla\omega, \nabla v\rangle + v^{-1}u^{\alpha-1}\langle\nabla q, \nabla v\rangle.\end{aligned} \tag{2.3}$$

Proof. Since $u = Pe^{-v^2}$, we compute

$$u_t = -2Pve^{-v^2}v_t = -2uvv_t \tag{2.4}$$

and

$$\nabla u = -2Pve^{-v^2}\nabla v = -2uv\nabla v.$$

Further, we get

$$\begin{aligned}\triangle u =& \nabla\nabla u\\ =& -2u|\nabla v|^2 - 2v\langle\nabla u, \nabla v\rangle - 2uv\triangle v\\ =& -2u|\nabla v|^2 + 4uv^2|\nabla v|^2 - 2uv\triangle v,\end{aligned}$$

hence

$$\begin{aligned}\triangle_h u =& \triangle u - \langle\nabla u, \nabla h\rangle\\ =& -2uv\triangle_h v - 2u|\nabla v|^2 + 4uv^2|\nabla v|^2.\end{aligned} \tag{2.5}$$

If we substitute (2.4) and (2.5) into Eq (1.2), it follows that

$$v_t = \triangle_h v - (2v - v^{-1})|\nabla v|^2 - \frac{1}{2}qv^{-1}u^{\alpha-1}. \tag{2.6}$$

Using the Bochner formula (see [5]) for ω, we have

$$\begin{aligned}\triangle_h\omega =& \triangle_h|\nabla v|^2\\ =& 2|\mathrm{Hess}v|^2 + 2\langle\nabla\triangle_h v, \nabla v\rangle + 2\mathrm{Ric}_h(\nabla v, \nabla v)\\ \geq& 2\langle\nabla\triangle_h v, \nabla v\rangle + 2\mathrm{Ric}_h(\nabla v, \nabla v).\end{aligned}$$

Hence

$$\begin{aligned}\triangle_h\omega - \omega_t \geq& 2\langle\nabla\triangle_h v, \nabla v\rangle + 2\mathrm{Ric}_h(\nabla v, \nabla v) - \omega_t\\ \geq& 2\langle\nabla(v_t + (2v - v^{-1})|\nabla v|^2 + \frac{1}{2}qv^{-1}u^{\alpha-1}), \nabla v\rangle\\ &+ 2\mathrm{Ric}_h(\nabla v, \nabla v) - \omega_t,\end{aligned}$$

where we used (2.6) in the second inequality.

A direct computation shows that

$$\omega_t = (|\nabla v|^2)_t = 2\langle\nabla v_t, \nabla v\rangle,$$

$$\begin{aligned}\nabla[(2v - v^{-1})|\nabla v|^2] =& (2 + v^{-2})|\nabla v|^2\nabla v + (2v - v^{-1})\nabla|\nabla v|^2\\ =& (2 + v^{-2})\omega\nabla v + (2v - v^{-1})\nabla\omega\end{aligned}$$

and

$$\begin{aligned}\nabla(qv^{-1}u^{\alpha-1}) =& v^{-1}u^{\alpha-1}\nabla q + (\alpha - 1)qv^{-1}u^{\alpha-2}\nabla u - qv^{-2}u^{\alpha-1}\nabla v\\ =& v^{-1}u^{\alpha-1}\nabla q - 2(\alpha - 1)qu^{\alpha-1}\nabla v - qv^{-2}u^{\alpha-1}\nabla v.\end{aligned}$$

We then arrive at

$$\begin{aligned}\triangle_h\omega - \omega_t \geq& 2(2 + v^{-2})\omega^2 + 2(2v - v^{-1})\langle\nabla\omega, \nabla v\rangle\\ &+ v^{-1}u^{\alpha-1}\langle\nabla q, \nabla v\rangle - 2(\alpha - 1)qu^{\alpha-1}\omega\\ &- qv^{-2}u^{\alpha-1}\omega + 2\mathrm{Ric}_h(\nabla v, \nabla v),\end{aligned}$$

which is the desired inequality (2.3).

This completes the proof of Lemma 2.4. □

3. Elliptic gradient estimates

In this section, on the basis of the key derivative inequality, by applying maximum principle, we establish Souplet-Zhang gradient estimates for positive solutions to Eq (1.2) with a Dirichlet boundary condition. In particular, we need to use Reilly's formula to deal with the boundary case. In fact, we obtain the following result.

Proposition 3.1. Let $(\hat{M}^n, g, e^{-h}dv)$ be an n-dimensional weighted manifold with the compact boundary $\partial\hat{M}$. Assume that $\mathrm{Ric}_h \geq -(n-1)K$ and $\mathrm{H}_h \geq -L$. Here $K \geq 0$, $L \geq 0$ and $N > 0$ are some constants. Let $u \leq N$ be a positive solution to Eq (1.2) with a Dirichlet boundary condition on $Q_{\rho,T}(\partial\hat{M})$. If $u_\eta \geq 0$ and $u_t \leq qu^\alpha$ over $\partial\hat{M} \times [-T, 0]$, then a constant C depending on n and α exists such that the following estimates hold.

(i) If $\alpha > 1$, then

$$\begin{aligned}\sup_{Q_{\rho/2,T/2}(\partial\hat{M})} \frac{|\nabla u|}{u} \leq & C\Big(\frac{1+\sqrt{D}}{\rho} + \sqrt{K} + L + \sqrt{\alpha}N^{\frac{1}{2}(\alpha-1)}\|q\|^{\frac{1}{2}}_{L^\infty(Q_{\rho,T}(\partial\hat{M}))} \\ & + \frac{1}{\sqrt{T}} + N^{\frac{1}{3}(\alpha-1)}\|\nabla q\|^{\frac{1}{3}}_{L^\infty(Q_{\rho,T}(\partial\hat{M}))}\Big)\sqrt{1+\log\frac{N}{u}},\end{aligned} \tag{3.1}$$

where $D = 1 + \log N - \log(\inf_{Q_{\rho,T}(\partial\hat{M})} u)$.

(ii) If $\alpha \leq 1$, then

$$\begin{aligned}\sup_{Q_{\rho/2,T/2}(\partial\hat{M})} \frac{|\nabla u|}{u} \leq & C\Big(\frac{1+\sqrt{D}}{\rho} + \sqrt{K} + L + \bar{N}^{\frac{1}{2}(\alpha-1)}\|q\|^{\frac{1}{2}}_{L^\infty(Q_{\rho,T}(\partial\hat{M}))} \\ & + \frac{1}{\sqrt{T}} + \bar{N}^{\frac{1}{3}(\alpha-1)}\|\nabla q\|^{\frac{1}{3}}_{L^\infty(Q_{\rho,T}(\partial\hat{M}))}\Big)\sqrt{1+\log\frac{N}{u}},\end{aligned} \tag{3.2}$$

where $\bar{N} = \inf\{u(x,t) | (x,t) \in Q_{\rho,T}(\partial\hat{M})\}$.

Proof. Let $\psi\omega$ reach its maximum at $(x_1, t_1) \in Q_{\rho/2,T/2}(\partial\hat{M})$, where ψ denotes the cut-off function in Lemma 2.3 and ω is the function in Lemma 2.4. We divide the arguments into two cases.

Case 1. When $x_1 \notin \partial\hat{M}$, without loss of generality, we may assume that $x_1 \notin \mathrm{Cut}(\partial\hat{M})$ by Calabi's argument [17]. At (x_1, t_1), we know that

$$\triangle_h(\psi\omega) \leq 0, \qquad (\psi\omega)_t \geq 0,$$

and

$$\nabla(\psi\omega) = 0.$$

That is

$$\nabla\omega = -\frac{\omega}{\psi}\nabla\psi.$$

A direct computation shows that

$$\triangle_h(\psi\omega) - (\psi\omega)_t = \psi(\triangle_h\omega - \omega_t) + \omega(\triangle_h\psi - \psi_t) + 2\langle\nabla\omega, \nabla\psi\rangle. \tag{3.3}$$

Combining (2.3) with (3.3), and using the condition of $\mathrm{Ric}_h \geq -(n-1)K$, then at (x_1, t_1), we get

$$
\begin{aligned}
2(v^{-2}+2)\psi\omega^2 \leq & 2(2v - v^{-1})\omega\langle \nabla\psi, \nabla v\rangle - v^{-1}u^{\alpha-1}\psi\langle \nabla q, \nabla v\rangle \\
& + [2(\alpha-1) + v^{-2}]qu^{\alpha-1}\psi\omega + 2(n-1)K\psi\omega \\
& - \omega(\triangle_h\psi - \psi_t) + \frac{2\omega}{\psi}|\nabla\psi|^2.
\end{aligned}
$$

Since $0 < \frac{2v^2}{2v^2+1} \leq 1$, $0 < \frac{1}{2v^2+1} \leq 1$, and $0 < \frac{v}{2v^2+1} \leq \frac{\sqrt{2}}{4}$, then

$$
\begin{aligned}
2\psi\omega^2 \leq & -\frac{v^2}{2v^2+1}\omega(\triangle_h\psi - \psi_t) + (n-1)K\psi\omega + \frac{\omega}{\psi}|\nabla\psi|^2 \\
& + \frac{2v(2v^2-1)}{2v^2+1}\langle \nabla\psi, \nabla v\rangle\omega + \frac{\sqrt{2}}{4}u^{\alpha-1}\psi|\nabla q||\nabla v| \\
& + |(\alpha-1)q|u^{\alpha-1}\psi\omega + |q|u^{\alpha-1}\psi\omega.
\end{aligned} \tag{3.4}
$$

Next, we estimate every term on the right-hand side of (3.4) at (x_1, t_1).

$$
\begin{aligned}
-\frac{v^2}{2v^2+1}\omega\triangle_h\psi = & -\frac{v^2}{2v^2+1}(\psi_r\triangle_h r + \psi_{rr}|\nabla r|^2)\omega \\
\leq & \frac{(n-1)K\rho + L}{2}\omega|\psi_r| + \frac{1}{2}\omega|\psi_{rr}| \\
\leq & \frac{|\psi_{rr}|}{2\psi^{1/2}}\psi^{1/2}\omega + \frac{(n-1)K\rho + L}{2}\psi^{1/2}\omega\frac{|\psi_r|}{\psi^{1/2}} \\
\leq & \frac{1}{7}\psi\omega^2 + C\left[\left(\frac{|\psi_{rr}|}{\psi^{1/2}}\right)^2 + (K^2\rho^2 + L^2)\left(\frac{|\psi_r|}{\psi^{1/2}}\right)^2\right] \\
\leq & \frac{1}{7}\psi\omega^2 + \frac{C}{\rho^4} + CK^2 + \frac{CL^2}{\rho^2} \\
\leq & \frac{1}{7}\psi\omega^2 + \frac{C}{\rho^4} + CK^2 + CL^4,
\end{aligned} \tag{3.5}
$$

where we used (2.1) in the first inequality.

By the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
\frac{v^2}{2v^2+1}\omega\psi_t \leq & \frac{1}{2}\omega|\psi_t| = \psi^{1/2}\omega\frac{|\psi_t|}{2\psi^{1/2}} \\
\leq & \frac{1}{7}(\psi^{1/2}\omega)^2 + C\left(\frac{|\psi_t|}{\psi^{1/2}}\right)^2 \\
\leq & \frac{1}{7}\psi\omega^2 + \frac{C}{T^2},
\end{aligned} \tag{3.6}
$$

$$
(n-1)K\psi\omega \leq \frac{1}{7}\psi\omega^2 + CK^2, \tag{3.7}
$$

and

$$\begin{aligned}\frac{|\nabla\psi|^2}{\psi}\omega &= (\psi^{1/2}\omega)\Big(\frac{|\nabla\psi|^2}{\psi^{3/2}}\Big)\\ &\leq \frac{1}{7}\psi\omega^2 + C\frac{|\nabla\psi|^4}{\psi^3}\\ &\leq \frac{1}{7}\psi\omega^2 + \frac{C}{\rho^4}.\end{aligned} \tag{3.8}$$

By the Young's inequality, we get

$$\begin{aligned}\frac{2v(2v^2-1)}{2v^2+1}\langle\nabla v, \nabla\psi\rangle\omega &\leq 2v\frac{|2v^2-1|}{2v^2+1}|\nabla\psi||\nabla v|\omega\\ &\leq 2v|\nabla\psi|\omega^{3/2}\\ &= 2v\Big(\frac{|\nabla\psi|}{\psi^{3/4}}\Big)(\psi\omega^2)^{3/4}\\ &\leq \frac{1}{7}\psi\omega^2 + Cv^4\frac{|\nabla\psi|^4}{\psi^3}\\ &\leq \frac{1}{7}\psi\omega^2 + \frac{CD^2}{\rho^4},\end{aligned} \tag{3.9}$$

where $D = 1 + \log N - \log(\inf_{Q_{\rho,T}(\partial\hat{M})} u)$.

We now estimate the terms that contain the parameter α and divide the arguments into two cases.

Case (i). If $\alpha > 1$, then by using the Young's inequality

$$\begin{aligned}\frac{\sqrt{2}}{4}u^{\alpha-1}\psi|\nabla q||\nabla v| &= \frac{\sqrt{2}}{4}u^{\alpha-1}|\nabla q|\psi\omega^{1/2}\\ &\leq \frac{\sqrt{2}}{4}N^{\alpha-1}|\nabla q|\psi\omega^{1/2}\\ &= \frac{\sqrt{2}}{4}(\psi^{1/4}\omega^{1/2})(|\nabla q|\psi^{3/4}N^{\alpha-1})\\ &\leq \frac{1}{7}(\psi^{1/4}\omega^{1/2})^4 + C(|\nabla q|\psi^{3/4}N^{\alpha-1})^{4/3}\\ &\leq \frac{1}{7}\psi\omega^2 + C|\nabla q|^{4/3}N^{4(\alpha-1)/3},\end{aligned} \tag{3.10}$$

and the Cauchy-Schwarz inequality

$$\begin{aligned}|(\alpha-1)q|u^{\alpha-1}\psi\omega + |q|u^{\alpha-1}\psi\omega &\leq \alpha|q|u^{\alpha-1}\psi\omega\\ &\leq (\psi^{1/2}\omega)(\alpha|q|N^{\alpha-1}\psi^{1/2})\\ &\leq \frac{1}{7}\psi\omega^2 + C\alpha^2q^2N^{2(\alpha-1)}.\end{aligned} \tag{3.11}$$

Combining (3.5)–(3.9) with (3.10) and (3.11), for all $(x, t) \in Q_{\rho,T}(\partial\hat{M})$, we have

$$\begin{aligned}\psi\omega^2(x,t) \leq& \psi\omega^2(x_1,t_1)\\ \leq& C\Big(\frac{1+D^2}{\rho^2} + K^2 + L^4 + \alpha^2 N^{2(\alpha-1)}\|q\|^2_{L^\infty(Q_{\rho,T}(\partial\hat{M}))}\\ &+ \frac{1}{T^2} + N^{4(\alpha-1)/3}\|\nabla q\|^{4/3}_{L^\infty(Q_{\rho,T}(\partial\hat{M}))}\Big).\end{aligned}$$

Noting that $\psi(x,t) = 1$ in $Q_{\rho/2,T/2}(\partial\hat{M})$ and $\omega = |\nabla v|^2$, where $v = \sqrt{\log(Ne/u)}$. It follows that

$$\begin{aligned}\frac{|\nabla u|}{u} \leq& C\Big(\frac{1+\sqrt{D}}{\rho} + \sqrt{K} + L + \sqrt{\alpha} N^{(\alpha-1)/2}\|q\|^{1/2}_{L^\infty(Q_{\rho,T}(\partial\hat{M}))}\\ &+ \frac{1}{\sqrt{T}} + N^{(\alpha-1)/3}\|\nabla q\|^{1/3}_{L^\infty(Q_{\rho,T}(\partial\hat{M}))}\Big)\sqrt{1+\log\frac{N}{u}}.\end{aligned}$$

Namely, we get the desired estimate (3.1).

Case (ii). If $\alpha \leq 1$, then by using the Young's inequality

$$\begin{aligned}\frac{\sqrt{2}}{4}u^{\alpha-1}\psi|\nabla q||\nabla v| \leq& \frac{\sqrt{2}}{4}\bar{N}^{\alpha-1}|\nabla q|\psi\omega^{1/2}\\ =& \frac{\sqrt{2}}{4}(\psi^{1/4}\omega^{1/2})(|\nabla q|\psi^{3/4}\bar{N}^{\alpha-1})\\ \leq& \frac{1}{7}(\psi^{1/4}\omega^{1/2})^4 + C(|\nabla q|\psi^{3/4}\bar{N}^{\alpha-1})^{4/3}\\ \leq& \frac{1}{7}\psi\omega^2 + C|\nabla q|^{4/3}\bar{N}^{4(\alpha-1)/3},\end{aligned} \tag{3.12}$$

where $\bar{N} = \inf\{u(x,t)|(x,t) \in Q_{\rho,T}(\partial\hat{M})\}$.

Using the Cauchy-Schwarz inequality

$$\begin{aligned}|(\alpha-1)q|u^{\alpha-1}\psi\omega + |q|u^{\alpha-1}\psi\omega \leq& (2-\alpha)|q|u^{\alpha-1}\psi\omega\\ \leq& C(\psi^{1/2}\omega)(|q|\bar{N}^{\alpha-1}\psi^{1/2})\\ \leq& \frac{1}{7}\psi\omega^2 + Cq^2\bar{N}^{2(\alpha-1)}.\end{aligned} \tag{3.13}$$

Combining (3.5)–(3.9) with (3.12) and (3.13) for all $(x, t) \in Q_{\rho,T}(\partial\hat{M})$, we get

$$\begin{aligned}\psi\omega^2(x,t) \leq& \psi\omega^2(x_1,t_1)\\ \leq& C\Big(\frac{1+D^2}{\rho^2} + K^2 + L^4 + \bar{N}^{2(\alpha-1)}\|q\|^2_{L^\infty(Q_{\rho,T}(\partial\hat{M}))}\\ &+ \frac{1}{T^2} + \bar{N}^{4(\alpha-1)/3}\|\nabla q\|^{4/3}_{L^\infty(Q_{\rho,T}(\partial\hat{M}))}\Big).\end{aligned}$$

Using $\psi(x,t) = 1$ in $Q_{\rho/2,T/2}(\partial\hat{M})$ and the definition of ω, we obtain

$$\begin{aligned}\frac{|\nabla u|}{u} \leq& C\Big(\frac{1+\sqrt{D}}{\rho} + \sqrt{K} + L + \bar{N}^{(\alpha-1)/2}\|q\|^{1/2}_{L^\infty(Q_{\rho,T}(\partial\hat{M}))}\\ &+ \frac{1}{\sqrt{T}} + \bar{N}^{(\alpha-1)/3}\|\nabla q\|^{1/3}_{L^\infty(Q_{\rho,T}(\partial\hat{M}))}\Big)\sqrt{1+\log\frac{N}{u}},\end{aligned}$$

which is the desired estimate (3.2).

Case 2. When $x_1 \in \partial\hat{M}$, we only consider the case $\alpha > 1$ because $\alpha \leq 1$ is similar. In the case, the estimate (3.1) still holds. Moreover, at (x_1, t_1), we get

$$(\psi\omega)_\eta \geq 0,$$

namely,

$$\psi_\eta\omega + \psi\omega_\eta = \psi\omega_\eta \geq 0.$$

Hence

$$\omega_\eta \geq 0.$$

Since $\omega = |\nabla v|^2$, where $v = \sqrt{\log(P/u)}$, by assumption, we know that ω also satisfies the Dirichlet boundary condition. It follows from Lemma 2.2 that

$$0 \leq \omega_\eta = (|\nabla v|^2)_\eta = 2v_\eta(\triangle_h v - \mathrm{H}_h v_\eta). \tag{3.14}$$

Because u satisfies the Dirichlet boundary condition, then

$$|\nabla u| = u_\eta.$$

Since

$$\nabla v = -\frac{\nabla u}{2u\sqrt{\log(P/u)}},$$

$$v_\eta = -\frac{1}{2u\sqrt{\log(P/u)}}u_\eta = -\frac{|\nabla u|}{2u\sqrt{\log(P/u)}} = -\omega^{1/2}.$$

We directly compute

$$\begin{aligned}
\triangle_h v =& \triangle v - \langle\nabla v, \nabla h\rangle \\
=& -\nabla\Big(\frac{\nabla u}{2u\sqrt{\log(P/u)}}\Big) - \langle\nabla v, \nabla h\rangle \\
=& -\frac{\triangle_h u}{2u\sqrt{\log(P/u)}} + \frac{|\nabla u|^2}{2u^2\sqrt{\log(P/u)}} - \frac{|\nabla u|^2}{4u^2(\log(P/u))^{3/2}} \\
=& \frac{1}{2uv}(-u_t + qu^\alpha) + \Big(\frac{2v^2-1}{v}\Big)\omega,
\end{aligned} \tag{3.15}$$

where we used (1.2) in the fourth equality.

Substituting (3.15) into (3.14), we arrive at

$$0 \leq -2\omega^{1/2}\Big[\frac{1}{2uv}(-u_t + qu^\alpha) + (2v - v^{-1})\omega + \omega^{1/2}\mathrm{H}_h\Big],$$

that is

$$\frac{1}{2uv}(-u_t + qu^\alpha) + (2v - v^{-1})\omega + \omega^{1/2}\mathrm{H}_h \leq 0 \tag{3.16}$$

at (x_1, t_1). The condition that $u_t \leq qu^\alpha$ over $\partial\hat{M} \times [-T, 0]$ yields

$$\frac{1}{2uv}(-u_t + qu^\alpha) \geq 0.$$

It follows from (3.16) that

$$(2v - v^{-1})\omega + \omega^{1/2}\mathrm{H}_h \leq 0.$$

Since $v \geq 1$, $2v - v^{-1} \geq 1$, and we get

$$\omega + \mathrm{H}_h\omega^{1/2} \leq 0$$

at (x_1, t_1), which implies

$$\omega(x_1, t_1) = 0 \tag{3.17}$$

or

$$\omega^{1/2}(x_1, t_1) \leq L \tag{3.18}$$

on $Q_{\rho,T}(\partial\hat{M})$, where we used the condition of $\mathrm{H}_h \geq -L$.

If (3.17) holds, then u is constant and the conclusion follows.

If (3.18) holds, then for all $(x, t) \in Q_{\rho/2,T/2}(\partial\hat{M})$, $\psi(x, t) = 1$, and we have

$$|\nabla v|^2(x, t) = \omega(x, t) = \psi(x, t)\omega(x, t) \leq \psi(x_1, t_1)\omega(x_1, t_1) \leq L^2.$$

It also implies the conclusion by using

$$|\nabla v| = \frac{|\nabla u|}{2u\sqrt{\log(P/u)}}.$$

We complete the proof of Proposition 3.1. □

4. Proof of the main theorem

In this section, applying the Souplet-Zhang gradient estimates for positive solutions to Eq (1.2) with a Dirichlet boundary condition, we complete the proof of the main theorem.

Proof. We only consider the case $\alpha \leq 1$ because $\alpha \geq 1$ is similar. The arguments can be divided into two cases.

Case 1. When $\alpha < 1$, by the estimate (3.2) in Proposition 3.1 for $K = L = 0$, we know that

$$\begin{aligned}\frac{|\nabla u(x,t)|}{u(x,t)} \leq & C\Big(\frac{1+\sqrt{D}}{\rho} + \frac{1}{\sqrt{T}} + \bar{N}^{\frac{1}{2}(\alpha-1)}\|q\|^{\frac{1}{2}}_{L^\infty(Q_{\rho,T}(\partial\hat{M}))} \\ & + \bar{N}^{\frac{1}{3}(\alpha-1)}\|\nabla q\|^{\frac{1}{3}}_{L^\infty(Q_{\rho,T}(\partial\hat{M}))}\Big)\sqrt{1+\log\frac{N}{u}}\end{aligned} \tag{4.1}$$

for all $(x, t) \in Q_{\rho/2,T/2}(\partial\hat{M})$.

Fixing (x_0, t_0) and using (4.1) to u on $Q_{\rho,\rho}(\partial\hat{M}) := B_\rho(\partial\hat{M})\times[t_0-\rho, t_0]$ and the assumption conditions, we get

$$\frac{|\nabla u(x_0,t_0)|}{u(x_0,t_0)} \leq C\Big(\frac{\sqrt{o(\rho+|\rho|)}}{\rho}+\frac{1}{\sqrt{\rho}}+o(\rho^{\frac{1}{2}(\alpha-1)})o(\rho^{-\frac{1}{2}(\alpha-1)}) + o(\rho^{\frac{1}{3}(\alpha-1)})o(\rho^{-\frac{1}{3}(\alpha-1)})\Big)\sqrt{o(\rho+|\rho|)-\log(u(x_0,t_0))}.$$

Letting $\rho \to \infty$, we have $|\nabla u(x_0, t_0)| = 0$. Because (x_0, t_0) is arbitrary, $\nabla u(x, t) \equiv 0$ and u must be constant in space, namely, $u(x, t) = u(t)$. We now prove $u(t)$ is a constant by contradiction.

Let $\tilde{q} := q(x)$, and thus by Eq (1.2), we have

$$\frac{du(t)}{dt} = \tilde{q}u^\alpha(t). \tag{4.2}$$

Integrating (4.2) in the interal $(t, 0]$ with $t < 0$, we get

$$u^{1-\alpha}(t) = u^{1-\alpha}(0) + (1-\alpha)\tilde{q}t.$$

Using the condition of $\tilde{q} > 0$ and letting $t \to -\infty$, we have $u^{1-\alpha}(t) < 0$, which is impossible, since u is a positive solution. Hence $\tilde{q} = 0$ and $u(x, t)$ is a constant.

Case 2. When $\alpha = 1$, by the same arguments as in Case 1, we easily find that u must be constant in space, namely, $u(x, t) = u(t)$. Similarly, we have

$$\log u(0) - \log u(t) \leq -\tilde{q}t$$

for all $t < 0$.

Hence

$$u(t) \geq u(0)e^{\tilde{q}t},$$

which is a contradiction to the condition that $u(x, t) = e^{o(r_{\partial\hat{M}}(x)+|t|)}$ near infinity.

This proof is completed. □

5. Conclusions

In this paper, we prove a Liouville-type theorem for positive ancient solutions to a weighted semilinear parabolic equation with a Dirichlet boundary condition on complete noncompact weighted manifolds with a compact boundary. The proof technique is based on Souplet-Zhang gradient estimates for positive solutions. This result can be viewed as an extension of Dung et al.'s [10] work on a linear heat equation.

The weight of the equation is expressed by a smooth function $q(x, t)$ in this paper. But for technical reasons, a Liouville-type theorem is obtained in the subcase $q(x)$ (i.e., it is time-independent). A natural question is whether there is a similar Liouville-type theorem when the weight is $q(x, t)$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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